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THE POPULARIZATION OF MATHEMATICS

by A. G. HOWSON, J.-P. KAHANE and H. POLLAK

Unlike other sciences, mathematics, or at least some parts of it, is taught to all schoolchildren; it is this which makes mathematics teaching and mathematics education so important. On the other hand there are few, if any, sciences which arouse such negative reactions or are as badly understood as mathematics. Most people, for example, would not even consider mathematics to be a living science. This study will be concerned with the public image of mathematics and mathematicians. It will seek to identify specific needs and to suggest ways in which mathematics can be more effectively popularized. Some of these needs and ways are not particular to mathematics;they also concern the popularization of any individual science, or of sciences in general. However, the popularization of mathematics has special features; obstacles, constraints and difficulties on the one hand, important possibilities and opportunities on the other. The present situation and record of past achievements differ from country to country and there is a need for international discussion in order to compare experiences, to clarify issues, and to promote further reflections and actions. This study, then, marks an important step forward, for it is intended that there should be a major gathering of those interested (in Leeds, England from 17-22 September, 1989) and that this should be accompanied by a nationally-organised, yet international, "event" comprising a major exhibition, films, videos and lectures.

The Leeds meeting, therefore, has two aspects: a national event and an international study. Each aspect will benefit from the other and the planning of the two will be closely coordinated. The present discussion document is the first contribution to the international study. We hope that, like discussion documents issued in connection with previous ICMI studies, it will stimulate written

contributions from all over the world. Such contributions, together with the present document, will form the basis for reports and discussion in Leeds. The resulting *Proceedings* of the meeting will then be published as ICMI Study 5.

1. A GENERAL FRAMEWORK: NEEDS AND METHODS FOR THE POPULARIZATION OF SCIENCE

Let us begin by making a few simple observations.

Advances in science and the day-to-day lives of humans are indirectly but, nevertheless, intimately connected. Strategical choices by states relating to economic, military and environmental matters, are fashioned by changes in technology and give rise to new technological challenges. Chains of relationships are then built up affecting all types of employment, the environment, public health, communications, home and family life, ... An informed citizen, whatever his or her occupation, should have some understanding of the crucial points on which these strategical choices are taken, some knowledge of the scientific advances appertaining to the technologies under consideration. Such a general scientific understanding is a democratic and economic need in every modern society and the provision of it may well be one of the decisive social challenges in the future.

However, there is now an increasing divergence between the advancement of science and the general understanding of the vast majority of human beings. Though science is universal and should help promote unity amongst people, we see that scientific research and scientific education may actually be organised in ways which increase inequalities and frustrations. Although scientific concepts are involved in every modern device used in everyday life, too many people are unable to grasp scientific ideas, do not know what a scientific way of thinking is, and, as a result, are too frequently pushed into irrational modes of thought. Even those who were well educated and equipped with some scientific knowledge all too often lack the time and incentive to enlarge their scientific understanding and to keep abreast of modern developments.

This, then, is the situation to which those involved with the popularization of science must respond. On the one hand there is an exponential increase of scientific knowledge produced by, and circulating amongst, small groups of specialists. On the other hand, there is a general, social need for a popular understanding of scientific discoveries, scientific achievements, scientific ideas, and scientific modes of thought. Any efforts to bridge that gap are part of popularization in its widest sense. In a more restricted sense, and that which this study will consider, the popularization of science involves all efforts made,

or which might be made, to bridge the gap between scientific advances and public knowledge and information, apart from those which take place within school systems and in higher education.

The process of popularization involves three factors: the topics to be considered, the sections of the public it is wished to interest in the topics, and the media to be used in the processes of communication. To help in the making of consequent choices there will be a clear need to identify specific aims and criteria for decision-making.

No topic should be excluded *a priori*. Whenever there is a real advance in science it has to be known outside the small circle of specialists which participated in that advance — or it risks becoming lost. Any effort to make it known, to explain its meaning to a wider audience, is part of the process of popularization which can take place at a number of levels. At the highest level, the dissemination of advanced topics (through, say, expository papers) is an extreme, but an essential, stage in the general process. Yet there are many other topics of interest apart from contemporary research: for example, the history of a subject, its applications (particularly any of a novel character) and an understanding of the type of people involved in that science and of their motivation.

Similarly, no section of the public should be excluded. Children of all ages, workers, citizens, all types of professionals, even other scientists. All motivations have to be considered: professional interest, curiosity, general knowledge, ... , but also prejudices and fears.

All channels, too, must be exploited: books, newspapers, periodicals, films, exhibits, TV and radio programmes, software, ... Education and continuing education will play a decisive rôle complementary to that of popularization. Games and competitions will have a part to play — particularly in mathematics. Whatever the medium, popularization will be analogous to translation, and its quality will depend upon the skills and experience of the translator. Some of these are professionals: scientific writers and journalists. These may well have a catalytic rôle to play in involving scientists, teachers and other professionals in the general process of popularization.

2. Special features of the popularization of mathematics

The popularization of mathematics gives rise to certain special problems. First, many people's relationship to mathematics is governed by what happened to them in school. The affective consequences were often considerable: love,

interest, dislike, hatred and, all too often, fear. It has to do with success in school mathematics and with the common belief that mathematics needs a special kind of mind and attracts only those of a particular disposition.

Mathematicians may reinforce this belief, either by refusing to participate in the subject's popularization, or by the way in which they behave or explain things to laymen.

"Mark all Mathematical heads which be wholly and only bent on these sciences, how solitary they be themselves, how unfit to live with others, how unapt to serve the world".

This view of mathematicians, expressed by Roger Ascham, 16th century educator and tutor to Queen Elizabeth I of England, is one which is echoed in many later writings. Blaise Pascal, who was himself intimately concerned with mathematics, used to contrast "esprit de géométrie" (a mathematical mind) with "esprit de finesse" (an accurate mind). The latter was an attribute of "honnêtes gens" (nobility and the high bourgeoisie), whereas the former was poorly regarded. This contrast has been a favourite theme for dissertations in French high schools, and has contributed to the view of mathematicians as strange characters, divorced from the real world.

Mathematicians may well reinforce this view when they speak or write about themselves and the mathematical world. As H. E. Robbins, himself a noted popularizer, puts it in his review of Ulam's *Adventures of a Mathematician*: if mathematicians appear as "thinking machines on the make, without discernible relation to parents, spouses or children, and oblivious to the human concerns of our times, ... if mathematical intelligence is strongly associated with emotional deprivation and social alienation, then ... we ... are in for trouble".

Let us raise a few questions for discussion. What is the popular view of a mathematician? To what extent does that view influence both the wish to study mathematics, or, should the possibility arise, to support mathematicians in their work? To what extent do books or films about mathematics or mathematicians reinforce unfortunate beliefs?

Given the importance of the affective relation of individuals with mathematics, can we agree that one purpose of popularization must be to create a favourable mental association with mathematics whenever and wherever it might arise?

A second special feature of mathematics which hinders popularization is the kind of topics on which mathematicians work.

Even the most abstract parts of physics or biology have a direct connection with some practically important and familiar subject, such as energy, space, the environment, or health. Topology in 3- or 4- dimensional spaces, finite groups,

or properties of $\zeta(s)$ in the critical strip cannot be connected as easily with important, real life problems. (And attempts to link them with unimportant real-life situations may well prove counter-productive.) As L. A. Steen has pointed out ("Mathematics; our invisible culture") it may well be that the research frontier of mathematics is yet another order of magnitude more difficult to communicate than other frontiers of science, and that in many instances not even a professional scientist will attempt to comprehend a new mathematical direction.

This apparently contradicts our previous ruling that no topic should automatically be excluded from popularization. It raises the question: "In the present state of mathematics and mathematical research, are there topics which can be explained only to an audience of mathematicians?"

Even at the level of an expository article for mathematicians there is another difficulty. Science is never a mere accumulation of results, but this is even more the case in mathematics than for any other science. When a theorem is produced, the most important result may be the lemmas. When a problem is solved, then it immediately loses its interest — the new focus of interest are the methods used in the solution. Theorems and problems have, in the main, but a limited time in the spotlight. It is the lemmas and methods which provide the matter for new theories, new concepts, new definitions.

How is it possible convincingly to present the real dynamics of mathematics as a living science?

The public image of mathematics and mathematicians and the esoteric character of advanced topics make its popularization extremely difficult. Yet other features of mathematics may ease our task.

(a) *The rôle of problems*

Problem-solving is a part of school mathematics, as well as a part of the activity of professional mathematicians. In no school activity can the activity of the professional researcher be more closely mirrored. "How to solve it" is a natural and powerful introduction to results and methods. Popularization, then, is not concerned solely with transmitting information, but also includes the involvement of the public in mathematical activities.

(b) *Historical and cultural links*

No other science can boast such a history nor can exhibit so many cultural links. For example, ICMI Study 1 *(The influence of computers and informatics on mathematics and its teaching)* showed how these historical links can be reinforced by the use of computers, for under their influence many parts of

mathematics have come to life again after a long period of lying dormant. To trace the history of a topic may be an easy and useful approach to popularization at every level. Alternatively, to see how the same demands in different societies have led to similar, even if superficially different, mathematical ideas can show the extent to which mathematics is culturally based.

(c) *New applications*

In the past twenty years mathematics has been recognised as a useful, indeed essential, tool in many disciplines and technologies. ICMI Study 3 *(Mathematics as a service subject)* considers the implications of this within higher education. Yet the implications are equally great for continuing education and for popularization. The interest of the public in the applications of mathematics — in their contribution to societal well-being — can well stimulate an interest in the mathematics involved.

What other "positive" features are there to be considered?

3. THE METHODS OF POPULARIZATION

The methods used must depend on the kind of public on which particular efforts are being targetted. We want to set the switches so that people will look forward to mathematics, and to the use of mathematics, in a great variety of circumstances. If one is young, this means that one looks forward to mathematics in one's own education; if older, to the use of mathematics in everyday life, in one's job and in civic responsibilities, and to the part mathematics will play in the education of one's children or grandchildren.

Popular lectures, television, museums, travelling exhibitions, films, plays, ... may all be used in order to create this favourable mental association with mathematics. We hope that one outcome of this study will be the collection of a set of good examples coming from different parts of the world. We suggest that there should be a careful study of specific displays, films or books about mathematics or mathematicians from different points of view: their aims and objectives, their quality, the positive impact they have made ("favourable mental association"), their negative impact ("mark all Mathematical heads ...") and, in general, the reactions of the target audience.

Many people, through their careers and professions, are provided with important motivation for renewing contact with some areas of mathematics. Popularization may provide a "second chance" for those whose previous

educational experiences of mathematics have not been successful. Many "popular books" on mathematics can serve this end. Popularization may satisfy a specific need in relation to new technologies (robotics, computer graphics, computer assisted design, ...), statistical methods in social sciences, agriculture, biology, ... , operations research in management, ...; part of it may be included in continuing education, in self-educational software, or in the general scientific and technical information contained in professional journals. How is this kind of popularization best organised? What are the potential traps to be avoided? How can one estimate the needs of the users and their reactions towards the books, etc., which they use?

Scientists are a particular case, as is the community of professional mathematicians and mathematics teachers at all levels of education. Are we satisfied with the expository papers and books on new trends in mathematics? If not, what can we suggest?

Involving others in mathematical activities is a very special way of popularization to some extent unconnected with trends in modern mathematics, for much use can still be made of classical concepts and puzzles. The wolf, goat and cabbage problem has entertained countless people for over a thousand years and will no doubt continue to do so. Mathematical columns in newspapers, mathematical puzzles such as Rubik's cube and many games, for example mweso or wari in Africa, have excited the interest and curiosity of millions. How can we make best use of these opportunities for popularization? Can we analyse the relation between "savoir faire" in puzzles and games, and mathematical modes of thought? If we use such methods of popularization, how do we prevent mathematics from being associated with the solution of inconsequential problems?

Recently mathematical competitions have developed and attracted public attention in many countries. What is the impact on society of such competitions as the very selective International Mathematical Olympiads and of competitions which are open to a much wider section of children, such as the Australian National Competition?

The links with history and culture are not always used as they might be. There are vast mines to explore. The history of mathematics is beginning to be treated as part of general human history and references now appear in books or collections. Greater emphasis is being placed on the study of mathematics in different societies and cultures. How can this new knowledge be exploited? Are there good examples of popularization which can be described and commented upon? In what ways can the multicultural aspects of mathematics be used as a stimulus for its study? As we have written above, new technologies provided new

stimulation and new tools. Computer graphics have enabled new and advanced mathematics to be introduced to vast numbers of people: think of the interest aroused because of the great beauty of the graphics associated with Julia and Mandelbrot sets. A new range of mathematical activities can also be introduced through the computer. How can the micro best be used in the popularization of mathematics? What software exists for this purpose? How effectively does it involve the user in mathematics, rather than, say, in art?

Not all of these questions will be appropriate for those in developing countries. Yet there is a rich amount of mathematical experience in each ethnic group, often described as ethnomathematics. To what extent is this experience related to the public image of mathemtics and how can it be employed in popularizing the subject?

Methods are nothing without practitioners. This study provides an opportunity to gather personal experiences and views, to appreciate the specific rôle of a few gifted personalities (adept popularizers or popular figures from the mathematical world), and to stimulate the participation of all mathematicians and mathematics teachers in the process of popularization. In particular, the responsibility of professional mathematicians for popularization must be more carefully spelled out. What personal part should each play? How can mathematics teachers be best involved in the process?

How can writers and dramatists be encouraged to develop mathematical themes? How can reading and publishing be stimulated? How can we build on the very best examples of popularization which can be seen, read, heard and participated in today?

CALL FOR PAPERS

We hope that readers of this discussion document will respond to it by writing papers on specific themes or questions. These will be welcomed both from those who cannot participate in the closed international seminar and from those who would like an invitation (the number of which will be limited) to do so. Papers should be submitted *no later than* 30 April, 1989. Copies should be sent to

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Remember that, by themselves, descriptions of attempts at popularization will have little value. There is a need to put the attempt within a particular context: to describe the target audience, the choices made (relating both to material and medium), and to provide some type of evaluation — however subjective — what works, what are the traps to avoid.

Those wishing to submit films, videos, ... for possible exhibition or to nominate books which might be included in a display should write to G. T. Wain, School of Education, The University, Leeds LS2 9JT. Please send a full description including technical details (length, subject matter, intended audience, ...).

There will be financial assistance available to bring some participants from developing countries to Leeds. Other participants, however, will, in general, be expected to pay their own travel and subsistence costs. There will be no conference fee for the international seminar.

PREVIOUS ICMI STUDIES

The influence of computers and informatics on mathematics and its teaching, Cambridge University Press, 1986.

School Mathematics in the 1990s, Cambridge University Press, 1987.

Mathematics as a service subject, Cambridge University Press, 1988. (See also, *Selected papers on the teaching of mathematics as a service subject,* Springer-Verlag, 1988.)

In preparation: *Mathematics Education and Cognition.*

A reply to Howson, Kahane and Pollak

Michel Mendes France

University of Bordeaux

A.G. Howson, J.-P. Kahane and H. Pollak have written an interesting article in a recent issue of Enseignement Mathématique on Popularization of Mathematics [1]. I appreciate and share their concern for the lack of expansion of scientific and mathematical knowledge. Yet I feel I cannot agree with some of the points they bring up.

Games

The authors claim that in order to promote mathematics one should attract people through games. Should we really disguise mathematics to make it appealing? Do parents put sugar on spinach to fool their baby?

All mathematic exhibitions I have seen in recent years look like game parlours, something like Disneyland, or maybe Las Vegas. That's the sugar. As it happens, the kids lick the sugar and leave the spinach. They swallow the bait and avoid the hook. They manage to avoid the mathematics. No, I do not believe in teaching mathematics through games.

Problem solving and competitions

Nor do I believe in popularizing mathematics through problem solving and competitions. Are mathematicians problem solvers? Yes, according to Howson, Kahane and Pollak. I do not share that view. My point is that some mathematicians are problem solvers and some are not. I know of many mathematicians who hate puzzles and who would say - paraphrasing Goebbels "when I see a Rubik cube, I pull out my gun." (I think I'm one of them.)

Describing mathematics is difficult. One feels what it is not. I definitely would claim that it is not problem solving.

Mathematics and beauty

We all believe in the aesthetic component of mathematics. The question is, where does it lie? According to a popular myth, mathematics is harmonious and, as a consequence, mathematicians are assumed to be musicians or music amateurs. I do not see the connection. Tomato ketchup goes with hamburgers yet I see no necessary link between both. Who has even heard of a musician who likes to read mathematics? (I actually do know one ... but one only!). So beauty lies elsewhere.

I can mention many beautiful results and proofs (my favourite is Cantor's demonstration of the inequality $\chi < 2^{\chi}$). I feel deeply moved by beauty which I would tend to confuse with intelligence or depth. I would certainly not say that a computer drawn picture has much to say. The first time you see one you may express surprise and delight. The second time, you say "Oh yeah, I've seen one already". Nothing to do with a Picasso painting. So, back to the question, where does aesthetics lie? I cannot answer. I just know where it does not lie.

I must add that by look at a colourful Julia or Mandelbrot set one hardly gains any insight of the underlying mathematics unless one happens to be a mathematician. What knowledge of the cloth industry does one gain by appreciating a beautiful dress?

Mathematics is a difficult topic which indeed must be taught at school at an early age. It must be well taught by qualified people, and for that I am afraid there is only one recipe and one only: MONEY. More teachers are needed, more schools, more buildings, more room. Quality is expensive. There are no two ways about it.

I am not saying that we mathematicians have no rôle to play. It is our prerogative to write more popular mathematics books. It is our duty to convince journalists to speak about our discoveries, to describe our community and maybe to reveal our history. It is our responsibility to ask publishers to reprint the writings of famous mathematicians (I am thinking of H. Weyl, E. Bofel, ...) It is our job to visit high schools and to lecture on our own research, thus illustrating how lively our field is. This can be done: many of us have experienced it.

My approach to mathematics may be somehow different from most of my colleagues. I consider mathematics as a language, not a game! It describes a certain simplified reality. Poetry and philosophy are not far away. And indeed, like a poet or a philosopher, I wish to be understood by the public, and not only by the happy few. Communication is the skeleton of humanity. We need to communicate whatever the language, French, Bantu, Yiddish or Mathematics. Each language has its own qualities, its own objects, its own personality, its own paradigm. In particular, Mathematics has its own beauty and I dare say its own humour.

I too with Howson, Kahane and Pollak wish to see mathematics promoted and to be acknowledged as part of the culture. With them I cry out for help.

Reference

[1] A.G. Howson, J.-P. Kahane, H. Pollak.
The Popularization of Mathematics, L'Enseignmement Mathématique 36, 1988, p.1-9.

POPULARIZATION OF MATHEMATICS

1 The most important location for the popularization of mathematics is the school classroom. If the subject is unpopularized at school (for example, by unsympathetic teaching or by too narrow a treatment in textbooks and examinations), there is little prospect of reversing that attitude in adult life. It is important that mathematics should be presented in school as a purposeful human activity, drawing its motivation both from within itself and from its potential usefulness in a wide range of application.

2 In any discussion of popularization it is necessary to make a distinction between those who (from whatever motivation) actively wish to extend their knowledge and understanding of mathematics and those who will only come into contact with mathematics through accidental encounter.

3 There is a long history of provision for the first of these groups: in the UK one thinks of the Gresham lectures, of a line of popular books such as <u>Calculus made easy</u> and <u>Mathematics for the million</u>, of the availability on regular television channels of Open University transmissions. This interest in mathematics may or may not be well focussed, but in either event the mathematical community has a duty to recognise and respond to it. In general, the arts of exposition and interpretation are not highly regarded or rewarded within universities; this aspect of popularization seems likely to suffer so long as higher status is attached to the pursuit and publication of original research than to its communication to a wider public.

4 It is not obvious that the mathematical community has any responsibility towards the second group, although in a democratic society it may see it as in its own interest to keep the general public informed in order to justify its own continued existence. To some extent there is pressure from other disciplines to be active in this respect: in a world where television is the most powerful medium of mass communication, mathematics has lost out in public esteem in comparison with more glamorous pursuits such as archaeology, geology, ecology, history and drama. There must be concern that this will have an effect in shifting the recruitment of young people from mathematics into such superficially more attractive fields of study.

5 It could also be argued that, with personal economic prospects more secure, and with the enlargement of global communication, has come a strengthening of the public social conscience; and the continuing appeal of mathematics rests on our ability to demonstrate that it has a contribution to make in the creation of a better society. This is a role which the subject plays in close association with other disciplines; and if this is to be appreciated by a wider public, then

the problem of popularization needs to be tackled collaboratively and not just by mathematicians in isolation. The elimination of acid rain, the control of flooding in Bangladesh, the exploration of Mars, the prediction of earthquakes and the reversal of drought in tropical Africa are all projects in which the intervention of mathematics may be crucial; we need to find ways in which this contribution can be made explicit outside the scientific community. It should be possible, using the resources of computer graphics, to find a middle way of communicating the power of mathematical models - to a population which has had the benefit of ten years of mathematical education at school - between the glib "now we put this on the computer" and the impenetrability of simultaneous partial differential equations. These issues are, for the most part, international; and this suggests that the resources to tackle the problems of public information might also be mustered on an international scale.

6 There is no merit in attempting to give wide publicity to the results of mathematics per se (for example, the latest largest prime number, or the four-colour theorem) unless there is some prospect of arousing thereby an active curiosity about the means by which they are arrived at and an appreciation of their wider significance. To concentrate on the results by themselves ("hey! look at this!") is to trivialize the subject with no compensating benefit. If there is any place for the disseminiation of pure mathematics to a wide public audience, then priority should be given to the processes by which mathematical knowledge is developed, and to the logical certainty which is the unique outcome of mathematical reasoning. That a result can be proved without risk of contradiction (or, even more remarkable, that a hoped-for outcome can be proved to be unachievable) is a quality which distinguishes mathematics from every other discipline; and any move to popularize mathematics should have as a major goal an appreciation of the central place of proof.

7 There is one special opportunity for enhancing the public understanding of mathematics which deserves attention: the renewal of interest shown by many adults when as parents they become aware of mathematics as a central component of the education of their children. There is some experience in the UK which could be drawn on: the active involvement of parents in some infant school classes, parents' evenings initiated by schools or parent-teacher associations, books written under the auspices of the Mathematical Association and the School Mathematics Project.

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11 Sep 1988

SEVEN GENERAL PRINCIPLES FOR THE POPULARIZATION OF MATHEMATICS

by

Claudi Alsina

Abstract. Some basic principles for an optimal popularization of Mathematics are stated and discussed.

The aim of this paper is to state and discuss seven basic principles which may clarify the goals and developments of any action aiming at popularizing Mathematics.

1. Popularization of Mathematics must be developed at the academic, social and professional levels, distinguishing in each case the goals to be achieved and the most efficient methods to be adopted.

At this moment, Mathematics is an "unpopular" subject. So the first job is to start a positive popularization at the different School levels. In this context several actions may be built upon the cycles 4-8 (Early Childhood Years), 7-12 (Elementary Schools), 11-16 (Junior Secondary Schools), 15-19 (Senior Secondary Schools), as well as the post-secondary level. The students' age, their intelectual powers and their social context must be taken into account. But popularization may be helpful in the clasroom as well as outside. On the other hand the increasing important role of Mathematics in our social environment yields the necessity of enrolling all math educators in these actions of popularization, extending simultaneously the program to the different social levels. It is hard to believe that there is a popularization for all, independently of the curriculum planing or the cultural backgrounds.

2. Popularization of Mathematics must be adapted to the particular cultural, historical and linguistic setting.

While at some professional levels one may be convinced that some general popularization techniques may be efficient (international journals, T.V. movies in English,...) at the school and social levels the impact of the popularization will increase if there are neither linguistic nor cultural difficulties. Familiar codes may be decisive. A movie on math problems arising in traffic control or money will be of interest if it can be followed in the mother tongue and applied to the particular country. It is quite doubful the result of New York kids looking for traffic solutions in Barcelona and Barcelona kids learning to count in dimes.

3. Popularization of Mathematics must use all the media and all ways of communication, exploring in each case the most adequate languages.

In principle any math topic (news, games, recreations, applications, biographies,... etc.) may be presented through any media (book, journal, magazine, T.V., video, movie,...) but the particular characteristics of the communication alternatives must be considered. Reading, learning and watching require different approaches. So it is important not to be wrong with the media and the message: a biography may be fun in a radio program, a theorem in a book and some geometrical models in a video,... but a blackboard lecture cannot be successful on T.V. If our language (voice, imatges, comics,...) is adequate we may try to be "normal". And normality in popularization has shown to be a chief concern (at least in order to avoid the possible anti-effect of a non contextual use).

4. Popularization of Mathematics should be a joint task of math educators and professionals of the different media.

If a score needs musicians and a piece of theater needs goods actors, popularization needs the joint works and efforts of people dedicated to communication and people devoted to Mathematics. This is a crucial point: math people are not familiar with the media and professionals have normally no idea about Math.

5. Popularization of Mathematics yields rich materials which have a great interest for the learning and teaching of Mathematics.

Clearly if good materials on popularization are produced, it is possible to use them in the clasrooms with an additional value: the active participation and the careful analysis.

6. Popularization of Mathematics must select the topics taking into account their interest and possibiliities of communication.

While most subjects involved in the school curriculum are important and easy enough to be popularized there exist abstract topics wich many have limited interest or may present problems of popularization. We cannot give false ideas or promise fiction applications or explain technicalities, e.g., people feel terribly confused with names like "catastrophe theory" or "chaos" or "artificial intelligence" and may be dissapointed in front of a report on the Bieberbach conjecture or the last achievements of the Field's medalists. On the contrary the idea of making exhibits has been shown to be quite fruitful .

7. Popularization of Mathematics may be an interesting subfield of Mathematics Eduaction.

In view of the panorama of the 90's, where information and images will be basic blocks of our societies it makes sense to believe that popularization may be a topic of research in the context of Mathematics Education. Perhaps math educators may start their popularization action beginning with themselves: our subject will be socially popular if our job is made known, if we do not remain isolated and we involve ourselves in the process of popularizing. We need to show the human dimension of mathematics: biographies of mathematicians and math educators, histories of concepts and results... to show that Mathematics is a human product where rigour, implications and deductions are mixed up with cries, smiles, complaints and happiness.

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MATHEMATICS FOR THE PUBLIC

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With some astonishment Hans discovered how different all things looked to his friend than to him. Nothing was abstract for Heilner, nothing he could not have imagined and colored with his fantasy. When this was impossible he turned away, bored. Mathematics, as far as he was concerned, was a Sphinx charged with deceitful puzzles whose cold malicious gaze transfixed her victims, and he gave the monster a wide berth.

- Hermann Hesse, Beneath the wheel (Bantam, 1970)

Bridge is much more a game of inference and logic than of mathematics.

- Steve Becker, Contract bridge (The Globe & Mail, Toronto, February 25, 1989)

Finally, we call attention to one additional aspect of the preceding analysis which may be of interest to teachers of mathematics. This is the fact that our result provides a handy counterexample to some of the stereotypes which non-mathematicians believe mathematics to be concerned with.

Most mathematicians at one time or another have probably found themselves in a position of trying to refute the notion that they are people with "a head for figures", or that they "know a lot of formulas". At such times it may be convenient to have an illustration at hand to show that mathematics need not be concerned with figures, either numerical or geometrical. For this purpose we recommend the statement and proof of our Theorem 1 [There always exists a stable set of marriages]. The argument is carried out not in mathematical symbols but in ordinary English; there are no obscure or technical terms. Knowledge of calculus is not presupposed. In fact, one hardly needs to know how to count. Yet any mathematician will immediately recognize the argument as mathematical, while people without mathematical training will probably find difficulty in following the argument, though not because of unfamiliarity with the subject matter.

What, then, to raise the old question once more, is mathematics? The answer, it appears, is that any argument which is carried out with sufficient precision is mathematical, and the reason that your friends and ours cannot understand mathematics is not because they have no head for figures, but because they are unable to achieve the degree of concentration required to follow a moderately involved sequence of inferences. This observation will hardly be news to those engaged in the teaching of mathematics, but it may not be so readily accepted by people outside of the profession. For them the foregoing may serve as a useful illustration.

- D. Gale & L.S. Shapley, College admissions and the stability of marriage Amer. Math. Monthly 69 (1962), 9-15

1. Public perception of mathematics

It is not much of an exaggeration to say that most of the population perceive mathematics as a fixed body of knowledge long set into final form. Its subject matter is the manipulation of numbers and the proving of geometrical deductions. It is a cold and austere discipline which provides no scope for judgment or creativity, and therefore carries no meaning for many. Certainly, there are difficult questions and arcane ideas which arise from time to time, but these are the preserve of a special breed of geniuses and beyond the range of ordinary mortals.

It seems particularly perverse that a bridge expert should set mathematics against inference and logic as though they bore no relation to each other. Is bridge mathematical only when you are counting cards or points or estimating probabilities? Rather, bridge is a microworld of mathematics. Beyond the technical application of combinatorics, it imposes the same requirement of conscious orderly thought, the same formulating and testing of conjectures and the same rewarding of clear and precise reasoning as any serious area of mathematics. Within a given framework (rules of the game), one gains power through an astute formulation of concepts and conventions (card play and bidding), just as in the evolution of a mathematical theory. Just as a large segment of our fellow citizens can gain a level of understanding and skill at bridge, so they should be able to enter to some degree into the world of mathematics.

What is the good news that we want to tell people about mathematics ("The Gospel according to St. Math", as I titled a talk I once gave)? That it is like music. It is alive, beautiful and accessible. It has power to awe and inspire. It is creative. Certainly, many areas of mathematics are deep and require great talent and years of training to master. But, as in music, it should be possible for a majority of the population to participate in mathematics or at least appreciate its more straightforward processes and results. The public should be brought to an understanding that the pace of mathematical creation and development is stronger than ever, and that its subject matter and application is ever more diverse, reaching into almost every area of human endeavour.

To pursue the analogy between music and mathematics further, both are concerned with structure and the relationship of ideas. Both seek out a natural expression which evokes a resonance with some aspect of our innermost being. But in order for this to happen, both require an understanding and an attention to technical detail which can be taught to and learned by someone with a proper mindset. Algorithms, standard results and proof schemata are the scales and arpeggios of mathematics; they provide a basis of skill and experience which allow a probing into the inner character and essence. Finally, mathematics is a source of tremendous intellectual power through its use of symbolism, its specification of concepts and theorems, its mustering of technique, its baring of essential structure and its cogent reasoning.

It is not just mathematics itself which has to be explained, but its central position in the education of the young. If we cannot move the discussion beyond its presumed utility through arithmetic tables and processes, then the fear and boredom engendered by much of the current curriculum will make us vulnerable to the proposition that in this world of calculators and computers, humans have been superseded by machines. It is not generally realized the extent to which a competent use of these devices forces us even more to be in tune with the underlying mathematical structure and to be able to monitor computational schemes both subtle and complex.

But whatever purpose one can state for teaching <u>anything</u> in school has its analogue for mathematics. We send children to school so that they might become autonomous citizens in a democratic society. They must be able to make informed judgments about and bring into effect policies that govern their daily lives. Mathematics permeates a surprising amount of their world, not only the usual mundane transactions, but also the models that underlie economic and social development, and the measurements that are used to classify and categorize. Without going into the technical details, citizens must learn what sort of contribution mathematics can and does make, and what sort of questions they ought to be asking about its limitations and strengths.

Mathematics is part of our cultural heritage. What the Greeks and our more immediate European ancestors did with and thought about mathematics governed their world view and ultimately fueled the tremendous intellectual advances of the Renaissance, the Enlightenment and the modern world. Euclidean geometry is not just an interesting or amusing fossil, but a way of thinking which has left its mark on the present day. We need to see mathematics integrated with science, with literature, with sociology and with philosophy.

Mathematics is a human activity which is continually being increased and refined. We should celebrate the genius of Newton, Euler, Gauss and von Neumann, as we celebrate that of Shakespeare, Milton, da Vinci, Bach and Beethoven. At the same time, we should encourage children to come up with a nice solution, as we encourage them in the creation of art, music or poetry. We need a mathematical curriculum which provides a basis for leisure and personal growth for those in all walks of life. Mathematics can bridge barriers of culture, creed, social standing and race.

Probably the most important perception to convey in school is that mathematics can be understood, that it is a way of thinking about and relating ideas which can lead us to secure conclusions. While we must ever guard against an improper transferance of mathematical processes to other aspects of life, we can still recognize that some universal judgments can be made about discrimination, objectivity, richness and resolution of disputes from a mathematical experience in which these issues seem to rise in a relatively pure form.

It is especially sad that mathematics, which brings so much joy to its practitioners, is so ill understood by the greater part of the population. Since this ignorance is especially rampant among the humanistic intelligentsia, literature and the media which can so well reflect many other human issues become the conduit through which the image of abstraction and soullessness become impressed upon the larger society. But if mathematics cannot be "imagined and colored with ... fantasy", much of the blame rests on the mathematical educational establishment which has managed never to enthuse teachers-in-training with experiences they can carry to the next generation nor reallyto takethe considerable percentage of the people that have come within its grip into its confidence about the real issues. The conclusion that mathematics should be a prominent part of the school curriculum is true. The usual premisses of its universal utility and its ability to sort out intelligence are sufficiently false that they threaten to undermine the future of mathematics and its application in the not too distant future.

2. The problems

Mathematics is difficult. But it is not so in quite the same way that most people think. It is not that there is so much to remember (although having a good memory for detail helps) or that one has to be able to do long involved computation (though this cannot be gainsaid). It is rather than one has to be able to follow a chain of reasoning and develop a a flexibility that allows a switching from one to another mode of thought, whether intuitive, formal, geometric, algebraic. Doing mathematics requires a proper perspective on the situation and a willingness to make judgments which will tend to increase one's insight and efficiency. There is no a priori reason that most people cannot attain these to some degree, but it does require experience and practice. Because the nature of the difficulty of mathematics is not understood, much teaching, learning and public exposition is bound to fail because appropriate strategies cannot be developed to deal with it.

An article about science or the arts in the public press is designed to acquaint the reader with the latest developments. The writer can generally count on a base of experience on the part of the reader which will permit the conveying of the essential ideas at least about analogy. It is hard to write a completely opaque article about the habits of a wombat, the nature of the moons of Saturn or the architecture of the last five years. On the other hand, the most recent developments in mathematics are hard to convey to a specialist in another mathematical area, let alone the intelligent layman. If mathematics is to have a place in the public press, it cannot be through the reporting of "news".

A possible alternative might be through games and puzzles. But these will never get beyond a limited public if they are simply posed and answers printed. There must be a teaching function, whereby the reader is enabled to understand how the answer can be found or why some trick works, and thereby to see a little of the mechanism of more advanced mathematics.

Thus, there are severe limitations to getting coverage of mathematics in the media. What can be made intelligible is generally not what is current. Because of the ignorance of the public about mathematics, the material has to be developed carefully and sensitively, and this takes space and time. In the perceived absence of public desire to be informed about mathematics and the lack of a suitable "hook", mathematical stories are bypassed by all but most courageous editor.

Even if a story is run, there is the problem of finding someone to tell it. There is little in the background of many specialist writers, let alone ordinary reporters, that will prepare them to handle a mathematical issue, to ask the right questions and tease out the significant factors. Mathematical specialists find it difficult to get the right level of discourse, and many have turned away in frustration from any contact with the public press whatsoever.

What does get published is very often the public interest story, which serves to reinforce stereotypes about mathematics and mathematicians. Mathematical "whizzes" (to use the word favoured by a Toronto newspaper) who win contests or get very high grades are presented as singular oddities, more in the category of someone who can wrap his ankles behind his ears than of the local winner in a music festival or basketball star. Articles about mathematicians focus on eccentricities (Paul Erdős is a recent subject of such treatment). Indeed there is public interest, but what is missed are the accounts about the innovative work done in odd corners by teachers and club organizers who by dint of inspiration, hard work and great personal skill bring a more authentic vision of mathematics to their charges because those in the press do not have a real understanding of the significance of such work.

So why bother? Apart from the missionary zeal felt by any professionals to involve others in his interests, there is the responsibility by professionals in both mathematics and the public press to provide an honest appreciation of a field which touches the lives of members of the public in ways that can hardly be enumerated.

3. Some ideas

I would like to present a few suggestions based on my own experiences as a contributer of a short series of radio broadcasts on the CBC (Canadian Broadcasting Corporation), as well as a few brief items on a Science magazine programme for CBC Radio entitled "Quirks and Quarks", and as the author of a 200-word column in my university's alumni magazine. My intention is to encourage the listener or reader to experiment and to realize that there are systematic tools which can be brought to bear on a mathematical problem, and these are of a similar character to those used in advanced mathematics. This can, of course, more easily be done with a live audience and a half to full hour time slot, where some interaction and development of ideas is possible.

I have discovered a large reservoir of talent for the more instinctive side of mathematics, such as dissection of geometric figures, recognition of patterns and the solving of puzzles and games. On the other hand, it is difficult for many people to appreciate the sparseness and economy of mathematical reasoning. For example, it is difficult to grasp the strength and generality of the pigeonhole principle; it is common for people to envisage only the possibility that each pigeonhole receives a letter before the doubling up occurs.

Here are a number of examples:

(a) Two-person zero-sum games of perfect information: Discussion of this topic can be built on experiences which almost everyone has had. Comparing the players in a game to the actors in an economic system can convince the reader or listener that there is a serious application for this topic, but it does not take long to get into questions of interest in their own right. The game of Sim is played as follows. Two players move alternately with pencil and paper on which there are initially six dots. A move consists of joining two dots not already joined (so that 15 is the maximum number of moves); the first player to complete a triangle with three edges he has drawn wins. With this game one can do a little combinatorics (Ramsey's Theorem assures that no game ends in a draw), mathematical reformulation (reduce the game to one move on either side - pick a strategy) and some sophisticated argument (the first player has a winning strategy). It can then be discussed that the problem of actually finding a winning strategy is both difficult and only recently solved.

Another two-person game of special interest has alternate moves the choice of a number between 1 and 9 inclusive which has not already been selected. The first player to find among the numbers he has picked three which add up to 15 is the winner. This game is new to almost everyone, and it is by no means clear what is the best way to proceed. However, the magic square is the key:

4	9	2
3	5	7
8	1	6

Three numbers add up to 15 if and only if they are in the same row, the same column or the same diagonal. Thus, it is not hard to see that our game is isomorphic to noughts-and-crosses, with which almost everyone is familiar. In this way, one can convey the idea of a mathematical structure and the strategy of transforming a given system into another which is more familiar or more easy to deal with. This is an idea central to mathematics which is untouched by the school curriculum.

There are other games which can be used to make a point in number theory, combinatorics or topology, such as Whytock's game, Hex or Sprouts.

(b) Using the imagination in topology and geometry: Two mathematical objects which appeal to many of the public are four-dimensional space and the Möbius strip. Starting with the idea of building up a square from a segment, and a cube from a square, we can then go on to the hypercube. By analogy, the audience can discover how many vertices, edges, faces and cells it has and devise strategies for visualizing it. A discussion of the Möbius strip leads on to the Klein bottle and the projective plane, and their representation by identifying sides of a rectangle. We can begin to develop the idea of a manifold and the way in which familiar objects can be used to help us think about those which are more abstract.

(c) Coding: A recent area of systematic mathematical development is ciphers. There are several topics which lend themselves to public exposition, but my favorite is the following. We have a number of symbols which we wish to represent by code words consisting of 0s and 1s (no blanks are allowed). We not only require that distinct symbols have separate code words, but also that the code word for one symbol is not the same as the beginning (prefix) of that for another symbol. The reason for this restriction is that at any point in the reception of the coded message, we should know whether we have completed a code word or are in the middle of one (without having to read ahead). If we try to specify the lengths of the code words in advance (symbol A is coded by a word of length a, symbol B by a word of length b, and so on), what conditions on the length will ensure that a suitable coding is possible? The answer is that the lengths a, b, ... are characterized by the Kraft Inequality: $1/2^a + 1/2^b + \dots \leq 1$. This is a bit of recent mathematics which is not difficult to make intelligible to an intelligent lay audience.

(d) Getting the right perspective: A public used to the idea that mathematics is formal and manipulative is not prepared for the flexibility of thought which will endow a seemingly difficult problem with clarity. Some of these problems are generally familiar, and an audience can be led to think about them in a more insightful way. Two such are the bird-and-train and the water-and-wine problems. (Two trains initially a kilometer apart and travelling towards each other, each with a speed of 20 kilometers per hour; a bird, starting at one, flies back and forth between them at a rate of 30 kilometers per hour. How far has the bird travelled when the trains meet? / You are given a litre of water and a litre of wine. One millilitre of water is transferred to the wine and the mixture stirred; one millilitre of this mixture is transferred to the water vessel. At the end, is there more water in the wine vessel than there is wine in the water vessel?) Another problem involves maximizing the area of an isosceles triangle with its equal sides given. If you take one of the equal sides as base and recall the half-base-times-height formula, the answer becomes inevitable.

(e) The systematic approach: There are many other problems well established in the public domain which will serve as vehicles for illustrating organized mathematical thinking. Most people attempt them relying on a mixture of guesswork, inspiration and luck, and it comes as a revelation that one can approach them in a more systematic way. The use of tree diagrams to delineate possibilities, for example, is a well-used mathematical technique. Problems in this category include:

- river-crossing problems such as the one about the wolf, goat and cabbage, or about the three jealous men and their wives;
- use of vessels with given fixed capacities to get an exact specified amount of water;
- equal-arm balance puzzles.

For example, you are given twelve billiard balls, eleven identical and the twelfth either heavier or lighter than each of the others. Using an equal-arm balance as seldom as possible, determine the odd ball and whether it is heavier or lighter. An analysis might note that there are 24 possibilities (ball #1 is heavy, ball #1 is light, &c.) and three possible outcomes of an experiment with the balance. After one use of the balance, the best one can hope for is to have no more than eight (24 ÷ 3) outstanding possibilities, although there will be at least eight in the worst case. By means of such minimax reasoning, one can quickly realize that at least three applications of the balance may be required and hone in on a strategy which will lead to no more than three.

(f) Mathematical truth and power: I am indebted to John Conway for the following example. Start with an ordered pair of positive numbers. We extend this to a sequence by taking the last number obtained so far, adding 1 to it and dividing its predecessor to get the next entry. An example is:

5 7 8/5 13/35 6/7 5 7 8/5 13/35 ... .

I invite each person to do the same with any two starting numbers of his own choosing; the periodicity is no accident. In the discussion, one can get into the following issues:

- are all such sequences of period 5? how do we know for sure?
- if we try to avoid the use of vulgar fractions by using a calculator, are there any pitfalls? (the calculator truncates the decimal expansion, so a cumulative error is introduced)
- not only are these sequences dandy practice for children in manipulating fractions, the question of detecting mechanical errors and getting the right answer is internal - there is no need to agree with an answer at the back of the book or to elicit a check mark from teacher in order to validate one's work;
- it being impossible to establish the generality of the periodicity by looking at ever more examples, one introduces variables as stand-ins for numbers and imputes to them operational rules satisfied by numbers; thus, algebra appears as part of a proof technique;
- the algebra becomes unmanageable unless one simplifies the fractions obtained (at one point, by performing a factorization); thus, technical manipulations are important for having access to information which might otherwise be hidden.

(g) Experimental mathematics: Pattern recognition is one thing that seems to work well with a lay audience; its open-endedness invites participation from a variety of people. There are a number of numerical equations involving squares, such as the pythagorean equation, which are ideally suited for demonstrating the art of conjecture and experiment. A sequence which is almost as good as the Fibonacci sequence for its richness is the following:

$1, 6, 35, 204, 1189, 6930, 40391, \ldots$.

Taking sums and differences of consecutive terms allows us to generate close-to-isosceles pythagorean triples; for example, $6 + 35 = 41 = 20 + 21$; $35 - 6 = 29$ and $20^2 + 21^2 = 29^2$. Since $1 = 1x1$, $6 = 2x3$, $35 = 5x7$, $204 = 12x17$, ..., one can broaden the discussion to approximating the square root of 2 and finding solutions of Pell's equation $x^2 - 2y^2 = \pm 1$. With such examples, the fecundity which fascinates mathematicians about their field can be illustrated.

(h) There is an Indian lady who travels from city to city giving demonstrations of her rapid calculating prowess. She begins her routine by finding cube roots of numbers of up to ten digits in her head. The skill of finding the cube root of a number of no more than nine digits is not too hard to teach. The realization that knowing that the number is a perfect cube gives quite a bit of information and that no more than three digits are sought cuts the job down to size. The last digit of the cube root is immediate; the first can be quickly determined by estimation by anyone who has memorized the cubes of the nine nonzero digits. The middle digit can be found by a combination of casting out threes and estimation. If an audience can be encouraged to look at the properties of numbers with a discriminating eye, it can master other mental airthmetic stunts and gain some confidence and satisfaction.

These are just a few of the ways in which we can couteract the abstruseness of most frontier mathematical research and yet convey some of the essence of mathematical work. We should look for common experiences and mathematical material which requires no more technical background than a typical thirteen-year-old can muster, and see how they can be used to make points about use of similarity of structure (isomorphism), use of symbolism and expression of ideas, and creation of algorithms and schemata. The mathematical community must begin systematically to record topics for public presentation, to analyze them for their mathematical content and ability to give an authentic picture of mathematical values and activities, and to reflect on and refine their experiences with the public and its media.

Pour faire de la vulgarisation, il faut être capable de dire ce qu'on fait et pourquoi on le fait. Cela contraste avec une certaine pratique des mathématiques - combien de chercheurs ne s'élèvent pas à une vision globale, et se contentent de résoudre les problèmes qui passent à leur portée.

Une erreur commune parmi les mathématiciens est de croire qu' un résultat difficile est nécéssairement important. Pour faire de la vulgarisation, il faut être capable de discerner ce qui est fondamental - c'est bien souvent décapant par rapport à la pratique courante de la recherche.

En fait, je ne pense pas qu'il y ait de discontinuité entre la vulgarisation et l'enseignement.

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THE POPULARIZATON OF MATHEMATICS: PHILOSOPHICAL CONSIDERATIONS

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I will argue in this paper that the source of the inability of mathematicians to popularize their subject is the fact that mathematicians themselves have not honestly addressed the question of the nature of mathematics. Not only does the schizophrenic view of mathematics exhibited by most mathematicians, whereby, as Reuben Hersh [1] would put it, they are platonists in private but formalists in public, inhibit their ability to communicate their ideas to others, but neither a platonistic nor a formalistic viewpoint is conducive to creating public interest in the subject. For neither the formalist's games nor the platonist's multiple ideal worlds appear to say much about the world about us. To most of the population, the manipulations of the formalist are dry and best left to machines, while the worlds of the platonist are fanciful and best left to absentminded dreamers. But neither of these views reveals the true nature of mathematics as an active creative science which has intimate connections with the way man views the world. This is what needs to be communicated to the public before one can hope for success in popularization. In the last section I will add a note on a course I have taught with this view in mind.

The public view of mathematics: Formalism

Since the early part of this century mathematics has taken on a public persona of formalism. The search for foundations in geometry and analysis had led to the arithmetization of mathematics; all of mathematics was to be rigorously deduced from a handful of axioms for set theory in combination with numerous well chosen definitions. Although this is not intrinsically detrimental, nor even an invalid philosophical position, the dominance of the formalist approach, from research presentations to elementary textbooks, has seriously harmed the public perception of mathematics. For the last 40 or 50 years almost every research monograph and research paper has been written in a terse axiom-definition-theorem-proof format, with most textbooks following suit. And many of the textbooks that do not follow this style nevertheless have presented mathematics as a series of formulas to be manipulated in certain set ways. In both cases, meaning, life, and creativity are replaced, at least as perceived from the outside, by symbolic manipulation.

From the formalist perspective, mathematics is not about something, unless that something is a set of symbols on paper. Hence there is nothing to popularize. A mathematical statement only says something about the particular symbols involved in the statement and their interconnections with the rest of the mathematical web of symbols. It cannot be simplified or explained in other terms because it is what it is, and nothing more (or less). Contrast this with the situation of the natural scientist. Even the most theoretical physicist speaks as if she is describing the real world, the way things are. The mathematics involved may be of the highest level, but a physicist will not hesitate to say that such and such a mathematical statement corresponds to such and such a physical object. A natural scientist may have various epistemological and ontological points of view, but she will invariably speak about her subject as if the objects were a fact of existence. In this way the natural scientist can relate her subject to an external reality and thus to the experience of every man, a path which thus leads naturally to popularization.

The private view of mathematics: Platonism

While few mathematicians publicly proclaim themselves to be platonists, René Thom and Kurt Gödel being a couple of the notable exceptions, among themselves mathematicians tend to speak as platonists. When explaining a concept or justifying a point they act as if the objects of their mathematical universe have substantial existence. Of course, this is not the universe of our sensual experience; it is a universe of which we may have fundamental intuitions, but no physical contact in any form. Our intuitions into this world help us formulate our basic axioms about the nature of this reality and aid us in seeking out its truths. These intuitions become clearer over time, and we become more sure of some axioms and less sure of others. Actually, it may be the case that there is not just one mathematical universe subject to our intellectual gaze; perhaps there are many such universes, one for every form of the basic axioms.

From this perspective, mathematics is about something. But is it possible to popularize that something? If this universe of mathematical objects exists, outside of our sense experience, can we show others, without years of training, how to see it? It seems unlikely. Plato recognized long ago that the one who is removed from the cave without years of acclimation will be blinded by the sunlight. Mathematicians may rave about the beauty of their world, but this is a world that has no existence to most of humanity and is but a blur to those who bring themselves to look at it without a proper background.

Popularization: A philosophical suggestion

Of course, the argument that neither formalism nor platonism is a productive philosophical basis for the popularization of mathematics is not an argument against either view as a philosophy of mathematics. If these were the only reasonable alternatives then perhaps it is simply not in the nature of mathematics to be easily popularized. I do not believe that to be the case. Rather, I see formalism and platonism as simplistic answers to complex questions; the mathematician uses one or the other in order to sidestep these questions so she can get on with her mathematics. That is not to say that formalism or platonism are simplistic philosophies, but that they are used in simplistic ways. Formalism provides a simple answer to the problem of foundations. Mathematicians feel that their work is secure and rigorous if it is presented in the rigid axiom-definition-theorem-proof format, devoid of intuition or reference to any entities other than the marks on the page. On the other hand, platonism provides a simple answer to the question of intersubjective agreement, why it is that mathematicians in diverse situations are led to consider similar questions and obtain similar results. In the platonic vision of mathematical reality one is able to find the objectivity that is missing in the formalist approach; one sees why some questions are of more interest than others and why the answers to these questions must agree. Hence platonism is more than just a convenient way to communicate on an informal level. Every scientist wants to speak about reality; platonism supplies the mathematician with hers.

Formalism is a serious philosophical reply to the problem of mathematical foundations and platonism is a deep philosophical system with a long history; neither may be dismissed lightly. But I am suggesting that mathematicians have accepted them lightly. I am also suggesting that it is time for mathematicians to take a hard look at their discipline and its relation to the physical world. For I believe that mathematics is intimately connected to the world of our sense perceptions. Our present day mathematics is the result of man's attempts over thousands of years to organize and rationalize the observations of the human senses. The foundations of mathematics are empirical, grounded in sense data. In an effort to model this sensual reality, we have created a mathematical universe of stunning complexity and subtlety. This is neither the formalist universe of marks on paper nor the platonist universe of ideal objects in a nonsensual reality; it is a universe of mental constructs within our own human minds. It is a remarkable feat of human creativity, rooted in physical reality but far transcending the data we take as inputs. Mathematics may not have the security and rigor sought by the formalist, but it has a connection with the physical world which provides its reason for being. It may not have the objectivity sought by the platonists, but in its stead it has the marvel of human creation. Questions of intersubjective agreement lose their simple answers, but the deeper answers involving the human psychological makeup, the way we interact with the physical world, and the ways in which we communicate among ourselves are far more fascinating to ponder, and much closer to the truth of the matter as well.

In the preface to his book Geometrical Methods in the Theory of Ordinary Differential Equations [2, page ix], V. I. Arnold says

> ... the author attempts to avoid the axiomatic-deductive style, with its unmotivated definitions concealing the fundamental ideas and methods; similar to parables, they are explained only to disciples in private. The axiomization and algebraization of mathematics, after more than 50 years, has led to the illegibility of such a large number of mathematical texts that the threat of complete loss of contact with physics and the natural sciences has been realized.

I am arguing that this loss of contact with the physical world which Arnold laments is at the heart of the failure of the popularization of mathematics. Mathematicians need not pretend, as the natural scientists do, that they are actually speaking about the real world, the way things are. But we need to communicate what it is that we are speaking about and that means facing up to the epistemological and ontological issues surrounding mathematical thought and mathematical objects. Philosophers have been debating these issues for years, with current views ranging from the platonic realism of Michael Resnik [see, e.g., 3, 4] to the nominalistic rejection of mathematics by Hartry Field [4] to the empiricism/naturalism of Philip Kitcher [6, 7]. I believe it is important at this time for mathematicians to join in this debate, as we would have 100 or 200 years ago. And I would argue that we should recognize the empirical, or naturalistic, basis for a sound philosophy of mathematics. Otherwise, we will lose all contact with our physical world. If that happens, not only will we have trouble popularizing our discipline, in fact we will have no reason for doing so.

Addendum: A practical suggestion

Supporting a view that the study of mathematics is an essential ingredient in becoming an educated person, many colleges in the United States require all of their students to take at least one course in mathematics before graduation. The ideal is that that course be calculus, but the reality is that most of the students outside of the sciences are not prepared for calculus. The standard alternative is to invent courses which contain material not found in the standard high school courses, yet easy enough for almost any college student to pass. From these considerations spring, in part, courses in "finite math" or "recreational math". I believe these courses do not further the cause of the popularization of mathematics, but in fact hinder it. The finite mathematics courses tend to be "cookbook" courses which serve to reinforce the formalistic view that the body of mathematics is nothing more than a compilation of formulas; the recreational mathematics courses tend to concentrate on unimportant specialized mathematical games and tricks which lie outside the mainstream of modern mathematical processes. Hence both courses lead to a distorted view of what mathematics is and what mathematicians do.

I suggest that a far more appropriate course for students in the humanities would be a survey of the nature and history of mathematics. I taught such a course a few years ago at Santa Clara University. The goal of the course was to give an overview of the important contributions that mathematics has made to our culture and to hopefully provide some insight into the nature of creative mathematical thought. The emphasis throughout the course was on mathematics as a system for organizing and rationalizing the physical world, in contrast to mathematics as a formal language of computation or as an abstract study of some nonphysical reality. I believe the students in this course left with a far deeper appreciation for the role of mathematics in the world of human knowledge than students leaving a finite mathematics, recreational mathematics, or even a first calculus course. A dance major in the class, having just finished reading A Mathematician's Apology [8], remarked that Hardy's lament over the loss of his creative powers had made a deep impression on her, and for the first time she had a full appreciation for the importance of creativity in mathematics. We must seek to foster this type of understanding and appreciation if we hope to further the popularization of mathematics.

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THE BUSINESS OF MATHEMATICS IN AMERICA

by

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ABSTRACT. This communication to the ICMI conference on the Popularization of Mathematics is written from the perspective of an applied mathematician, describing problems and perceptions surrounding mathematics and mathematicians in the American commercial workplace. Its objectives are to delineate obstacles grown there that impede mathematicians as a class, to suggest actions that could place the profession and its sanctioned practitioners in a better public light, and to increase prospects for mathematical research.

KEYWORDS. Mathematics, commerce, prejudice, survival

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BACKGROUND. For me, the occupation "Mathematician; GS-9" was viable in a freely funded and loosely run U.S. Naval lab twenty-five years ago. There was often good collaboration with scientists and time to bend one's mathematical head to interesting work-related problems. But, in the course of five years, funding and direction at the lab came to be tightly controlled by project managers I could neither respect nor work for. Thus, at the peak of the U.S. production of Math Ph.D.s and a bit older then most, I became alternatively employed in the commercial world as a consultant. I then knew nothing of this other world's codes and enjoyed assignments even if research was way beyond the scope of work; the change from school was good. Since then, my consulting stint has provided entre to a variety of entrepreneurial and government workplaces across the U.S.A. At this point I speak from seventeen years of postdoctoral experience with the social problems and social perceptions that affect mathematics and mathematicians. In the interim I have tried fervently to communicate and interact with other professionals, which leads me to this writing. My observations, of course, are anecdotal even if I'd like to think they are of a more universal character, and there will be exceptions to everything I say.

INTENT. I want to emphasize that in any effort to popularize math (Ref. 1), perceptions of math and mathematicians do not come only from lore, high-school experience or a peek at academic mathematicians. Further, I believe that the perceptions of those who might practically matter most, the potential government sponsors or richly funded scientists, are substantially formulated in the commercial marketplace; that the problems of applied mathematicians are now visited upon the class of mathematicians as a whole; and that an exposition of these particular problems comes best from one who has directly encountered them.

PROBLEMS. In my environment, the employment of mathematics is an essential but subordinated activity. I have met no mathematician at any degreed level or by any commercial name who has clout; most with Ph.D.s work in the basement. Those transferring purposefully from academics or those coming with MBAs are more likely to move upstairs, but they are neither representative nor particularly appealing. Mathematicians with a B.A. or M.A. survive so long as their rates, attitudes and titles are amenable to the tasks at hand, and their work need not be trivial. But it is rare that a mathematician with an advanced degree negotiate middle age in my workplace without several cultural adjustments and it is often tempting to forsake the mathematics altogether or become small and isolated. One would be forewarned that a technical ladder in a corporate hierarchy is usually a fiction and an employment gimmick. Those who try to climb one will find themselves instantly on the managerial rungs dominated by politicians, spending most time in transit. In reality, a schism between manager and mathematician is natural, the joining (especially in one body) is not. So a mathematician is generally a technician without a career ladder.

The infusion of analytical software at personal levels is a fait accompli that has more than one ragged edge. While the subject "math" is now bent to serve "real" purpose by all manner of cheap laity, such as the ubiquitous programmer/analyst, the mathematician of any age or degree unfortunate enough to live where his subject is electronically plied can be highly "unbillable". This phenomenon follows a widely accepted economic philosophy that minimizes cost and lets quality float. It's true that mathematicians can and do use such software, but to others this seems redundant, while the creation of such software has employed relatively few experts once and for all.

The streamlining of commerce has other consequences for the mathematician. The detailed technical report, written on paper and to be searched for substance or error, is almost a relic. The dominant form of communication between worker and client is the "briefing", a press conference with pictures that comes in hard or soft copy. Borrowed from the military, it promotes open and profitably unresolved debate. Mathematicians, prone to detail, cannot even comfortably sit in the audience.
Publishing can be a problem, beyond the creation of a contending manuscript, when the employer deems the material to be irrelevant, proprietary or "sensitive". The first situation means you've wasted their time but can treat the paper as your own; the second situation is hopeless; the third can be finessed by generalizing the context of a problem. Still, one may have to risk unemployment in order to contribute to mathematics in the traditional way, as a matter of principle.

If there were a natural law that said "wherever math is practiced, a mathematician would be present", our vital statistics would soar. From my experience a mathematician is absolutely required only in litigation that calls for such, which is infrequent. Contrary to conventional wisdom (Ref. 2), physicists and engineers are <u>not</u> our allies in the workplace. I have worked for and tried to work with these people and found them to be exceptionally prejudiced against mathematicians and unduly manipulative with mathematics. With entrepreneurial motives, they regard applied math as their domain, snatched up after school in a no-show contest for propriety over that portion of the mathematical entity seen as useful. At best we are now competing for control over new applications and are heavily outnumbered and outranked, while their weakness is a tendency to jam problems into familiar contexts with no particular spirit of adventure and little sense of error. Medical doctors are consummate owners and practitioners; mathematical doctors are at the other end of the proprietary spectrum, freely dispensing solutions that frequently have no immediate application. From a commercial perspective, we are terribly naive or, worse, absolutely bent on going bottoms-up.

PERCEPTIONS. An obvious image problem is the absense of an association of "success" in some publically acceptable form (acclaim, wealth, knowledge, medals, etc.) with mathematics or mathematicians. In truth, it is easy to associate the subject and its followers with anonymous frustration, and one wonders if the negative mass so eagerly sought by celestial mechanics doesn't exist in the urge to do mathematics. Too much worth-weight is given by our culture to superficial high profile activity. The relatively isolated and quiet activity of thinking, especially analytically, is scorned by those who cannot do it or cannot see a profit in it. The NRC's new report (Ref.3) says three out of four Americans stop studying math before they know enough to be useful in industry or pursue math on the college level. From my experience, that statistic does not preclude the three from being employed or rising to the top in my environment, with the fourth working for them. If the toys in the sandlot of youth could be more mathematically oriented, one might expect a cultural change from action toward thought. But, right now, young Americans are more physical than logical, with the cultural momentum toward movement and noise. Their parents and others, whose success is readily assessed, are too often empirical evidence that the high road is the way to go.

In my workplace, the practicing mathematician has a reputation for getting too involved with a problem and for being user-unfriendly. The real issue is management's control over an employee's open mind. I'll give the profiteers this: the uncompromised mathematician differs innately from a businessperson in objective, ethic and mode of operation; indeed we are natural antagonists. But, the bent head posture is terminal. To survive, ex-mathematicians will wrap themselves metamorphically in the fabric of commerce, head erect, and disavow the profession and its problems. There is little trust or camaraderie here. It is not a proud profession.

The status of the mathematician at work is no secret. Most students have some sense of the job market and will steer their education away from a void. Since the professional employment section of any newspaper and the placement bulletin at any school will show few jobs for mathematicians, per se, students keen on an income will not consider the profession by name. It's a subject, not an occupation. Indeed it can be argued from all corners that "mathematicians" with Ph.D.s generally do not belong in the workplace and our vital statistics may manifest such sentiment. The AMS-MAA (Ref. 4), in its dwindling count of new native mathematicians, includes those who graduate in management science, engineering-economic systems, industrial engineering, OR, computer science, public health science, actuarial science and biostatistics. You'd think such commercially amenable titles would aid employment and to an extent they may. But in this diversified educational process "mathematics" has become even more isolated as the impratical side of things. The diversification diminishes math in the public eye, just when it could enhance it.

In commerce the traditional rap about math and mathematicians is amplified. The young mathophobic's query "Do I really need this?" is reiterated more assertively and from a position of greater authority by self-righteous nonscientific professionals. The practicing scientist's corresponding attitude that mathematicians are impractical, with personal experiences to illustrate their point, is self-serving. Unfortunately, the actual loosening of government purse strings is performed by a politician or a manager/scientist not unallied with business. They will either expect a "meaningful" return or be persuaded to donate money for math. At best they will be dewy-eyed about it. At worst, they will add their personal impressions to what they've heard about academic mathematicians and regard such expenditures as a waste if it involves research. The funding of math education can be seen as an obligation, while the funding of research is probably a small act of kindness akin to feeding the homeless.

IMPERATIVES. The most important long term goal in the popularization of mathematics in the U.S. is to convince Americans that their cultural attitude against analytical creativity, against "mathematics", is not shared by other cultures and will injure their standard of living. But it is wise not to confront the public. Let them get the message through familiar sources. Math is indeed making the news here these days. As I write, a math professor from Amherst is on a radio talk show. I see a

Boston Globe article on "innumeracy" (Ref. 5) and the latest NRC study (Ref. 3). A report by the Educational Testing Service showing South Korean teenagers to be twice as mathematically savvy as their American cohorts was just (Jan. 31, 1989) summarized on national TV by Dan Rather. It is important that similar books and reports continue to be produced and find their way persistently to the media. There are risks of misinterpretation. The citation of decreasing counts of indigenous math degrees conferred per year can suggest to an innumerate public that math is an overcrowded profession. An appeal emphasizing the service aspect is good PR but could be fatal to research when Federal support for math research now constitutes only .0033% (Ref. 6) of the monies spent by governments on math education. Can a superficial official response to the crisis fool a large number of students into becoming "mathematicians" when there is really no future for them? Yes, it happened in the U.S. in the 1960's and I suppose it could be replayed to the same sour conclusion. But it's preferable that recruitment, renewal and commitment be based on a cultural turn. So it is important that social and other scientists make judicious cause/effect commentary on the status quo. Teachers, books and students should not bear the heaviest responsibility for the brainpower shortfall. The public, its officials, its commerce, must be made to see their faults. To assist, we may have to help assemble an historical validation of mathematical effort in plain english, however difficult (lacking a Nobel prize for mathematics) or subjective. Ultimately, we must sell math as a vital research as well as educational endeavor to keep math alive on a geologic time scale.

Academic and practicing mathematicians must work together to reclaim applied mathematics, at least prospectively. Academic institutions are the likeliest source of applied research in mathematics in the near term and should be encouraged to support this work. More U.S. math journals, like SIAM's, should be devoted to applied math research, to build prestige and to give IEEE some much needed competition. We need to find friendly collaborators, who offer the greatest chance for partnership and success. New math and a harder science could evolve from joint efforts in the "softer" sciences, geology, biology, meteorology, economy, etc., where there must be opportunity and need (cf Ref. 7). There is potential new footing there and an attachment to phenomenon the average person could begin to appreciate. It is critical, however, not to let a science simply absorb our contributions along with those we educate. So we must create well-named disciplines: mathematical geology, mathematical biology, etc., that stand as distinct connections between math and science. In a collaboration, the scientist will bring equipment that might be expensive. The mathematician will come relatively naked, with or without a computer, and their low overhead should be a positive factor in selling research once a case is made for their presence. An allied effort at drawing mathematics and mathematicians into a protective circle would be to better advertise the AMS-MAA sanctioned professionals as our own. Perhaps all derivative degrees should read "Mathematics, with a specialty in _________". At the least, these professionals and their personnel departments could use the term "mathematics" more frequently in conversation and job descriptions.

The intersection of commerce and science, however warped, is one of the richest and least tapped sources of mathematical inspiration, a place where difficult problems surface and are routinely given short shrift. To mesh problem with problem solver, a "positive cash flow" must be associated with mathematical enterprise. Significant new financial incentives such as corporate tax credits are required in order that the U.S. business community allow math research to be pursued on its premises. Another route would have the NSF extend its SBIR concept to bring innovative research funds to mathematicians in very small or very large companies. The NSF could also strive to increase the portion of such funds sent to mathematicians above the current level of approximately five percent. Of course, these are only practical means to an end, conditioned on a willing government and a willing people. A lever of sorts is the promise that research in the consumer's camp can show applicable publishable results. The rock we have to budge will move more easily if the labor were paid for with public monies.

A sound educational system, as well as proper intellectual motivation, is required to bring mathematics to any prominence. For most college-bound students and their parents, the study of math is endured to get high SAT scores. The educational system can also sucumb to that attitude, and there is a newsworthy solution to the declining SAT math scores in the U.S. (cf MIT): lower the standards. An underscored loss in the battle to produce brainpower might help garner resources for the war. In college, math material in its "discrete" form, as for business majors, etc., is not atypically approached by rote rather than reason, in both text and class. The student loses the significance of math to bad motives, bad instruction, and a fading memory. In the workplace math's redemption is lost almost completely by dilution with a different ethic. From begining to end, the educational process for many students who would not major in math is intellectually corrupt in its approach to mathematics. For those who are bound for the sciences or even mathematics, the process is just tedious since e.g., it can take four semesters to get through calculus alone while basic physics takes only two. There is no doubt that the system needs an overhaul, in philosophy as well as mechanics.

Some earlier exposure in the arts has at least not been terribly distorted nor negative, and I would suppose that the arts could be useful in the effort to popularize math. Dustin Hoffman played the dramatic role of mathematician in Strawdogs (1971), a sexist cinematic rendition of a novel The Seige of Trencher's Farm by a British writer G. Williams. In it, three or four country rogues (the engineers) rape and bugger Hoffman's wife (his math) and ultimately must be butchered to save home (profession) and honor, though the home is burnt down and Hoffman leaves his wife in the process. In Tangled Up In Blue, Bob Dylan describes some mathematician friends in beseeching lyrics: "They're an illusion to me now" and "I don't know what they do with their lives". Rubik's cube and fractal images were popular, but a positive public link with math or mathematicians was ironically lost to the tactile and visual senses. Autobiographies such as Mark Kac's Enigma of Chance (1985) appear to be written for and read by other mathematicians, curious about their own kind. Drama, in an entertaining form (cf Ref. 8), could be the most effective way of touching the general public, with the mathematician as the good guy.

CONCLUSION. Down the road from my home is an old building with a sign "Parke Mathematical Lab" on it. The company is listed in one of AMS's recent pubs as an employer of mathematicians, while it has been essentially defunct for some fifteen years. The building is really the birthplace of a successful engineering firm that eventually left it and the misleading sign behind in the wake of growth. This place embodies the plight of mathematics to me; it exists on paper, it is used by others to great advantage, but those who would bend their heads to it outside Academe are hard pressed to make a living because they are not seen as commercially fit.

Further, the academic mathematician, even in privacy with their own criteria for success, are viewed as I am viewed. We are approaching the same collective fate. To thwart it and be true, we must seek to put an aura round our precious bent heads.

Artists survive in cooperatives where a hired manager need not impose conceptually on the artists. Mathematicians are not unlike artists. Their profession is offbeat. Their work is often abstract and its value is mercurial. I could imagine mathematical cooperatives and can cite one, Wagner Associates, that is successful in the U.S. Beyond, I could imagine a more radical union in its meanest sense, where mathematicians and students at the age of majority in all classes taught by mathematicians pledge to protect their turf...and it is amusing. The fact is we cannot stand alone. Today, whenever mathematicians seek a room for intellectual freedom, with a chair or two, they are following their nature and simultaneously going to their grave. That's not amusing. At the least a sense of community must be engendered, encompassing the variety of individual mathematicians, to fight (as hard as we think) for our share of the bread and the gold. At the most, we will have company and succeed.

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SOME ASPECTS OF THE POPULARIZATION OF MATHEMATICS IN CHINA

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Chinese mathematics has its age-old tradition. The western mathematics was taught in schools only after 1910's. Now there are around two millions mathematics teachers and mathematicians in China, who are also in a way taking the responsibility of popularizing mathematics. It is probably the largest group of mathematics teachers and mathematicians in a country in the world.

1. Public honours

Some Chinese mathematicians by winning public honours have helped to make mathematics more popular in China. The following are two examples.

Hua Loo-Keng (1910-1985) is a household name in China. Colleagues in the mathematics community know the name through Hua's works on number theory. However most of the people in China heard of him because of his relentless effort in popularizing mathematics. Starting from the mid 60's, and apart from continuing his theoretical research, he was actively involved in the popularization of some operational research methods to the general public in China. He and his team visited thousands of factories and farms and met with millions of people over a period of ten to twenty years. As a result, "the optimization method" became a common word in China. Professor Hua had reported some of his experience in popularizing mathematics in China at the ICME 4 in Berkeley 1984.

The legendary story of how Hua started as a village shop assistant and later became a professor in mathematics, was made into a film script in 1984. Then a television programme on "Hua Loo-Keng" was produced. So far Hua is the only Chinese scientist who becomes a leading character in a film during his life time.

Professor Chen Jing-Ren, a student of Hua, is another well-known mathematiciann in China. He also specializes in number theory, in particular, the Goldback conjecture. That is, it is conjectured that a large even number must be the sum of two odd prime numbers. In 1973, Chen made an important step toward solving this conjecture. More precisely, he proved that for a given large even number N we could find odd prime numbers P', P'' or P_1, P_2, P_3 such that $N - P' + P''$ or $N - P_1 + P_2P_3$. Five year later, Xu Chi, a well-known Chinese writer, published a report on the Goldbach conjecture. He described the conjecture as a problem "1 + 1" and Chen's result as "1 + 1" or "1 + 2". Xu depicted vividly how Chen approached the problem which is known as the crown jewel in number theory. This report was later reprinted in all the newspapers throughout the country. Even up to now Chen is still regarded as an emblem of hardwork. Many gifted students follow Chen's example and study mathematics diligently. So much so that the department of mathematics became a commodity in great demand at the time and many students joined the department of mathematics.

Mathematicians are regarded as a symbol of intelligence in China. Believe or not, most of the presidents of the well-known universities in China are mathematicians. It is certainly a marvel that this is so.

Some mathematicians are also political figures. A well-known senior goemeter Professor Su Bu-Qing was elected to the vice-chairman of the Chinese People's Political Consultative Council. Furthermore, at least twenty mathematicians are deputies to the national people's congress at the moment.

2. Mathematical competitions

During the past ten years, the public has paid increasingly greater attention to the mathematical competitions. It is a small "hot spot" of social life in China.

The first mathematical competition was held in China in 1956. At present, there are various competitions organized at the levels of country, city and province. The three nation-wide competitions are :

Hua Loo-Keng Golden Cup competitions (age 11-13)

Junior school mathematical competition (age 13-15)

Senior school mathematical competition (age 16-18).

About 30,000 pupils take part in each of the above-mentioned competitions. The first 50 participants with the highest marks in each competition will be given the awards by the Chinese Mathematical Society. Among the group of age 16-18, 20 students are selected to join a winter training camp for IMO (the International Mathematical Olympiad). After the training camp, six of them are finally selected to form the Chinese IMO team.

The following table shows the results of the Chinese contestants.

1986			1987			1988		
Gold	Silver	Copper	Gold	Silver	Copper	Gold	Silver	Copper
3	1	1	2	2	2	2	4	0

The 17 winners include 3 girls. Like a torch, every winner sparks off the flames of mathematics in his or her home town.

Another exciting event is a competition between children of age 7-8 and the bank clerks. The questions asked are those involving addition, substraction and multiplication of two-digit numbers. The children using

F. S. Shi's fast counting method answer the questions more quickly than the clerks with the aid of calculators. Mr. F. S. Shi had given a performance on his method of calculation at the office of UNESCO. It was said that all the audience applauded the performance.

3. Publishing.

Publishing is still the most important means of circulating mathematics in China. We have not really produced any mathematics films or videos. The television programmes on mathematics so far available present only the classroom scene on the screen. More than 1000 books in mathematics are published every year. It is a pity that only those books connected with entrance examinations for high schools and universities might be sold out. A great deal of mathematical publishing incurs a deficit. Some extra-curriculum readings are unmarketable.

Recently, some foresighted publishers edited quite a few excellent mathematics series. One of them is "Modern series in science and technology," in which we find booklets with the entitles such as "secret codes", "Networks", "Computer graphics", "Distance", "Database system".

Another series is "series in the appreciation of famous mathematics problems". There are 14 books included in the series, for example, "Gödel theorem", "Goldbach conjecture", "Riemann conjecture", "Fermat's last theorem", "The fixed point theorems", "Fibonacci numbers", "Salesman problem", "The problem of KirKman's girls", "Hilbert's tenth problem".

The condition under which the popularization of mathematics in China could develop smoothly is not favourable. The size of the population has given us many serious troubles. Since 1985, students of high ability gradually lost interest in mathematics. In view of this, a master plan of how to popularize mathematics is presently under discussion. The Chinese NSF (National Science Federation) would support the plan.

POPULARISATION AND CULTURAL RELATIONSHIPS

Alan J.Bishop

1 Introduction

This ICMI Seminar is to be particularly welcomed at this time because the evidence from our colleagues in countries around the world is that the increasing influence of matheatics on the lives of millions of people is being matched by an increasing alienation from the subject felt at a personal level by many of those same people. Many of the contributors to the Fifth-day special on "Mathematics, Education and Society" at the International Congress on Mathematics Education held in Budapest, Hungary in August 1988, referred to this point. (See Bishop et al. 1989).

It is also no accident that there is an increasing awareness of the need to explore the ideas of ´culture´ in relation to mathematics, in order to address these essential issues of influence and alienation which lie behind the current call for ´popularisation´. Popularisation is a response to a socially characterised problem, and cultural relationships always underlie social practice. Furthermore if popularisation is to have any real effect then it should not be thought of as an instrumental process. In this paper I will address some of the important cultural relationships which I believe must be debated if any specific strategies for the improvement of the situation within any one country are to succeed.

2 Mathematics has different cultural traditions around the world

The idea that mathematics is a cultural product is becoming widely accepted. Less widely accepted is one of the evidential corrollaries, that different cultural groups have not necessarily generated the same mathematics. While we can find similarities, we can also recognise differences, and at different levels. Most obviously, for example, and relatively trivially, we know that many different systems of numeration exist around the world. Less obviously, but much more significantly, there are also wide differences in cultural values, cosmologies and epistemologies which can have profound implications for beliefs about knowledge in general and mathematical knowledge in particular.

We in mathematics education have become aware of several particular situations where cultural alienation has been detected, and they illustrate well the relationships between 'influence' and 'alienation' which can affect all people, whether in formal education contexts or not. First, in countries of the world which were former colonies, there is an increasing awareness of the tensions between their own indigenous cultural values and beliefs and those imported along with the colonial mathematics and science education which they received. The educational context is where these conflicts are most easily recognised, but they have undoubtedly contributed to the sense of cultural alienation felt by many people in those countries (see for example, the discussions by Levidow 1988, and Joseph 1987). In other countries also, for example those which are predominantly Islamic in religious belief, there is a questioning of the assumed superiority of the 'western science' model (Sardar 1984).

Second, in the continents of America and Australasia in recent centuries, the waves of immigrants coming mainly from Europe, but also from Asia, have created a cultural domination over the indigenous and aboriginal people still living there. Once again, the cultural alienation has been most obviously exposed by educators in relation to children but it clearly exists for all indigenous people in those countries.

A third group of countries, in Europe and Asia, has a dominant indigenous cultural group with several minority immigrant groups. The potential cultural alienation of the immigrants is obvious and the process of acculturation is inevitably a painful one. Once again, a recognition of cultural differences in mathematics implies also that we should recognise the importance of cultural influence and alienation in many people's understanding of Mathematics as a phenomenon.

In all of these societal situations, the popularisation of mathematics demands an awareness of the fundamental and basic mathematical activities which permeate all cultures' knowledge. It is important to facilitate and encourage research which helps to increase our understanding of mathematics in other cultures. Just as we are now aware of different languages, different religions, and different laws, in different societies, so there is an urgent need for us to become more aware about differences in our own subject - we need to know what different mathematics exist in order to be able to relate them to what has become the dominant 'version' - that which we now call "Western Mathematics". The primitiveness of this term, and the confusions surrounding it, only serve to illustrate just how ignorant we have become of our own subject, in its various cultural forms. Our own general ignorance is one root cause of cultural alienation, and it must be addressed if popularisation is to mean anything significant in the international context. (Indeed we should now perhaps ask, is it the popularisation of Western Mathematics with which the conference is concerned?)

The work of the Aschers (1981), of Closs (1986), Gerdes (1986), Harris (1980), Lean (1986), Pinxten et al. (1983), Zaslavsky (1973) and others deserves to be encouraged and built upon. So far the value of this work has been recognised by mathematics educators but there has been little recognition of it by the mathematical community itself. This must be a priority. Initiatives like the establishment of the Commission on the History of Mathematics in Africa and the International Study Group on Ethnomathematics are to be welcomed. This work must be better publicised, encouraged and developed so that it will produce resources which all can use in their popularisation strategies.

3 Mathematicians as a supra-cultural group

While it is important to become more aware of different cultural traditions in mathematics, and to recognise the need for attacking the internationalised roots of the alienation problem, it is equally important to attend to the relevant issues within our own national societies. The cultural perspective offers two ideas here which are highly significant. The first concerns mathematicians themselves, and their role in the alienation problem.

In the Western cultural tradition historians point to the Greeks as the inventors of the genus 'mathematician', and the history of Western culture since then contains many examples of the Mathematicians who have played a significant role in the development of Western thought and civilisation. Part of the explanation of their significant role in the history of ideas lies with the well-established 'exclusivity' principle - to do the best mathematics, you must become an exclusive person. The recent account by Davis and Hersh (1981) of 'The Ideal Mathematician' describes this exclusivity in its present-day form:

> "The ideal mathematician's work is intelligible only to a small group of specialists, numbering a few dozen or at most a few hundred. This group has existed only for a few decades, and there is every possibility that it may become extinct in another few decades" (p.34). "He is labelled by his field, by how much he publishes, and especially by whose work he uses, and by whose taste he follows in his choice of problems" (p.35). "He studies objects whose existence is unsuspected by all except a handful of his

fellows. Indeed, if one who is not an initiate asks him what he studies, he is incapable of showing or telling what it is. It is necessary to go through an arduous apprenticeship of several years to understand the theory to which he is devoted. Only then would one's mind be prepared to receive his explanation of what he is studying" (p.35).

From the cultural perspective, mathematicians in the Western cultural tradition could be considered to form a sub-cultural group. (see Wilder, 1978, for some of the discussion.) In fact however the common-sense feelings connected with the word 'sub-culture' do not really seem appropriate (e.g. underground, inferior, subversive, secretive and shady) in relation to another point - namely the mathematicians' influence on society. In Westernised societies we can see their influence everywhere through scientific, technological and environmental uses of mathematics, which really can be subsumed under one heading - technological. Westernised societies are now rapidly becoming mathematico- technological societies. Technological needs drive mathematical developments and mathematical ideas in turn help with technological advances. In mathematico-technological society, 'problems' are perceived in technological terms and therefore technological solutions are sought. Mathematico-technologists are the high-priests of mathematico-technological culture. (See Hodgkin, 1986.) Mathematicians therefore should to my mind be designated as a 'supra-cultural' group in Westernised societies, in recognition of both their exclusivity and their influence.

This position of influence should however also carry with it an attitude of responsibility to society. We are familiar with calls for greater social responsibility among scientists, and perhaps mathematicians should also be responding to those calls. For example it is a well-known fact that research into military technology has fuelled much mathematical development in the past, and it continues to do so, albeit under the present pseudonym of 'defence research'. Whether mathematicians are number theorists or wave-form analysts, the 'defence' technology industry supports their research to an extent worth questioning by those possessing an attitude of social responsibility. So where are the papers calling for such questioning of defence contract sponsored research? If society in general is questioning the morality of certain technological research and development, surely another aspect of the popularisation of mathematics is the way mathematicians as a group should respond to these questions? The need for popularisation surely indicates a concern over just these aspects of exclusivity and influence which now need to be addressed.

Mathematicians as a ´supra-cultural´ group could thus be charged with causing much of the alienation felt by many people from the subject. Popularisation cannot be merely seen as improving communication from mathematicians to the people. It is, like any communication process, two-way, and must include fcilitating the ability of people and society to influence the work of mathematicians.

If the concern for popularisation is genuine, we therefore need to address the issues of exclusivity and influence. Are the current institutional structures adequate to ensure that mathematicians´ exclusivity can be challenged? Are the accountability procedures adequate to ensure that their influence can be responsive to societal preferences? What institutional codes of practice will develop a social responsibility among mathematicians? Do we, as Philip Davis says need a Hippocratic Oath for Mathematicians? And will that of itself achieve the desired goal, or is it merely the first stage of a developmental process? (See the account of Hearing 4.2 ´How does Mathematics education relate to destructive technological development´ in Bishop et al. 1989, for some relevant points.)

4 Similiarities exist between the culture of mathematics and the culture of everyday life

A third idea from the cultural perspective which relates to the popularisation process is the notion of culture conflict. It is clear that in relation to the existence both of different cultural traditions, and also of the 'supra-culture' of mathematicians, there is a great deal of latent culture conflict in existence in any society. Any mathematically-oriented popularisation attempts which do not recognise this fact are probably doomed to failure.

At the roots of this culture conflict is an awareness of <u>differences</u>, from which is generated feelings of ignorance and fear. Any popularisation strategy needs therefore to be aware of and to stress <u>similarities</u>, and I have characterised these in the title as those which exist between the culture of mathematics and the culture of everyday life.

In my own research (see for example Bishop, 1988), I have tried to understand mathematics as resulting from six different yet interacting activities which all humans engage in, and which relate to their physical and social environment. These activities are:

counting - the use of a systematic way to compare and order discrete phenomena. It may involve tallying, or using objects or string to record, or special number words and names.

locating - exploring one's spatial environment, and conceptualising and symbolising that environment with models, diagrams, drawings, words or other means.

measuring - quantifying qualities for the purposes of comparison and ordering, using objects or tokens as measuring devices with associated units or 'measure-words'.

designing - creating a shape or design for an object or any part of one's spatial environment. It may involve making the object as a 'mental template' or symbolising it in some conventionalised way.

playing - devising, and engaging in, games and pastimes with more or less formalised rules that all players must abide by.

explaining - finding ways to account for the existence of phenomena, be they religious, animistic or scientific.

The important point about these activities is that not only do they offer a cultural base for understanding mathematics but that all people engage in them, and all people understand them. Whether the people being considered are street trader children or sugar cane farmers in Brazil, tailors in W.Africa, homemakers in England, legal or illegal gamblers (to name some groups who have been studied) or people at work or leisure anywhere, these six activities are present. To recognise mathematics as the product of activities which all of us engage in by virtue of our human interactions is the first stage of reducing culture conflict, and must be the first aim of any popularisation strategy.

Other similarities can also be emphasised - common value systems, the mathematician as just another worker, social and political pressures on all people, and personal satisfactions and pleasures. Any overall popularisation strategy needs to begin with stressing similarities before exploring the differences between mathematics culture and everyday culture.

Conclusion

The popularisation of mathematics, as a social process, must attend to the cultural relationships underlying the perceived problems. In this brief paper I have drawn attention to three important areas of cultural relationship, and have indicated some of the issues which I believe need to be considered. I welcome the opportunity provided by this ICMI Seminar to discuss these issues further.

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Remarks on mathematics in popular education and the popularising of mathematics

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Notwithstanding the immediately contemporary meaning of the phrase "popularisation of mathematics," it seems appropriate to consider the grounding of the implied problem within the history of popular education. The juxtaposition of these two words, with similar phonology but different meanings, will help interpret the necessarily remedial characteristic of the phrase "popularisation of mathematics" against a background of popular education systems which have not been completely successful. In other words the attempt to popularise mathematics in the last 150 years or so through the establishment of popular education systems, must, in some way, be responsible for the need, now, to give an impetus to a popularisation programme. The paradox of non-success being a significant contribution to our work is central to the argument that follows.

There is a wealth of difference in the practical consequences of, on the one hand, establishing popular education and on the other, popularising mathematics. Popular education is not particularly concerned with that enjoyment which is normally associated with things being popular and to which a popularisation programme is directed. It is to do with accessibility. The success of a popular education system in the modern sense, that is, the last two hundred years, can be measured in terms of access. This belief in an object called education which can become accessible to those who do not possess it arises from a particular manifestation of the basic, universal and necessary teaching through which anyone learns to belong to the society, group,tribe, family, which gives them succour and to which they subsequently contribute. This basic characteristic of being human has given rise, inter alia, to the notion of education as a necessary drawing out, broadening and preparing for the diversity of tasks facing educated people in their capacities as people of status.

Popular education

The history of popular education in England, for instance, has involved perceiving the virtues of the leader as an 'educated man' and his consequent effectiveness, together with the possibility of others competing for such virtues. The Industrial Revolution so altered the economy of the country that the balance of trade and agriculture which had preserved in some form a strong hierarchy with 'educated' people in control became distorted. The extension of power meant that those who, through their skills, established industrial organisation in the eighteenth and nineteenth century wanted their children to be 'educated'. That is to be able to compete and co-operate with those others who took education as their right and who exercised power.

The importance of the concept of an educated person as one which naturally divides society hierarchically sits uneasily with the use of the word education to refer to a natural universal right for everyone. Without some understanding of the distortions involved here the idea of 'popular education' will continually risk being misunderstood as we can see by the simultaneous demand for 'education' to be a right and the insistence that the richer the country the longer the period required for 'education' to be compulsory.

This confusion is compounded by the linguistic usage which allows the differentiating word 'education' also to be a generalising word: 'education'. The desire to give everybody equal opportunity is as confused as 'not everyone can benefit' and rests on the failure to find a discourse which was able to monitor necessary modifications in teaching as changing circumstance speeds up the obsolescence of what schools try to do. We finish with the paradox of a 'popular' education which is 'compulsory'.

Stable sub-systems at variance

The principle 'that everyone should be able to benefit from education' is stated in terms which do not stand up to close scrutiny. It is rhetorical and begs several questions, but while it is used as an unchallenged principle it gives people something to work towards. The fact that it refers to a differentiating phenomenon (why otherwise invoke the principle?) means that without clarifying the differentiating nature of 'education', the thinker is left with an apparently realistic guide to a situation whicn can never actually be resolved, except trivially. As a consequence of the confusion of the right of access to one's society with the differentiating act of educating, the function of an education system is liable to be lost in a welter of obscurantic discourse. The impact of linguistic philosophy on contemporary philosophers of education has been possible because of this confusion. It is not sufficient however to point out the confusion. What is significant and ironic is that the implicit limitations are actually welcome. As the system cannot satisfy all its principles, it will produce a result at a level of efficiency which will be acceptable from some points of view and from others it will be a goad to do better. In other words it works so long as it is not perfect. Its non-success implies its usefulness but this carries with it a certain unpredictability in what is not achieved. Given that the system does not have to be perfect it can therefore carry as many "imperfections" as it is socially possible to tolerate. Add to this that one person's imperfection is another person's perfection and we have a recipe for a highly flexible, anything goes, real time system to which an infinite range of criteria can equally apply, all with satisfactorily acceptable outcomes as non-perfectible normative states. This notion of non-perfectible normative state is embodied in the productive idea of a non-efficient system. There is consequently a need to monitor how any particular such system is governed by the appropriate limiting characteristics of its range of non-perfectible normative states.

In arriving at one or other of the possible normative states, 'satisfactorily acceptable' ensues when a sufficiently powerful section of the appropriate population are satisfied with a particular state. The shift to another state will follow adjustments of the power conditions and 'satisfactorily acceptable' will occur for the new state but in response to differing criteria and a different power condition. Totally ordered systematic thinking cannot operate efficiently when the big system is a collection of partially ordered systems working in one or other of a set of dynamic relations with each other. The discourse of appropriate systemic analysis is not as well understood as well-ordered systematic discourse where every term has the potential of being well-defined.

The trouble is that the language of debate obscures this basic structure especially by building in the assumptions of systematic discourse. A child's right to education is one such. If we mean by this that a child's right to belong to a particular community implies developing the particular virtues valued by that community then there need be no argument. The problems arise when the community distinguishes its members in some way. Once a community has established such discrimination then the fact of the discrimination can become the basis for different interpretations from different power bases in the now disrupted community. The statement, for instance, that a child 'be educated according to age, ability and aptitude' embodies possible discriminations. How these discriminations take place will depend on which partial system is powerful enough to ensure that its interpretation holds sway at the time.

Power in the System

In England the advent of popular education in an already discriminating system embodied in a diverse community, necessitated the extension of modes of examining to the point that the examination system exists as a partial sub-system with considerable power. It is possible to argue that the ramifications of the examination sub-system in terms of the correspondence of its means and outcomes with real phenomena of desire and purpose found in a modern disrupted community gives it powers, powers which are only partially under the control of those outside the sub-system. Examinations matter as a means of demonstrating the measure of accessibility which the popular system has achieved. They actually work as non-efficient systems through the appropriate range of non-perfectible normative states. No one expects 100%. This state of a modern divisive community needing to find stability amongst its range of sub-systems through the failure, severally, of any of the sub-systems to be individually successful, is known in the vernacular as' playing the percentages'.

In current behaviourable objective thinking it seems that reasonable people are attempting reasonable things where the language of the percentage game is not overtly acceptable, even though the normative processes are fundamental to the continuation of their work. We hence have the picture of a necessarily non-perfect system being held by a mythology of perfectibility. This is a recipe for guilt about the inadequacy of personal action in the face of failure and for weakness in response when external critics recommend change.

The flexibility of 'failure' systems

It is in this sense of an understanding of partially ordered sub-systems that I maintain that 'popular education' has 'failed' to popularise mathematics. This 'failure' would be irrelevant, that is, is not a real failure, so long as what has been made effective is acceptable. Mythologies develop to explain and justify the state of affairs. The belief, for instance, that people had 'ceilings' to their mathematical ability was a powerful instrument in maintaining the stability of the failure of an earlier system. To some extent some version of it still operates to explain 'failure'.

The fact that ICMI has agreed to a conference on the popularisation of mathematics implies a recognition of a problem. This recognition is an indicator that we can no longer afford to accept the level of 'failure' of the appropriate sub-system. What matters therefore is that if this concern is taken as a serious manifestation of a de-stabilising process we must somehow attend to the factors that have created the system which has nurtured some of us and rejected the rest. I can argue that since it is those whom the system has nurtured who are least likely to want to agree that the system is a failure then it is probable that we shall not attend realistically to a solution to the problem. Similarly those for whom the system was a failure are certainly in a position to understand that fact but with what authority could they recommend solutions?

The first thing is to recognise this 'double bind' and to know that any proposal for action which has any chance of success must question more deeply than people may like. This involves revealing the pre-suppositions on which the sub-system has hitherto relied for its stability. Revealing pre-suppositions is always uncomfortable because they are precisely not suppositions or assumptions of which we are conscious and their revelation implies that things could be other than what they seem. One of the early victims of this process is the belief that "the popularising of mathematics" is a worthy activity to pursue for itself. The existence of the conference is a manifestation of a malaise in need of attention. It is the malaise that needs to be identified. Otherwise the chief danger is that proposals for popularising do no more than popularise what is already in existence. Some recent attempts in the press and TV have the effect of confirming the specialised and esoteric nature of the object called mathematics. I am never sure in these circumstances whether this is because the mathematics specialists involved implicitly confirm it or whether it is the non-mathematical journalists and presenters who preserve their integrity by exercising a 'shut-out' technique which confirms their personal non-involvement. These attitudes precisely reflect the double-bind situation described earlier. There is of course nothing wrong with this if we accept that the effect of such attempts at popularisation can only be marginal but that the marginal effects are worth the effort. They are nevertheless not going to overcome those factors which are leading to the kind of de-stabilising caused by the failure of the sub-system of popular mathematical education adequately to provide for the functioning of all the sub-systems which need mathematical human resources.

Offerings and outcomes

The teaching and learning processes in 'popular education' have developed under the influence of adaptive forces. Taking the prior notion of making education accessible there is always going to be a limiting constraint of benefit. Who may/can benefit? The second limiting constraint is in the mode of what activity in the learner is thought to invite benefit. The dialectic between the two forces which acts consequent on the existence of these constraints, produce forms which then become the media through which the constraints are understood. For instance, an offering can be seen in an interpretation of Euclid's Books which gives rise to the way in which the geometry of the Euclidean plane is experienced. Interpretations such as these have been different in different countries but in each the particular mode acts as a limiting constraint. The first constraint of who may/can benefit always has dimensions relating intention to outcome. In other words access is offered to all who may benefit. The outcome will be related to the mode of offering and will govern how those taking responsibility (teacher/examiner) will identify the people who can benefit ('may benefit' is now accidental). A third element is now introduced which is the theory which explains the outcome as related to the offering. A bifurcation operates which distinguishes those who are taking responsibility. The conservative arm finds explanations which support the necessary qualities built in to what has been done. This in essence will always be reducible to the stark proposition that those who do benefit are demonstrating their fitness to do so; those who do not, demonstrate the reverse. This theory then becomes transformed into instructions for ensuring the continuity of the system by knowing that, of those offered access, only some can benefit. There is thus developed a normative sub-system with built in failure for some.

The radical arm of the bifurcation is often a small minority of those taking responsibility which, although recognising the circular nature of the interpretation of the adaptive cycle, does not accept the self-preserving theory even though it may accept the temporal state. The consequent action of this radical arm depends on how far it accepts the necessary arbitrariness of any particular reductive process which makes mathematics accessible.

If we look at processes of development of popular education with respect to mathematics over the last hundred years, there are detectable occasions when radical actions have been taken to alter the modes of offering with corresponding alterations in the expectancies of outcome. Changes have followed. These changes are not necessarily the ones envisaged by those few who took the radical initiative. The need to monitor the relation between offering and outcome itself develops as a dialectic between the original demands for extending access to education and the modes referred to which attend such extensions.

The corresponding development of a public system of examining as a necessary adjunct to a popular movement in education has already been referred to as power developing within the system. A properly conservative system hence develops whereby the examination offers information in the form of results which allow action to be taken with respect to the outcome and, as backwash, in the modes of offering. An example of the latter is the publishing of textbooks which explicitly are offered as a mode for satisfying the examination system. Currently in England the process of stylising 'mathematical investigations' to fit the examining system is ensuring a transformation from their existence as a radical change in offering (the alternative arm of a bifurcation) to a conventional element of a conservative examining system. This natural stabilising activity could still lead to an improved functioning of popular mathematics education but not necessarily as envisaged by the innovators.

The trouble is that the development of the examining sub-system in all its ramifications as a consequence of the need to monitor the relation between offering and outcome, has not commonly been explicitly recognised. The mechanism of examining, developing quite naturally as a means of demonstrating the success of popular education, has established itself as so basic a characteristic that its real effects go unnoticed. By this I mean that when critics refer to the necessary evil of examinations and go on to list the negative consequences, they rarely refer to the part played by the existence of examinations in structuring the programmes of work undertaken by teachers, and which give rise to the stable forms to and from which any given teacher works when employed in any institution of popular education.

All the forms of mathematics which exist in institutions of popular education are there because of a history of dialectic between offering and outcome in the form of the examining system. Typically "school" mathematics differs from mathematics found outside educational institutions in that its forms are governed by the examination systems used to monitor the relation between offering and outcome. There has been a powerful negative feedback as a consequence of the outcome becoming the examination result. The deviations in expected results have been fed back to the mode of offering and a highly stable conservative sub-system has developed. It can,rather frighteningly, be said that the examiner is more important than the teacher in a popular education system. It is important to observe this because of the difference between popularisation and popular education. In the former there is a 'process of presenting abstruse information which is popularly intelligible' (Sir Roy Shaw) whilst in the latter, abstruse information is offered in a form which allows the receivers to announce their understanding as outcome. In the former the outcome is free, in the latter the examining sub-system takes care of the receiver's ability to make the announcement. However, the 'free' outcome tends to be influenced by the power of the good story whereas despite being restricted by the form, there is always room for the development of probity in the 'receiver' of popular education.

Non-efficiency and the benefits of failure

What then of the relation between the establishment of 'popular education' and the emergence of 'school mathematics', amongst other forms, consequent on the satisfying of the need to monitor the offering-outcome couple by the examination sub-system? There has never been any whole hearted encouragement to examine the dynamics of the system, largely because the logistical reliance placed on the existence of examination results as a means of signalling information at a distance seems to be a necessary function in societies where community has been disrupted and trust diminished. In the short term credibility is governed by the mythology of acceptance of any given system. Witness the anxiety in the UK at the disappearance of GCE 'O' levels and the enormous strength of the systemic ways in which GCE 'A' level is preserved.

The mythologies of school mathematics and its consequences need considerable analysis if we are to begin to unravel one system and construct a more productive and popular basis for the teaching of mathematics. I know of no national or international proposals for enquiry whose basis is as deep rooted as this. The difficulty is that action cannot be taken unless the system is sufficiently disturbed and even then it is difficult, and often impossible, for people experienced in the system to imagine corrections to the disturbance which demand insights from outside the system, and whose application would alter fundamentally the character of the system.

All publicly funded enquiries presume the characteristics of what is known as school mathematics and show very little sign of questioning at a level which, by demanding revolutionary change in criteria, can alter significantly the way mathematics can be come more popularly accessible.

It is because we operate the sub-systems with only partial success that some acceptable overall outcomes are achieved. But nationally and internationally we are beginning to fail at this 'percentage game'. The emphasis on discrete outcomes on which extant examination systems rely, serves to preserve the myth of mathematical truths being independent of cultural and national distinctions. Whether this is a reasonable myth or not is more debatable than is usually accepted but it is increasingly clear to some of us that by monitoring more closely the dynamics of the relation of offering to outcome that cultural phenomena are increasingly seen to have significance. This need be no more than the fact that so much mathematical activity depends on ordinary language and imagery for its existence as a communicable phenomenon.

Conclusion

My thesis has been that the development of systems of popular education with the concomitant development of examining systems has led to closed sub-systems ensuring that partial success is the accepted outcome. This is true whether it be teachers formed by the system; engineers, scientists, economists, or politicians produced by the system; all those rejected by it; parents informed by it; and the consequent ways in which power is exercised to manoeuvre with the system.

It is essential to enquire into the ways in which the necessary conservation of stable sub-systems can be so modified as to engage with new insights into the ways in which human beings can respond and in that response create more productive, but still necessarily conservative, sub-systems.

This means that the questions we ask must be formulated so that they take into account the ways in which the history of popular education shows evidence of how these partial stable sub-systems develop and what mythologies preserve them. When popular education satisfies the criteria necessary for its effectiveness in a particular milieu there will be no need for movements to popularise mathematics.

The effective discourse for this to happen is one which includes the understanding of successful stable systems available to change and development through their necessary non-efficiency. Idealistic aspirations automatically limit their effective application.

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CULTURAL ALIENATION AND MATHEMATICS

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Introduction

The aim of the popularization of mathematics is to influence the perception which people have of the subject. Since this perception differs in different sections of the community it is firstly important to identify a target audience and then to seek to understand the nature and origins of the views which they have of mathematics. Blanket attempts at popularization based on the perceptions which mathematicians have of their subject are unlikely to succeed.

In this paper, some factors associated with the popularization of mathematics among the Maori people, the original inhabitants of New Zealand, are considered. It is likely that similar, but not necessarily the same, factors will pertain to other ethnic minorities with no strong formal mathematical tradition.

It will be argued that the Maori people have been culturally alienated from mathematics and that attempts to overcome this must go beyond the superficial introduction of elements of Maori culture into a traditional presentation of mathematics. Initiatives, by the Maori themselves, firmly based in their own culture have much more potential.

Background

The Maori have been in New Zealand for about 1000 years. It is agreed that they derive from those Polynesians who first settled in East Polynesia. About 200 years ago the first European contacts were made, initially through explorers, sealers, whalers and missionaries, and then through systematic settlement from Britain from about 1840. Before this European contact the Maori had a stable and coherent culture and lifestyle.

This Maori culture had no written language and, as with many, if not all, oral cultures there was no formal mathematical tradition. A very highly developed intuitive sense of geometry is evident in Maori art (Knight, 1984,1985, Donnay and Donnay, 1985) but any quantitative element in the culture was very limited. With the arrival of the traders, however, the Maori soon recognised the need for elementary mathematics and proved to be able pupils. One of the early missionaries (Duncan, 1853) comments on their aptitude:

> "many of them thoroughly understood the simple rules of arithmetic and could calculate readily."

This ability was not universally welcomed by the traders some of whom suggested that the missionaries should restrict their teaching to religious matters. Their motives are clear, it was becoming more difficult to cheat the Maori. This is perhaps a simple illustration firstly of the empowering nature of mathematical knowledge and secondly of an attempt by people of one culture to deny that power to people of another.

It might be argued that where the intentional attempts of the traders to deny mathematical knowledge to the Maori failed, the domination of the Maori culture by the introduced culture has succeeded, largely unintentionally.

The problem is most evident in New Zealand schools. The country prides itself on its egalitarian society and on its equality of educational opportunity. There are certainly no legal barriers to Maori access to education. However, the education system has not been successful in catering for the needs of the Maori population. Evidence of this is provided by the national School Certificate Examination available to students at about 15 years of age. This examination plays a critical role in New Zealand education. It opens, or closes the way to opportunities in higher education and, particularly in mathematics, has a very significant role in giving or denying access to employment. New Zealand Department of Education statistics show that a non-Maori student entering secondary school is of the order of 3.5 times more likely than a Maori student to leave school with an acceptable grade in School Certificate mathematics.

It would be a mistake to attribute all of this difference directly to ethnicity, there are other factors involved such as socio-economic status. However, Garden (1984) found in a study of ethnic factors in the New Zealand data from the IEA Second Study of Mathematics that there was a significant difference in performance which he could only attribute to ethnicity. The social and political implications of this situation are obvious and are evident in the situation of other ethnic minorities around the world.

Cultural alienation and mathematics

The results of the failure of the New Zealand education system to cater successfully for Maori students and to capitalise on the Maori aptitude for mathematics which the missionaries found are deep seated and no easy remedy is likely to be found. An important first step, however, is to acknowledge this mismanagement and to recognise that one of its major effects has been that Maori students feel culturally alienated from mathematics.

Mathematics is, and for the last 150 years has been, taught almost entirely by non-Maori teachers, in English, using syllabuses and textbooks reflecting the dominant non-Maori culture, in institutions whose structures and values take no account of the structures and values of Maori society. It is little wonder that generations of Maori students have come to regard mathematics, along with other school subjects such as science (Stead, 1984) as 'pakeha' or non-Maori subjects.

This cultural perception then interacts with any cognitive difficulties in a distressing way not dissimilar to the interaction of cognitive and affective factors associated with mathematical anxiety. When Maori students are not succeeding, whether or not this is due to cultural factors, their alienated view of mathematics is reinforced. Why should they expect, or even want, to succeed in this 'pakeha' subject. In attribution theory terms we have a clear case of 'learned helplessness'.

The major problem of the popularization of mathematics with the Maori is one of overcoming this cultural alienation.

A Maori perspective on mathematics

For an educator with a western cultural background, the obvious way to tackle this situation is to introduce elements of Maori culture into the presentation of mathematics. Such an approach has the label 'taha Maori' (Maori perspective) in New Zealand. It has included trying to eliminate cultural bias in textbooks by including Maori people in illustrations, using Maori names for people in word problems, and even numbering the pages with Maori numerals. This has many parallels with attempts to eliminate gender bias in mathematics teaching. Opportunities have also been taken to introduce Maori examples in the content. Polynesian navigation methods may be discussed and elements of Maori art are introduced into geometry lessons.

Such initiatives are, of course, well intentioned and similar approaches have been tried in other parts of the world. Unfortunately, in New Zealand at least, they do not seem to work. In fact they seem to benefit non-Maori students more than Maori and almost certainly fail to really address the problem of cultural alienation. For non-Maori students the approach gives them another perspective on their already secure view of mathematics and may serve to promote cultural awareness. The Maori, however rejects the initiatives as 'tokenism' and many regard the approach as positively dangerous since it salves the conscience of the pakeha without confronting the real issues.

In a study of the attitudes of Maori students to science Stead (1984) argues that the rejection of 'pakeha knowledge' by the Maori is a reaction to the rejection by the dominant culture of New Zealand of the knowledge and values considered important by the Maori. This view is supported by Cummins (1986) who stresses the importance of status and power relationships in any account of minority group performance in education.

If this is the case, in order to popularize mathematics with the Maori it must be presented, not as 'pakeha knowledge' to which elements of Maori culture have been attached, but as 'Maori knowledge'. Ideally, this might involve the same techniques which gave the subject its current cultural image. Maori teachers teaching mathematics in the Maori language, using syllabuses and resources which reflect Maori culture in institutions whose structures and values are those of Maori society. Although far from this ideal, current initiatives by the Maori people themselves are very much along these lines and have considerable potential for success.

Maori status and prestige in relation to mathematics

The fundamental principle of these Maori initiatives is that instead of starting with mathematics and introducing a Maori perspective, one must begin with Maori culture and introduce a mathematical perspective. In this way, the status, power and prestige, the 'mana', is given to Maori rather than to mathematics. There is no other way in which mathematics can be accepted as 'Maori knowledge' and the cultural alienation be overcome.

This focus on Maori culture rather than mathematics as the origin of any popularization enterprise applies at least as much to the context and process of presenting mathematics as it does to the mathematical content.

At the very heart of Maori culture is the Maori language.

'Ko te reo te mauri o te mana Maori'
(The language is the life principle of Maori power and prestige)

For many years after the arrival of the European, the use of the Maori language was very strongly discouraged. It was assumed that it was in the best interests of the Maori to forsake their old ways and to learn the ways and values of the colonists. Many children were punished for speaking the Maori language at school. As a result of these assimilation policies the language was in serious danger of being lost altogether. In recent times, however, there has been a renaissance of cultural awareness by the Maori centering on a revitalization of the language. Increasingly Maori is being taught in schools and being spoken on radio, television and even in the courts.

One of the most important features of this renaissance has been the establishment, by the Maori people themselves, of a pre-school total immersion language programme called Te Kohanga Reo (the language nest). The success of this enterprise has led to an increasing number of bilingual programmes in primary and secondary schools. It is within these programmes that the potential for dealing with the cultural alienation which Maori students experience in relation to mathematics exists. At secondary school, bilingual programmes have been operating only for a very short time and the mathematics teaching within them is hampered by a lack of Maori speaking teachers. There are also difficulties associated with the constraints of working within an examination dominated system when the examination reflects the knowledge and structures of the dominant society.

In spite of these difficulties, a visit to any one of these bilingual units will indicate just how powerful a change of context of this kind can be in influencing students' attitudes to school in general and to mathematics in particular. Many of the students find for the first time that their Maoriness can be an advantage, not a disadvantage, in learning mathematics and that mathematics can be enjoyable. This is surely an indication of the reversal of cultural alienation. Research by Wagemaker (1988) also indicates that the bilingual approach is increasing retention rates at school for Maori students.

The constraints of examination prescriptions make changes in content more difficult to achieve and there is still a good deal of work to be done here. A proposed relaxation of these constraints should help considerably. The most promising approach to content is described in Begg (1988) and involves a theme based approach in which topics are taken from the Maori language syllabus and the mathematics associated with these topics is studied alongside the language study.

For example, one of the language topics is 'Kai' (food). In the language class the students discuss favourite Maori foods, looking for and growing foods, preparing and cooking food, and the Maori custom of feasting. In the mathematics class they might discuss budgeting (arithmetic), volume, weight, cooking temperature etc. (measurement), growth curves for plants and animals, cooking times etc. (algebra), analysis of foods, sampling (statistics), and shapes of foods and packaging (geometry).

This approach to learning which blurs the boundaries between subject areas is very much in line with the traditional Maori ways of learning.

Conclusions

There are a number of general lessons for the popularization of mathematics among culturally alienated people which can be drawn from the New Zealand experience which has been described.

Firstly, the initiative for the changes came from the Maori people themselves. Sympathetic non-Maori mathematicians have been able to contribute but it is vital that they stay in a supporting role. Any bid by the dominant culture to take over and control the enterprise would change the critical status relationship of the enterprise and render it ineffectual.

Secondly, not only are status relationships between people important but also those between values and different kinds of knowledge. Mathematics must take the subordinate role and fit into Maori culture, not the other way round. This has implications for both the content and the means of transmission. The content must come from the culture and be transmitted through the culture if the target audience are to identify with it as 'their' mathematics.

Thirdly, the New Zealand experience provides support for the view expressed by Bishop (1988) that mathematicians confuse the 'universality of truth' of mathematical ideas with the cultural basis of mathematical knowledge and expression. The solution to the problem of cultural alienation from mathematics is totally dependent on the acceptance of this view.

In response to their desires to once again be involved and competent in mathematics, the Maori people are pioneering a new relationship between mathematics and culture which has implications for other subject areas and other cultures.

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ABSTRACT

UNEASY BED-FELLOWS IN SOUTH AFRICA: MATHEMATICS AND POPULARITY

In South Africa, with all her First and Third world problems in the teaching and popularisation of mathematics, plus additional difficulties due to apartheid, it is rather a question of looking at the accessibility and relevance of the unpopular subject.

An historic background is interwoven with the present position with most of the discussion within the educational framework.

Within the existing curricula, illustrative examples of two cognitive initiatives are outlined: one from the field of Inservice Training of Teachers, outside the formal government system and the other concerning pupil accessibilty, also outside the formal school system.

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UNEASY BED-FELLOWS IN SOUTH AFRICA: MATHEMATICS AND POPULARITY

In 1987 high school pupils in Alexandra, near Johannesburg, protested and were boycotting mathematics and science classes, as they saw these subjects as being used as a tool to compound the Establishment and status quo of apartheid education. It is from some of this background that we must view popularisation. In fact, it is more in keeping with the South African situation to be discussing the accessibility and relevance of the unpopular subject of Mathematics. South Africa has all the First World problems in the teaching and popularization of maths - a traditionally difficult subject with its (unique?) cognitive demands. South Africa also has, for the majority of her population, all the Third World problems in terms of numbers, materials and cognitively disadvantaged background. South Africa further has a whole host of additional problems which result from a long history of inferior educational provision and apartheid. Outside of the school system there is little or no mathematical clubs, societies or activities. INSET, Maths Olympiads and The Mathematical Association of South Africa (a historically white biased group) have roots buried in the academic. Any true attempt at popularisation should aim at involving the informal sector. Are we at that stage yet? Most discussion, of necessity now, has to be within the educational framework.

A HISTORIC BACKGROUND

> "When I have control over native education, I will reform it so that natives will be taught from childhood that equality with Europeans is not for them."
>
> H F Verwoerd, 1953.

Verwoerd was Minister of Native Affairs at the time when Bantu Education was introduced.

> "Educational policies in South Africa must be dictated by the apartheid philosophy."
>
> F Hartzenberg, 1980 when he was Minister of Education and Training.

Whilst there is some measure of 'own affairs' in white, indian and coloured (Department of Education and Culture) education, the Department of Education and Training for blacks is still controlled by whites. The insertion of the word 'training' is not insignificant.

> "Education cannot be seen as separate from the wider society. In South Africa education is part and parcel of a racist and capitalist system. Schools reproduce class, colour and gender inequalities. The main aim of the educational system for the opressed is to create a cheap, black labour force, with the skills and attitudes that will benefit capitalism."
>
> 3rd National Forum, 1986.

THE PRESENT POSITION

The Alexandra pupils saw then this role of education, and mathematics education in particular, as reproducing the existing political, economic and social order. It can be argued from this viewpoint that

o Mathematics is used to restrict entry to certain job categories

> "At university the only faculties not requiring maths are Art, Music, Drama, Social Work and Law. At the Cape Technikon, all faculties require maths except the faculties of Creative, Interior and Textile Design, Photography, Jewellery, Fine Art, Food and Clothing Technology and Secretarial Studies."
>
> Supplement to Weekend Argus, Jan 28, 1989.

A graphics artist at the University of the Western Cape was denied opportunity to go on an overseas video course, because he was not in possession of a Standard 10 school leaving Mathematics certificate.

o mathematics is used to label people as 'bright' or 'dull'

o maths is used as the basis for destructive technologies

o it is mostly taught in an authoritarian fashion. With the large pupil/classroom ratio (sometimes 50/1 to 70/1) primary school mathematics is often taught purely by the lecture-method.

Coupled with these comments, the Alexandra pupil would be aware of bias in material presented in the school system. The syllabus in History for example provides minute details of the Treks and the 1820 Settlers, but pays scant attention to African histories.

> "There exists a widespread Eurocentric bias in the production, dissemination and evaluation of scientific knowledge."
>
> Joseph, 1987

Breen (1986) identifies amongst others, the following biases in existing South African textbooks:

o The capitalist and sexist underpinning

o The racist underpinning

o Vested sport interest as opposed to popular sport

o Class discrimination

o Preoccupation with recreational activities that are foreign to the reality of the majority of South Africans.

WHY IS MATHEMATICS A FAILING SUBJECT?

From grassroots interviews, come the folowing comments:

* Pupils find mathematics intimidating. This very fact often results in a negative attitude which invariably leads to poor performance in the subject.

* Pupils underestimate their own ability. They do not even attempt to do more than they have to do.

* The large pupil/teacher ratio has many effects, one of which is insufficient stimulation in the finer points of mathematics.

* Pupils do not spend sufficient time on mathematics.

> "The standard demanded by the South African Examination system is roughly the equivalent of a U S high school diploma, although the standard demanded is met only in the best American schools." (Mehl and Rhodes, 1988).

* Pupils, who feel that mathematics is too difficult, are often preoccupied with finding an answer. This is sufficient and getting the solution is the most important thing in mathematics.

* The teachers never made mathematics seem useful.

* When concepts such as the solution of an unknown was introduced, tempers flared and outcries as given below were common.

> "Why solve for x! What are we going to do with all these xs? Just useless xs!"

UNDERESTIMATION OF COGNITIVE DEMANDS

Until relatively recently there has been an underestimation in the complexity of cognition required for both

> "mathematical thinking and the body of knowledge (content and techniques) described as mathematics". (Burton, 1984)

"In contrast to the factor-analytic approach to the study of intelligence, another trend that has become apparent in recent years is an increasing focus on process. The need that seems to have been increasingly felt among investigators is that of identifying processes that underlie intelligent performance. Two examples of this trend are the views of Feuerstein and of Sternberg." (Nickerson, Perkins and Smith, 1985)

Feuerstein and his coworkers (1979 and 1980) have contributed key concepts of cognitive modifiability, and mediated learning experiences based upon the cognitive map and on insight into deficient cognitive functions shown by disadvantaged learners.

Sterberg, within his Triarchic Theory distinguishes five different components (or processes) each of which "is an elementary information process that operates upon internal representations of objects or symbols". (Sternberg, 1986)

Processes are built up by Klahr & Wallace (1976) into a production system.

"Each production has two parts: a condition and an action."

(Ericsson and Simon, 1984)

Davis (1984) further refines the language available in both schema theory and information processing to extend the concept of procedures.

Landa (1976) extends the rules in guiding thinking processes.

Apart from the cognitive demands, as elsewhere, there are many complex cycles in South Africa which contribute to why maths is a failing subject. Two are briefly regarded here.

In the nineteen sixties many parents of today were subjected to functional mathematics and arithmetic as the only mathematics taught to them. It was sufficient to see that the labels on the bottles were the right way up and that there were 12 in a case!

Thus parents of present pupils have a lack of background. Indeed often there is evidence of reverse parental pressure, coupled with almost pride in failing maths, following it to a given Standard only or never using maths in their everyday lives.

A second cycle concerns the availability of specialised teachers. In DET schools all pupils take maths up to Standard 8 and beyond. It is compulsory up to Standard 8 in DET schools, as opposed to other Departments where other choices are possible at the Standard 7 level. Principals therefore have a large demand for mathematics teachers at a reasonably high level, but there is not the supply. In 1964 around 2 000 Black students took the school-leaving examination, also called the matric. By 1984 the number of Black candidates exceeded 82 000. Today only 2% of Black teachers have university degrees. Any teacher, despite subject-specialization, if in possession of Standard 10 maths, is liable to be teaching mathematics.

Under these circumstances, it is little wonder that there is revolt and increasingly violent debate about eduation and the teaching of mathematics.

> "...we resolve to actively strive for people's education as the new form of education for all sections of our people, declaring that people's education is education that - equips and trains all sectors of our people to participate actively and creatively in the struggle to attain people's power in order to establish a non-racial and democratic South Africa."
>
> Resolution from the Soweto Parents' Crisis Committee Conference, December 1985.

The SPCC later became the NECC (National Education Crisis Committee), which together with a number of its affiliates were declared restricted organisations in February, 1988. In terms of slogans LIBERATION BEFORE EDUCATION had a limited popular following for a time, but the PEOPLE'S EDUCATION FOR PEOPLE'S POWER continues to live on, as does a study of ethnomathematics.

However, overt and 'natural' processes of reform are subject to repression. For example, DET inspectors are seen as rigid. One DET teacher made a heart-felt comment to a DEC teacher, who was describing how she had explained her lesson to her subject advisor, "With us there is no 'why', only 'must'.".

In this milieu, certain questions concerning the popularization of mathematics have to be considered.

WHY POPULARIZE MATHEMATICS?

It is not the intention of this paper to address immediately the many possible answers to this question. Love, enjoyment, interest, the applications to real life situations, as a service tool for other subject disciplines, the discipline, the logic, societal needs, technological needs, etc., all point one way. Mathematics is here to stay.

HOW POPULARIZE MATHEMTICS?

It is often less effective to take an eclectic approach: use apparatus; use more of the human senses; regard maths as a human activity; emphasise the pleasure that can be obtained from doing mental activity; introduce more in preschool as number work is less strain to the eyes; be less dependent upon arbitary rules; do not teach concepts the pupils already know and fully . understand, than it is to try to operate from a more widely established, research-based, metacognitive structure.

Only two mechanisms are considered in any depth, within the constraints mentioned above.

WHICH PARTS OF MATHEMATICS AND WHICH TARGET AUDIENCE?

These two questions are interlinked: Given the South African 1st, 3rd and apartheid world experience, the two mechanisms are taken as examples from the field of Inservice Training of Teachers, outside the formal government system, and pupil accessibilty, outside the formal school system.

Those areas addressed are within the existing curricula.

According to Dr Ken Hartshorne at an opening of a TOPS (Teacher Opportunity Programmes) national methodology conference in January 1989, three lessons have been learned.

1. Teachers do not exist in isolation. As we talk of a democratic, open, just, equitable future, we must be prepared to work towards that post-apartheid South Africa now.

2. Within this broader framework, TOPS is trying to enpower the teaching community to participate in its own development.

3. Democracy is a state of mind: the process and organisation of INSET must bear this in mind.

"There is general agreement among all analysts and observers of the education scene in South Africa that Inservice Education for teachers is one of the most important areas of need."
(Ashley and Mehl, 1987)

INITIATIVE 1: TOPS (TEACHER OPPORUNITY PROGRAMMES)

The Alexandra pupils were also aware of the second-rate quality of their teachers' own educational experiences. The HSRC (De Lange) Commission which investigated the state of education in the country in the early eighties identified the un- and under-qualified black teaching corps as a major target in the effort to improve education. At the same time, the private sector, coming to the realisation that it had to involve itself more fully in education in both its own and the national interest, was seeking appropriate projects to support. TOPS was such a project, begun after guarentees of sponsorship by a major oil company in partnership with the Urban Foundation, with the teacher in the senior standards of the primary school as the programme's focus.

In 1983 the minimum teaching qualification was declared to be Standard 10. Overnight, teachers in permanent positions found themselves to be subject to temporary employment. Teachers organisations added their weight and the TOPS Academic component was born.

By 1989 significant progress has been made: TOPS is the largest non-government INSET agency, with an enrolment of 2 500 in the academic component. However, it was never envisaged that TOPS would only deal with formal underqualification. The need to improve classroom teaching was recognised. Thus a Teaching Methodology component was provided for. In order to break a teaching tradition which strongly favoured chalk and talk transmission teaching, (Here comes the teacher, who knows what we must know) emphasis was placed on pupil-centred, enquiry-based, interactive and process-orientated learning.

A school management component was also added, which would support institutional innovation.

In the Mathematics methodology, kits to emphasise a practical approach as well as printed materials were available nationally.

The approach was a much more democratic one, with groups of teachers in different regions taking the materials provided, reviewing them critically and then developing them in a manner that is appropriate to their local needs, working from their own experience.

Within the primary field, an activity currently engaged upon is groups of teacher developing their own tried and tested materials. This is also being extended to the secondary level.

TOPS MATHEMATICS METHODOLOGY CERTIFICATION

Current work on the subject of certification again underscores the genuine attempt for democratisation with a view in mathematics to contribute to teacher-driven Inset, instead of an "expert" initiative filtering down to the grassroots level. The mere fact of teachers having the major input to an Inset syllabus is a major break with a South African tradition. The work plan for syllabus collation involves consultation with the widest audience regionally aimed at meeting teachers' actual needs.

> "There was a strong indication from early workshop discussion that teachers present deemed content knowledge to be only one of the factors necessary to improve teaching and that there was demand for a more holistic professional approach to teacher training."
>
> Overview from TOPS National Methodology Conference, January, 1989.

The certification situation addresses two key issues: those of incentive and engagement on both the educational level and the political level.

In terms of the popularization theme, the target audience defines which part of mathematics to concentrate on. The mode has been indicated and the media of transmission would be group meetings, activities with pupils, more advanced modules being available through distance learning and existing texts.

INITIATIVE 2 : UWC OUTREACH/CURRICULUM AND MATERIALS DEVELOPMENT

The Alexandra pupils are also aware of their paucity of materials with which they have to work, and the technological advances which have brought possibilities of other teaching/learning media. The University of the Western Cape (UWC) is the world's largest user of the mainframe computer system employing PLATO and PLM. It is fitting that, with the research into the cognitive skills of disadvantaged students in various subject disciplines headed by Professor M C Mehl at the Gold Fields Mathematics and Science Resource Centre, UWC is the home of a large scale implementation of computer based instruction at matric level in mathematics and the physical sciences.

Perkins gives a report card for maths:

> "MATHEMATICS. Often strong on argument, presuming the arguments are understood. Falters frequently for lack of vivid models. Very explicit on the structure of the content, but often neglects the structure of how to do things - the problem solving process. In serious trouble with purpose, because of a genuine, but partially surmountable, difficulty of describing the later role of concepts and results just being introduced."
>
> (Perkins, 1986)

Views on mental models become important as outlined in Norman (1983), who models a mental model, and Johnson-Laird (1983), who argues that people reason in terms of models rather than by means of rules of inference.

If the role of models and analogies are to be considered as one of possible strategy for partially surmounting the purpose issue, then other concepts become crucial.

The visualisation process has to come under close scrutiny: reveiwed by Bishop (1988).

Attitudes and beliefs are also vitally important in the South African context.

Recent work on Transfer e.g. Perkins and Salomom, 1987, 1988 has suggested that more attention should be paid to general and local knowledge, and high and low road transfer.

> "So general and local knowledge are not rivals. Rather, they are members of the same team that play different positions. Proper attention to transfer will make the best of both for the sake of deeper and broader knowledge, skill and understanding."
> Perkins and Salomom, 1988

Larkin (1979) further points out that instructional units should be detailed, but related to larger frameworks: remote connections need to be established.

Within this framework, teams of teachers and educators are trying to take advantage of the latest research results in learning theories and infuse thinking skills identified from the study of cognitive science into the high school curricula. Some experimental work with Study Guides, Supplemental (peer-group) Instruction, the Knowledge As Design paradigm, and Schoenfeld (1980) managerial strategy and heuristics has been done with teachers and students at UWC.

Regular research group meetings provide a big input here, as does cooperation between other researchers and institutions.

Cognitive initiatives are applied both to the production of teacher strategies and pupil materials. In the latter case,there is a contribution to be made to CBE (Computer based eduaction).

Since 1984 about 2500 pupils per week are bussed, after school hours, to take part in the main programme on campus with hundreds more taking advantage of facilities offered by Mobile Plato. This has been widely reported on in the past (Jones 1986, Mehl and Sinclair, 1984).

However, such is the student's life-long exposure to transmission teaching, academic content and thrice yearly examinations, a group of high school science pupils at the beginning of 1989 refused to leave their terminals to take part in practical work.

When the shirt is buttoned wrongly, all the buttons have to be undone, before the shirt may be buttoned correctly. Schoenfeld (1987) reminds us that

"The focus of process analyses is on the means used to obtain a particular result; of product analyses, on the results obtained."

As the example immmediately above shows, our target audience are at the beginning of the process versus product debate, to which they need more exposure. To do this, we must continue to be in touch with world trends.

CONCLUSION

In the struggle for schools, the aggressive Alexandra youth have still the taste of power from pupil action in 1976. Teachers are in the middle. Hartshorne (1989) had this to say about the position of the black teacher in South Africa:

* They exercise their profession in a framework with more demands and pressures; with less freedom to relate to the country's youth.

* They are blamed by the Government and the Department (DET) for not being strong.

* They are first, men and women; second, members of the community; third, teachers and lastly, employees or civil servants.

* They are blamed by parents for lack of discipline, not providing the education required, and not taking part in the struggle for liberation.

* They are charged with the responsibilty of calamitous matric results, which are products of years of neglect.

This is a gloomy, but realistic picture in which the need to make mathematics accessible and relevant to the Alexandra pupil exists. At our present situation the popularization of mathematics is surely linked with the professional development of the teaching fraternity.

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Notes Toward a Mathematics Exhibit

by Louis H. Kauffman

58. *Print Gallery*, lithograph, 1956

Mathematics Exhibit Description

by Louis H. Kauffman

Introduction

A mathematics exhibit should be accessible, useful, enjoyable. It should give its visitors an experience of the beauty of mathematics ,a sense of the connections of ideas within mathematics, and an understanding of the relationship of mathematics to other endeavors.

A person who works with this exhibit can receive the direct practical benefits of an enlarged ability to use and reason with basic mathematics, and an expanded horizon with regard to what is mathematics.

Ability to do arithmetic (counting and calculating), plus some geometry, forms the common background for most persons who would enter an exhibit of mathematics. Thus we intend that anyone over the age of nine would be able to use our offerings!

From this base - arithmetic and some geometry - we intend to take people on a journey that leads to an expanded vision of what mathematics really is.

Mathematics is the study and classification of pattern.

This statement holds the core of our viewpoint. Numbers represent patterns of measurement and collection. Calculations themselves are patterns of process.

New worlds arise when a level of description becomes a domain to be observed.

Thus numbers, whose utility is related to their reference to what you want to count, can become a fascinating realm to investigate on their own grounds. Problems arise (such as the fact proved by Euclid that there are infinitely many prime numbers) that seem remote from the origin of ordinary tally and calculation.

And mathematics lives in this seemingly airy realm of the study of patterns formed by patterns.

Airy though it may seem, it is through this path of abstraction that all the practical benefits of mathematics arise. Thus primes become used in the construction-and breaking of codes. And the introduction of a symbol to represent emptiness (zero) turns out to be the key to rapid and efficient calculation.

Modern mathematics is a complex and fantastic texture of patterns and techniques. We intend to give a flavor of that interconnection and also to present many simple situations that allow visitors of the exhibit to expand their direct knowledge and ability.

To this end , the exhibit is divided into a main room, an arcade room (with computer-instructional recreations), and a theatre for mathematical films. A central exhibit in the main room is a "mirror box". This is a room lined with mirrors that people can enter and play with. The mirror room is directly related to our theme of mathematics as the study of pattern. The experience of "infinity" that one obtains by standing between two facing mirrors is an experience of the repetition of the framework of the room within itself. It requires the comparison of "near" and "far" (mirrored) copies of the basic space. In standing in such a room we can begin to see how infinity arises so naturally in mathematics through the possibility of repeating a pattern. This simple idea of repetition underlies the infinitude of the counting numbers (always one more). The success of computers is based in their ability to repeat the patterns we give them (through programs).

Thus the mirror room forms a touchstone for the idea of pattern. We spin it out in many directions with exhibits about symmetry and art, weaving and topology, arithmetic and numbers, applications of the geometry of reflections to lasers and finding the distance to the moon, cryptography, fractal geometry, probability and statistics, moire patterns, proofs (these are patterns too!), and many microcomputer demonstrations.

Mathematics is the study , classification and invention of patterns. It is not taxonomy! That this invention is deeply related to invention in many other fields can not be surprising (surprising though it may be!). All that is required for mathematics is that it be possible to lift the form of an activity away from its particularity. Thus numbers refer to the properties of collections , but not to the particular properties of any individual collections. Such forms can then bring forth connections between apparently distant domains. The nautilus sea shell and the Andomeda galaxy share in the pattern of the logarithmic spiral. This astonishes us and it arouses our curiosity for a deeper connection beyond the formal. But this goes beyond mathematics into a wonder that is ours alone.

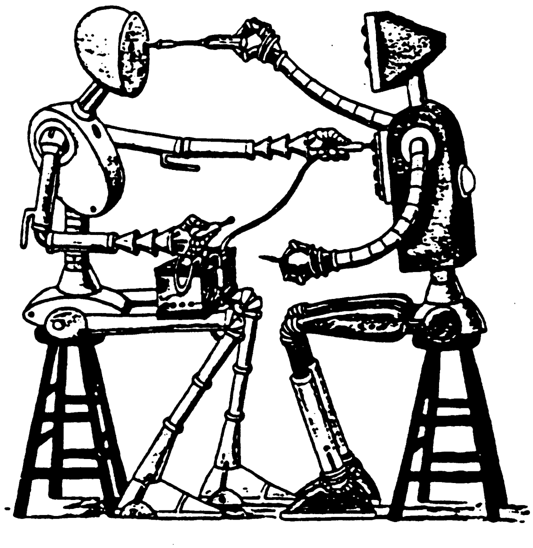
[The root of the word alone is all - one.]

This document is a summary of discussions with many people. Particular thanks go to Marshall Ash, Maria Mezzina, Richard Graff, Jim Flagg, Steven Jordan, Richard Miller, Catherine Murphy, Paul Sally, Dan Sandin, Kim Hicks, Diane Slaviero, Penn Stallard, Naomi Caspe,Doug Kipping,Steve Bryson,Carmit Bat-Shalom, David Solzman, Zalman Usiskin, Carol Becker, Louis Crane, Avrum Weinzweig, and Ray and Raquel Mireles and their children.

$$i = \sqrt{-1}$$

"The Divine Spirit found a sublime outlet in that wonder of analysis, that portent of the ideal world, that amphibian between being and not-being, which we call the imaginary root of negative unity."

Gottfried Wilhelm Leibniz
1646-1716



A broad outline of the exhibit

There will be a main exibit room coupled to three special rooms. These are the mirror room, the arcade, and the theatre.

The mirror room is lined with mirrors, providing experiences of reflections, infinity, geometry and curvature. The arcade has mathematical games and demonstrations (using computers or other devices). Movies demonstrating mathematical principles are shown in the theatre.

The main room contains demonstrations and interactive exhibits; it is devoted to bringing forth themes that unify separate parts of the exhibit. The main room will be devoted to a number of specific topics and their relationships. Suggested topics are-- counting and growth, probability, symmetry, recursion, geometry and topology. The body of these notes give specific ideas for the introduction of these themes.

We have emphasized applications whenever possible, with the intent that
surpising connections become apparent!

It is difficult to trace all
connections in a summary document. Nevertheless, here is a scenario for a hypothetical visitor (whom we shall call Albert):

Albert's Journey

Albert begins in the mirror room, with the virtual space of reflections. He tests and modifies curvature of the reflection tunnels through the handles provided (a slight change in the angles between opposite mirror walls will result in peceived changes in the curvature of the reflected space).

After leaving the mirror room Albert finds himself at the symmetry exhibit, and he sees a demonstration of how a kaleidoscope is constructed, and that it follows the same principles as the room he has just left. He also understands more deeply how aspects of geometry can be generated through reflections. The symmetry exhibit shows patterns that are generated through repetition and spatial movements. It contains artistic examples from sources such as Escher's drawings and the mosaics of the Alhambra. By reading short texts and playing with computer demonstrations, Albert comes to see thatthe problems of design and the problems of classification of patterns are closely interlocked.

In the remarkable case of the Alhambra, all the basic designs had been found by the ancient Moorish artists long before mathematicians invented/discovered the concept of the group (a generalization of symmetry). Yet the abstract classification of the possible patterns (the 17 planar symmetry groups) lent new insight into the ground from which these artists worked.

Other placards remind Albert that the situation with these patterns is not so different from language itself -- with the origins of speech patterns buried in deep antiquity. These patterns and their limitations are articulated much later by the grammarians and linguists. In this case the role of mathematics in linguistics is only recent. A placard introduces the idea of deep and surface structures (Chomsky), and a computer demonstration invites Albert to see a decomposition of a sentence that he types, into a tree and corresponding deep structure.

Another program converses with him, generating its sentences through mathematical linguistic structures. Here we reach the limits of present knowledge, in that the quest for a classification of possible languages has only begun, unlike the complete story that is available for the simpler wallpaper patterns.

If Albert continues to work with the symmetry exhibit , he will encounter the simplest regular tiling patterns for the plane (squares,equilateral triangles, hexagons) and will learn about their relation to the construction and classification of the regular solids (cube,tetrahedron,octahedron,dodecahedron,icoshedron).

Next Albert walks into the arcade room where he is attracted to a computer game involving driving a car at high speeds (simulated) over a course of curves and dangerous turns. Sign posts on the screen remind him how far he has travelled. His speed at any time is also shown, along with a graph of this speed as a function of time throughout the course. After play is complete, the computer demonstrates how the area underneath the speed graph gives the total distance travelled. <u>This is a direct illustration of the fundamental theorem of the calculus.</u>

Albert plays with other games involving guessing number sequences, and then with one that lets him figure compound interest and mortgage payments. He is about to try a stockmarket simulation when his attention is caught by a sound like hundreds of falling marbles from the main room.

The sound was the sound of the probability demonstration being recycled. This demonstration shows how statistical pattern arises from random behaviour. Balls fall randomly through a regular lattice. At each collision of a ball with a strut in the lattice there is a binary choice of left (L) or right (R) for the ball. The choice is randomly (indeterminately) decided by the collision. After 20 choices the ball ends up in one of 20 bins arranged from left to right. The left-most bin corresponds to 20 left-choices , while the right-most bin corresponds to 20 right choices. The bins in-between correspond to intermediate choices. After many balls have fallen through this lattice, the distribution becomes very close to the standard normal distribution. Placards near the lattice explain the ideas , and the expected behaviour. The lattice goes through its cycle every twenty minutes.

Albert watches the probability lattice begin to fill up once more. With all his work in the mirror room and with symmetry, he begins to wonder what would happen on an even larger, or on an infinite lattice! His eye is drawn to a placard near the lattice which addresses this very question. The placard explitans how, in the limit, the distribution pattern for a very large lattice approaches a curve described by an exponential function [exp(-x*x)]. The placard goes on to explain applications to statistics.

But Albert starts to wonder about the paths in an infinite lattice. They would be endless sequences of L or R choices (for example LRLRLR...). Then he sees another placard that addresses exactly this idea! And here it is explained that this excursion to infinity leads back to numbers. For decimal numbers (like .3333333...) are also infinite sequences, and by using the binary system , there is a correspondence between left-right sequences and real numbers expressed using zeroes and ones. This placard refers Albert to another part of the exhibit telling about real numbers and the concept of infinity.

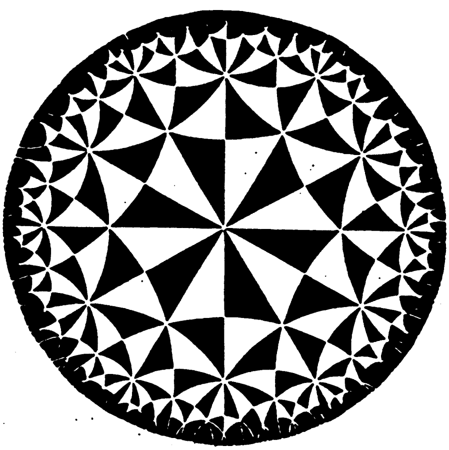
Back to walking, Albert encounters a probability demonstration on a computer screen. It invites him to make a series of choices and watch how his actions influence the proportions of an outcome.

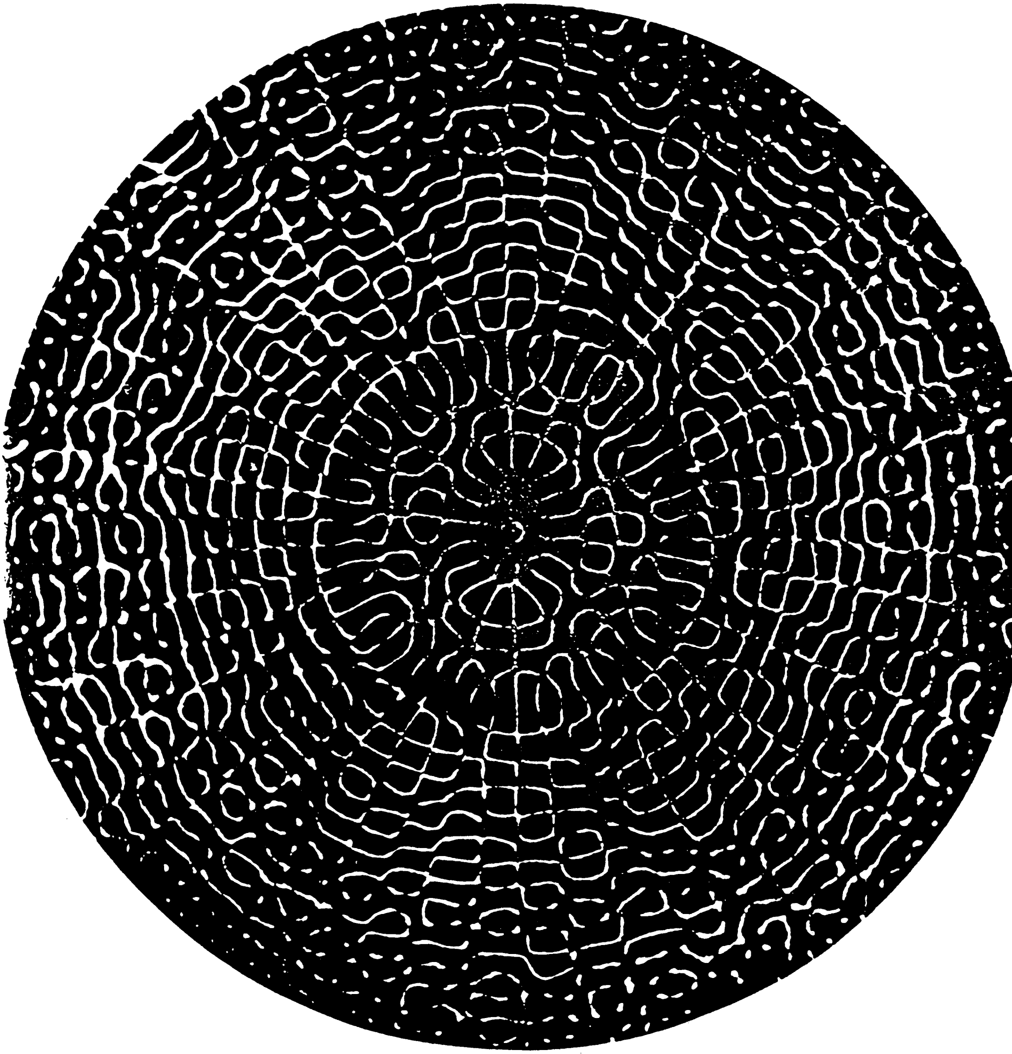
He plays with another demonstration that shows how weaving patterns are generated on a loom. This exhibit lets him create his own weaves on a computer screen. This topic involves the ideas of coding information, and also symmetry. Albert remembers the mirror room and the beginnings of his tour.

He decides to leave topology and the theatre for another day. On his way out he spends a few minutes interacting with a demonstation of video feedback (showing patterns that arise when a video camera records the image that it is itself producing.). Albert is a bit startled to find how closely related are the phenomena of mirrors, symmetry, weaving , feedback and recursion. Enough for one day!

<u>Comment</u>

This imaginary journey through the exhibit only scratches the surface of possibilities. There are many other ways to use the exhibit, and many more interelationships of ideas available. The remainder of these notes give capsule summaries of our ideas for exhibit design.





Specific Exhibit Descriptions

Rooms

There will be a main room and three smaller rooms adjacent to this main room. The main room will house a number of interrelated exhibits; the small rooms serve specific purposes.

Mirror Room

The first small room is an alcove lined with mirrors - including the ceiling and the floor. It will hold up to five people at a time. The purpose of this room is to give a direct experience of the sense of infinity through multiple reflections. It provides an entry to other parts of the exhibit dealing with symmetry , reflections, kaleidoscopes, counting and recursion. And its fun!

In the mirror room viewers are restricted to a pathway leading to a raised platform in the center of the room. This is protected from the rest of the floor by railings. Lighting is provided through the cracks between the mirrors, along the edges of the walls and ceiling.

Viewers may interact with the level of the lighting, providing dramatic changes in the quality of the space.

Another form of interaction involves manipulating handles that control angular tension between opposite walls. This action transforms the "curvature" of the visual tunnel, providing direct experience of the concept of a space curved through multiple reflections.

Peep-holes allow persons from outside the room to experience its reflective geometry.

Arcade Room

The second room is devoted to mathematics as an activity. Of course the entire exhibit has this theme, but this room is aimed at involving the visitors in specific activities. To accomplish this encouragement, we will have puzzles posted on the walls , an abacus riveted to a table. Suggestions for activities posted nearby. There will be interactive computer terminals throughout the exhibit. Here they will be in the form of games. Some examples:

Guessing number series and patterns. This can be just the usual fill in the missing number; the computer gives a sequence like 3,6,-,24,48. Exhibit-goer guesses. If correct, a bell rings and the rule is described verbally and algebraically. If incorrect, a hint is given. (E.g. 6 is double 3.) If you get five in a row, you get a prize.

Interactive number sequences: A person puts $2000 into an IRA. The exhibit-goer gets to pick the interest rate and the first and last years at which the money is put in. The idea is to guess how much money there will be at some time picked by the exhibit-goer. The computer tells whether the guess is too high or too low until the guess is changed to be accurate.

Mathematical theorems can be involved in the games. For example we can involve the fundamental theorem of calculus in an arcade game consisting in driving a car along a highway with obstacles. The driver changes speed by tightening or loosening his/her grip on a lever. The speed of the car is displayed at each instant of play, along with the graph of that speed over time. It is then seen that the area under the graph displays the distance the car has travelled.

A graphics activity involving manipulating objects in 3D using joysticks. Here it is possible to bring in different perspectives and to challenge spatial intuition.

A computer blackboard. This would allow drawings and doodling with the aid of joysticks. Also movement and duplication of patterns is available in paint programs or simplified logo.

Conway's Game of Life. This is a game that borders on serious mathematics. It is, technically speaking, a cellular automaton based on a two-dimensional grid. The grid is like a large checkerboard (say 100x100 squares). Play begins by choosing an initial pattern of occupied squares. An occupied square is said to contain a marker. In a series of time-steps the pattern evolves according to predetermined rules. (A marker with 2 or 3 neighbors survives. A marker with more than 3 or less than 2 neighbors dies. An empty square gives birth to a marker when it has three neighbors.) All kinds of patterned phenomena occur in these evolutions. The game provides a superb arena for exploration. Since Conway's Life has been extensively explored, this particular game can have colleral information and suggestions.

The Life Game affords the possibility of speculation about the relation of recursion and biology. Simpler examples of this theme occur elsewhere in the exhibit in relation to the Fibonacci Series , and to another computer demonstration of cellular automata in one dimension (e.g. Stephen Wolfram's examples) where the exhibit-goer gets to choose the rules of evolution!

The experiments with cellular automata are particularly striking instances of simply designed mathematical processes that share specific and understandable analogies with natural processes of growth and form.

We also suggest one or two standard games such as are found in an arcade. Pinball machines illustrate physics. One armed bandits illustrate probability. In this context the latent mathematical content will become apparent.

To repeat, the second room has the theme of mathematical activity, disguised (thinly) as an arcade.

<u>Theatre</u>

The third side-room is a theatre where mathematical movies will be shown. I envisage short films that are repeated at intervals. There should be a library of such films so that the offerings change from day to day. The films can range from the elementary to the advanced (such as the film of turning the sphere inside out). While some of this material will be available through computer graphics, we strongly suggest the theatre as well. It provides an arena of concentrated ideas presented in a dark space.

Exhibit Room

Groups, Symmetry, Kaleidoscopes, Mirrors

There will be an exhibit about groups and symmetry leading outward from the mirror room. This exhibit displays the inner workings of a simple kaleidoscope , and shows other examples of reflection symmetry.

By using semi-transparent mirrors the symmetry exhibit will demonstrate how a rotation is obtained via two succesive reflections- a fact that can also be seen in the mirror room and with a kaleidoscope.

Forms of symmetry from art (e.g. the Alhambra) will be shown, and the notion of symmetry as a unifier of patterns will be demonstrated for wallpaper patterns, crystals, regular solids.

Another important ingredient in this exhibit will be a "corner mirror" and its applications. A corner mirror consists in three flat mirrors at ninety degrees to one another, forming a corner. Light sent into the corner mirror reemerges parallel to its angle of entry. This can be shown directly with a beam of light, and it can be seen in the circumstance that an observer's eye is fixed in the corner for that observer. Because of this geometry the corner mirror has been used to measure the distance from the earth to the moon via laser beam and corner mirrors on the moon. This is an extraordinary application of simple geometry and light.

Probability

There will be a probability demonstration. This shows how the normal distribution arises from balls falling through a lattice structure. This exhibit is important for its graphic nature, its connection with applications and its physical and visual impact.

Counting and Growth

There will be an exhibit relating counting, number concepts, and patterns of growth. This will start with the simplest notions of counting as illustrated (historically) by pebbles, knots on rope, direct notations (such as /// representing three). A person walking through this part of the exhibit can watch a redevelopment of what he/she already knows about arithmetic — starting from the beginning! The exhibit will, for example, illustrate the familiar laws of arithmetic via geometrical groupings (two groups of three becomes three groups of two). With this introduction it is possible to continue, illustrating figurate numbers and numerical relations that have their explanation in geometric patterns. A simple example is the fact that 1+3+5+7+9+11=6x6 understood by appropriately dissecting a square of six by six dots. Patterns of numbers relate both to calculation and to other topics in the exhibit as a whole (for example to tiling and symmetry). There will be a presentation of the Fibonacci Series (1,1,2,3,5,8,13,21,34,...) relating it to patterns of growth both mathematical (golden rectangle, exponential growth, compound interest) and biological (nautilus shell,sunflower,spirals). (There will be two nautilus shells, one whole,one split. Also a movie (see theatre room) about growth.)

Recursion and Fractals

The Fibonacci sequence provides an entry into the subject of growth and recursion in natural structures. A computer demonstration of one dimensional cellular automata will show another aspect of this: By choosing the rules for the evolution of the automaton the exhibit visitor can produce patterns that have a striking resemblance to forms seen in leaves and shells. This in turn leads through other demonstations of branching growth and fractal (self-similar) patterns to the view that a multitude of natural forms are modelled qualitatively through simple recursions. Here we have digressed into another exhibit on the "fractal geometry of nature" (cf the book by that title by Benoit Mandelbrot.). Beautiful computer demonstrations of fractals are available.

Primes and Coding

The exhibit on the natural numbers will also play with primes,their infinitude, the search for large prime numbers, and the relations with coding and cryptography.

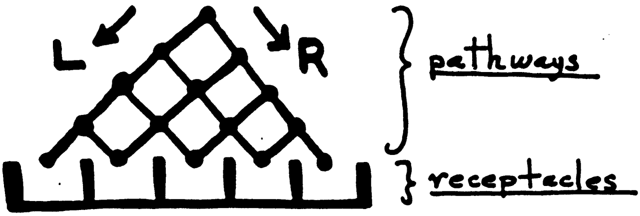
Famous Numbers

The discussion of integers leads into an exhibit about other famous numbers such as pi and e , and to the extensions of the number concept to rationals, irrationals and imaginaries. We will illustrate the irrationality of the square root of two, [placards with two explanations — one due to the Greeks showing incommensurability, the other the modern demonstration using divisibility]. The remarkable effectiveness of the square root of negative one will be demonstrated via the "treasure hunt game" invented by George Gamow in his book One Two Three Infinity. This illustrates the geometry of complex numbers. Again we have the theme of the deep interrelation of arithmetic and geometry.

Nearly all of the material related to numbers will be complemented by interactive computer programs.

Infinity

Infinity is a theme that recurs throughout the exhibit. It is inherent in counting through the concept "next". It occurs in geometry through the endlessness of the circle. It occurs in connection with large numbers through limiting behaviours, and this is graphically illustrated through the probability lattice.



Next to this lattice whose basic structure is as shown above, there will be a discussion of the pattern of a corresponding infinite lattice, and of the pathways that balls could take down through such a structure. At each juncture, the ball is presented with a choice to go left (L) or right (R). Thus a pathway down the infinite lattice is represented by an infinite string of L's and R's such as LLLRRRLLLRRR... . The infinity of lattice pathways will be compared with the infinity of real numbers(decimals). This leads to the matter of comparing infinities, and to Cantor's diagonal argument proving the uncountability of the reals.

The mirrors provide another pathway to infinity. A major theme here is that the properties of the infinite are accessible through a combination of graphic illustration, reason and reflection.

Video Feedback

Another exhibit involving pattern and infinity: A television camera is set to record its own output on a TV monitor. The results are remarkable patterns tending toward infinite repetitions of frames, spirals and dynamics. This is a fascinating interactive exhibit, and it connects with the ideas of reflections, pattern, recursions, feedback, stability and instability, infinity, topology.

This is a particularly good example of how a single exhibit can be related.to a large number of themes.

Topology/Biology

Another exhibit will introduce topology by way of the twisting nature of your telephone cord, and its relation to the linking and twisting of circularly connected DNA. Twisted telephone cord twists and also supercoils when the tension on it is relaxed. Pull the cord and the supercoiling becomes more twist. Relax it and the higher order twisting appears once more. There is a conservation law:

Twist + Supercoiling = Constant

And the same considerations that apply to the telephone cord apply to the geometry of large molecules. There is more to say about the molecular biology here, but this does provide a very vivid entry into both topology and an important application.

From the telephone cord - DNA toplogy we will lead outward into the mathematics and lore of knots and weaves, to other interactive geometric situations.

Topology/Soap Films/Minimality

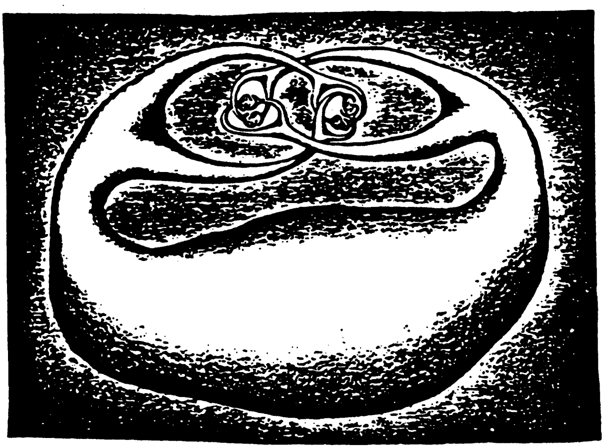
There will be a soap film minimal surface demonstration, where visitors to the exhibit can dip wire frames of various shapes into soap solutions to observe the resulting surfaces. This topic of minimal surfaces will be interrrelated to minimality principles and phenomena. Thus we will make the connection to the ray optics model of reflection used in the mirrors and kaleidoscopes section. (Here light follows a shortest path.) Minimality properties of a hanging chain (the catenary - related to design of suspension bridge) and corresponding situation for tentlike surfaces can be demonstrated.

Weaving/Computers/Knots

Knots will also be introduced directly through weaving. A tapestry will hang on the wall , and next to it will be an explanation of the basic weaving process — including the coding of lift sequences used to produce the patterns. This will be accompanied by a computer demo that produces patterns corresponding to a given lift sequence. The idea of coding weaves is then related to both computer programming and to geometry! Knots are studied by topolgists through a coding of the weaving patterns not unlike that used by weavers. Thus we shall draw a clear relation between practical arts and the development of abstractions and mathematical models.

Topology/Networks/3D

In relation to the recursive patterns of weaving (note that this harks back to symmetry) it is possible to have a three-dimensional model of the horned sphere and to explain its significance. The Alexander Horned Sphere is an example of a wildly embedded sphere in three-space whose "self-weaving" creates a topologically complex complementary space.



In order to expand this context ,we imagine a simple exhibit illustrating the Jordan curve theorem in the plane. Here a loop in the form of a circle is deformed through a series of pictures to a convoluted curve in the plane , that nevertheless divides it into two simple regions. [There will be a corresponding interactive demonstration in the computer grahics section - yet to be described.]

There are a larger themes here. The Jordan curve theorem is about inside-outside two-valued ness. Hence it connects with basic binary on-off yes-no. And the closed curve in the plane is the form underlying the Venn diagram.

There will be another computer demonstration that teaches the ins and outs of Venn diagrams- These are extensions of the single Jordan curve. Another theme is the difference between two-dimensional geometry and three dimensional geometry. The horned sphere illustrates this, but the same considerations show that any finite network can be embedded in three dimensional space. This is of great practical importance , and it can be illustrated in parallel with the horned sphere , by showing three dimensional embeddings of non-planar graphs.

Famous (unsolved) Problems

In regard to illustrating famous results and/or unsolved problems, we are working on how and what can be incorporated into the exhibit. Suggestions are: Fermat's Last Theorem, the Poincare Conjecture, the Collatz problem (a simply stated problem involving multiplication by three and division by two) , the Four Color Map Theorem,the apparent infinity of twin primes (twin primes are pairs of prime numbers that differ by 2, such as 3,5 or 29,31) , the three body problem, The Continuum Hypothesis (a problem about the "sizes" of different infinities).

In regard to the three body problem, there are available beautiful films by Richard Miller, of computer simulations of (thousands of) masses moving in space under Newtonian gravitation. These films illustrate aspects of dynamics, geometry , and relations with astrophysics and cosmology.

In regard to the matter of unsolved problems in general: They can often be very simply stated, and sometimes it is easy to produce empirical evidence. Thus, in the case of the twin primes, it is great fun to watch a computer program generate pair after pair of twins!

In some cases we can state a technical problem in an engaging way. the Poincare Conjecture is a case in point: The conjecture states that given a three dimensional space in which every closed loop of (possibly) knotted rope can be contracted into an arbitrarily small ball (without tearing the rope), the this space is topologically identical to ordinary euclidean three space. The conjecture remains unproved, while its analogs for all higher dimensional spaces have been shown true.

Moire Patterns

These are the visual interference patterns produced by overlapping regular patterns (such as wire mesh). The effects are very startling, and they can be related to projective geometry. The same idea in another context produces X-ray interferometry to reveal the atomic structure of a crystal lattice.

Physical Demo of Binary Count

There will be a small exhibit demonstrating counting via balls falling through a network of mechanical flip-flops. This can be operated by the visitor. It teaches counting as a dynamic process, and it shows the fundamentals of digital computing. Next to this exhibit will be a decimal system counter powered by a crank to be turned by the visitor. Just as the flip-flops demonstrate binary carry, the crank will show the familiar carry in the decimal system. Note relation to speedometer and odometer.

Analog Computation

A short history of analog devices will be given, ranging from the measurement of volumes by the displacement of water, to timekeeping devices, to the slide-rule and its implementation of multiplication (through logarithms and physical addition of lengths).

A water displacement device will be demonstrated, that can solve ,quadratic and cubic equations by dipping cones and other objects in liquid , until a desired water level is obtained.

Cymatics

The tern Cymatics was coined by the experimentalist Hans Jenny. The theme involves the patterns prodiced in physical media through vibration. A classical example is the nodal patterns delineated by sand on a vibrating plate. For this exhibit we suggest the patterns induced on a liquid through vibration. (An easy kitchen experiment-- drag a plastic cup of coffee along a formica table top. Watch the liquid surface.)

The Cymatic experiments illustrate deep relationships between simple natural phenomena and mathematical structure. Just as in watching the movement of water in a stream, the play of light on a surface, or the movement of trees in the wind, these experoiments raise more questions than they answer.

Microcomputer

There will be a microcomputer(s) with a large viewscreen for onlookers , and a regular screen for its operator. The computer will be supplied with a friendly menu and a near-infinity of activities ranging from games ,to teaching programs, to interactive demos of various topics. Along with computer demonstrations that we have already mentioned, here are some further ideas:

Graphic illustrations of the recursive soutions of quadratic and cubic equations. Relation to imaginary numbers and their geometry.

Real quadratic iteration and the period-doubling bifurcation to chaos. (Feigenbaum's constant.)

Geometry of complex iteration. Julia Sets (a particularly beautiful class of fractal patterns, generated via complex numbers).

A simple recursion that approximates the golden rectangle by whirling squares.

Demonstrating Fourier series with computer, oscilloscope, microphone and music.

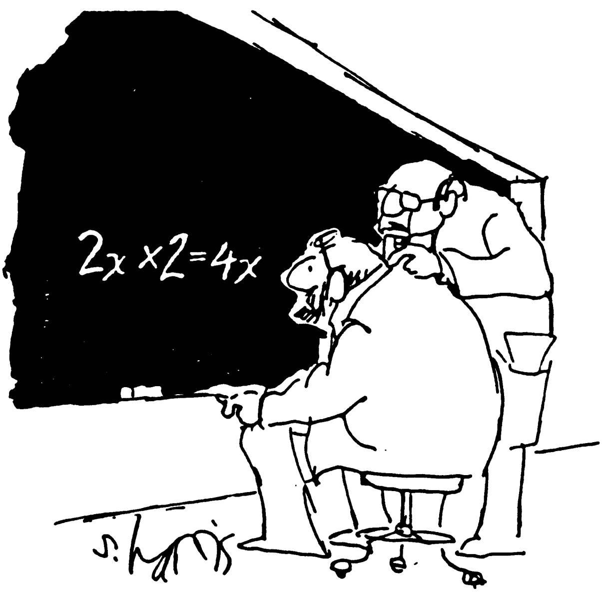
Demonstration of image focusing using the Fourier transform (this technique was used in the Mars Lander photography.).

Length measurement and fractal dimension. The length of a coastline depends upon the size of your measuring rod!

Computer drawings and illustrations of conic sections. Relations with planetary orbits.

Ray reflections from curved and straight one-dimensional mirrors. Exhibit visitor chooses contour and gets to see the pattern of reflections. In regard to this we can have a physical exhibit showing patterns of light caustics.

Geometry of inversions and hyberbolic geometry (computer films available from Maria Mezzina).



"But Gershon, you can't call it Gershon's Equation if everyone has known it for ages."

Proof

One major feature that distinguishes mathematics from other sciences is its concept of proof. There will be discussion of the nature of proof, the idea of axioms, and some simple examples of proofs. It will be pointed out how attitudes toward axioms have changed historically from the position that axioms are self-evident truths to the position that axioms are statements of structural relationship.

Proof in mathematics is not only characteristic, it is neccessary! Thus phenomena such as the irrationality of the square root of two (it is not possible to find integers M and N such that (M*M)/(N*N) = 2) demand a convincing argument. One counterexample can spoil the best of theorems!

Proof is deeply tied with the notion of infinity. Most proofs actually encompass an infinity of special cases (there are infinitely many right triangles , and only one Theorem of Pythagoras). The power of a theorem lies in its capacity to encapsulate an infinity.

We propose an exhibit that leads the viewer on a "walk through the Pythagorean Theorem" , going through the steps and images of Euclid's classical proof. Here the interest is both historical and geometrical.

The limitations of formal proof and the relation with numbers and coding forms the subject of Godel's Theorem. This topic is deeply related to the themes of paradox and recursion. Since these themes occur throughout the exhibit, we plan a series of placards about paradox,proof,self-reference and language that are self-explanatory. These placards will create relationships among ideas of coding, mirroring, and the structure of Godel's unprovable theorem. (Godel showed that any formal system that has sufficient structure to handle ordinary arithmetic also has enough power to mirror itself and thereby contain a theorem(statement) whose content is "I have no proof in the system.". Such a theorm is seen to be true from outside the system, but is neccessarily unprovable within it.)

Godel's Theorem, like no other before it, has led to deep reevaluation of the notions of proof and provability.

Pythagoras

A good subject for looking at proof is the theorem of Pythagoras. [The area of the square on the hypotenuse of a right triangle is equal to the sum of the areas of the squares on the other two sides.]

There are many proofs of the theorem, some old , some new. The ones involving tesselations , or dissections are appropriate for a museum exhibit. I am particularly partial to using tesselations and thereby relating the theorem and its proof to the study of symmetry.

The theorem has become a cornerstone of geometry, forming the background for the concept of distance in Cartesian geometry, and the backdrop for the new distance relations in Relativity. In its own day, the Pythagorean theorem was central to the "paradox" of the irrationality of the square root of two (since by this result the diagonal of the square of side one has length square root of two).

We continue to be delighted by proofs of this theorem.

And that is an indicator that the search for understanding is incomplete, and must go on. In joy and light!

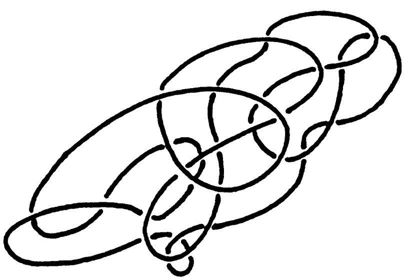
Epilog

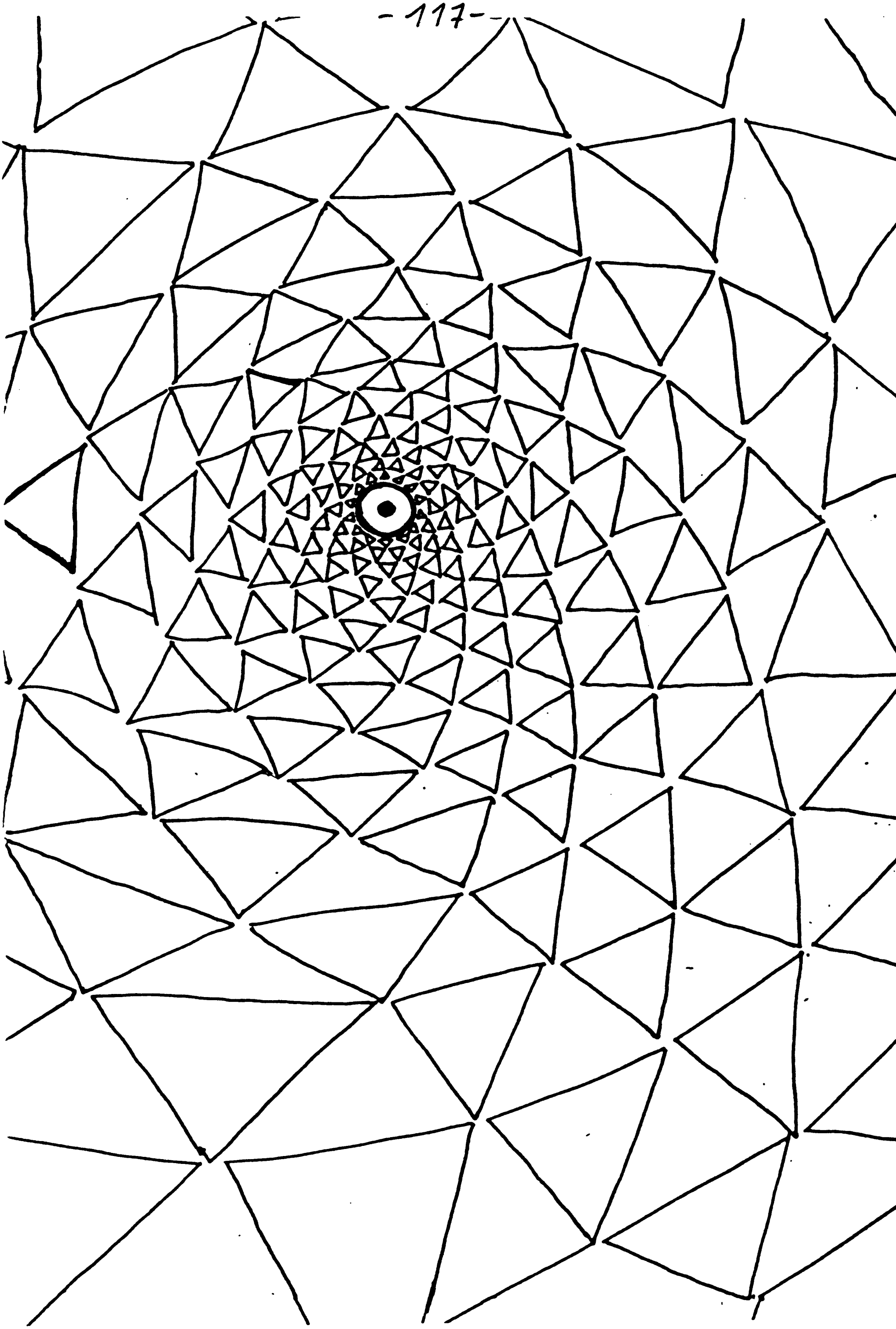
We'll stop here.

These have been notes on the possibility of a mathematics exhibit.

Mathematics is the study of pattern.
Mathematics is the creation of pattern.
And we are all mathematicians.

The Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago
Chicago, Illinois 60680
June 30, 1985





MAKING A MATHEMATICAL EXHIBITION

by **Ronnie Brown and Tim Porter**
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Abstract: We explain how we came to be involved in making a mathematical exhibition; some mistakes we made; and the philosophy we evolved on the principles which underlay our presentation. The main conclusion is that the exhibition should tell a story and aim to give an impression of the character of mathematics.

For the last three years, a team in the School of Mathematics at the University of Wales, Bangor, have been designing the exhibition Mathematics and Knots, which is exhibited at this ICMI89 meeting at Leeds. In this paper we would like to explain what we were attempting to achieve, and the problems we had in getting to this stage.

We do not claim to have achieved all our aims, or to have reached a final version. The exhibition will be useful if it is enjoyed by the public and by mathematicians. We hope it will also stimulate others to think about the problems of exhibition design in mathematics and to encourage them to prepare for themselves presentations of mathematics in a variety of topics and media.

1. How it began

Our involvement in making a mathematical exhibition came about in the following way.

One of us (R.B.) was invited to give a London Mathematical Society Popular Lecture on knots, one of two lectures in an evening, for June, 1984. It seemed a good idea to have material to display in the foyer for people to view when they arrived and in the coffee interval, so coloured enlargements were made of slides of knots in art and in history, and also a few of the overhead transparencies used in the lecture were enlarged to A3 size. All this material, and some models of knots made of copper tubing, was rather randomly distributed over some display boards.

During the next year we accumulated more material, which was used successfully for an Open Day for the Centenary of the University College of North Wales, in Royal Institution Mathematics Masterclasses for Young People in Gwynedd, and was also shown as an accompaniment to a Mermaid Molecule Discussion (again by R.B.) in London in November 1985, again on "How mathematics gets into knots". In the Summer of 1986, it turned out that there was some money left over from the grant by Anglesey Aluminium to our Masterclasses. Permission was given to use this money to develop the exhibition in a more professional way, by bringing in a designer. We also set up a design team of R.B. (Chairman), T.P. and Nick Gilbert (now at Queen Mary College). It was the continual development of ideas from this team, with a full and frank range of suggestions and criticisms, from within the team, from our designers, and from others who have seen previous versions, which has led to the exhibition you now see.

Versions and parts of the exhibition were shown in 1986 at the Royal Institution in January, and at the British Mathematical Colloquium in April; in 1987 at the Eisteddfod in Pwllheli (a Welsh version of a part) and at the British Association for the Advancement of Science at Belfast in August; on the occasion of talks by me and others at various universities and schools; in 1988 at the Royal Institution to accompany a Discussion by Sir Michael Atiyah on "The geometry of knots". On the last occasion there was also an exhibition of some John Robinson sculptures, of Celtic Interlacing by Lady Wilson from the British Museum, and of knots by Geoffrey Budworth of the International Guild of Knot Tyers.

2. Major problems

The design team and the designer had not realised what we were taking on. The design work has already extended over almost three years, and has involved many drafts. The reasons for this were of several kinds.

2.1. Novel aims

We set ourselves some novel aims for the exhibition (see section 3). In so doing, we were setting out into uncharted territory, and had to learn as we went along.

2.2. Lack of design experience

We knew not even the elements of the mechanics of producing an exhibition, yet our aims required a close marriage of content and medium. So we had to learn some of the problems and techniques as we went along. We were not able to employ the designer for the continued help we needed because of the lack of funds.

2.3. Insufficient funds

In order to finance the exhibition further, we wrote to a number of firms and organisations, and obtained a lot of help. A list is given at the end of this article. Also, when the Committee for the Public Understanding of Science started in 1987, we made an application to them and were fortunate to obtain a grant of £2,000. Without this total sum of about £3,860 the exhibition would not have been able to be produced.

To produce anything to a professional standard is expensive. An estimate we had recently from a member of a Government department was that a 30 square metre exhibit for Olympia had cost £20,000, of which half was for design and half for construction. In fact our designer gave us a lot of help for very little financial reward. He suggested the use of polystyrene boards with aluminium surround for durability, lightness, and transportability. However it gradually became apparent that his proposed layout with two columns per board was not appropriate for the exhibition, where the graphics had to predominate. The layout was considerably revised with the help of another designer. Finally, on the advice of the head of a local Arts course, we paid for the layout to be redesigned yet again using a grid approach, in which all modules of text and graphics fit into a centimetre square grid, and for all the knots to be redrawn to have a three dimensional effect and so as to fit into the modular layout.

A severe problem is printing costs. To produce an A2 board one first produces an A4 board with the text and graphics. This is photographed to an A2 negative from which the print is made. The enlargement from A4 to A2 (four times the area) means that the typesetting has to be with 1200 dots per square inch: the usual output of a standard office laser printer is quite unsuitable for this job. Commercial computer typesetting is expensive. The only way of proceeding within the budget then available was to use facilities in the University system.

At that time, the only system available was the Oxford Lasercomp system. This is an excellent system for producing a book in one or another standard text format. It also has a good variety of fonts, though not so many as commercial printers. It works however on what is called a "markup" system, where all control is by inserted commands in the text. This has the advantage of ease of sending files electronically. At first we simply sent down the text, so that the layout was achieved by pasting. Then we put the commands into the text to achieve the desired layout. However, the disadvantages of a markup system for complicated page layouts are great, since you do not see whether or not the layout commands have been correct until the proofs return days or, at difficult times, a week or so later. Further, the combination of graphics and text can only be produced by photographing large drawings to a small size for pasting on the A4 page, so that the enlargement produces a good quality product. Thus the design has got to be completely decided, and there is little room for experiment.

In the end the current production would not have been feasible without the use of a Desktop Publishing system, in this case Pagemaker®, with Postscript output. The graphics were drawn, scanned, and then positioned within the text, allowing tight control of spacing and layout. However there is at present a smaller range of fonts available than in the Lasercomp system. For example, the standard range of fonts does not include an extra bold. A further problem was that in each case the techniques had to be learnt from scratch. The current boards are typeset at the University of London Computer Centre.

2.4. Stand-alone exhibition

The key difference from the design viewpoint between the foyer exhibition and our new aims was that we intended a travelling exhibition independent of any lecture. This meant that the boards had to be self-explanatory, and not just titillation for an explanation to be obtained later in the lecture. So we had to decide what story we wanted to tell. It was at this stage that we began to formulate and clarify our aims, while we were designing the text and graphics for each board.

2.5. Visual impact

The visual impact in an exhibition has to dominate. Each board has to tell a story, but the story has to be told mainly through the eye rather than through the text. So the exhibition format is one of the hardest to get right for the conveying of ideas, rather than simply the presentation of images.

2.6 Use of language

It is very easy for a mathematician to use words and phrases which mean something to him but which convey nothing to the general public. As an example, we found ourselves using the phrase "uniquely up to order". In fact, this is a sophisticated idea, which we eventually conveyed by specific examples with numbers. The aim was to use a simple and clear language. The stripping of inessentials and unclear language was a part of the design process. On the other hand, we also found that some ideas needed a more leisurely exposition than we gave them at first.

3. Aims

The decision as to the aims of the exhibition was crucial. The issues involved are part of "exhibitology". This term has been coined by Len Brown (R.B.'s brother). He learnt the basics of this study in his period as Head of the Engineering Section of the Science Centre at Toronto, and suggested the following simple example as illustration of some basic principles. Suppose that you wish to produce an exhibition on "Bridges". There is certainly a lot of material available. However a decision has first to be made on the point of view to be taken. Is the exhibition to be about transport? structure? geography? history? rivers? trade? Each of these themes could lead to an acceptable exhibition, the various exhibitions would have much common material, but in the end each would be telling a different story, and give the visitor a different impression of the subject of Bridges.

In our case, we had to decide what impression of mathematics we were intending to convey, and then seek to find the means to do so.

The aims we set for our exhibition fell into two kinds: structure, and content.

3.1. Structure

We agreed that the exhibition should be:

a) self contained
b) easily transportable
c) reasonably cheap to produce
d) reproducible in several copies
e) able to be set up and managed without continual supervision .

3.2. Content

We agreed that the exhibition should:

a) suggest that the making of mathematics is a natural human activity, part and parcel of the usual methods by which man has explored, discovered, and understood the world

b) present each item with a purpose and context, and not just because it was something that could be shown or demonstrated

c) convey an impression of some of the key methods by which mathematics works

d) show mathematics in the context of history, art, technology and other applications.

4. Consequences of the agreed aims

4.1. Structural

Our requirements tended to rule out hands-on material, at least for the moment. Such material is expensive to produce and maintain; if it does not work it is worse than useless; it can be stolen, and indeed this is more likely the more attractive the material. In any case hands-on material can also suffer from being a *gizmo*, designed because it is hands-on rather than to make a point which illuminates the themes of the exhibition; nice to play with, but superficial. The participant is expected to exclaim "Wow!", but there is still a question as to what he or she has learnt. Of course this tension between the requirements of entertainment and arousing interest, and the requirements of instruction and information, is basic to the whole activity of exhibition design.

Our requirements also meant that we were initially intending a static exhibition: something to be looked at, and enjoyed, but not involving an activity. Once the structure and the content had been decided, it would still be possible to design hands-on or animated material which would advance our overall mathematical aims and which could be used as occasion demanded and allowed.

4.2. Content

It was in terms of content that we felt we were taking the more radical line, and the various features of these aims deserve separate paragraphs and discussions.

Mathematical content

1. Mathematics and discovery

The novelty and excitement of mathematics is conveyed by some of the major mathematical exhibitions (Horizons Mathématiques in Paris and also as a travelling exhibition, and the Mathematika at the Boston Science Centre). However we felt that it is helpful to analyse the basis for mathematical novelty and excitement, in order to clarify what we were intending to present.

We felt that this excitement and interest comes from the vision of new relations, and new kinds of order or patterns. It is not the whiz! bang! excitement of the amusement arcade. For example, it is extraordinary that the number π is involved not just with the ratio of the circumference of a circle to its diameter, but also with the description of population distribution. It is extraordinary that whereas we think a negative number cannot have a square root, such does exist if we allow a new kind of number, and even more extraordinary that these new numbers should have applications to the study of prime numbers, to the design of electronic circuits, to cosmology, and to the study of elementary particles.

An exhibition should convey some flavour of the real achievements of mathematics. If instead it simply presents an assortment of, for example, strange polyhedra, and states that these are the wonderful things mathematicians study, then it will be very easy for the public to be convinced that mathematics is hard or weird or both. Each exhibit should have a mathematical point and should explain its relations with other parts of mathematics and with other disciplines.

In the case of this exhibition, we felt the most surprising idea that could be conveyed in a way related to everyday experience was the analogy between knots and numbers revealed by the notion of a prime knot. It is for this reason, as well as for the needs of exposition, that we devote more than one board to this topic.

Surprising applications are also important for conveying some of the excitement of mathematics. In this case, we stress recent applications, such as to knotted orbits in weather systems, and to knotted DNA.

Key aims of mathematics are to show new perspectives, views, and order in what seems initially a tangle of unanalysable phenomena. This is one impression of mathematics that we wish to convey to the viewer.

2. Normality of mathematical methods

Here is where we feel we are really breaking new ground. Mathematics lacks an adequate discussion of methodology. A majority of students of mathematics do not know what they are doing or why they are doing it; they know only that they have to learn how to do certain things. Very few university courses attempt to explain the reasons for the development of a particular piece of mathematics, in some cases because the teachers are unaware of these reasons. However, experience shows that an analysis of the particular methods used in mathematics, and a relating of them to standard methods by which we explore and manage the world, is welcomed with relief by school teachers and students, who often seem starved of a global viewpoint.

It has been said that the difference between a professional and an amateur is that an amateur can do things, in many cases as well as a professional, but a professional also knows how he or she does things. It is this knowledge, based on tradition, experience, perception, judgment and analysis, which gives the professional the confidence to produce work on demand and to certain standards.

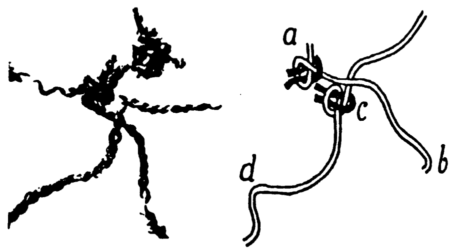
Of course, in this exhibition we cannot hope to convey the whole gamut of mathematical ways of working. We are not interested in conveying technique. What we want to express is the mathematical equivalent of musicality - perhaps we should call it mathematicality? This is a horrible word, but its derivation should at least convey what is intended.

Often, mathematics is presented as a completed body of knowledge, whose development has been unrelated to the activities of human beings. The questions which motivated the whole theory in the first place are in teaching often simply omitted, and students and pupils are asked to appreciate the methods and the theory without context, without relevance to other mathematical or scientific activity, one might even say, without meaning. For example, how many books on group theory are there which mention the range of applications of group theory, from crystallography to modern physics, and which show how the exposition given fits into the wide mathematical and scientific context? The dehumanising of the presentation of mathematics has gone very far.

Our aim was to use the theory of knots to illustrate some of these basic methods of mathematics. Our listing and analysis of these methods carries no claim to finality. However such a listing is useful as a systematisation, and, more crucially, as a way of relating these mathematical methods to standard methods of exploration and analysis. Thus we illustrate the claim that the peculiarity of mathematics lies not so much in its methods, but in the material and the objects with which it deals.

3. The context of history and art

Our aim here was to remind the viewer of how knots have enormous richness and importance in the history of man. It has even been suggested that the Stone Age should be called "The Age of String". We were fortunate to have been told by Joan Birman of the oldest known knot: the Antrea net, dated 7,200BC, from the Helsinski National Museum. The net, found in a peat bog, was 30 m. by 1.5 m. with a 6 cm. mesh. It had stone sinkers and bark floats, and was made of willow bark. The knot used then is still used today.(See the paper *The net discovery of Antrea* by J.-P. Taavitsainen and M.Huurre, National Board of Antiquities and Historical Monuments, Helsinki.)



One can only speculate on the social organisation and lives of the people who constructed this net, and on the length of time such a technological achievement took to evolve. One can also sense that the early understanding of the form of knots, and its link with survival, is an expression of an early but by no means primitive geometrical feeling, an understanding that the form not only can be so but has to be so, by virtue of its logic.

We had planned a series of boards on knots in history, practice and art. However we found that a full range of pictures was best presented as a slide show, and also most of the design effort has so far gone into the mathematical boards. One result of this interest, though, was the exhibition of John Robinson sculptures at this Congress [B]. We were also helped by information on Celtic interlacing from Nancy Edwards, who wrote a Ph.D. thesis on this subject [E].

5. Some mathematical methodology

We believe the following list of basic methods to be useful for our purpose:

a) Representation
b) Classification
c) Invariants
d) Analogy

e) Decomposition into simple elements

f) Applications

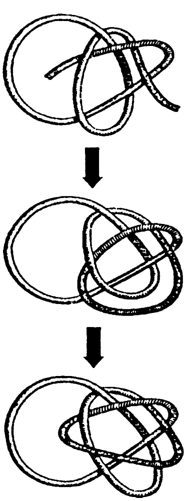
We now analyse these in detail.

5.1 Representation

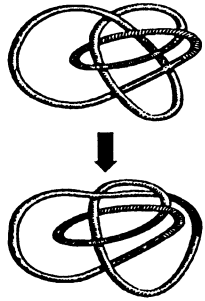
In the case of knots, we have to show them and present them. This is usually accomplished by a planar diagram of the knot. But the difficulty of this step should not be underestimated. For example, we have found it helps children to make the jump from the diagram to the knot and back again by giving them pieces of string and asking them to make the knots according to specific diagrams. In the exhibition, we use shaded drawings to give a 3-dimensional effect.

5.2 Classification

Making lists is a basic human activity. However, in view of the complexities of the world, you cannot make a list of everything, at least not if you are to lead a sensible life. You may recall the Memory Man described by Luria in [L] who was unable to forget anything, and consequently was not able to lead a normal life. One of the diagnostic features for autistic children is that they remember nonsense patterns as easily as other patterns [W]. So in making lists we impose or find order: we classify. For example, a zoologist does not list all the animals in a game reserve, he lists antelopes, elephants, lions and so on. In order to do so, he needs criteria for saying that two animals are the same. In mathematics the notion of equality, or, in more precise mathematical terms, equivalence, is basic.



Knot theory presents interesting mathematical points in this area. Firstly, "When are two knots the same?" is a non trivial question. Secondly, to make an initial list of the first elements of an infinite family involves some classification into the "simplest" elements. Thirdly, the presentation of even such a simple list is likely to suggest the need for some further order and classification.



We address the first question by showing in our boards how diagrams of knots can be transformed, without changing the knot.

This is one area where computer animated graphics could greatly improve the presentation. However, we are limited by the cost of producing the graphics, and the cost of presenting the graphics at an exhibition display. The latter is not so hard to overcome, since a video could be made and easily displayed, for example at most schools. However, it is more important for us to show the context in which any computer graphics would run, as this would dictate what graphics should be produced in order to support the overall themes.

One of the goals that we set ourselves was to explain the meaning of a list of knots. Although such a list is apparently simple, the explanation involves the following ideas: *when are two knots the same; crossing number; mirror images; the arithmetic of knots; prime knots*. These themes give interconnections between the different boards.

5.3 Invariants

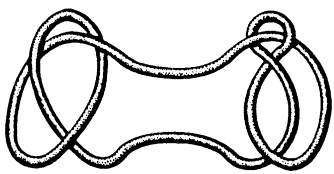
Classification of knots involves two aspects: when are two knots the same? and when are they not the same? The first usually involves transforming one diagram of the knot into another. The second involves the more subtle point of deciding when such a transformation is not possible. Such a decision involves the notion of invariants.

We deal with four invariants in our presentation: *crossing number; unknotting or Gordian number; colouring number; bridge number*. We also mention briefly the new knot polynomials which enable one to distinguish easily between a trefoil and its mirror image. The advantage of the four invariants we deal with in detail is that they can be easily presented at this level, and that they suggest many detailed exercises and examples which people can try for themselves.

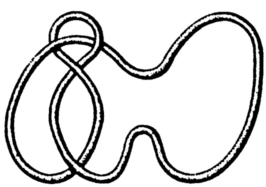
The further point made by the discussion of invariants is that we do not claim to give a complete set of invariants, that is, we do not have some method of distinguishing all possible knots. Thus many problems remain in the theory, and this again is a point which is easily conveyed. We do want the reader to see that mathematics is, and will continue to be, an open-ended activity.

5.4 Analogy

In order to explain our list of knots, we have to describe the notion of *prime knot*. This is done by explaining a basic composition of knots.



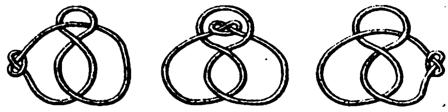
There was a decision to be made here: whether to call the composition of knots *addition* or *multiplication*. The literature uses both terms. We chose the term addition for two reasons. One was that the notation 0 for the unknot is more descriptive than the notation 1.



K + 0

The more important reason was to emphasise that the analogy is not between things but between the way things behave, between their relationships. For this it is helpful to have a different notation for the two operations which are being taken as analogous. Instead of making an analogy between two multiplications, we make an analogy between an addition and a multiplication. This, we hope, is more striking and also illustrates a general point, that such analogies may be available in other situations.

Although we use the term "The arithmetic of knots", we also use algebraic notation and emphasise laws.



K + L = L + K

This makes the point that it as the study of analogies that algebra obtains its generality and power. By using the commonplace word "analogy", we aim to demystify, and to show that one aspect of the method of algebra is as a standard process by which we understand and try to make order of the world. The other point to be made is of course the excitement of an unexpected analogy, of "That reminds me of ...!".

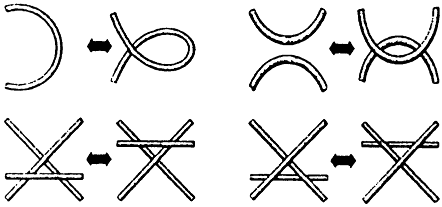
We are also able to state the deep fact that *knots have a decomposition into a sum of prime knots, and this decomposition is unique up to order*. The appreciation of this led one small boy after the Mermaid Molecule Discussion by R.B. to ask: "Are there infinitely many prime knots?". It so happened that the lecturer had not previously formalised the question for himself, so had really to think in order to be clear that all torus knots are prime, and that there are an infinite number of them. However the proof that torus knots are prime is not so easy. It is good to have something to state which is comprehensible and believable, but which it is not at all clear how it might be proved. This is one of the great advantages of knot theory for expositions at this level.

5.5 Decomposition into simple elements

Decomposition into simple elements is a basic process in mathematics, or indeed wherever one deals with complicated matters. In knot theory the process crops up in a variety of guises.

a) We have already mentioned the prime decomposition of knots. Here the prime knots are the simple elements and the fact that any knot can be expressed uniquely (up to order) as a sum of prime knots is clearly an important fact about knots.

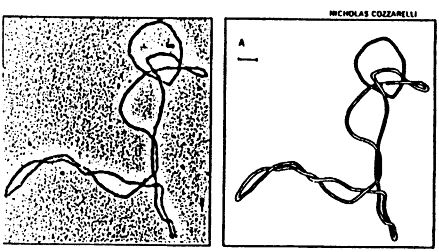
b) The process of transforming one diagram of a knot into another may be quite complicated. It is therefore of interest that such a complex process can be resolved into a sequence of simple moves, the Reidemeister moves.



We illustrate this in the process of changing the Bowline, and also in illustrating why the colourability of a knot is an invariant.

5.6 Applications

The exhibition starts with a picture of the sculpture Rhythm of Life, by John Robinson. It ends with an indication of some applications, including how knotting is involved in DNA, one of the building blocks of life itself.



This section on applications could be expanded, and made more vivid in a variety of ways, given funding, space, and so on. But all we hope to do in this exhibition is to catch the imagination of some people. The exhibition is intended to be small scale, and unambitious in its use of technique. Indeed, this is imposed on us by the criteria which we outlined at the beginning. Such limitations are still compatible with high expository aims.

6. Conclusion

We believe that if the exhibition is successful on the terms initially laid out, then it should be possible to build out from it as wider funding and staffing, and more ideas, become available. The honing of ideas and presentations, the discarding and developing of innumerable drafts, the criticism and comments from many, all have been valuable in clarifying our aims and our methods. In particular, we give our thanks to the long suffering designers, Robert Williams, Jill Evans, and John Round, who have been involved at various stages.

The existence of various drafts of the exhibition has enabled Heather McLeay to start designing a set of worksheets for young pupils [M]. Drafts of these worksheets have been used successfully at Royal Institution Mathematics Masterclasses in 1988 and 1989 at Bangor and Cambridge.

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Ronnie Brown, Nick Gilbert, Tim Porter

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Assistance

John Derrick, Jill Evans, Gwen Gardner, Tony Jones, John Round, Philip Steele, Robert Williams.

The Popularization of Mathematics

Mathematics Trails

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Mathematics has a negative image among a large proportion of the general public, usually governed by earlier school experiences, and it was with this in mind that a mathematics trail was set up around the centre of Melbourne to support an attempt by the Australian and New Zealand Association for the Advancement of Science (ANZAAS) to popularize mathematics and science during their annual congress in 1985. The trail was designed and implemented by members of the Mathematics Education Centre at Monash University with the help of Diploma in Education mathematics students. The trail was mainly intended for parents and children and was run during one week of a school holiday. It was such a success and caught the imagination of the education and wider community to the extent that many similar trails, based on this early model, have been produced throughout Australia and South East Asia in a range of venues and with a variety of themes.

The philosophy behind these early mathematics trails was that they should not be overly competitive, certainly not like orienteering, and that they would use the environment to stimulate the learning of mathematics. Above all they were intended to be fun. They were seen as a way of developing an appreciation and enjoyment of mathematics in everyday non-threatening situations and, in the case of school children, as an extension of work more usually carried out in the classroom. In the event they seem to have achieved not only their initial limited aims but have created the beginning of a new philosophy and methodology for teaching and learning mathematics in the broadest sense. For those involved in their design and implementation they have become part of a planned process of popularizing mathematics for people of all levels of ability and all ages.

Typical of the activities for the "Mathematics Trail Around the City of Melbourne" (Appendix A) was an investigation of the value of π (pi) in one of the main shopping malls using a pattern of concentric circles made of bricks. This very simple investigation, which is almost standard in most school textbooks, not only created a great deal of interest among the general public and the media, many of whom "discovered" something that had been a mystery since their school days, but also provided an insight for a number of the mathematics graduate student teachers who were helping with the trail. In particular they gained an insight into the use of informal units of measure in ratio and the way in which mathematics can be learned and understood outside the classroom. Other activities in this trail such as the "Challenge Question" to find the approximate height of the Cathedral spire using reflection in a pool, the height of the observer and pace lengths also aroused a gratifying amount of involved interest and activity.

During the week that it was officially open about two thousand participants completed the trail and within the next school term the remainder of the ten thousand copies of the trail, printed by the Age newspaper as one of the sponsors, were distributed to local schools on request and used by them. Since then many more copies have been photocopied for use by schools and by family groups.

From the subsequent experience of constructing further trails, many of the types and style of questions used in the earliest examples would not now be used in the various trails currently being developed around Australia. The early activities were very much in the mould of textbook questions translated to a different environment. Many of them were interesting and stimulating but an evaluation of the earlier work, together with recent experience, suggests that there is a great deal more potential than had first been thought in this type of mathematical activity. The earlier work on trails did, however, provide a catalyst and the impetus for a whole range of attempts to popularize mathematics, particularly through the generous financial backing of a large industrial concern, CRA Limited. This sponsorship made it possible to carry out a series of workshops and in-service activities throughout Australia with teachers and parents interested in gaining a better insight into the exciting possibilities presented by mathematics trails.

Many different groups have now produced trails, many of them around shopping complexes and town centres. While these have often been useful first attempts have been limited in both the possible mathematical examples and their level of motivation. It was not until an enthusiastic group student teachers were given the opportunity to create mathematics trails around Sovereign Hill, a recreated gold mining town near Ballarat on the site of the Eureka Stockade rebellion that some of the more innovative possibilities were discovered and a new level of involvement and motivation, particularly in the use and learning of mathematics across the curriculum, were developed.

The following is an example taken from a mathematics trail produced by a student teacher for school age children as part of an assignment during a visit to Sovereign Hill. The role playing has a distinctly Australian flavour with the participants being put in the situation of an old-time gold miner:

> "You've decided to try your luck at gold panning in the creek. But before you have a go, you will need to buy a Goldmining License. Go to the post office:
>
> a. How much does a license cost? (old money)
> b. How much would this be, when compared to todays costs? That is, would the license have cost the miners a lot of money?
>
> Congratulations! You found a small nugget of gold in the bottom of your pan.
> Hurry to the Colonial Bank and Gold Office in Main Street to find out just how rich you've become:
>
> a. How much per ounce are they paying for gold?
> b. How much will you get for your two ounce nugget?

> You have left your nugget in the Bank's safe. You feel like treating yourself to a new outfit. Go to the shops in Main Street........"

There were no prizes for guessing the gender of this goldminer.

Since then the Mathematics Education Centre has been asked to design and set up a mathematics trail as a permanent feature at Sovereign Hill for the general public as well as for visiting school groups.

At another historic location a question that caused both interest and amusement, as well as some mathematical calculations, concerned the possibility of firing a 30 kilogram cannonball from an ancient cannon mounted on a cliff top across a seaside bay to hit a newly built hotel on the far shore. After some experiments comparing the mass of a cannonball with well known objects and with the aid of a scale map a number of concepts, including estimates of 1.6 kilometres, were reinforced for the children on that particular trail.

Most recently the possibilities of setting up problem solving and investigative situations have been explored at sites like these. Rather than moving around a trail to complete a number of fixed activities participants are able to select one or two projects that they may pursue in depth, possibly as part of a more formal project requirement in the case of school pupils. In addition trails are still often used at these sites both for making people aware of the possibilities and style of these types of activities as well as to create interest and motivation.

As part of a local initiative from teachers and the local civic authority in another Australian coastal town, a mathematics and science trail has been developed in the existing children's adventure park. The local Tourist Board in recognising the potential of this gave financial sponsorship for the trail enabling among other things the addition of three new bridges to the existing four over the waterways which criss-cross the park to re-create the classical Bridges of Konigsberg problem. The opportunity to try out this problem as part of a practical experiment before resorting to paper and pencil and the more formal aspects of the exercise indicates a good example of some of the possibilities and potential for mathematics trails. In pedagogical terms it meets many of the requirements for understanding and gaining an insight into mathematics while at the same time it provides a good example of the way in which mathematics can be popularized by making an interesting classical study both familiar and accessible to different groups of people, including families.

In the same town a re-created historic harbour has been used to provide an opportunity for another trail and project work in mathematics. One of these reflects the notoriety of this dangerous coastline for the early sailing ships. Based on "The Shipwreck Museum" located at the site a set of activities on "The Mathematics of Sailing Ships and Shipwrecks" has been developed by a local school to help meet the project work requirements of the school leaving examinations in mathematics. At the same time a trail has also been developed at the site for the participation of younger children and the general public.

Other notable Australian venues that have been used to develop mathematics trails include the new Parliament House in Canberra. The very unusual design and geometrical patterns to be found in this controversially designed building and its surrounds inspired a group of local teachers to develop three mathematics trails which, with sponsorship from a textbook publisher and the help of the Parliamentary Education Office, resulted in some attractively produced trail booklets. These have been flexibly designed to allow their use for visiting school parties or family groups and they can also be used as starting points for more in-depth projects.

Currently a mathematics trail is being developed around a large zoo in New South Wales which has revealed further exciting possibilities for popularizing mathematics. From a discussion with a young veterinary surgeon employed at the zoo it was discovered that all the feeding, as well as the medical treatment, is based not only on the type of animal but also on individual body weights. This became even more mathematically interesting when it was revealed that traditionally vets take various simple measurements from certain animals which are then substituted into algebraic equations. For instance the weight of an elephant can be accurately estimated by measuring around its foot and substituting the result in a particular equation. From this an activity for the mathematics trail was developed which first requires a visual estimate of the weight of a particular elephant followed by measuring around a plaster cast of the elephants foot, made available on the wall of the viewing area. Similar activities have been developed, where appropriate for other animals, including the popular question on how to weigh a Joey. Another activity developed at the zoo for younger mathematicians allows them to compare their running speed with those of a number of animals in the context of whether they could escape if chased. A path has been marked out with the distances each animal can run in ten seconds and participants in the trail are encouraged to compare their own running prowess with the animals.

A number of other mathematical and innovative ideas have been incorporated into other trails. One that has real applications of mathematics involves the designing of a "Survival Trail" between a child's home and school. Based on the premise that there is a safest route for each pupil to travel to school this is investigated by assessing traffic flow and other variables at critical points. This is sometimes carried out by individuals, often with the help of older members of their family. As a mathematical exercise this has been found to be extremely motivating and while not strictly a trail does give some point to the well known school activity of counting all the vehicles passing a certain point.

The evaluation of the effectiveness of mathematics trails is difficult to measure in a formal way. At one level they can be assessed by observing the smiles on the faces, the level of involvement and the comments at the end of the trail about it not being like "proper" mathematics and requests for more activities. At another level they can be evaluated by some of the feedback such as the young boy who after spending a day trying to find a clock with the Roman Numeral IV on the face, rather than the usual IIII, wrote to say that he had spent several weeks investigating this phenomena, which had been discovered on earlier trails. His voluntary self-imposed project and written paper was based on his research in libraries and clock repairers and which resulted in a very plausible explanation of this historical and mathematical oddity.

Other interesting feedback has come from schools and other groups who send in a continuous stream of copies of new mathematics trails. One favourite arose out of a workshop on mathematics trails run at a primary school which resulted in a trail around the school grounds for the children. This example was then used by the children to develop individual mathematics trails for their parents to complete on Open Day. The inventiveness of young children as young as six and seven years old when asked to develop mathematical activities can be quite sophisticated when the right motivation is found.

Teaching and learning strategies involving the environment and mathematics can fall into a number of categories. Learning mathematics in and from the environment appears to increase motivation for the student. Mathematics can be seen to be part of the real world and an awareness can be developed of things which may previously have been ignored or missed. Using mathematics to learn about the environment is also valuable as well as the possibilities of using mathematical problem solving, modelling and decision making to find ways of improving the environment. Mathematics trails are one way of allowing a wide range of people of all ages to become involved in mathematics in a non-threatening way and as such deserve to be included among the range of activities used to popularize mathematics.

One particularly gratifying aspect of developing mathematics trails has been the level of imagination shown in producing even better and more innovative examples once the process has been taught. Another interesting benefit has been the improved perception shown by the general public towards university academics and school teachers who have attempted to make mathematics more accessible and interesting through the design of mathematics trails. The interest of the media, both television and newspapers, in the concept of mathematics trails has been most rewarding, owing in no small part to the use of well known personalities like Sir Edmund Hilary who have been prepared to not only launch mathematics trails but to have been seen taking part in them. As the then Governor-General of Australia, Sir Ninian Stephen, said on television when opening a series of trails in Canberra, "I've launched a number of things in my time, but never a maths trail. I'm not sure how you do it!" Some time later, still with full media coverage, he was prised away by a harassed aide from an animated discussion with two young girls on the mathematics of growth in grape vines. His farewell request was that he should be sent a book explaining the Fibonacci sequence which he had just discovered for the first time.

APPENDIX A

EXTRACTS FROM THE MELBOURNE TRAIL

A. STATE BANK CENTRE

From the State Bank Centre cross Bourke St and then Elizabeth St carefully so that you are on the corner diagonally opposite the State Bank Centre and look back at it.
Without calculating write down a quick estimate of the number of small windows you can see in the side of the tower facing you A1

Now calculate the number of windows A2
(Hint: Count how many there are in each row and the number of floors.)

A window cleaner takes ¼ hour to clean each window. If he works for 8 hours each day, 5 days per week, how many weeks will it take to clean all the small windows on the building? A3

B. GENERAL POST OFFICE (GPO) — OUTSIDE

Walk about 50 metres up Elizabeth St. (North) and find the stamp machines outside the GPO at the top of the steps. The one on the left takes coins up to 50c (1, 2, 5, 10, 20, 50).
To be able to purchase the lowest value stamp provided by this machine, calculate:

(i) the greatest number of coins you could use? B1

(ii) the least number of coins you need to use? B2

In how many different ways could you put coins to the value of 10c into the machine? B3

On your left, beside that machine you will see details of posting times. If you post a letter to Perth on a Tuesday at 5.00 pm, would you expect it to arrive on the next working day? B4

If you post a letter to Sydney at the same time, will that one arrive on the next day? B5

C. GENERAL POST OFFICE (GPO) — INSIDE

Enter the GPO Building from Elizabeth Street. On your right is the Daily Weather Report.
From this:

(i) What is the rainfall for Melbourne so far this year, in millimetres? C1

(ii) Is this above or below the average for the past 129 years? C2

(iii) Estimate the average monthly rainfall so far this year C3

(iv) If rain continues to fall at the same rate for the rest of the year, estimate the total rainfall for this year C4

(v) What might cause your answer to be different from the actual answer?

....

....

.... C5

The River Report is also on this wall. From the information given, calculate:

(i) The height of the Murray River at Bringenbrong.. C6
(NB. Use another town on the Murray River if the Bringenbrong figures are not available.)

(ii) How far, in metres, does the Murray have to rise at the town that you used before it floods? C7

CHALLENGE QUESTION?

If the change in the height of the river for the past 24 hours continued at the same rate for some time, calculate how long it would be before the river:

(i) floods? (if the change is positive) ....

(ii) dries up? (if the change is negative).. C8

Moving further into the GPO from the entrance, you will see hundreds of small private mail boxes on your left. Look at the top five rows on Board A. If we use a co-ordinate system with 1 to 36 along the bottom and 1 to 4 up the left hand edge, then Box 5A is at position (1, 1).

What box is at (17, 3)?.. C9

Give the co-ordinates of box number 27A.. C10

Leave the GPO by the same door and walk down Elizabeth St. and enter the Bourke St. Mall. Re-enter the GPO through the Mall entrance. In the passage there is a plaque on the wall on your right. Assuming that Victoria is 150 years old in 1985, calculate how old the State of Victoria was when the Postal Hall was established?.. C11

Behind you is a painting of the only English Captain who came to Australia and failed to take a wicket or score a run.

Write down his name C12

D. BOURKE STREET MALL

Go back down the steps into Bourke St. Mall and walk past the David Jones Store.

Move to the front of "K-K-K-Katies", watching carefully for trams. Find a circular pattern in the paving.

How wide across the middle is the smallest circle in brick widths? (This is called the diameter (D)) ....D = D1

POPULARIZING MATHEMATICS AT THE UNDERGRADUATE LEVEL

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This is a personal account of an attempt, and failure, to popularize mathematics in the undergraduate community of a university. We hope to convince you that this was an exciting adventure, with much scope and promise, and to warn you of the political problems that arose and finally aborted the project. The idea was to offer a mathematics enrichment course for students who had completed freshman mathematics but were pursuing careers in other disciplines. Two particular target groups were in mind: the first consisted of students and alumni we meet occasionally who are still enthusiastic about mathematics. Although they never went beyond freshman mathematics they can still wax nostalgic about their former romance with mathematics. They want to know more about mathematics. The second was a more nebulous group, students who might be destined for positions of influence for funding mathematics at the national level, and therefore presumably would obtain university degrees and see something of mathematics along the way.

To the first group we hoped to offer a course that was exciting, satisfying and gave a broad perspective on the role of mathematics. Mathematical in nature, but naturally not a standard, technical mathematics course. To the second group, we hoped to show the centrality of mathematics, both historically and presently, and to convey a picture of a living, evolving subject with many questions and issues yet unresolved.

The catalyst for the course came three years ago, when it was mathematics' turn to choose a speaker for the annual science lecture at our university. Sir Michael Atiyah accepted our invitation, and proposed to speak on "Geometry and Physics: Euclid, Einstein and Elementary Particles"[1]. The auditorium was unexpectedly packed, with people sitting in the aisles (my wife arrived at the last moment and was turned away for lack of space). In previous years, talks in other areas of science drew slightly better than half this size. A mathematics talk outdrawing talks in computing, bio-engineering, or nuclear physics? How unexpected; how delightful! Flushed with such success, we sat down to draw up a course outline based on the topic of geometry and physics. It would trace the evolution of mankind's ideas on the geometry of the universe.

Here is a description of the course and an outline of the thirteen weeks it would be offered (three hours per week).

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[1] a copy of the poster (designed by mathematicians) advertising this talk appears in the appendix

Week 1: Euclidean Geometry: Pythagoras; Unique parallels; sum of angles in a triangle; locus equidistant to a line; Greek outlook on the cosmos, history and attitudes of Greek geometry.

Week 2: Spherical Geometry: bounded lengths, bounded areas, angles and lengths equivalent; no parallels, no similarity; spherical trigonometry and navigation; history of spherical geometry, non-acceptance as a Non-Euclidean geometry.

Week 3&4: Analytic Geometry: Descartes and higher dimensions, concept of distances; cosmology of Kepler and Newton, elliptic orbits; "The Elements" and the "Principia".

Week 5&6: Hyperbolic Geometry The Poincaré model; many parallels, no similarity; angles and length unique natural length element. History of the parallel postulate: development of facts and attitudes. Three dimensional hyperbolic space.

Week 7&8: Minkowskian Geometry: New metrics on R^4; constant speed of light and other background to special relativity and space-time; the paradoxes. Relation of Minkowskian Geometry to Hyperbolic Geometry.

Week 9,10,11: Manifolds: Definition; local vs. global properties; scalar, vector and tensor fields; the metric tensor, relation to classical geometries. Curvature: extrinsic and intrinsic; the curvature tensor.

Weeks 12: Geometry of the Universe: Interpretation of a force as curving space-time; energy-momentum tensor; Einstein equations. Higher dimensional manifold to accommodate all forces of nature; singularities in space-time; black holes.

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Texts

The book by Lanczos covers all the ideas of the course. It may be heavy going for a student with the bare prerequisites, but would be suitable for the instructor and some students. The book by Gray covers essentially the same material and would be useful to all students but would not have a high enough mathematics content. We would like a book halfway between. This could be achieved by taking three of the books in the reference list (Greenberg, Penrose and Fredericks) but not in any one yet found.

As our enthusiasm grew for the subject matter of this course, we came to see how ideal it was as a vehicle for popularizing mathematics. Geometry of the universe begins with the most approachable of topics - geometry - and links it with a higher-order desire, our need to understand and have a general picture of the universe in which we live. Surely, this is the foremost task of a popularization course: to be able to show the importance and relevance of some area of mathematics in the science and culture of humanity, and yet at the same time maintain some easy familiarity and accessibility for the student. For example, it seems unwise to rely heavily upon algebra, a less friendly topic for most, nor should it attempt to aim low, at simply explaining how some technological advance depends upon mathematics. In these aspects, the geometry of the universe is more apt then, say, number theory and cryptography, just to take one obvious alternative. Students at university, despite dry academic courses, really are looking to make their way in a realm of deep questions, questions of identity, relationships, career, and their place in the universe. They recognize the uncovering of the geometry of the universe as a serious, deep, and legitimate problem. Many of our colleagues, upon first hearing of our proposal agreed immediately that this was an important subject matter that we ought to be offering.

The evolution of geometry historically has had many significant interactions with philosophy, culture and how we view the world (see for example J. Grabiner [1] or M. Kline [2]). A major side-effect of this course would be to expose the student to this rich context in which the geometry developed, adding enormously to the meaning derived from this course. Tracing this evolution will also make it clear that our present-day models are likely not the final ones, and that our quest continues. The audience must be left with the impression that mathematics research is a lively and vital area of human activity. Notice that the major thrust of this course is not the rationalization of the importance of the milestones of mathematical development, such as projective geometry of the calculus. Certainly these would be major biproducts, but the students' eyes would be looking higher, at the quest to explain our universe. And that makes a perfect vehicle for these other objectives, of showing the centrality, relevance and vibrancy of mathematics.

There is a very delicate problem here of how this course should be taught. The purpose is to expose students in some forty hours of lectures to a series of increasingly sophisticated mathematical ideas (such as hyperbolic and Minkowskian geometry, curvature, manifolds, and present-day models), with an emphasis upon leading them up to an overview of the current issues. As background preparation, we assume only freshman calculus and linear algebra: a tenuous familiarity with differentiation, integration, Euclidean space and complex numbers. (As an aside, we mention that this is required in all programs in science, engineering, and commerce so this is a reasonable requirement.) How can this gap be bridged? It is necessary to find keys that will bring advanced topics to such students and avoiding paths that would get ensnared in the material from intermediate-level mathematics course. Attempting to work with the standard logical development from carefully worded and irrefutably accurate definitions via nonexplanatory but slick proofs would spell disaster. The teacher for this course must see, confront and overcome the culturally-defined mystique about how university mathematics courses ought to be taught that prevails so widely today. Otherwise the popularization of mathematics at this level is doomed.

This would not be an easy course to teach. No textbook exists to our knowledge and even the reference books are not organized as textbooks, with carefully chosen exercises. In many ways, this is closer to a course in the Arts than a technology course. It deals with a central intellectual question, with a huge and relevant literature. The teacher must bring to bear almost all he/she knows in this vast area to be effective in teaching. Most challenging is this need to be prepared to work with a wide variety of deep relevant student questions that should arise while teaching each topic. And the students must be seen, and their immediate reactions, to the teaching used as a tool for developing the right keys to open advanced topics to them.

Is it possible? Can the gap be bridged between advanced topics and unsophisticated students? Hopefully, this is a major facet of what popularization of mathematics is all about. We have experience over ten years of a weekly seminar with high school students, and another with undergraduates, presenting them with exciting, sometimes unorthodox, but invariably advanced material from automata theory, number theory, set theory, geometries of many types, and dozens of other areas. Also, working on <u>Expository Mathematics: an annotated</u>

bibliography (by John Poland, M.A.A., 1989, to appear) led to the discovery of expository articles at all levels of difficulty and varying from a few pages to book length: surely such variety indicates that ways exist for this course to be taught successfully. There is a skill, and it can be honed, to presenting this material legitimately to "underprepared" students.

So, you are convinced that this course looks excellent. Why then did it never get approved by our department?? For one, we must remember that the age profile of most mathematics departments, including ours, reveals a primarily conservative group, trained in classical mathematics departments, and who never have seen such a course before. We tried to be quite explicit, even in the description intended for the university course calendar, that the material was to be primarily descriptive in nature and aimed at non-mathematics majors. But we deliberately did not want to see this course mired down in the safe but less vital discussion of just pre-twentieth century geometry. The reaction of many of our colleagues was that the advanced material simply could not be treated adequately without much more preparation or more time spent in detail in the course. In reply to our reminder that this was intended to be descriptive, colleagues argued that it followed then that this course was not primarily mathematical in nature, but rather closer to the history of science. We replied (both to our mathematics colleagues and colleagues in history and philosophy) that the design of this course was to expose the student to the intellectual heritage that mathematics represents. It had as its prerequisites freshman calculus and linear algebra in order that the students could come to grips with some of the mathematics involved, but a delicate balance between mathematics and its wider effects would have to be struck throughout the course. In the end many of our colleagues in mathematics became convinced both that some of the topics were too advanced and that the course was not mathematical enough in nature. It would have to treat these topics superficially and that just was not how mathematics courses should be taught. Everyone realized that the course was dealing with central issues in civilization that ought to be taught, but at the same time it was important to many of our colleagues that this subject be dealt with properly and not just superficially. We had hit a central nerve, it seems!

Perhaps it is worth noting that more recently, in discussing possible changes to freshman calculus in our department, many colleagues argued that, acquiring technical facility is the keystone to successfully doing mathematics. Alternate approaches such as more conceptual questions, should only be addressed once this technical mastery has been demonstrated - otherwise the students in mathematics courses would be short changed.

Mathematics has ideas and techniques, and some mathematical ideas are embedded in technicalities while others stand alone. This is true in any discipline. It is possible to have a mature sophisticated understanding of some ideas without being in a position to use it creatively (Natural Selection in biology might be a familiar example). Our debate in the department was to some extent about whether a mathematics course can legitimately teach ideas without techniques. Do we have a tendency to not allow our students to listen to Beethoven quartets in class until they have demonstrated that they know how to compose; they cannot really appreciate the quartets or even understand them otherwise? What then is music appreciation?

The need to popularize mathematics to a mathematically-literate audience of non-mathematicians seems vitally important. A university course seems the natural vehicle (our university even offered the possibility of a televised lecture series locally). And what could have as compelling relevance, importance and abiding historical and cultural interest as the geometry of the universe?

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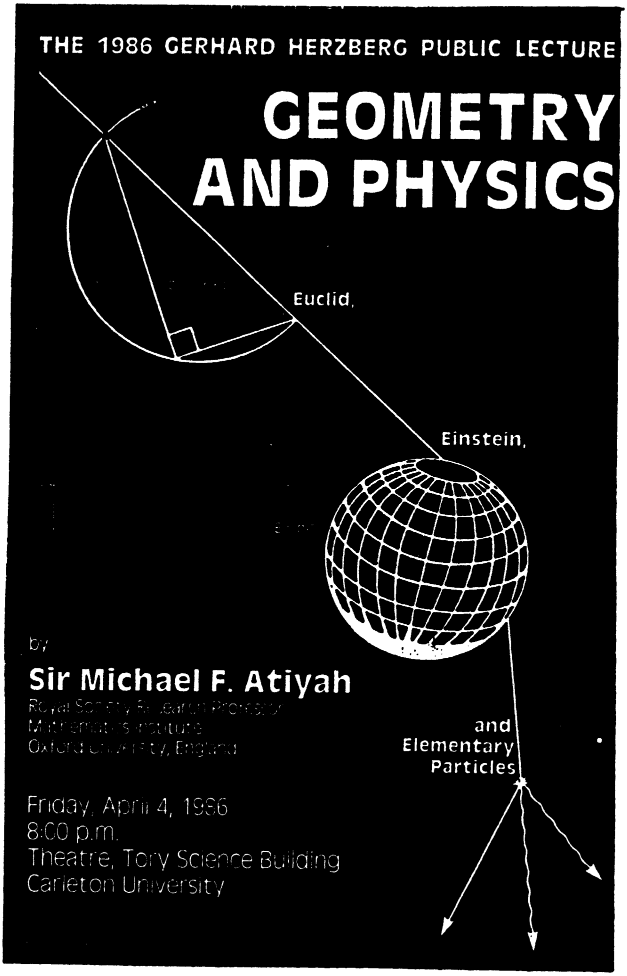
THE 1986 GERHARD HERZBERG PUBLIC LECTURE
GEOMETRY AND PHYSICS
Euclid,
Einstein,
and Elementary Particles
by
Sir Michael F. Atiyah
Royal Society Research Professor
Mathematical Institute
Oxford University, England
Friday, April 4, 1986
8:00 p.m.
Theatre, Tory Science Building
Carleton University

SOME EXPERIENCES IN POPULARIZATION.

It has been suggested to me that, as I have spent a large part of my life in an effort to popularize mathematics, an account of some of my experiences might be useful to others.

THE THINKING BEHIND MATHEMATICIAN'S DELIGHT.

The starting point was the question - what do I believe to be possible? What would we regard as a totally successful outcome? Some educational theorists assert that anyone can do anything. This seems extremely unlikely. Admittedly the potentialities of most human beings are vastly underestimated, but that is a different proposition. I believe there is some truth in the view that mathematics (at the everyday level) is a high-g ability; that is to say, it depends on the general level of intelligence, and each of us should do mathematics as well as we do most other things. Where this is not so, something has gone wrong. The usual reason is that an emotional block has been created, sometimes so strong that no cure is possible.

Complete success would mean that every individual felt,"I enjoyed the mathematics that I had time to learn. If I ever need or want to learn some more, I shall not be afraid to do so."

We cannot expect to achieve this overnight, but there is nothing inherently impossible in this aim. Fear or dislike of mathematics is not inborn. In fact the opposite is true. Margaret Drummond in The Gateways of Learning urges teachers of young children not to use fairy stories to make arithmetic more interesting. She says that to young children numbers are more interesting than fairies.

It would require considerable organization to ensure that everyone continued to have this pleasure and interest in mathematics. We would have to make sure that no one was introduced to arithmetic by a bad-tempered teacher, and that everyone worked at his or her natural pace. Too slow can be as disastrous as too fast. Supremely important is the way mathematics is taught.

Bad teaching can be done with the most admirable motives. A man I knew went, around 1950, to a school that prided itself on getting working-class boys through examinations and into Oxbridge. The teachers' attitude was "Learn it! Remember it! Do it!" To ask why was regarded as impertinence.

It is not surprising that he was left with a sense of insecurity. If you are taught in this way you are unable to imagine the subject you are learning. If your memory distorts a rule, you have no way to recover it in its original form.

The ban on rote learning is not absolute. For instance you might well learn the formulas for the surface area and volume of a sphere some years before you were able to derive these. But such things should be islands of rote in a sea of understanding.

Years ago it struck me when I heard children arguing in the street that they seemed to show much more intelligence than they did in arithmetic lessons. The reason was that they knew what they were talking about, while school arithmetic seemed to be something quite apart from the world outside, so you tried to guess what the teacher wanted you to say.

I met an extreme case of this divorce of arithmetic from reality, when I tried to convince a class of very bright children that 3x2+4x2 was not 7x4. (They had added together the numbers before the multiplication sign, and similarly the numbers after it.) I argued and drew pictures; surely if you saw 3 couples and 4 couples in a ballroom, you would not say,"Look, 7 quartets." This had absolutely no effect. They agreed with me only after working out the expression and finding - to their surprise - that it was 14.

There can be astonishing separations of theory from practice. During the war I did some work with Royal Engineer cadets. They used to plan some enterprise in the classroom and then go outside and carry it out. Once they planned to stretch a rope at a fair height and then one of them was to go along it hanging from a pulley. The spectacular effect was destroyed as soon as he got going, for the rope ceased to be level and straight, and his feet came only about a foot above the ground. When they seemed surprised, I said,"Have you never heard of the triangle of forces?" "Oh, yes sir, but we did not know it had anything to do with ropes."

Evidently there has been much bad teaching in the past, for so many people react to any mention of mathematics by saying,"I could never do that."

There is an entirely different emotional reaction to a real life situation with mathematical elements, such as deciding on the shortest way from one point in a town to another. No one says"I could not possibly do that". There is an intellectual difference too. A ridiculous idea, such as an extremely circuitous route, would be rejected immediately. In short, people think most mathematically when they do not know they are doing mathematics.

This was the guiding idea in Mathematician's Delight. All problems were to be presented in terms of actual things.. To an experienced teacher this might seem obvious. It was not so to a wider public. Some readers were astonished by the furniture and bricks in Chapter 2. "What, is this geometry?"

A NEEDED PROJECT.

We can go a stage beyond talking about things and drawing pictures of things by arranging for the actual handling of things. There is evidence that this greatly increases the proportion of the population capable of learning mathematics, and this evidence is on a mass scale. A Scottish report of 1947 speaks of the amazement of teachers at the interest in mathematics and the progress made in the armed forces by former pupils. "The approach there was usually from the practical problem to the mathematics involved, and the problem secured the interest". The report spoke of "the dullness and futility of much school teaching", and the need for "a drastic overhaul". "It is the great central mass of boys and girls, ranging from the C's well up into the B group, who have fared badly" in school. (1)

To the best of my knowledge, a drastic overhaul of mathematics in secondary education has not taken place either in Scotland or anywhere else. There are obviously many problems to be overcome. I would like to see some experimental project, outside the schools and unfettered by timetables, by which pupils whose interest was in things rather than ideas, would be given the opportunity to learn mathematics in a real setting with a real purpose. To many such pupils customary teaching is totally meaningless and they feel their lives are being wasted. They should be brought into the project before they have become totally browned off and turned to vandalism.

THE CHRISTCHURCH SIXTH FORMS CLUB.

The most effective instance I have ever known of the spreading of good teaching occurred with a sixth forms mathematics club we had in New Zealand. It was open to all sixth forms within motoring distance of Christchurch. There was no subscription, no membership list, no paper work. We met at each secondary school in Christchurch in turn. There would be a talk lasting an hour, and then the school would provide light refreshments and the sixth-formers would usually stay for about another hour, discussing and arguing informally.

We felt that two bad meetings would kill the society, so we had a committee of pupils who would only recommend teachers known to be very good and talking on subjects they did extremely well. As a result our members came to regard as normal teaching that was in fact quite exceptional. Sometimes pupils gave a talk for all or part of the hour and it was clear that a tradition of mathematical oratory had developed. It was not enough to set out the purely mathematical argument. The talk had to have some emotional element, to provoke interest and curiosity.

We held 6 meetings a year at monthly intervals. Two of these were challenge meetings. In one of these

I chose the problems, in the other one school would challenge the rest. The problems were handed out a month in advance. The rules were that the whole school could work on solving them, if they wished to, but there was to be no help from teachers, parents, or old boys and girls. At the meetings we would find out, for each problem, which schools claimed to have solved it. The names of these schools would be put into a hat, and one or two representatives of the school drawn would expound their solution, which others were then free to supplement or discuss. Usually there was considerable excitement. Parents were very puzzled. The idea of teenagers sitting on the edge of their seats while mathematics was discussed was one that had not occurred to them.

In one girls' school, after the sixth-formers had solved as many problems as they could, they were asked to explain their solutions to the fifth form. They had the experience we all do when we start teaching. They gave what they thought was a crystal clear account and found that not a word of it had been understood. They would try again later and go on until they succeeded in getting the message across. After this, when they came to address their peers in the meeting, their exposition was superb.

An extremely important idea is involved here - that pupils, while still at school, should be exposed to good teaching and given opportunities to acquire teaching skills themselves. If they eventually become teachers this experience will be invaluable. Ideas absorbed in adolescence are far more potent than doctrines met for the first time at training college, when attitudes may already be firmly fixed.

The club improved the morale of teachers. They realized that they were presenting a subject that could arouse genuine enthusiasm, and they were astonished how well their pupils did in solving problems and the confidence shown in presenting the solutions to a large audience, usually 150 or more. The recruitment of mathematics teachers in Canterbury province showed a significant increase, which, as far as anyone could tell, seemed to be due to the existence of this society.

It is quite possible that unusual circumstances existed in New Zealand at this time. There seems to have been an unsatisfied demand among the youth for intellectual stimulation, which this club was able to meet.

MATHEMATICS CLUBS IN TORONTO.

When I came to Toronto I was faced with a drastic change of scale in comparison with Christchurch. I formed a club, but as there were 60 or so schools within reach, they had to be rationed to 4 pupils apiece. These could be in any class in the school, if the school believed they had enough mathematical talent.

The audience thus consisted of 60 teachers with 4 students each that they had brought in by car.

Picking 4 pupils out of a very large comprehensive school is clearly going to produce a rather gifted audience. It is sometimes urged that meetings arranged for the gifted should give enrichment rather than acceleration, that is, that one should studiously avoid anything that may be in the standard curriculum for the years ahead. This is totally and utterly false. No curriculum intended for the majority can possibly meet the needs of the very able, who should be encouraged at the first opportunity to work and read ahead on their own. (I myself was fortunate, as I grew up under such a system.) Incidentally, in Toronto I heard of an excellent scheme by which an exceptional pupil would be put in touch with a graduate student at the university for occasional advice.

I therefore used these meetings to lecture on parts of mathematics that I thought a good young mathematician should meet early in life, without any concern for the standard curriculum. No school ever complained about this. In some schools, those in the higher grades, who had attended these meetings, gave an account of the material covered to the rest of the class.

I was not able to persuade pupils in this club to give talks themselves.

Clubs also played a part in teacher training. I asked my student teachers to identify some defect in the existing educational system and start a project to improve the situation. Some recognized the frustration of the brighter children in elementary schools, and ran a club for 10-year-olds to introduce them to algebra. The principal of the school where this took place said you could hear the children rushing towards the club's room when school ended. Others were concerned with the plight of the physically active child who has to sit at a desk for long periods. They offered to start a Club for Restless Boys. The idea was have athletic events with elaborate records of achievements, graphs of improving performance, the correlation between throwing at a target with the right and with the left hand, and so forth. One student wanted to run a club for slow learners; he said he had always been a slow learner himself. He had the impression that they would respond to his great concern for them. Such a concern is of course highly desirable, but it turned out to be insufficient by itself, as might have been expected. He was advised to experiment with games and other methods of making the work more interesting.

THE "MODERN MATH" MOVEMENT IN U.S.A.

Two distinct issues were involved in this campaign. One was to convey some idea of 20th century mathematics. I tried to mke a contribution to this in A Path to Modern Mathematics, which assumed some knowledge of calculus and complex variable. The other aim was to get away from rote learning in the schools. For this I wrote Vision in Elementary Mathematics, which assumed practically nothing.

Most of the confusion arose from these two objectives being fused, instead of appropriate considerations being brought to the notice of two quite separate audiences.

It was hoped to make 20th century mathematics accessible to those who applied mathematics to practical ends. To do this two things were essential - to give examples of practical applications, and to sketch the historical process that led to recent work, for without such a background modern mathematics is extremely indigestible.

Functional analysis is an excellent example of applicable modern mathematics and its origin is well documented.(2) Hadamard pointed out that we have an intuitive picture for the continuum of real numbers and for that of complex numbers, but we have no such picture when the unknown is not a number but a function, as in the calculus of variations. It would be very desirable to construct some geometrical object, on the points of which functions could sit in the same way that real and complex numbers do on the line and the plane. His student Fréchet carried this program out in a celebrated thesis.

Fréchet worked on the basis of analogy . In both real and complex theory, z(n) tends to k when the distance of z(n) from k tends to zero. The properties of distance used in classical analysis are simple and very few in number. If we can find a way of defining the distance between two functions with these properties, we shall be able to carry over many of the classical theorems about sets of numbers to sets of functions - and indeed the theory can cope with other mathematical objects, such as matrices, transformations and integral operators.

Such an account would not require much space and would make the acquisition of recent methods much easier for applied workers. Unfortunately this approach is far from universal.

STUDENT JOURNALS IN U.S.A. AND CANADA.

In U.S.A. I edited "The Student Mathematics Journal", which came out several times a year, for the N.C.T.M. In Canada I started and myself ran "Student Mathematics", which appeared only once a year.

Accounts of original mathematical work by pupils in elementary and secondary school, together with their photographs, were a main feature of both journals. The photographs helped to dispel certain stereotypes, for many of the girls were beautiful and the boys looked normal. I was very glad when I got a picture of someone very young, as I felt, then and now, that reasonably intelligent children had to wait far too long before meeting mathematics with substantial content.

Although written for, and largely by, pupils, this journal was able to give recognition to good teaching. A pupil's article often began, "My teacher suggested I might investigate..." We had, for instance, many items from Monarch Park school in Toronto, where the head of mathematics had decided that every first-year pupil should do original work on some problem suited to his or her ability.

CAMBRIDGE,1977-.

Since 1977 our home has been visited on Saturday mornings by 4 or 5 secondary school pupils. Some of these have been extremely brilliant and capable of work many years ahead of the school curriculum. However they seemed almost paralyzed when algebraic manipulation arose incidentally. Manipulation is clearly not the most important mathematical ability, but its absence can be almost crippling. Clubs for bright students should make great efforts to overcome this weakness in places where it exists,(3,4).

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1959. A Concrete Approach to Abstract Algebra. W.H.Freeman. Dover reprint, 1978. There is often an acute communication problem between those who work on mathematics at an advanced level and learners at an earlier stage. This book helped undergraduates both in U.K. and U.S.A. to understand the drift of lectures on modern algebra.

1961. What is Calculus About? Mathematical Association of America.

1964. ***Vision in Elementary Mathematics. Penguin Books. It is very frustrating that this book is out of print. I have some evidence that it makes algebra and some co-ordinate geometry accessible to children and adults who would not be likely to learn it any other way. If a non-profit scheme could be devised to reprint it, I would be glad to forgo any claim to royalties.

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SERVICE CLUBS AND THE POPULARISATION OF MATHEMATICS

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1. INTRODUCTION

Staff at this institution try to popularise mathematics in a number of ways. One is by visits to high schools, particularly those in rural areas, where talks are given on careers in mathematics other than teaching and students are involved in a problem solving exercise in the style of Polya's film on guessing. Another method is Open Days where the public can view and touch and have explained some of the research and consulting projects of the staff.

A third approach is by guest speaking at meetings of service clubs such as Rotary, Apex, Lions and so on. Since the members attend their meetings for reasons other than listening to the guest speaker, they are not all put off by a topic involving mathematics, and one has an audience of influential local community and business leaders, hard-headed, but receptive.

Clearly the choice of speaker and topics is important. The latter should be something to which the audience can relate in some way, preferably with something tangible to show them, and the speaker should be able to address them with clarity, knowledge, enthusiasm and a touch of humour. The following are some topics which the present writer has used, though they are outlined here for a mathematical audience rather than a lay audience for whom lots of diagrams would be necessary, as well as working examples of the devices in question.

2. Three Examples

2.1 X-ray Camera

A Government department wanted a Gandolfi X-ray camera sample holder to oscillate through 90° in a vertical plane while it rotated through 360° in the horizontal plane to avoid coincidences of the crystal planes.

The solution was to utilise a property of the hypocycloid which is the locus of a fixed point on the circumference of a circle rolling (without slipping) on the circumference of a larger circle. When the radii are in the ratio 2:1, this locus is a diameter of the larger circle. The first condition could then be satisfied by a hypocycloidal gear mechanism in which a rod on the smaller circle was attached to a mechanism which could oscillate while the larger circle could rotate.

If the respective angular velocities are $\dot{\phi}$ and $\dot{\theta}$, then while a point on the larger circle does one revolution in $2\pi/\dot{\theta}$ seconds, the rod takes $2\pi/\dot{\phi}$ seconds to traverse the groove. Coincidences occur when integer multiples of the two times are equal: $m\dot{\theta} = n\dot{\phi}$. By choosing relatively prime numbers for these integers, we can satisfy the second condition. (Two versions of the camera have been patented).

2.2 Credit Creation in NBFIs

The process of credit creation in the banking system is well known to economists. The conditions under which the deposit taking non-bank financial intermediaries (NBFIs), such as building societies, can create credit are less well known.

We shall assume that banks observe a constant liquid assets to deposits ratio *(r)* and that this ratio is always attainable. The requisite liquidity ratio is governed by legislation or convention or commercial prudence. Also assume that the banking system will gain s of any potential deposit while the NBFIs gain $(1 - s)$ of the deposit, where $O < s < 1$. Thus, any deposits, whether from the banking system or the NBFIs, will be distributed between the banks and the NBFIs in the proportions s and $1 - s$. Assume again an extraneous injection of liquids into the banking system of D, and that the required liquidity ratio of the banking system is r_2 while that of the NBFIs is r_1, where $O < r_2 < 1$ and $O < r_1 < 1$. A bank approached us to determine the ultimate increase in deposits in the banking sector from this initial deposit D.

If a deposit D is made to a bank at stage 1, the required liquids will be Dr_2. If the bank is prepared to grant overdrafts to customers to the full extent of its liquidity ratio, then it can make an advance of $D(1 - r_2)$ which we suppose is redeposited . At the beginning of stage 2, $D(1 - r_2)s$ will be deposited in the banking system and $D(1 - r_2)(1 - s)$ in the NBFIs. The process quickly becomes unwieldly unless we recognise a pattern. The key is to use suitable notation and set up appropriate relations. If this is done, we can prove that the required answer is $D + Ds(1 - r_2)/(r_1 - s(r_1 - r_2))$.

2.3 Insulin Dosage Meter

The full extent of diabetes in Australia is not known. Mass screenings have been going on at shopping centres in recent years with mixed success. What is known is that the cost to Australia of diabetes is around 1.2 billion dollars annually.

According to a study at Sydney University Medical School "few diabetics were able to adjust their insulin dosage to overcome problems caused by intercurrent illness or differing dietary and exercise patterns. These problems in insulin administration present a serious barrier to a flexible lifestyle for patients with diabetes".

A drug company invited us to develop a computerised meter which would estimate a diabetic's daily short-acting and intermediate-acting insulin requirements based on information about blood sugar levels, exercise and carbohydrate variations and state of health. The devices we have developed do this and also modify the base doses of insulin. They also lock and store the information so that the physician can print it out later.

To do this, we initially set up a two compartment model for insulin-glucose dynamics *in vivo*. Later, other compartments had to be added to match experimental data. These enabled us to learn more about when insulins peaked in the body according to the injection site, and to look at the effect of varying mass indices.

The device we have produced is essentially a bar-code reader for scanning foods and their quantities for dietary analysis. This has a cap which can be connected to a commercial reflectance glucometer for blood glucose readings and to a modem for connection with a hospital computer.

3 DISCUSSION

It is important to bear in mind in such talks that one is not trying to teach mathematics. Rather, we show that there are problems to which mathematics can make a significant contribution. We may know this, but even a member of the Board of the Commonwealth Scientific and Industrial Research Organisation recently expressed doubts that mathematics had made any significant contribution to humanity in this century!

The examples chosen can strike chords with most audiences. The X-ray camera is particularly useful in analysing metal fibre and the uses of this can be mentioned. Money matters make most of them sit up and listen, and many know someone who is a diabetic, and mostly there is at least one diabetic in the audience.

There is usually a discussion during which someone nearly always says "I never knew what mathematicians did, other than teach mathematics". If there are tangible results, especially gadgets, for people to handle informally later this also helps. A simple poster on each topic can then be displayed too.

While there is always the danger of attempting too much in 20 to 30 minutes, these audiences do not want to be talked down to, and once they have the problem explained they are happy to see the end result without too much detail in between. A useful by-product, though not actively sought, has been commercial help for some of our research and development activities. This, and invitations to talk at other venues and enquiries about buying the products, are tangible indications of success in this approach to the popularisation of mathematics amongst groups of business and community leaders who are, in Australia, increasingly becoming involved on committees which advise governments on educational policy issues.

POPULARIZING HIGH SCHOOL MATHEMATICS: A UNIVERSITY APPROACH

John Webb
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The Department of Mathematics of the University of Cape Town has three linked programmes for popularizing high school mathematics. A magazine is published, circulating throughout Southern Africa, a regional mathematics competition is held for schools within a 100 km radius of Cape Town, and series of enrichment lectures and activities are held on the campus for local high schools.

MATHEMATICAL DIGEST

MATHEMATICAL DIGEST is a quarterly magazine which was launched in 1971. Today it circulates throughout South Africa, Namibia, Botswana, Lesotho, Swaziland, Zimbabwe and Malawi. A free copy of each issue is sent to some 1800 high schools, and individual subscriptions and bulk orders from schools push the total circulation to about 6000. The magazine provides a wide range of enrichment activities for enthusiastic young mathematicians, who can be identified when they submit entries to the many problem solving competitions in every issue.

The magazine features a wide variety of articles.

- news of mathematical developments from all over the world
- articles on mathematical topics not in the school curriculum
- new ways of looking at school mathematics
- stories of famous mathematicians, past and present
- book reviews
- problems from both rounds of the South African Mathematics Olympiad
- news of the UCT School Mathematics Competition and other mathematical events in South Africa
- careers in mathematics
- information about bursaries
- the International Mathematics Olympiad
- competitions with cash and calculator prizes
- mathematical games - and how to win them
- puzzles and problems, both serious and frivolous
- anecdotes, quotable quotes, poems, jokes and cartoons

A couple of Editorial Do's and Don'ts are worth mentioning.

DON'T publish long articles. The attention span of this type of reader may be broken by having to turn the page.

DON'T whinge in print about low response. It's YOUR fault if nobody wants to read the magazine. (I remember a Punch cartoon in which an obviously dissatisfied customer is marching out of a shop. One saleswoman is saying to another "The trouble with customers today is that they don't understand modern sales techniques."

DON'T criticise or make fun of textbooks or official examinations or curricula. These are terribly tempting targets, especially in South Africa, and may deserve everything they get. But teachers may feel threatened, and you need the cooperation of teachers in getting your magazine to the pupils.

DO arrange exchanges with similar journals. That way you will never be short of ideas.

THE UCT SCHOOL MATHEMATICS COMPETITION

MATHEMATICAL DIGEST works at long range. For high schools in the Cape Town area, we have an annual mathematics competition, which was started in 1977 by a small group of enthusiastic teachers. At first held in a local school, by 1980 it had become so large that it had to move to the UCT campus. This year nearly 4000 high school pupils of all races converged on the campus on a Monday evening in April to write a multiple-choice competition paper.

So far, so conventional. However, there are some features of our competition worth mentioning. As there are about 250 high schools in the Cape Town area, we have to restrict entry. Apart from a maximum of five entries per school in each year of study, we allow three teams of two to compete, for those who might feel wary of going it alone against the experts. The results of the pairs and individuals are classified separately, and being able to compete in a pair has proved to be a very popular option.

Why hold the competition at the university? Surely if the papers were simply posted to the schools we would be able to allow everybody to enter?

The first answer is that pupils enjoy visiting the university, and competition night at UCT is a major social occasion for pupils and teachers alike. There is an obvious benefit to the University in terms of public relations and student recruitment. In return, the University offers a truly non-racial venue, for UCT has a long record of outspoken opposition to apartheid. Because of deep sensitivities to past humiliations in communities which the South African Government classifies as "non-white", it would be difficult to find an organizational structure and a venue for the competition which would be as broadly acceptable. An incident in 1988 is worth mentioning. During the week of the competition a number of "coloured" schools in Cape Town were experiencing class boycotts. On the night, however, the pupils all turned up at UCT to write.

Why not hold the competition in two rounds? The first round could be held in the schools, and the top students could then be invited to a second round at the University. The problem with this structure is the danger of the second round being dominated by a handful of top schools, mainly (though not exclusively) white. Because there is so much enthusiasm within schools, some schools run elimination rounds, using the previous year's papers, to select their entry. For those schools, the Competition is a already a two-round event.

How does the University academic community view the competition? There is always the danger that high-powered researchers will dismiss a mathematics competition as "kids' stuff", while more socially conscious academics may be wary of excessive elitism. However, the academic benefits of the event to the University are becoming more and more obvious.

Apart from its obvious value in promoting enthusiasm for mathematics, not to mention talent-spotting, the UCT School Mathematics Competition is providing useful data for the University's "Alternative Admissions" programme. Although all high schools in the country work to the same mathematics syllabus and write equivalent (but separate) school-leaving examinations, black schools still suffer the consequences of years of educational deprivation. Statistics show that performance in final examinations by whites is a reasonably good predictor of success at university. For blacks, however, the corresponding statistics are unreliable. The University of Cape Town's Alternative Admission Programme recognizes this fact, and tries to identify educationally disadvantaged students for a programme of Academic Support. In 1987 a pilot programme was begun, and data gained from our competition has been of great help in constructing suitable problems for the Mathematics Test, which is an important feature of the selection process.

Thus the competition is favourably viewed by the ivory tower researchers as a way of identifying mathematical talent and persuading the top students to study mathematics at UCT, while at the same time it is seen to provide opportunities for alternative approaches and affirmative action in mathematics education.

The competition is followed up each year with the publication of a booklet containing all the question papers, full solutions, detailed statistics and names of prizewinners.

THE UCT MATHEMATICS SERIES

The UCT School Mathematics Competition is certainly effective in identifying mathematical talent, but as it occurs just once a year it cannot be expected by itself to convert the enthusiastic young mathematician to the notion that further study in mathematics is desirable. Nor does the competition provide much opportunity, except at the Prize Giving, for UCT Mathematics lecturers to get to know the entrants personally.

In 1986 a programme of "Saturday Afternoon Mathematics" was begun at UCT by Dr Christopher Gilmour. Twice a year, all secondary schools in the Cape Town area are invited to send five "mathematical enthusiasts" to three successive Saturday afternoon programmes on the University campus. About 200 pupils usually register for the series, whose success can be gauged by the dropout rate: about twenty from the first Saturday to the third. The programmes quickly settled into a format which has proved popular, judging by questionnaires filled in at the end of each series.

We begin at 2 p.m. each Saturday with a lecture. The opening lectures of each of the three successive Saturdays cover a single theme such as "Constructible and Non-Constructible Numbers". These lectures are kept short (about 30 minutes), and usually end with a problem for the pupils to solve before the next Saturday. The second and third Saturdays then commence with the pupils dropping their answers into a large box, from which three correct entries are drawn. The lucky winners receive book prizes.

After the opening lecture there is a competition, usually lasting an hour. Each week the competition takes a different format. In the first week, teams of four compete in a relay race. All teams receive Problem 1, and only when they hand in their solution is Problem 2 given to them. This calls for strategy: if the problem is hard, should they give up, hand in and get the next (hopefully easier) problem? Here I should add that we allow teams to form any way they like, with the sole restriction that no team may contain two members from the same school. Thus the pupils get to know fellow enthusiasts from other schools, and inter-school rivalry is not an issue.

As in the Competition, we enjoy substantial support from "non-white" schools, as well as schools which would not regard themselves as "academic" in any way. The range of ability of the pupils who attend is very wide, and we encourage "mixed ability" teams.

In the second week we have an individual competition. We hold this in a large lecture theatre, and the problems are projected on two screens, using overhead projectors. On one screen we project a series of easy problems, which stay up for about two minutes each before being replaced, while on the other screen appears a harder problem, for about ten minutes. Bells and whistles are used to alert the pupils to a change of problem.

In the third week we revert to a team format, but this time all the problems are handed out at once, and the team has to decide how to distribute them and work them out. In addition to the rule that no two members of a team may come from the same school, we now require that no two members of the team should have been in the same team in the first competition.

The verdict of the pupils, as revealed by our questionnaires, is that the competitions are a very popular feature of the Saturday programmes.

We encourage teachers to attend the series. Indeed, the relay competition would not be possible without their assistance. About twenty teachers usually attend, and many of them are keen to take part in the other competitions.

After the competition it is time for a refreshment break. Generous sponsorship from a Life Assurance company, Southern Life, enables us to provide Coke and doughnuts for all, without charge. Refreshments are very important, not just because pupils enjoy them. This is where a good deal of interaction between the pupils, teachers and mathematics lecturers takes place.

The final part of the programme is another lecture, usually of about fifty minutes. These lectures have included a survey of the mathematical work of Isaac Newton (an obligatory topic in 1987), a discussion of problem-solving techniques, and lectures with titles like "Prime Time" and "A Funny Thing Happened on the Way through the Theorem".

At the end of the last Saturday afternoon we have a short prize giving. The prizes (popular mathematics books by authors such as Martin Gardner) are presented by a representative of the sponsors.

The lectures, problems and lists of prizewinners are published in a booklet which is sent free to all participants.

Some features of the series need mentioning.

The contents of the lectures and the nature of the problems set in the competitions are firmly in the direction of enrichment rather than acceleration into university mathematics syllabuses.

These Mathematics Series are seen by the University as an excellent way of attracting to UCT the best mathematical talent. At the same time, the sponsors keep on the look-out for potential actuarial bursary students.

Since some of the lectures we give are very much in the style of a first-year university lecture, we can give many of our future students an idea of what to expect when they come to university, helping to bridge the gap between the educational approaches of school and university.

TEACHER ENRICHMENT

All our activities - MATHEMATICAL DIGEST, the UCT School Mathematics Competition, and the Saturday programmes - are directed at the mathematically talented high school pupil in the 14-17 age group. Nevertheless, teachers play an important role. We rely on teachers to identify their best pupils and to encourage them to attend our Mathematics Series. We rely on teachers to sign up their pupils for the UCT School Mathematics Competition, and to turn up on the night of the competition to help invigilate. We rely on teachers to bring MATHEMATICAL DIGEST to the attention of their pupils, and to encourage pupils to subscribe. A teacher's judgement on exceptional mathematical talent may not always be reliable, and teachers sometimes send their favourite, rather than their best, pupils to our activities. It is, however, important to us to have the confidence of teachers in what we do, and not to bypass them. We hope that this attitude has some spin-off. It is an unfortunate fact of life in South Africa that a large number of mathematics teachers are poorly qualified. Thus it is a bonus if teachers read MATHEMATICAL DIGEST, discuss the Competition problems in the classroom, and attend our Mathematics Series. We hope they benefit from the mathematical enrichment we offer.

"L'OUVERT"

Un périodique pour la vulgarisation des mathématiques

Jean LEFORT

Responsable de la publication

Dans le texte qui suit, je présente les différents points dont il faut tenir compte pour l'édition d'une revue de vulgarisation des mathématiques. Je parlerai des difficultés qu'il faut résoudre, des réussites et des échecs qu'a connu '*L'Ouvert*' au cours de ses quinze années d'existence.

Un titre

Le choix du titre d'un ouvrage quel qu'il soit est fondamental. Tout journal, tout mensuel, tout périodique annonce d'emblée son but à travers son titre. Un titre tel que "*Bulletin de la Société* ..." ou "*Revue de mathématiques de* ..." n'est pas spécialement accrocheur. C'est un peu mieux avec quelque chose comme "*Mathématiques amusantes*" ou "*Le Monde des mathématiques*". Mais pourquoi vouloir à tout prix placer le mot "*mathématique*" dans le titre? Un terme mathématique fait aussi bien et même mieux l'affaire puisqu'en plus de son clin d'œil à la science on peut, si le choix est bien fait, jouer sur toutes ses connotations. En cela, '*L'Ouvert*' est une réussite. Inutile de préciser ce qu'est un "*ouvert*" à nos lecteurs mathématiciens, mais rappelons quelques expressions de la langue courante :

— avoir un esprit ouvert;
— laisser une porte ouverte à la discussion;
— pratiquer une politique d'ouverture;
— parler à livre ouvert;
— garder l'œil ouvert;
— l'ouverture d'une œuvre musicale;
— à bras ouverts ou à cœur ouvert ...

Un logo

Le logo du titre lui-même fait partir de l'image de marque d'un périodique. Il est important que l'image ajoute un plus au contenu du titre. La forme des caractères, leur taille ... sont des éléments importants qui visent à préciser l'information contenue dans le titre.

Depuis 1984, un livre, ouvert bien sûr, remplace le "*v*" d'"*ouvert*" du titre. Ce livre fait référence à la citation de GALLILÉE sur l'univers qui serait un livre ouvert écrit dans la langue mathématique.

L'OUVERT

JOURNAL DE L'A.P.M.E.P. D'ALSACE ET DE L'I.R.E.M. DE STRASBOURG
I.S.S.N. 0290 - 0068

Une couverture

Les mathématiques ont une réputation d'austérité. Pas question de faire une couverture neutre ou de mettre le sommaire complet directement sous les yeux des lecteurs. Par contre un dessin, illustrant éventuellement un article, est le bienvenu. Il s'agit de montrer que les mathématiques partent du concret et y retournent, que l'image fait tout autant partie des mathématiques que le symbolisme habituel, que les mathématiques sont source d'inspiration pour les artistes, ... C'est pourquoi la couverture de '*L'Ouvert*' est sans cesse renouvelée ce qui n'est pas sans poser des problèmes de recherche et de choix. Les gravures d'ESCHER, les dessins de Max ERNST, des œvres de Max BILL, des surfaces ou des polyèdres reviennent souvent. Mais on trouve aussi des portraits de mathématiciens, des figures de géométrie ... Chaque couverture justifie d'une présentation ou d'une explication donnée sur la première page intérieure.

La présentation

Titre, logo et couverture sont des éléments accrocheurs, mais il faut aller au delà et offrir un contenu à la fois intéressant et agréable à lire. En ce qui concerne la forme, depuis l'adoption de TEX, (en 1986), '*L'Ouvert*' propose une présentation très sophistiquée. Chaque article débute en haut d'une page et la fin de l'article est éventuellement complétée par un encart qui peut être une présentation d'ouvrage, une information, une blague mathématique, ... En haut de chaque page de gauche, le nom de l'auteur et de chaque page de droite le rappel du titre. Au bas de la première page de chaque article on trouve le "*copyright*".

Cette présentation a pour but de faciliter les photocopies dont l'usage est restreint par la loi mais dont tout le monde use abondamment!

Le contenu

"*Qu'importe le flacon pourvu qu'on ait l'ivresse!*" C'est vrai que l'emballage, ici l'aspect extérieur, est nécessaire pour attirer le lecteur, mais le contenu (en ce qui concerne le fond) est primordial pour le retenir. Et c'est vraiment là que commencent les difficultés. En effet, si chercheurs et enseignants de mathématiques écrivent régulièrement, soit pour une publication, soit pour leurs cours, des textes de mathématiques, ce ne sont pas, le plus souvent, des textes de vulgarisation. Plusieurs cas peuvent se présenter :

— Il s'agit d'un résultat classique que le mathématicien présente dans son cours, mais la compréhension de ce résultat nécessite une introduction plus ou moins longue qu'a certes eu l'étudiant mais que n'aura pas toujours le lecteur.

— Il s'agit d'une découverte dans un domaine très spécialisé où les prérequis sont trop importants.

— Il s'agit, et c'est souvent le cas le plus simple, de la trouvaille d'une nouvelle présentation de résultats élémentaires; cela s'apparente alors plus à la didactique.

— Il s'agit d'une notion qui transcende les mathématiques comme l'histoire, l'infini, la notion d'opération, ... et l'on trouve plutôt ces textes dans le paramathématique.

Ceci permet de comprendre les difficultés extrêmes auxquelles se heurte la

recherche de textes pour '*L'Ouvert*' :
— éviter la paraphrase d'un cours de mathématiques;
— éviter la trop grande technicité.
Au sein d'un même article il se pourra d'ailleurs que la rédaction tombe alternativement dans les deux travers tant il est difficile de bien vulgariser.

Le niveau

Tout ceci implique la référence continuelle à un niveau bien déterminé. Il s'agit que '*L'Ouvert*' soit lisible par un bon bachelier scientifique ou par un étudiant des deux premières années d'études universitaires (DEUG-A). Cela ne veut pas dire que l'on ne s'intéresse qu'à ce qui est enseigné à ce niveau mais que les explications devront être données en références à ce niveau. Cela permet de s'adresser aussi à un public qui a poursuivi des études plus longues mais qui, faute de pratique, a oublié certains résultats. A ce public il reste une culture mathématique que les articles de '*L'Ouvert*' ont pour but d'entretenir et de développer. '*L'Ouvert*' est actuellement la seule revue française dans ce créneau.

Les auteurs

Trouver de bons auteurs est pratiquement une quête sans fin puisqu'il faut les renouveler à chaque sujet (il paraît impossible d'avoir une personne compétente en tout domaine), mais aussi s'assurer de leur talent de vulgarisateur. De plus écrire un texte de vulgarisation mathématique n'est pas encore valorisant ce qui fait que beaucoup d'auteurs potentiels remettent sans cesse à plus tard la rédaction d'un tel article. Il faut toutefois noter que certains prennent goût à l'écriture de ce nouveau genre mathématico-littéraire et n'hésitent pas à proposer des papiers de leur cru. Malheureusement la quantité de sujets proposés ou suscités n'est pas si grand que '*L'Ouvert*' puisse se permettre de refuser des articles trop techniques ou au contraire de qualité médiocre (sauf exception). C'est pourquoi le contact avec l'auteur est essentiel afin que celui-ci admette l'une ou l'autre modification pour rendre son article abordable par le plus grand nombre de lecteurs. Il est agréable de rencontrer des mathématiciens qui acceptent facilement cette remise en question. Actuellement, de par son implantation, la majorité des auteurs provient de l'Université de Strasbourg, mais petit à petit il est fait appel à des mathématiciens extérieurs à l'Alsace, ce qui ne peut qu'améliorer le niveau de la revue.

Le financement

Actuellement le financement de '*L'Ouvert*' est assuré par l'Institut de Recherche sur l'Enseignement des Mathématiques de Strasbourg (IREM). Ce financement comporte essentiellement : 1) la dactylographie (en TEX) et l'expédition qui sont assurées par la bibliothécaire de l'IREM; 2) le tirage offset à 700-800 exemplaires par numéro avec 4 numéros par an, tirage assuré par le département de mathématiques de l'Université Louis Pasteur de Strasbourg. La vente, presque exclusivement par abonnement, couvre l'ensemble des frais d'édition et de diffusion. Dans quelques rares cas, une participation supplémentaire est demandée à la régionale de l'Association des Professeurs de Mathématiques de l'Enseignement

Public (APMEP) d'Alsace, puisque la revue dépend de ces deux institutions.

Le fait d'assurer l'édition au sein du département de mathématiques nous restreint dans nos possibilités. Par exemple '*L'Ouvert*' n'est pas en couleur. Cela n'est pas trop gênant au sein des articles, mais au niveau de la couverture, nous laissons échapper de nombreuses possibilités, faute de pouvoir faire appel à la quadrichromie ou même ne serait-ce qu'à une couleur qui donnerait une dimension supplémentaire tant au niveau de l'impact visuel qu'au niveau de la lecture de l'image.

D'un autre côté, nous n'avons jamais cherché à attirer la publicité ce qui nous prive de recettes publicitaires et nous permettrait, peut-être, de nous développer davantage. Mais c'est sans doute manque de compétence et de connaissance des circuits publicitaires de la part de l'équipe qui s'occupe de la revue.

La diffusion

La diffusion n'a jamais réussi à dépasser 800 exemplaires et tourne actuellement autour de 700. Poutant nous touchons 17 pays différents répartis sur les cinq continents. Il faut cependant noter que 15 % des abonnements concernent des institutions (bibliothèques, établissements d'enseignement ...) ce qui amène une notoriété certaine à '*L'Ouvert*' sans influence positive sur son développement.

Par ailleurs, si le lectorat est international, 60 % de la diffision a lieu en Alsace ce qui veut dire, encore une fois, que '*L'Ouvert*' a du mal à avoir autre chose qu'un succès d'estime en dehors de sa région de naissance.

Les raisons de ces difficultés sont peut-être à rechercher dans le manque de compétence dans le domaine économique et publicitaire de l'équipe animatrice. Comment atteindre un public non-enseignant ? Comment susciter des abonnements individuels dans les universités autres qu'alsaciennes ?

Et l'avenir

Les réussites, les échecs et les difficultés que je viens de passer en revue montrent bien la voie à suivre pour assurer un développement accru de '*L'Ouvert*' comme périodique de vulgarisation des mathématiques. Cela se résume en une volonté d'accroissement de la diffusion de la revue. Reste à trouver les moyens de cet accroissement :

— étoffer l'équipe animatrice de compétences en matière économique;

— passer à un stade plus professionnel en s'affranchissant de la tutelle de l'université;

— développer des actions de partenariat avec d'autres revues d'expression française ou non;

— multiplier les traductions de textes étrangers.

Peut-être se trouvera-t-il des personnes pour nous proposer leur aide? Qu'elles nous écrivent à:

'*L'Ouvert*'
Bibliothèque IREM
10, rue du Général Zimmer
67084 STRASBOURG CEDEX
FRANCE

GAMES AND MATHEMATICS

Miguel de Guzmán
Universidad Complutense, Madrid

Mathematics, art and games.

Mathematics is a manysided human activity. It is, of course, a science; even more it is the model and paradigm of all scientific activity. It is a powerful instrument for the exploration of the universe and for the appropriate use of the natural resources at our disposal. It is a model of thought which along the centuries has served as a privileged field for the study of the capacities of the human mind.

But mathematics has also been and continues to be an authentic art and game and this artistic and gamelike component is so consubstantial with the development of mathematics that every field of mathematical work that does not attain a certain level of aesthetic satisfaction remains unstable, reaching for a more polished expression that might convey a unitary, harmonious, pleasurable, amusing vision, in the same way as an unfinished symphony or poem stretches out in the mind of its author for the most beautiful possible form.

In what follows we shall briefly analyze the relationships of mathematics with games, leaving aside the study of the artistic components of mathematics, which has been carried out by many authors, among others Garret Birkhoff, Helmut Hasse, Andreas Speiser, Hermann Weyl,...

The nature of games.

Games have been analyzed in depth by the sociologist Johann Huizinga in his work Homo ludens. He emphasizes the following features as characteristic of a game:

*The game is a free activity, free in the sense of the Greek paideia, i.e. an activity which is exercised for the sake of itself, not for the profit derived from it.

*It has a certain function in the human development. The human cub, like the animal, plays and prepares himself for competition and for life. The human adult plays and by so doing he feels a sense of liberation, evasion, relaxation.

*A game is not a joke. Games have to be played with a certain amount of earnestness. The worst game spoiler is the one who does not take it seriously.

*The game, like the work of art, produces pleasure through its contemplation and execution.

*It is separated from ordinary life in time and space.

*There are certain elements of tension in it, whose catharsis and liberation cause great pleasure.

*The game gives rise to very special bonds among practitioners, a sort of deep brotherhood.

*Through its rules, the game creates a new order,a new life, full of rhythm and harmony.

A perfunctory analysis of mathematical activity allows us to check that all these traits are present in many of its forms. Therefore, mathematics is also, by its very deep nature, a game, although this game involves other aspects, like the scientific, the instrumental, the philosophical ones, that together make of mathematics one of the fundamental pillars of our human culture.

The practice of mathematics and games.

If games and mathematics have so many features in common regarding their ends and nature, it is no less true that they also share the same essential traits in what concerns their practice. This is particularly interesting when one is asking for the most adequate methods to transmit to a wide audience the profound interest and the enthusiasm that mathematics can generate and to convey a first familiarization with its usual ways and procedures.

Any game starts with the introduction of a set of rules, a number of objects or pieces, whose function in the game is defined by those rules, in exactly the same way as the objects of a mathematical theory are determined by implicit definition: "We are given three systems of objects. The objects of the first system we call points,..."

Whoever gets started in the practice of a game has to acquire a certain familiarization with its rules, relating the pieces with each other in the same way as the novice in mathematics compares and makes the first elements of a theory interact among themselves. These are the elementary exercises of a game or a mathematical theory.

The practitioner who advances in the mastery of the game is capable of acquiring a few simple practical techniques that, in circumstances which appear rather often, lead to a successful end. These are the basic lemmas and facts of the theory that usually are easily accessible in a first tackling of the easy problems of the field.

A deeper exploration of a game with a long history will give the practitioner a knowledge of the particular ways and procedures that the true masters of the game have left to prosperity. These are the moves and strategies at a deeper and more complex level that have required a special insight since they are far from the initial elements of the game. This corresponds in mathematics to the phase in which the student tries to assimilate and make his own the great theorems and methods which have been created throughout the history of the subject. These are the thinking processes of the truly creative minds which are now at his disposal in order that he also can find light in the middle of confused and delicate situations.

Later on, in the more sophisticated games, where the stock of problems never becomes exhausted, the advanced player tries to solve in an original way situations of the game that have never before been explored. This corresponds to investigation of the open problems of a mathematical theory.

Finally there are a few who are capable of creating new games, rich in interesting ideas and situations that give rise to original strategies and innovative styles of playing. This is parallel to the creation of new mathematical theories, fertile in ideas and problems and possibly with applications to face other open problems and to explore more deeply some levels of the reality that until now have remained in the shadows.

The impact of games on mathematics.

Very frequently in the history of mathematics an interesting question made in a gamelike manner or an ingenious observation about an apparently innocuous situation has given rise to new modes of thinking. This is the sort of spirit that makes science effectively advance, when one is able to look at the subject in an unconstrained and playful mood, away from the severe and earnest context in which official science is usually placed.

The beginnings of the combinatorial analysis are situated in the Book of Changes (I Ching) with its distribution of different divinatory symbols and in the construction also in China of the magic squares with mystic connotations.

The games with stones (psefoi) of the Pythagoreans gave rise to interesting theorems in the theory of numbers. Zeno's paradoxes should probably be read as a mockery against the prevailing ways of thinking among contemporary mathematicians. Euclid himself used a collection of fallacies in one of his lost books, Pseudaria, as a means to motivate his students in the correct thinking processes. Archimedes, with his Problema bovinum and his Sand-reckoner faces strange situations in a gamelike manner in order to sharpen his mathematical instruments.

The list of mathematical objects that have come to existence motivated by the spirit of games would be without end. It is enough to quote some of the names of important mathematicians who can be thought of in this context: Fibonacci, Cardano, Fermat, Pascal, Leibniz, Euler, D.Bernoulli, Gauss, Hamilton, Hilbert, von Neumann,... A short, but very rich sketch of the evolution of mathematical recreations can be seen in the article by W.L.Schaaf on Number games and other mathematical recreations in the Encyclopaedia Britannica.

Mathematics in games.

The richness of mathematical themes in classical and modern games is impressive. The best way to perceive this is to look through the classical works of Lucas or Ball (Ball and Coxeter) and through the bibliographical compilations made by W.L. Schaaf on the recent literature on games and published by the National Council of Teachers of Mathematics.

Besides arithmetic, geometry, number theory as the traditional sources of recreations one can name topology, combinatorial geometry, graph theory, logic, probability theory,... In all these old and younger fields there are uncountable open problems of an amusing and attractive appearance that are possibly as easy to state and as difficult to solve as, for example, Fermat's last theorem, probably waiting for the creation of new thinking processes that can throw some light on their solution. About many of them one could not say if they should be classified as serious mathematics or else as idle oddities or puzzles. One can surely affirm that any game or puzzle with enough depth can have very intense repercussions on interesting aspects of mathematics. In the creation of puzzles or games man can display his imagination with complete freedom without being constrained by the conceptual or methodological bonds of a traditional theory.

The spirit of games in mathematics

There is very much deep mathematics with the flavor of games. Among modern examples one can select a few in which this is very obvious. Some of them can be used in fact as a basis for amusing and entertaining games.

*Four color theorem. Every planar map can be adequately colored with just four colors.

*Ramsey's theorem (elementary version). Given six points on a circumference one joins each pair of them and paints the resulting segment blue or red. Then at the end there is always at least one triangle of such segments with the sides of the same color.

*Sperner's lemma.One triangulates a triangle ABC, i.e. one partitions the triangle ABC into smaller triangles so that each two of them are disjoint or have only a side in common or only a vertex in common. The vertices of the triangulation are given the names A,B,C, under the only restriction that there should not be a vertex C on the side AB of the big triangle, no vertex A on BC and no vertex B on AC. Then, at the end, there is always at least one triangle of the triangulation whose vertices are A,B,C.

*Kakeya's problem. To find the infimum of the areas of all plane figures in which a needle of length one can manoeuvre in a continuous way so that at the end it occupies the same position in inverse orientation.

*Fixed point theorems.

*The triangular billiards table.

*Helly's theorem.

*Hadwiger's conjecture.

*Borsuk's conjecture.

Mathematical games as an instrument for the teaching and popularization of mathematics.

Martin Gardner has assessed the situation quite rightly: "Surely the best way to wake up a student is to present him with an intriguing mathematical game, puzzle, magic trick, joke, paradox, model, limerick or any of a score of other things that dull teachers tend to avoid because they seem frivolous" (Mathematical Carnival, Preface).

The expert mathematician starts his approach to any question with the same spirit as a child starts playing with a new toy, open to the surprise, with deep curiosity before the mystery he hopes to illuminate, with the pleasant effort of the discovery. Why should we not use the same gamelike spirit in our pedagogical approach to mathematics? A well selected mathematical game can lead a student of any level to the best point of observation for each one of the subjects he has to face. The benefits of so doing are many: blockbusting, openness,motivation, interest, enthusiasm, amusement,...

On the other hand the similarity of structure of mathematics and games allows us to begin exercising in the games the same tools, the same thinking strategies that are useful in mathematical situations. Specifically the heuristic abilities in mathematics can be succesfully initiated with the practice of many different games, as has been beautifully shown in the work of Averbach and Chein, Problem solving through mathematical games, with a very rich collection of games.

But above all, this gamelike approach to the more serious mathematical subjects can deeply benefit the student and positively influence his whole attitude towards the diverse mathematical situations for the rest of his life, by showing him how to set himself in the right spirit in order to face mathematical problems.

From the point of view of the popularization of mathematics, the effectiveness of mathematical games is so obvious that I need not stress it. In the dedication of the recent masterwork by Berlekamp, Conway and Guy, Winning ways for your mathematical games, the authors write, with entire justification, "To Martin Gardner who has brought more mathematics to more millions than anyone else". Mathematics and games, as we have seen, are often indiscernible in their contents, but even much more in the common spirit with which they can be approached.

Mathematics is a great and sophisticated game that, besides, happens to be an intellectual work of art bearing at the same time an intense light to explore the universe and so having great practical repercussions. The attempts to popularize mathematics through its applications, its history, the biography of the most interesting mathematicians, through the relationships with philosophy or other aspects of the human mind can serve very well to let mathematics be known by many persons. But possibly no other method can convey what is the right spirit of doing mathematics better than a well chosen game.

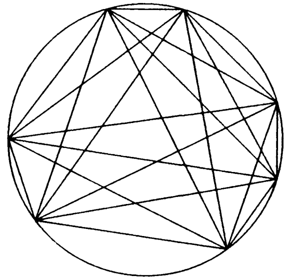

fig. 1

Solving the Problem of Popularizing Mathematics through Problems

MOGENS ESROM LARSEN

During the last three years I have been smithing puzzles by purpose for the popular science magazine *Illustreret Videnskab*, published monthly in Denmark, Finland, France, Norway and Sweden. I beleive that for many people it is pure fun to solve mathematical problems, if the task is voluntary, and the problems challenging. Hence supplying a popular magazine with entertaining problems is for me a great chance to lure mathematical methods of thinking into the minds of the readers. Even so, if they should never appreciate this aspect of their behavior.

I think from my own experience as problem solver and poser, that it is wizer not to over–estimate the forehand knowledge of the readers, and not to under–estimate their intelligence. So I hesitate to ask too stupid questions, but not to ask difficult questions which are easily understood.

The puzzles are at most classical matters from H. E. Dudeney, Sam Loyd etc., but I try to sneak in a little mathematics here and there. E.g. in the problem of the jeep crossing the desert, I added the question, "how big a desert can we cross?" Hopefully some reader will prove the divergence of the harmonic series. And according to letters some did.

To tease computer-freaks I like to ask questions with very large solutions. A source to such problems is the Pell equation. Some of these problems I have discussed in [2], but another will be the following:

"The pride of the republic of Inner Urdistan was the army. Each year every one of the 60 regiments send 16 soldiers to the parade. They marched in 60 squares. Then the general M. Urder joined the forces and all of them formed one big square together.

"After the revolution the new leaders founded a new regiment making a total of 61. But the general M. Urder wants to form the parade all the same. So he asked the 61 regiments each to send a square number of soldiers such that they could all together including himself form a new big square.

"How many soldiers did each regiment send?"

In this case we must solve the equation

$$x^2 - 61 \cdot y^2 = 1$$

The smallest solution is

$$51145622669840400.$$

Among the numbers up to 100, the biggest solution is required by 61.

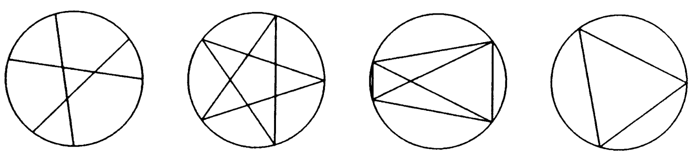

fig. 2

Counting is good, but formulas are better. One thing is to draw a hepta–gon with all diagonals and count its 287 triangles. (See figure 1.) Another is to find the number of triangles in an n-gon in general position with vertices on a circle, see [3], the solution is

$$\binom{n}{6} + 5\binom{n}{5} + 4\binom{n}{4} + \binom{n}{3}$$

to be found in [1].

As stressed by Richard K. Guy in his answer to [1] this formula is evident looking at figure 2 carefully.

It is even more fun to count the number of triangles in a regular triangular lattice, see figure 3., an old reference is [5], but see [4] on the general formula:

$$\left[\frac{n(n+2)(2n+1)}{8}\right]$$

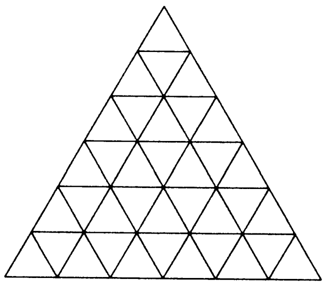

fig. 3

Theorems are still better than formulas. Take a paper with the ordinary square lattice. Take any 5 lattice points and prove that there must be two of those 5 points, such that the interval from the one to the other goes through a lattice point. (The pigeon–hole principle).

As computation can give the security of a result, and this is second to the understanding obtained from the logical deduction of a theorem, the latter is second to the *insight*, the experience of seeing right through the problem as can happen in geometry. Draw two intersecting circles and ask for that line through one of their cutting–points which is longest: in figure 4 we ask for the choice of line through A, such that the segment BC is longest.

The solution is surprisingly simple. Draw the triangle ΔBCD, where D is the other cutting–point of the circles. See figure 5.

Then it is obvious, that the angles at B and C are independent of the particular choice of line through A. All the different triangles are

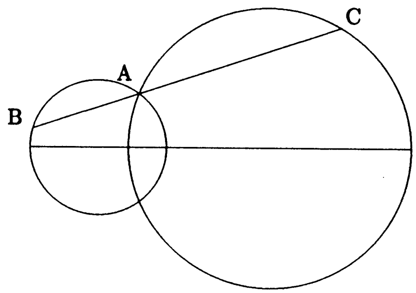

fig. 4

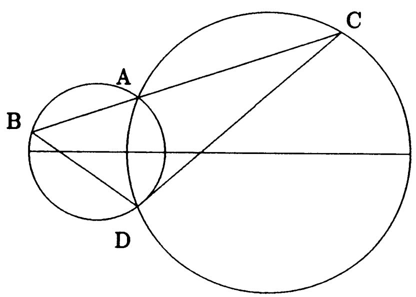

fig. 5

similar. This means that we obtain a maximal distance BC if we chose a maximal distance DC (or BD). And this is easy, we have to chose one (and then both) as the diameter from D. See figure 6.

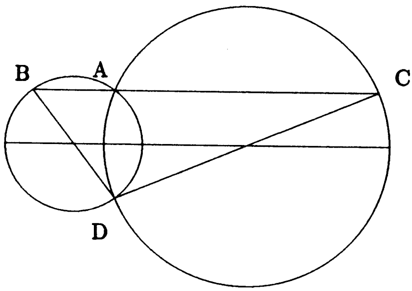

fig. 6

Then BC becomes parallel to the center–line.

Even more fun than geometry will be topology. Everybody knows the impossibility of joining three utilities with three houses in the plane without crossings. It is to say, the complete graph between two sets of three elements is not a planar one. But if we change the conditions to a non–planar surface, e.g. a torus, a Möbius strip, or the projective plane, then it might be solvable. And it is seen in figure 7 below.

This solution works equally well which one of the interpretations of the rectangle you prefer.

These are examples of what I like to do. Of course, it is a problem to continue the supply. I have published about 500 problems over 3 years, how can I do that? The easy answer is, that I do feel free to steal problems everywhere. The history of the problems shows that stealing them has been the custom all the time. So, I have a couple of problem collections and a couple of friends, with whom I exchange problems.

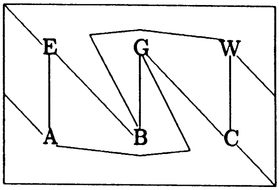

fig. 7

Whether my efforts are worth–while or not is hard to tell. I know from myself how much I enjoied doing such problems as a youngster, so I do naïvely assume, that others will enjoy the same. But I do know from teaching friends that a problem I have posed is immediatly challenging, so that all the pupils wants to see it solved. In this way my problems have made it easier to get a classroom getting started on the calculus of integration or whatever they must. And if I have contributed to do the teaching of mathematics just a little more entertaining, I find it worth–while doing so.

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Competitions for the masses

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The London Marathon has certainly captured the public imagination. Few people can deny the emotional appeal of watching, or joining with, other human beings who are stretching themselves to the limit of endurance. Mathematics evokes an equally strong emotional response; but most people respond *negatively*. And among those few who respond positively in the UK, relatively few have traditionally relished the harsh glare of mathematical *competition*.

How then can the Australian Mathematics Competition, after less than ten years in existence, attract over 400 000 entries in a country with less than a quarter of the UK's population? Over 80% of all High Schools join in, and more than one in four of *all* Australian High School pupils take part each year. These are staggering statistics! Could *any* single event in the UK -let alone a voluntary school maths competition- attract a comparable number of participants - say 1½-2 million? (The London Marathon, open to all ages, has around 20 000 participants; it seems unlikely that the total number of potential runners could be much more than five times that figure.)

Britain and Australia are very different countries. One is old and small, the other young and breathtakingly large; one is forward looking and (often naively) optimistic, the other tends to be backward looking and pessimistic (even defeatist); one has a class structure which helps to determine children's attitudes and aspirations from a very early age, in the other social mobility is taken for granted; in one the drop-out culture is fuelled by hopelessness, in the other by a surfeit of pleasure. Australia's size and political structure (in which the Federal Government has no control of education - each state designs the curriculum and assesses its pupils in its own way) ensures that most educational activities are locally based. In such a setting an 'unofficial' national celebration such as the Australian Mathematics Competition may have a peculiar appeal - especially if it is set up by a group of people with a clear vision of what can be achieved and who have the necessary political skills to achieve it. In the UK things tend to be more complicated, so it would be rash to imagine that one could simply copy the Australian model (or anyone else's). However, it would be even more foolish not to try to learn from their remarkable experience.

The last ten years have seen a number of developments in the UK which hold out some hope that we too are coming to accept the idea of mathematics competitions 'for the masses'. Frequent public examinations are an integral part of our curious education system. Moreover these examinations have traditionally been pitched at such a level that very few candidates experience the satisfaction of having performed well. (Pass marks of 30-40% ensure that very few candidates score more than 70%.) It is partly for this reason that many of those who have in recent years sought to stimulate a wider interest in mathematics among British schoolchildren have *avoided* timed written examinations, despite their administrative convenience, preferring to encourage other approaches to mathematics which have been relatively neglected. The result has been a profusion of *local* initiatives, adapted to *local* circumstances and interests, and constrained by the associated administrative burden to remain *local*. Examples include

(1) inter-school team quizzes (London, Tyneside),

(2) team project competitions (Scotland, Leeds),

(3) take-home sets of (normally six) problems (Merseyside, West Midlands, York).

Some of these have been more determinedly 'populist' than others. But experience on Merseyside and in the West Midlands suggests that there is indeed scope for a *national* competition in the UK aimed at 30-35% of the ability range, which might hope to involve say 50% of all secondary schools. Such an event should obviously supplement, rather than compete with, existing local activities.

If a national competition is to be administratively viable, it cannot hope to match the richness and strucural complexity of many local competitions. It must nevertheless learn what it can from their experiences. The UK Schools Mathematical Challenge seeks to do precisely this. It is based on a number of fundamental observations.

(1) If it is to be a large, national event, it must be based on a timed paper which all participants take on the same day.

(2) To minimise disruption in schools, the paper (i) cannot last longer than 1 hour, and (ii) must be accessible to a sufficiently large fraction of the ability range to allow schools to enter whole classes.

(3) Entering and administering the test in schools must make as few demands as possible on teachers.

(4) The organisers must therefore work very hard to keep things as simple as possible,

and to avoid complications, confusion and errors (though at the end of the day they cannot escape the constraint imposed by teachers' willingness to read, and to adhere to, instructions).

(5) To benefit mathematically from the experience, pupils need to spend time on the problems after officially completing the paper.

(6) The only way of *making* this happen is to set questions which are sufficiently intriguing for pupils to *want* to solve them, so that they discuss them spontaneously among themselves and with their teachers after the event.

(7) If one is concerned to influence *attitudes* to mathematics, then the 'apres-ski' of results and certificates is just as important as the problem paper itself.

(8) To build up a tradition with schools taking part *every* year, it is important that all schools receive sufficient recognition (e.g. in the form of certificates) for having taken part.

(9) But if the competition is to continue to appeal to the *whole* of its target group, it must not allow itself to be sidetracked into trying to single out a handful of 'prizewinners'.

(10) To guarantee accuracy and to accumulate reliable statistics on the suitablility of questions the paper must be (machine) marked centrally.

(11) Results and certificates should be received by schools before the end of the term in which the paper is taken - ideally within four weeks of the test date.

The idea of such a competition in the UK was first mooted in 1982. The first pilot run took place on 3rd March 1988, underwritten by the Mathematical Association. Publicity was mainly channelled through 'Regional Coordinators' in six regions (Cambridgeshire, Cornwall, Gloucestershire, Lancashire, West Midlands, and Wigan). Two thirds of the total entry of 16 000 (from 340 schools) came from these targetted regions. The one hour paper consisted of twenty five multiple choice questions, each with five options. Answers were marked on special machine-readable answer sheets. Certificates were awarded to the top 30% of participants (5% Gold, 10% Silver, 15% Bronze).

All sorts of things went wrong. But most of these were put right for the 2nd UK SMC, which followed very similar lines and which took place on 2nd February 1989, thanks to support provided by the University of Birmingham. Again publicity was channelled mainly through Regional Coordinators, this time in around twenty regions (the UK is divided into ninety such regions). Again the targetted regions generated around two thirds of the total entry of 32 000 (from 550 schools). The inclusion of an entry form in the Mathematical Association Newsletter generated entries from 100 schools. All of the 340 schools who took part in the 1st UK SMC received publicity about the 2nd UK SMC direct from the organisers as well as via their Regional Coordinators (if any). Of these, around 75 did not enter the 2nd UK SMC. Subsequent enquiries indicated that almost half of these were the result of a conscious decision in the light of their experiences with the pilot run, the rest being the result of oversight or overwork. The schools who did take part this year seem to have had far fewer problems, and the many letters received have been uniformly appreciative. (Their support has had one interesting consequence. When the continued existence of the UK SMC was threatened by a perverse interpretation of Section 5 of the Education Reform Act 1988, so many teachers wrote to their MPs that the DES had to climb down and admit that "Section 5 was never intended to apply to competitions".)

For the next three years the UK SMC will be sponsored by the National Westminster Bank plc. We plan to expand slowly. Only time will tell whether our goal of involving 50% of secondary schools is realistic.

1988 Q11. Quince, quonce and quance are three types of fruit. If seven quince weigh the same as four quonce, and five quonce weight the same as six quance, then the order of heaviness of the fruits (heaviest last) is

A quince, quonce, quance *B* quance, quince, quonce *C* quonce, quance, quince *D* quonce, quince, quance *E* quince, quance, quonce

1989 Q4. The names of the whole numbers from one to twelve are written in the order they occur in a dictionary. What is the fourth number on the list?

A four *B* five *C* six *D* seven *E* nine

FROGS and CANDLES – Tales from a Mathematics Workshop

A Mathematics Workshop is an event organised for a group of voluntary participants – usually children – to meet and engage collectively in a variety of mathematical activities. In this paper we describe the development of mathematics workshops, mainly in the United Kingdom, by members of the Association of Teachers of Mathematics (ATM).

Workshops can take a variety of forms, perhaps the simplest of which is one mounted on a Saturday morning, when typically the organisers are members of a local branch of the ATM. Each of the contributing teachers will arrive with a couple of their favourite activities which they will run all morning with different groups of children.

Sometimes families arrive as groups, so that a wide range of ages of children work together with parents joining in as well. For example, at a recent workshop a set of mathematical games, available at several levels of difficulty, was set out on a table. It was observed that the older children were initially happy to play a simple game which everyone could cope with and then to help the smaller ones to play a harder version. At no point did they appear even to wish to take advantage of the fact that they were more competent at what was involved.

Other groups of children come with a teacher and the willingness on both sides to expend "free" time on the workshop indicates the value and appeal of mathematics presented this way.

For more intrepid and devoted organisers, the residential workshop offers an extended chance to work in this creative way in a novel environment. This can allow not only the opening up of new mathematical avenues but, when teachers take their own pupils, the development of stronger pupil-pupil and teacher-pupil relationships.

The FROGS and CANDLES of the title are two activities which have proved perennially popular with children (and adults) in many workshop settings.

FROGS is the problem of two sets of frogs meeting as they use a line of lily pads as stepping stones to cross a pond. Each lily pad can support only one frog and the frogs pause when they see there is only one vacant pad separating the two parties. The aim is to interchange the two parties using the minimum number of permitted moves which are

(i) a glide (one frog moves to the adjacent empty pad), and

(ii) a leap (one frog leaps over another onto an empty pad);

and hence to discover the number of moves required by different sized parties. The activity is often initiated by two teams of children (boy frogs and girl frogs) enacting specific cases.

Whilst FROGS can be worked on anytime and anywhere, CANDLES is an out-of-door activity which needs the cover of darkness to achieve its effect, and is thus more used on residential workshops. Each participant carries a lighted candle, safely anchored in a glass jar, and moves to a position which satisfies the conditions given by the group leader. Several variations of increasing complexity are tried, and when the group leader sees that everyone is appropriately placed the participants place their candles on the ground and repair to an upstairs room from which to view the locus that has been mapped out.

Workshops and the popularisation of mathematics

As indicated above, mathematics workshops can take many forms but we believe that there are a number of characteristics which all successful workshops have in common. These include choice, enjoyment, activity, sharing and challenge.

Choice First of all the participants choose to come to the workshop, and, having arrived, find there is a variety of activities to choose from and a choice of leaders to work with.

Enjoyment A major aim of all workshops is for the participants to experience enjoyment in engaging in mathematical thinking. If the choice to participate is to be made there needs to be a promise of enjoyment, and to keep participants engaged, the promise needs to be fulfilled.

Activity Another emphasis is active engagement with the mathematics – workshops are based on the belief that mathematics is not a spectator sport. The starting activities are often very literally active starting from a "people maths" game or puzzle. There is usually plenty of physical apparatus around, but starters can be "in-the-head" problems.

Sharing Working in collaboration with others enables participants to contribute to the solution of problems which they could not tackle alone, and to share delight in group success.

Challenge The activities though enjoyable are not the end in themselves. There is typically a period of reflective follow-up to activities within the workshop itself, and usually further questions arise of the "I wonder what would happen if ...?" variety, which participants carry away and work on, sometimes long after the original event.

These attributes all contribute to the participants' developing image of mathematics. This image is "popular" both in the sense of being a shared image, evolving from the pooled perceptions of a group working together, and in the sense of participants being free to choose to engage in something enjoyable. They promote an image of mathematics as a dynamic and challenging field of activity, rather than a static body of knowledge.

The ATM Children's Workshop Group

In the late 1970s a number of members of ATM were experimenting separately with residential mathematics workshops for children. In 1981 a co-operative working group of members, interested in pursuing this idea further, was formed and met several times with financial support from ATM.

In July 1981 the group arranged and ran a five day residential workshop for children from two schools. The venue became popular and was the setting for several more similar, though often shorter events. These led to informal publications of children's work including the original "Maths at Simonsbath".

As the group collaborated with each other on developing their own expertise in setting up and running workshops, individual members were invited to give talks about their experiences to a variety of interested groups, often teacher groups who wished to run workshops themselves. It became evident that such teachers also welcomed practical support and so the group began to develop the notion of a workshop box. Two of these were set up, one in the north of England, and one in the south. They contained a variety of suggestions for workshop activities and the practical materials needed to support these. Teachers were able to borrow a box and use it as the basis of their own workshops, and were encouraged to add their own ideas to the contents.

By June 1982 the Children's Workshop Group offered interested members of ATM the following services, listed in their newsletter of that date.

> "We can
>
> 1 keep you up to date with a regular newsletter with
>
> – information about workshops that are coming up
>
> – suggestions of how to set up your own workshop.
>
> 2 Offer you opportunities to
>
> – visit a workshop in action
>
> – bring some children to a workshop
>
> – set up your own workshop with help from experienced people.
>
> 3 We are putting together A BOX of useful materials which will be available for you to borrow.
>
> 4 Two pamphlets have been produced, written by children, based on their experiences at Simonsbath.
>
> – Maths at Simonsbath
>
> – Mathematics at large."

The development of the boxes eventually culminated in the publication by ATM, of a pack entitled "Away with Maths" which presents an extensive resource for running workshops.

Evaluation

The ATM children's workshop group did not attempt a formal evaluation of their work. Instead they issued invitations to other teachers to observe and/or to participate, and hence discover the value of workshops for themselves. This theme will be taken further in the section entitled *Workshop as exemplar*. Here we simply remark that the group's success in spreading the workshop idea is some measure of value in itself.

Further evaluative evidence is manifest in comments made by participants. The children tended to remark on specific activities, (and it may be instructive to compare their descriptions of FROGS and CANDLES with ours), but often their written comments implied response to one of the five attributes of successful workshops which we have identified. For example:

> "The thing I enjoyed most on this weekend was ... when we were allowed to go into one of the classrooms and make or do anything we wanted. ... Clara, Raphaella and I made hexagons with hexagonal beermats. When we made a few we stuck them together. Then we tried to make stars out of matchsticks ..." (Choice, activity.)

> "The FROGS is a puzzle where you have a certain amount of space and in each one of these you have to put a counter except for the central space. The frog (counter) is allowed to jump over another frog (counter) or move into a space next to it. The objective of the game is to get all the counters on the opposite side to what they started on. We got the answer correct and had a lot of fun in getting the answer."

> "This was just one of the experiments which we had to work out. There were many other enjoyable puzzles which we had to work out. The weekend was very enjoyable and good fun."

> "I think my favourite thing of all was with the candles. Everyone of us took a candle in a jar. ... Next we got a piece of paper and took it and the candle outside to where there were two lighted flares. Then we got two edges of the paper and lined them up with the two flares and placed our candle down at that spot. Finally we went back inside and looked out of one of the upstairs windows. We found out that all the candles had formed a large circle. It was a terrific sight." (Enjoyment.)

> "The person in front had to put their right hand between their legs and join with the person behinds left hand and so on down the line. When that was done the person at the back had to go through everybodys legs and not break hands. Why don't you try it with a friend?"

> "I liked the tick-tock game because it was so stupid and everyone got muddled up with the tock going one way and a tick going the other and "a what!" somewhere in the middle." (Activity.)

> "One of my favourite problems was to do with cubes here with your group you had to imagine a cube, a white cube, and we had to count the number of sides on the cube the number of corners on the cube. We then imagined a line joining all the sides together through the middle of the sides and no more than one line on the side of the cube."

> "My favourite things on the maths weekend were games were we all got into groups." (Sharing.)

> "The four cubes puzzle is fairly complicated. I enjoyed this most of all because it keeps you buissy (sic) and makes you concentrate – hard." (Challenge.)

An older pupil, a lower sixth-former quoted in "Away with Maths", welcomed the opportunity to work at greater length and depth on some aspects of mathematics:

"The work has given us all a chance to go much deeper into complicated problems, formulae and mathematics in general. We have been able to forget other subjects for a time and concentrate all our efforts on the maths. We have covered a large and varied amount of work in a very short space of time."

Group leaders as well as participants recorded reflections on their experiences. For example a student teacher said:

"The informal, relaxed atmosphere of a maths weekend is something very special, but we have had the same "buzz" in lessons since then. ... A good and valuable time was had by all. Perhaps a weekend like this should be part of everyone's maths experience?"

A teacher who had reservations about taking part in any form of residential visit with children had her views modified when she took part in a well-organised residential Mathematics Workshop:

"The first time I went away with a group of children to spend a weekend at a residential centre was in my first year of teaching. ... I had no previous experience of the sheer energy of children of this age group, and after an exhausting – if enjoyable time in parts – vowed never to do it again. Little thought had been given to the structure of that weekend, and I had no real place.

I was 'captured' however after agreeing to spend a weekend away with a third-year tutor group. ... Much thought had been given to walks and activities and I had a place ... I was to do the evening sessions. A circle of chairs was placed for anyone who cared to join in. They were expectant and there was sufficient structure to demand involvement. ... I was aware that something different was happening here ... and of the possibilities for maths times ..."

A number of teachers who pooled their reasons for running residential workshops, came up with the following list:

- workshops take maths out of the school environment and give children and staff a chance to have a real mathematical experience in a relaxed informal atmosphere.

- going away is exciting.

- children have the opportunity to pace and organise their own work.

- there are no bells to interrupt. They have time to explore their ideas to a greater depth.

- workshops motivate teachers as well as children. This may well be transferred back to the classroom.

- "it gives me a chance to do some exciting maths that is not on my syllabus."

- "it gives me a chance to do some of my school syllabus in an exciting way."

- teachers have the freedom to experiment and exchange ideas.
- the relationship between children and their teacher is altered.
- there are more adults.
- it gives children confidence to have a go.

Workshop as exemplar

The ATM Workshop Group concentrated on working with other teacher groups who contacted them. Other ATM members have used workshops to share their ideas with specifically targeted groups of adults.

Over a period of many years Manchester Polytechnic has run an inservice course for teachers of "middle-years" pupils (age range 9-13). This course is unusual in that its second half consists of a sequence of workshops which the teachers resource and run and to which they bring a small group of their own pupils. These run partly inside school hours and partly outside. From the children's point of view they appear to represent an exciting chance to come to a large "senior" institution and do some interesting mathematics. From the teachers' point of view they offer a chance to see the similarities and differences exhibited within a group of children of varying ages and from different schools. A vivid memory is the picture of a group of "Junior 4" children (aged 10-11) directing everyone else in an immense game of FROGS in order to verify their solution. Their competence and confidence were impressive and gave many of the secondary teachers an entirely fresh view of what young children can achieve mathematically.

Workshops also form a powerful means of changing parental attitudes. Some element of deception is perhaps needed the first time this is used, as only the confident will choose to attend a workshop if it is advertised as such. However experience has shown that those who find themselves at a "talk" which proves to be a workshop rarely, if ever, walk out again in protest, but remain to join in with good humour. Indeed at a recent workshop of this kind, when alas no camera was available, the sight of about a hundred parents deeply involved in games, puzzles and investigations remains vividly recorded in the mind's eye. There was a threat from some of the fathers to instigate a betting system for some of the more complicated games – perhaps we could have made a profit for school funds! However what was most memorable was that when the time came to give a short talk about the educational ideas behind the evening's activities, it was almost impossible to separate this large group of adults from the mathematics, indeed some of them continued to work on surreptitiously like naughty children.

We have asserted that describing workshops gives only the merest flavour of what they are like, and that it is better to see one for oneself, better still to participate in one. ATM therefore determined to put on a children's workshop at ICME6 in Budapest in 1988. Those who volunteered to run this were experienced and enthusiastic about running workshops in the UK, but were aware that they could meet barriers of language and cultural expectation in transplanting the form to Hungarian soil.

So a year before ICME6 a small advance party (Jan, David and Gillian) went to Budapest to establish contacts and to work with children in a school noted for its teaching of English. They discovered that FROGS retained its almost universal appeal, and with just a little help at the start to set up the problem, they could work almost without need of words with a large group of children who knew only a little English. (They did however find it expedient to learn the Hungarian for "frog" and for "backwards"!) Indeed photographs taken of this large group of 80 or so children at work look almost disappointingly like any other workshop run anywhere, giving both a sense of anti-climax and of triumph.

Thus ATM ran its workshop at ICME6 with a core of some of the children they had met on the preliminary visit (who incidentally gave up four days of their summer holiday to participate). These children were joined by a growing band of children of conference members and together spent long tracts of time working on mathematics of all kinds, undeterred by the variety of languages being used. Typically to be found in the workshop, was a group of children of six or more nationalities, ranging in age from 5 to 14, working on activities. This usually also drew in the occasional parent, Hungarian teachers and other conference members who had come to see what was going on. "People games" (i.e. those games and puzzles which can be presented using people as counters) were particularly successful. The success of the workshop for the children is perhaps summed up by the very able boy, not a native speaker of English, who said at the end of the last day:

"Thank you for all the problems – at school they only give us calculations to do!"

There was only one aspect of ATM's ICME6 experience which was disappointing. Although we had been given plenty of space for both noisy and quiet activities, and for displaying our materials, we were neither in the main conference building nor the exhibition building, so many conferees passed us by. It was therefore decided that if the conference would not come to the workshop the workshop should go to the conference. So, ably led by "Jan the Magnificent" the children, and not a few adults, marched on the foyer of the main conference building and worked on some "people maths" activities there. This provoked more camera clicking than any other conference event we witnessed. The message went round the world. Doing mathematics is an international form of communication!

Summary

In this paper we have tried to describe the nature of mathematics workshops and indicate their potential effect on the image of mathematics held by all those who become involved in them. We reiterate our belief that the best way to learn about workshops is to take part in one. In an earlier section we listed the characteristics of a successful workshop, but these were somewhat abstract so we conclude by offering a more detailed list of typical elements of mathematics workshops.

(i) They are fun.

(ii) They can involve children of all ages and abilities.

(iii) They allow children and adults to work together on related activities.

(iv) They can involve activities which make sense to the smallest child but which can challenge the ablest adult.

(v) They are active and practical.

(vi) They offer the opportunity to work on mathematics for an extended length of time.

(vii) Existing, as they do, outside the confines of the normal school day they convey the unspoken message that doing mathematics is a worthwhile way to spend time.

Reference:

The ATM pack "Away with Maths" is available from the Association of Teachers of Mathematics, 7 Shaftesbury Street, DERBY DE3 8YB, UK.

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HANDS UP FOR HANDS-ON MATHEMATICS

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Introduction

Mathematics is not a spectator activity. You have to participate in it actively to appreciate the beauty and fun of the discipline. By popularizing problem-solving in the early 1980s an attempt was made to drag the formal subject studied in schools into 20^{th} century homes. But popularization cannot be achieved on a large scale by merely setting more popular pencil-and-paper problems. I believe that more emphasis is needed on hands-on mathematics.

Hands-on mathematics can take many forms. I first started to use it as a high school teacher thirty years ago to illustrate some aspects of Euclidean geometry. We folded cardboard triangles in the classroom to show that the sum of the internal angles of a triangle make 180°, and to illustrate that the straight line joining the mid-points of two sides of a triangle is parallel to the third side and equal to half its length. The students enjoyed this "break" from formal mathematics, and were more motivated to study the formal proof after they had performed the demonstration themselves. When we turned to trigonometry I couldn't resist getting them to make their own clinometer (theodolite) from a drinking straw, a protractor, a piece of string and a metal weight. The school building had many different measured heights that day according to the various calculations performed, but to conduct a mathematics lesson outside the classroom was a rare event in that era of structured chalk-and-talk lessons.

Later I moved to tertiary institutions and was amazed at the small amount of interest shown by academics in both primary (elementary) school mathematics and hands-on mathematical aids. Apart from some exceptional books by John Holt (1964, 1965) it was not until "Sesame Street" arrived on television that pre-school mathematics was popularized, but this was never extended for the general viewers to mathematics at any level above counting and basic shapes.

When my children began school in the late 60's and early 70's I decided to devote part of my research time to popularizing mathematics within the community. So here is an account of the directions I took and the progress made.

Mathematics in the Home

I became involved with the primary school's Parents' Association, and after a short while I arranged for the head of my department, Professor John Burns, to come and give a talk to the parents one evening on mathematics in the home. John had given this talk some six years earlier when his children were in primary school, and I was anxious for the next group of parents to hear it. He talked about the way he and his wife had involved their children in hands-on mathematics around the home, particularly in the kitchen with cooking or laying cutlery on the table. John emphasised the use of buttons, jelly beans, matches, coins and other objects to illustrate aspects of mathematics. He said that parents should encourage children to play games such as dominoes or cards where their school mathematics could be put into practical use. Indeed older brothers and sisters would naturally introduce more advanced ideas of mathematics, and so the mathematics involved could be considered at all levels.

John's main point in his talks was that while we look to our schools to provide children with formal instruction in the principal language, we certainly do not expect them to do the whole job. It should be the same for mathematics.

To the ideas suggested by John could be added the geometry of making a seed bed, the geometry and calculations needed in elementary carpentry (such as building a simple toy boat from wood) or more advanced carpentry (such as constructing a picture frame), and the mowing pattern needed for an irregularly shaped lawn.

Therefore, the teaching of mathematics should be a joint effort between school teachers and parents, and this is why it is so important for us to think of new ways of popularizing the subject. Hence John and I support the introduction of the Family Mathematics Program in Australia and the United States, but we are disappointed with the direction that it is taking in some regions. We feel that it is more important to show parents how to introduce hands-on mathematics in the home, than to educate them in the curriculum being taught to their children at school.

The A.C.T. Mathematics Centre

While on study leave at Cambridge in the United Kingdom in 1973 I had seen a mathematical obstacle course at the Stapleford Mathematics Centre nearby. I noted the enthusiasm of schoolchildren once again for hands-on mathematics using pieces of cardboard and counters. The obstacle course was designed to keep primary schoolchildren amused while their teachers visited another part of the centre looking at resource materials. After taking my children to the centre during the school holidays and seeing their delight and interest in this "new" way of presenting mathematics, I decided that a child-oriented hands-on mathematics centre was worth building.

In 1974 I applied to the Australian Government for a grant to put together a hands-on mathematics problem-solving centre for children from Year 3 (age 8) to Year 6 (age 11). I was successful, and hired a teacher, found an empty classroom in an inner suburban school and began building hands-on mathematical tasks. We opened in 1976 to classes of less than 30 from any school, public or private. At that stage, one hundred tasks had been constructed from wood, plastic and other colourful and durable materials. Each task had an instruction card, the associated hands-on materials, and was contained in a clear perspex kitchenware container.

Each class could come to the Centre for a session, which lasted $1\frac{1}{2}$ hours, during which the children worked in pairs and solved about four tasks. Verbal communication in mathematics within the pair was emphasised, as well as with the centre teacher when the task had been completed. Details of the centre and some of the now available 500 tasks are contained in de Mestre and Duncan (1980).

The A.C.T. Mathematics Centre was adopted by the ACT Schools Authority in 1978, after our trial run of two years, in which we showed them how exciting it was to teach problem-solving this way.

There are now two teachers at the centre, and it is open every school day of the year for three sessions a day. More than 120,000 children have experienced the fun of the Mathematics Centre. Although it has now been extended to include secondary schools also (Year 7 to Year 12, ages 12-17), we found in the first year that the 8 year olds (Year 3) could not comprehend simple written instructions, and so that group had access delayed for one year.

This centre has spawned a number of similar school-based centres in Victoria, teacher-training mathematics centre projects in Brisbane and Los Angeles, and a museum-based centre in Baltimore. A travelling hands-on mathematics centre is being planned for small Australian towns later this year, and I will shortly be endeavouring to set up a new centre on the Gold Coast of Queensland in association with Australia's first private university.

Parent groups have visited the ACT Mathematics Centre at night, and there is no doubt that after 13 years of community exposure it has helped significantly in popularizing mathematics amongst school children.

National Science and Technology Centre (N.S.T.C.)

Last year was Australia's bicentenary of white colonization. A new science and technology centre was funded and opened as part of these activities. Three years prior to the opening I began efforts to have mathematics incorporated prominently in the new centre. I became the Australian Mathematics Society's co-ordinator for mathematics exhibits at N.S.T.C. More than fifty ideas for exhibits were sent to me from mathematicians and other scientists all over Australia and some of these have already been turned into working prototypes (de Mestre, 1988a).

During 1988 I toured eighteen science museums throughout the U.S.A., Europe and Singapore looking for more ideas for mathematics exhibits. Another forty were collected.

Mathematics now occupies one of the four major galleries at the N.S.T.C. and we are ready with many new exhibits to replace the current sixteen on display. I spent the last three months of 1988 on leave from my university working with the design and construction teams at the N.S.T.C. on new mathematics exhibits. Although different exhibits will help popularize mathematics to the general public, it is the hands-on exhibits that are best. This is the emphasis at the N.S.T.C. Static exhibits behind glass cases are not used at the N.S.T.C. (one of its great successes) and those that I saw overseas attract about one passer-by in fifty (for example, the static exhibit commemorating 100 years of the American Mathematical Society in the Natural History Museum in Washington).

There is much still to be done to develop hands-on exhibits that will popularize mathematics. Such development requires the building and modification of improved prototypes. For example, one new idea that is being developed into an exhibit is concerned with the height distribution of visitors to the N.S.T.C. each day. Firstly their heights are to be measured (either directly or, as at the Singapore Science Centre, by similar triangles using the image of a light source in a plane mirror some distance away on the floor) and then these heights are to be recorded. We are proposing to have twenty-two vertical clear plastic rods each 4 metres long. An orange plastic ring will be issued to each visitor and as they walk up the ramp inside the entrance foyer they will be able to place their ring over the appropriate rod for their height-class. Visitors will then be participating in a hands-on experiment, and will be able to see the distribution of heights build up as the day proceeds. Design problems have been investigated and recently overcome on how to attach the rods to the wall given that there should be freedom of movement for the rings, and that they need to be removed from the bottom end of the rods at the end of each day.

The basic requirement of each hands-on mathematics exhibit is that it should illustrate a principle of mathematics, be attractive to touch, and be durable and strong. Exhibits currently on the floor include the cycloid, an elliptical billiard table, the Möbius strip, binary calculations and probability using dice. In the planning stage are exhibits involving regular plane figures using mirrors, Zeeman's catastrophe machine, trochoidal ocean waves showing particle circular paths, Moiré patterns, fractals, wallpaper pattern symmetry, the travelling salesman problem (Aldis, 1988) and Green's functions (de Mestre and Blake, 1988).

With thousands of visitors attending the centre each day there is no doubt that the mathematics gallery is contributing to the popularization of mathematics for the general public.

Mathematics and Sport

To popularize mathematics more successfully throughout the community it has to be removed from its isolated and abstract position, and connected to something that the community more readily identifies with. In my mind one of the most useful connections is with sport, since the media continually creates interest in this area.

Large sections of all communities are extremely keen about sport. People play it, watch it, avidly read about it, and it is a great conversation piece. Admittedly some people find it boring, describe it as a mindless preoccupation, and have other interests. But this is only a small group, and the majority of people enjoy it. I believe that we mathematicians can popularize our subject by capitalizing on this enjoyment of sport.

To this end I have written mathematical articles for sporting magazines (de Mestre, 1986b, de Mestre and Bretag, 1986) and sporting articles for mathematics magazines (de Mestre, 1985, 1986a). Talks have been given on mathematics and sport to teachers' groups in the hope that this will stimulate them to do the same for their students (de Mestre, 1987). I also give a section of a general mathematics course to first year undergraduates studying humanities subjects on the topic area of mathematics and sport.

I have already been involved with developing consultancy research on aspects of sport from a mathematical basis, using the experience gained recently in writing a book on the mathematics of projectiles in sport (de Mestre, 1988b).

The whole area is ready for popularizing, and my colleagues and I are about to launch into it even more deeply. Almost any sport can be examined from a mathematical point of view ranging from the distribution of the numbers on a dart board, through the staggering of lanes for a 200m track sprint, to the maximum distance for a football kick. Then there are the fluid mechanical aspects of windsurfing, the arrangement of rowers in an eight, batting averages in cricket and baseball, the air drag on front runners in middle-distance events or on cyclists, and the many different processes of scoring used in games.

To appreciate many of these sporting problems properly, and understand more fully their connection with mathematics, we need to participate, try them out or have a hands-on approach.

Conclusion

There are many ways to popularize mathematics.

The first step is to show the public that mathematics is fun, and that it is an interesting subject to explore. Once their interest has been aroused, some of them may then be ready to study mathematics in more detail. Indeed if adults can be made aware, through the popularization of mathematics, of the importance of this wonderful discipline then, by a filtering process, the attitudes of children to the subject studied in school may not be as negative as they now seem to be.

In this paper I have endeavoured to show that an emphasis on a hands-on approach is one of the best ways to popularize mathematics. It can be introduced in pre-schools (kindergarten), primary, secondary and tertiary institutions. It can be used in the home, in museums and in special centres. It can be incorporated in sporting activities. It only needs some confident and enthusiastic mathematicians to rise to the challenge. Will you join me?

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MATHEMATICS AND THE MEDIA: DIFFERENT METHODOLOGIES

Michele Emmer

INTRODUCTION

In recent years, there has been a significant increase in the amount of interest shown in scientific matters by the mass media, in particular by newspapers and television. The general public is showing considerable interest in scientific activities in the fields of physics, chemistry, biology and medicine. One only has to think of a recent example like nuclear fusion, and the enormous amount of interest that this subject generated in newspapers and on television all over the world. To what extent the information was scientifically correct is, of course, another matter.
Another highly topical subject is biotechnology, not just the scientific aspect but also the ethical implications of these new technologies. Then there is no need to emphasize the effects that developments in medicine have had on the general public, in particular concerning serious and wide-spread diseases such as cancer and AIDS. It is to be expected that the latest developments in these sectors should become general knowledge; in other words, that they should be trasmitted to the general public through the conventional mass media channels.
Another traditional occasion, closely covered by the mass media, is the annual award of the Nobel prizes. Therefore, in the fields of physics, chemistry, biology, medicine and natural sciences, we can say that there is costant coverage by the media of all the latest developments and the most recent researches. There is no doubt that one of the main reasons is that some of the results of scientific activity in these areas have immediate effects on the world community. One need only think of Chernobyl and the ozone layer as typical examples.

Another important factor to be taken into consideration is that these fields of scientific research require massive financial investment; so it is to be expected that the scientists themselves also wish to point out the necessity of developing one field rather than another.
So we reach the situation, as in the case of nuclear fusion, where the news of the experiments is announced to the press before the full scientific report has been published in the specialized journals. Research work is not simply a *private* aspect of the reseacher's life. Rather, it is something that comes under the heading of what can be called the *politics of research* involving not just different groups of researchers in a single country but research centres in all the major countries. Clearly, under such circumstances, it is the researchers themselves who are most aware of the need to announce the results of their work.
Another important aspect, in my opinion, is the attitude of the researcher workers towards the history of their particular scientific discipline. People working in the fields of physics, chemistry, biology and medicine have a very different attitude to those involved in mathematics. It is quite normal for a physicist involved in research to have an interest in the history of that particular sector.
In a recent article Patricia Clark Kenschaft, of the Montclair State College, has written [1] :
«Too often mathematicians are considered to be cold and insensitive people. This picture has a negative effect on our ability to acquire research funds and scholarships, to recruit teachers, and to convince primary school teachers to learn sufficient mathematics to prepare students for a technological world. We have to change our *image*. »

As pointed out in a recent motion presented at an AMS congress, the work of a mathematician is largely the work of a single researcher, or at most of small groups, who generally speaking do not need large sums of money for their research. In other words, mathematicians do not need to *get into politics* , unlike other groups of scientists who need large financial resources.
The result of this situation is that the world's scientific Museums have large departments devoted to the history and development of physics, chemistry, biology, medicine and natural sciences. The task of these museums is clearly made easier by the fact that they are able to display tangible *objects* representing the evolution of a particular discipline. Also, many of these museums have special areas where visitors can actually watch demonstrators reproducing classic experiments from the history of sciences. To give two examples: the Exploratorium of San Francisco and the Monterey Aquarium in California.
Having said all this, what then is the situation as far as mathematics is concerned ?

THE ROLE OF MATHEMATICS

First of all, I would like to make a few observations about mathematics and the attitude of the general public towards this subject. It is obvious that the only contact that many people ever have with mathematics is at school. Generally speaking (and unlike other disciplines), math teaching does not deal with the history of the subject, nor does it make any attempt to link the history and development of mathematics to that of other scientific and humanistic disciplines, thus showing their mutual interaction.

Mathematics is presented as a ready-built construction made up of a sequence of facts which are not open to discussion (since they are mathematically demonstrable!). While in other subjects it is normal to know the names of the scientists who have achieved fame in a particular field, in mathematics most people leave school knowing almost nothing about important mathematicians and their works (with only a few occasional exceptions).

Everybody knows that there are many different fields of specialization in, say, physics or medicine, very few people realize that there is not just one subject *mathematics* but a whole series of different specializations that flow into the wide river of mathematics.

This non-historical presentation of mathematics in schools means that mathematicians are considered to be academics who are cut off from the rest of the scientific world, without any exchange of information or mutual interaction. Then again, people get the impression that mathematics is a sort of science in which everything is already defined, in which the only activity is the addition of a few more theorems to those that already exists.

In short, mathematics is seen to be a discipline without a history, whose activities are sometimes interesting only because they are useful to other scientific areas. For most people, mathematics means simply arithmetic and the Euclidean geometry learnt at school. After leaving school, these people not only ignore mathematics but also question the role of mathematicians in today's world. What do mathematicians do in their research work ? Such a question would be ridiculous if asked about a physicist, a chemist or a doctor.

The result of math-teaching in schools is that most people know very little about what mathematics is today, about the different topics within mathematics, and about what mathematicians do. If one were to ask for the names of famous mathematicians, how many people would be able to give more than three or four

It is true to say that there are specific problems involved in the popularization of mathematics, problems that are different to those posed by other scientific disciplines. However, the teaching in schools is already an obstacle in many cases, because right from the start the students are clearly told that not everyone can understand mathematics, that there is a sort of barrier that not everyone can get past. In short, most people leave school with a marked aversion to mathematics as a whole, without really knowing what it is.

During recent years, the use of computers in schools has become widespread. So much so that if you ask someone what mathematicians do, most likely the answer will be « They work with computers ».

To all this must be added the sad fact that many mathematicians have little interest in the popularization of their science, and even less for its history, which is considered to be something almost irrilevant, something to be dealt with by math-teaching staff, people who are not *real* mathematicians but only *teachers*. This of course makes popularization even more difficult.

One aspect of the problem that should not be understimated in my opinion is that in many sectors of modern mathematics it is extremely difficult to give a *tangible* idea of the question involved. This is partly due to the fact that each sector has its own language and techniques that are only comprehensible to the mathematicians working in that sector (obviously, it is impossible to explain such concepts to someone who lacks even a basic knowledge of the subject); but, whereas in other scientific disciplines it is usually possible to explain the purpose of an experiment, in many mathematical fields even this is difficult to explain to non-specialists. In many cases, it is not possible to answer the question « What use is it ?» because this is not really the question.

It is difficult for an observer to understand that mathematics is indeed a science that seeks the solutions to problems, but that there is not much point in trying to distinguish between *real* (or applied) problems and *internal* problems of the discipline itself. In many cases the distinction between pure and applied mathematics is very contrived. In fact, theories and results that seemed to be highly abstract and specilized have sometimes been applied to very tangible problems, drawn from the physical world. There are numerous examples of this transfer.
It should also be noted that there is no Nobel prize for mathematics, and that the Field medal is only referred to in the specialized journals. Add to this that the names of contemporary and ancient mathematicians are unknown to most people, and one begins to realize how many problems there are facing the popularization of mathematics.

MATHEMATICS AND THE VARIOUS MEDIA

In the popularization of any discipline, it is fairly obvious that the methods used will depend in part on the nature of the medium - newspapers, films, exhibitions, conferences, etc.. Over the last 15 years I have had the chance to experience all these sectors and I would like to make a few comments about each one. However, first there are several comments of a general nature that apply to all media.
In keeping with what I said in the previous section, it is clear that, in order to popularize mathematics to a wide and varied audience, one has to assume that most of the people involved do not have even the most elementary knowledge of mathematics, nor of the language used to express its

concepts. Therefore, when working with the media, it is essential to provide the basic information and bibliographic references, without which any attempt at divulgation risks becoming a foolhardly undertaking. In addition, as I have already mentioned, it is important to select the subject matter carefully. Personally I am convinced that it is almost impossible to deal with some of the aspects of contemporary mathematics.
Therefore, when undertaking a project of this type, over and above the specific features and language of the medium employed, one has to bear in mind the problems created by the inherent difficulty of the argument, as well as the ignorance of the general public.
Now we come to the comments on the individual media.

MATHEMATICS AND FILMS

Films dealing with mathematics can be divided into two categories: brief movies, without soundtracks, used to illustrate elementary or simple phenomena; and longer movies with sound tracks and music. I feel that brief films have been completely superseded by personal computers. Nowadays, with computer animation techniques, one can create in real time, and directly on the screen, animation effects that used to be included in films with greater difficulty. So, I do not think there is much point in continuing to make films of this type. In any case, I suspect that their impact was minimum, except when the movies were made directly with the students.[2]
With so many images costantly surrounding us today, one of the problems we are confronted with is that of finding the right images to visualize situations in a wide range of scientific fields. In mathematics, we are dealing with ideas that are often abstract and difficult to grasp; obviously, we are not always able to find images that effectively clarify

the question. This leads, in my opinion, to the conclusion that it is useless to attempt to create a sort of movie library covering every topics of mathematics. The cinema medium, where images have precedence over words, is clearly not suitable for such a task.

In order to use cinema techniques for mathematical subjects, the two most important aspects (the scientific facts and the images used to illustrate them) must arise from the same source. One cannot hope to decide on the subject first, and then to search for the images with which to visualize it.

In my personal experience, the decision to link mathematical subjects to the visual arts (to architecture as well as to physics, chemistry and biology) seemed quite natural. One of the aims of artists is *to make visible the invisible* ; why not use the images that artists have created, starting from a more or less scientific base with the addition of a personal element, to talk about mathematics, and at the same time to talk about art while dealing with mathematics ?

My main idea with regard to producing math-movies is that of creating cultural documents seen through the eyes of a mathematician. In other words, as I have written elsewhere [3] :

« The movies are attempts to produce works wich are, at the same time, vehicles of information of a scientific and artistic nature on various mathematical subjects, and also to stimulate the observer towards further investigations of these same topics. The posssibility offered by cinema techniques are fully exploited. My intention was to use the full language of images and sound. The problem, of course, is to maintain a balance between entertainment and informative popularization in such a way that one aspects does not dominate the other. »

The movie should not be a lesson with pictures, but rather a new language that integrates the two ingredients. Compared with other media, the cinema has the great advantage of being able to provide a large quantity of information in a limited period of time.

Another major advantage is that the language of the cinema is universal. The language of images in movement is understood by people of all ages and all cultural backgrounds. This aspect has to be fully exploited in movies concerning mathematics, which have to catch the interest of an audience ranging from primary school children, to university students and the general public.

A part from the question of adaptations into different languages, one has to aim for a worldwide audience, just like a normal full-length movie. In other words, the spectators have to be able to find that point of interest that stimulates them to want to know more, to go deeper into the subject, to invent their own ways of pursuing the subject. The principal aim of a math movie is to help the viewers understand more, and to help them look closer at the material, in the light of their own knowledge and experience.

My own series of movies *Art and Mathematics* was based on these principles. Bearing in mind that the cost of making a 30 minutes movie is quite high, if it is not possible to reach an international audience (that is, by means of a presentation that is not linked to the educational programmes and cultural background of a particular country), then it is not advisable to use such an expensive medium.[4]

I should point out here that when I refer to movies, I mean movies made on film. Videocassettes and the reproduction of movies on videocassettes is another matter. These techniques lead to more efficient use of the material from the educational point of view, but much of the visual impact is lost. My overall view is that the ideal math movie should provide a certain quantity of information and stimuli (mainly visual) in a limited period of time, relating to a problem or a mathematical theme that is of potential interest for anyone.

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An important aspect is that the techniques used should be the best possible, consistent with the level of financing available. Nowadays viewers are used to watching highly sophisticated and eye-catching images; they would be disappointed if a math movie were not of a similar quality, not necessary at the same level, but definitely not a penny-pinching type.
In short, to be effective the math movie must be produced using specialized cinema technicians, with a high degree of precision, bearing in mind that the viewers are in some ways prejudiced against the subject matter in question. In this context, I think back to my first contacts, years ago, with the programming executives of the Italian Television Corporation (RAI) when I first proprosed making a series of movies on *The Cultural World of Mathematics*!
Of course, one has to find the right balance between the correct (though necessary incomplete) mathematical content and the spectacular effect. This is why I chose to make the series movie *Art and Mathematics* using the experience of both mathematicians and artists who had dealt with the same mathematical themes, visualizing some of them in their own ways.
One final remark: one should not fall into the trap of thinking that by just watching math movies one can learn the subjects dealt with. The movie works if it helps stimulate further interest, without boring the viewer.

MATHEMATICS AND NEWSPAPERS

In December 1987, a congress organized by the French Mathematical Society was held in Paris, on the theme: «Mathematics for the Future: what mathematicians for the year 2000 ?».

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Even by French standards, the press devoted an unusually large amount of space to the event. For instance, on the opening day of the congress, the daily newspaper *Libération* produced an EIGTH page special insert on mathematics. The front page of the paper carried a box giving high visibility and promotion to the insert. It's worthwhile reproducing the whole text of the box, under the title *Objectif Maths* :
«Mathematics is not well-known. But it is becoming more and more important in informatics, aeronautics, in the economy and in medicine. A group of French mathematicians, tired of seeing their disciplines misunderstood and used only as a mean of selection, have decided to become *seducers* during the congress *Mathematics for the Future* in order to catch the attention of politicians and industry chiefs. Explore the planet Math in our supplement.»
The eigth pages of the insert were illustrated with photos which one either liked or disregarded completely. However, they represented an imaginative effort to visualize what was written in the caption of the first photo : « The search for the unknown.» The photos formed a sort of picture story in which the leading character was the mathematician Sine.
« When one listens to mathematicians, their enthusiasm comes over with surprising force. Their words and discussions have provided the captions for our picture story: the adventures of a mathematician who sets out during the night to search for an unknown (in French *inconnue*).» In the first photo, it is night-time and we see professor Sine deep in thought; the caption says: « In the darkness, Professor Sine is half asleep; he feels that the unknown is within his grasp. The phantom of the solution wakes him up. He sets out to find the solution that he has been seeking for so long. »

The second photo shows inextricable mass of intersecting lines - the *ideas* of a mathematician. « Simple ideas do not come suddenly. On his flight of phantasy, Professor Sine comes across some alluring images, still unclear in his mind. But he does not trust them. The proof must not just be elegant, it must be true.»

But mathematicians are strange characters, creatures of the night. In the third photo of the picture story, Professor Sine is standing on his head surrounded by mathematical symbols.

« He trembles with enthusiasm. But the problem is to state the proposition with exactitude. So, Professor Sine adopts the pianist's position (standing on his head !) in the search for the purity of sound. The aim: calmness. »

One of the articles in the supplement, by Dominique Leglu and Selim Nassib, was an interview with the mathematician Jean-Pierre Serre, winner of the Field medal in 1954, when he was 28.

« Judging by his appearance, he is a normal person !» (" Well now, he looks like a mathematician, so what?-- I do not know; I have really no idea what a mathematician should look like ! -- Now that is a really sensible thing to say ! A mathematician does not have any special features ! In other words, he should look intelligent, in a general sort of way, without any particular qualities. Nowadays nobody looks the part, except for catholic priests, because we use our heads even more impersonally that our hands; but mathematics is the extreme case, it even ignores itself...." from Robert Musil, *The Man Without Quality*).

« Jean-Pierre Serre is completely happy when he goes to bed at 10 p.m.. Half-awake in the darkness and silence, this *Mozart of mathematicians* (as a colleague, Anatole Abragam has described him) achieves optimum concentration.

In fact, says Serre, it has never happenend that something perceived during the night turned out to be false....I have mixed feelings about beauty. It cannot be added to a proof as if it were an ingredient. One should not look for it. Instead, if one does something well and with precision, it is almost certain to be beautiful.»

A mathematician's task. Probably the only time a newspaper has tried to understand the subject and to help others understand it. During the Paris congress, a round table was devoted to the relationship between *Mathematics and the Media* : newspapers, television and scientific magazines. The roundtable was organized by the scientific journalist S. Deligeorges and amongst the partecipants, apart from several mathematicians concerned with divulgation, was the woman journalist who had prepared the *Libération* supplement on mathematics with other colleagues.

She said clearly that when she and her colleagues (all whom had some type of scientific or mathematical background) had proposed the idea, they were thought to be out of their minds. The main reason for such a reaction was that there is no such a thing as a generally accepted mathematical knowledge, so that when you deal with mathematics in a newspaper you have to start from the beginning every time. Another objection was: What are you going to talk about ? What illustrations can you use ? And the the crucial question: But will anybody be interested ? A question of some importance for a daily newspaper.

In spite of many difficulties and considerable perplexity, the editorial staff accepted the challenge of publishing this insert which represented a tangible attempt to help people understand and to invent a new way of talking about math and mathematicians.

I have given a detailed account of the case of this French newspaper because it seemed to be a unique example. For the last two years, I have worked as a freelance collaborator with an Italian daily newspaper which is distributed on a national scale.[5] It is the only paper that has an entire page every day given over to science. On average, I wrote an article every twenty days, with half the page of the newspaper at my disposal (more or less five, six standard pages). So my comments on the subject come from first hand experience. However, I would not deal with the monthly magazine, at a high level like the *Scientific American*, which many people have direct experience of.

One of the problems that arise when writing for a newspaper is that the subject has to be linked to some topical, day to day situation. This is not too difficult with other scientific disciplines, but with mathematics it can become very complicated. It is difficult to talk about a *discovery* in mathematics, and in any case it is difficult to make the problem comprehensible and to show how it was resolved. Also, there is no way that complicated symbols can be used in a newspaper. So, there is always the risk of over-simplifying everything and not communicating anything.

Articles about mathematics must necessarily be partly free from topical restrictions, even though they should have some reference to a recent event (a congress or conference, an importan result, etc.). They also have to long enough to give fairly detailed information including the basic bibliography on the subject. In my experience, an article should be about five or six pages long. Longer articles can be split up into instalments. The language should be accessible without over-simplification, one should not hesitate to use a few words that not all the readers will understand. The criteria are similar to those that apply to articles on literature and philosophy; it is better to be exact and difficult than approximate and over-simplified.

In short, one should avoid giving the impression that everything about the subject can be said in five pages! If the articles appear with a certain regularity, a useful relationship builds up with the readers, leading towards a degree of mathematical awareness, so lacking today. One should avoid being over precise, while still aiming for the right degree of exactness.

MATHEMATICS ON SHOW

A fairly new development, over the last few years, in the divulgation of mathematics is the itinerant exhibition. There are many examples of permanent exhibitions devoted to mathematics (in the USA, France, UK). But, over the last few years, we have seen the appearence of mathematical exhibitions linked to recent theories and results. To name a few examples: *The Frontier of Chaos* devoted to the fractals; [6] *Getting to the Surface* on the latest developments in the theory of minimal surfaces;[7] *The Etruscan Venus* devoted to topology and computer graphics.[8] Images play an essential role in this sector, especially those created by computers with sophisticated graphic capabilities. Then another interesting aspect is that these events have become not just scientific shows but *artistic* displays. A particular case is the topology exhibition *The Etruscan Venus* which combines the work of mathematicians and artists.
Here too, as I mentioned for films and as one could almost say for newspaper articles, the idea of building a bridge between scientific and artistic images enables one to deal, in a comprehensible manner, with a history of mathematics

parallel to the artistic and scientific events with which the audience is more familiar. It also enables one to illustrate the periods in which there was fuitful collaboration between mathematicians and artists; to show mathematical images through the artist's eye and, conversely, works of art that have been influenced by mathematics. It is clear that I am referring in particular to the Dutch graphic artist M.C.Escher, [9] but the question does not finish there, as can be seen in the exhibition *The Eye of Horus: a journey into mathematical imagination* held in various Italian cities in 1989.[10]
Another interesting feature is that visitors to these exhibitions are presented with problems which they are asked to resolve, sometimes with the help of live demonstrators. The organizers try not to use computers for this purpose, to avoid creating the impression that mathematics and informatics are the same thing. The French organized an interactive exhibition entitled *Horizons Mathématiques* at the Museum of Science and Industry in La Villette in Paris. Material from this show was included in *The Eye of Horus* in Italy.
However, one should avoid reducing the matter to a question of "*mathematical games*", whether complicated or less so. It is important not to give the impression that mathematics is merely a game; but at the same time it is equally important not to frighten the spectator. The aim should be to provide ideas and stimuli for looking closer at the subject.

CONCLUSIONS

I have tried to show the best uses of each medium for the divulgation of mathematics, pointing out the common features of the various languages, and basing my comments on my personal experience over a period of 15 years. In conclusion, I'd like to say that in the case of films and exhibitions it is the image that plays the essential role, awakening interest and stimulating the imagination. Also newspaper articles aim to achieve the same effect. We have to create an "*awareness*" of mathematics using every available medium. But we should not be afraid of not doing *Real Mathematics* , but only popularizing mathematics. For real mathematics we need other tools, although it is the computerized image that is becoming more and more important.

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SQUARE ONE TV: A VENTURE IN THE POPULARIZATION OF MATHEMATICS

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Mathematics is decidedly not popular in the United States of America. Course enrollments decline markedly as students progress through secondary school. Many of our students achieve little more than low-level computational skills. Adults often exhibit a narrow view of mathematics tinged with disregard, even hostility. Even though many adults recognize the importance of mathematics in work and careers, they often disavow its personal relevance. In fact, our businesses and industries spend billions of dollars each year on remedial programs. If one accepts the premise that mathematics is essential to a well-functioning citizenry, then popularization of mathematics is not only valuable but necessary. The general failure of our principal effort in the popularization of mathematics--pre-college education--suggests that popularization be attempted in alternative settings.

THE CTW MODEL FOR TELEVISION PRODUCTION

The Children's Television Workshop (CTW) produces educational television programs for children. With Sesame Street, its first production, CTW initiated the use of mass-market, commercial broadcasting techniques and styles for educational purposes. The CTW model for television production brings together three distinct groups: production, research, and content. The production group consists of the full range of specialists in television production. The researchers are specialists in child development, psychology, or communications. The content group consists of experts in the subject area of the field of the series at hand. The three groups work in concert under the leadership of an executive producer whose experience is grounded in television production, rather than any of the research areas or a content area. CTW has refined this model over 20 years in the production of Sesame Street, The Electric Company (dealing with the reading of English), 3-2-1 Contact (concerning science), and now Square One TV.

SQUARE ONE TV

Square One TV is a television series about mathematics. Its main audience are children voluntarily watching at home. Non-commercial stations broadcast its half-hour programs five days each week, Monday through Friday, usually late in the afternoon, after school. Square One TV follows the magazine format, that is, each program comprises a number of independent segments ranging in length from 10 seconds to 10 minutes. Within this format, the approach is to parody television broadcasting practices and conventions. A segment may be a parody of any of the programs and devices typical of the commercial channels: situation comedy, detective drama, music video, game show, news programming, commercial interruption, self-promotion, and so on. The rationale for using this format and approach is discussed below.

The Square One TV library now includes 115 programs. For these programs we have produced segments of several types:

Studio Sketches	235
Mathnet Episodes	65
Animations	188
Non-Studio Pieces	31
Game Shows	72
Musical Pieces	41
Total	632

Studio sketches are short pieces featuring a seven-actor repertory company. The pieces parody well-known television productions and television genres. Mathnet is a collection of five-part serials, an episode of which ends each program. Filmed on location in Los Angeles and New York, Mathnet parodies a popular detective drama of the 1950's. A variety of animation techniques appear in the library. Non-studio pieces include films and videos shot on locations outside of our studios. They typically take the form of a commercial interruption or an interview. The eight mathematically-based game shows are very much in the style of popular game shows on network television. Some musical pieces emulate the popular music-video format.

Forty more programs now in production will be broadcast for the first time in January 1990. In them we will introduce about one hundred more segments to the library.

AUDIENCES

The intended audience for Square One TV are 8- to 12-year old children. Secondary audiences include teachers and parents of children in the primary audience, and other viewers, both children and adults. Individual broadcasting stations independently schedule the program, typically in the late afternoon when children are likely to be at home. Nielsen ratings show that, on the average, about one million households view the program each broadcast day. Since its premiere on January 26, 1987, the show has been on the air 380 days at a cost of about six cents per household-viewing.

An interesting note from the Nielsen information is that Square One TV has a significant number of viewers younger than the 8- to 12-year-old target group. Square One TV is also finding an international audience through broadcast licenses in 22 countries other than the United States. The programs air in their original version, except in Australia where segments which significantly feature non-metric units of measurement are replaced.

Even though viewers are usually at home, some also watch the program in school. CTW permits the school community to record the programs on video-tape for use for three years after their broadcast. While half-hour programs are not often appropriate for formal instructional use, the magazine structure of the programs facilitates using shorter pieces. However, there are significant impediments to any use at all in schools. The necessary hardware is often not readily available. Many teachers are not prepared, nor do they have the time, to integrate the material into their programs. To encourage teachers, CTW provides three guides which describe instructional uses for a small number of segments. However, even though we have anecdotal evidence that some teachers are making use of Square One TV, there is no indication of substantial use among the nation's 1.2 million elementary school teachers.

GOALS FOR THE SERIES

Square One TV has three goals. Its first and primary goal is to support and stimulate interest in mathematics among its target audience. Interest in mathematics appears to begin to decline in the late elementary grades; hence our choice of audience. CTW's experience with earlier productions indicates the potential effectiveness of television in affecting attitudes. In our approach to Goal I, we try to show mathematics as a powerful and widely applicable and useful tool; to present some of the beauty of mathematics; and to convey the message that mathematics can be understood and used by non-specialists, even those in our audience. In addition, we try to help the viewer recognize mathematics in out-of-school contexts.

Goal II is to model good problem-solving behavior. Characters in the series' segments encounter mathematical problems and deal with them willingly and successfully. Their actions illustrate aspects of problem formulation, problem treatment, and problem follow-up, while they apply a variety of problem-solving heuristics.

Goal III is to present a broad view of mathematics. School mathematics in our country is concentrated on computational arithmetic. Without the constraints of a standard curriculum or testing program, we are free to present any mathematics which we can render interesting and accessible to our viewers. Thus, we undertake elements of geometry, probability, statistics, combinatorics, and functions and relations. Standard topics such as properties of numbers and counting, arithmetic, and measurement also appear in problem-solving contexts. A detailed statement of the goals is available from the authors.

We have made an extensive analysis of the content of the segments of Square One TV across the entire series library. (The analysis, as well as other documentation of the project, is available from the authors.) Of the segments making up the 115 shows, almost 60% address Goal II by explicitly presenting a problem for solution within the segment. The profile of the segments across seven mathematical areas of Goal III appears as follows:

Numbers and Counting	28%
Arithmetic	46%
Measurement	14%
Functions and Relations	25%
Combinatorics	3%
Statistics and Probability	17%
Geometry	25%

Since many segments involve more than one area of mathematics, the percentages sum to more than 100.

RESEARCH ACTIVITIES

We maintain an extensive program of formative research as an integral part of the production process. We show new segments to groups of target-age children to assess the material's appeal and comprehensibility. Moreover, we are currently conducting a major summative study of the effects of Square One TV. Its purpose is to examine in great detail, through extended individual interviews of eleven-year-old students, changes that might occur in their attitudes toward mathematics (Goal I) and in their inclination to use problem-solving techniques (Goal II) as a result of persistent viewing of the series. Results will be available late in 1989; a report will be available from the second author.

ISSUES RELATED TO POPULARIZATION

There are two arenas in which Square One TV can be considered relative to issues of popularization: external to CTW and internal to CTW.

Our main purpose is to convey to our audiences some notions about mathematics that are embodied in our official goals. As it happens, these ideas (e.g., mathematics as something that can be engaged in by non-specialists; mathematical problem-solving as something that is worthy of post-solution reflection; mathematical content as something that includes much more than the arithmetic that dominates school mathematics) are not widely shared by any of our viewing audiences--primary or secondary.

Having decided to attempt to convey such a view of mathematics, the next steps included choices of specific content, situations in which to embed the mathematical ideas, level of mathematical sophistication, frequency of repetition and reinforcement, degree of complexity in presentation of concepts, among others. This decision-making process is heavily dependent, in turn, on popularization in the smaller arena--that consisting of our own colleagues.

The Children's Television Workshop, as a company, is primarily engaged in television production. It has been producing television programs for children for more than 20 years; its top management is expert in, and almost exclusively concerned with, issues of television production. One principle of the CTW production model is to employ specialists in the many and diverse fields that contribute to a television show and have them cooperate under the leadership of an experienced executive producer, rather than to put the production under the direction of content experts.

At the peak of our work cycle about 55 people are involved in the various activities of production, research, and content. Several hundred more come in on an occasional basis in the execution of musical pieces, animations, and film work. These include members of our 18-member advisory board and other mathematical consultants on whom we call for occasional advice and assistance. Among this large group is a four-person Content Department. While we have substantial day-to-day input into the show, and responsibility for the mathematics, nonetheless it is humbling to keep in mind the fact that of all the many people who are directly involved with Square One TV, only the four in the Content Department come with a background in mathematics and mathematics education. In the main, the production staff and researchers, and so on, while eminently talented in their own fields, are outside of mathematics. Some have a casual interest in the field, but most fall along a continuum from neutrality through aversion and beyond. That is, even though skewed in terms of educational level and income, insofar as their feelings toward, and knowledge of, mathematics are concerned, they are much like any other random selection of U.S. adults. Of course, if the population at large were otherwise, there would be little reason for a project of the nature of Square One TV. Thus our first efforts in popularization are directed toward a small, clearly defined audience--our colleagues in television production.

To be successful, we must attract our principal viewing audience: children viewing at home. This is a largely discretionary audience. Many potential viewers have competing interests that might engage them as alternatives to watching Square One TV. In particular, there are other television shows that compete directly with Square One TV. Our children typically decide from themselves which programs to watch; a flip of the channel selector will expunge our show from the screen, no matter how mathematically worthwhile it is. To have any impact at all, then, we need first to attract an audience in a highly competitive environment. Hence the first criterion by which we judge any segment of our show is its appeal to our primary audience. Since our colleagues on the production side by and large do not find mathematics naturally appealing or attractive, they may find it difficult to perceive the mathematics itself as a source of appeal for the audience.

While this is frequently frustrating for us in the Content Department, there is a bright side to this picture from the point of view of effective popularization. To the extent that the staff of Square One TV is a microcosm of our external audiences, the Content Department's natural interest in and enthusiasm for mathematics is, in fact, unnatural. The fact that the Content Department must constantly promote, justify, and defend the mathematics which does appear on the show reminds us that we must be just as diligent with our viewing audience. To put it another way, one can imagine a popularization project in which all the staff were enthusiastic about, and well versed in, mathematics. Quite possibly the product could depend on the mathematics to be its primary attraction, and, as a result, fail to attract a broad audience.

The problem of producing an attractive, appealing series is exacerbated by the fact that our audience is greatly varied along several dimensions: age, location, experience, social sophistication, and mathematical sophistication, among others. This in part drives our choice of the magazine format and humorous parody as an approach. Our research shows that our audience is very knowledgeable of the conventions of the television. They watch large amounts of television programming. And they are attracted to broad humor and parody. Through the magazine format, we can aim to satisfy a diverse audience by producing individual shows whose segments vary in style, format, and tone, as well as in mathematical content and sophistication.

Other tensions in designing the series have to do with the mathematics itself. We regularly confront decisions about the amount and depth of mathematics in a segment or a show. One of the dangers of high mathematical density is in slowing the pace of the show to a crawl, especially in depicting arithmetic calculation. However, we do not necessarily avoid segments of high density. In fact, we have many pieces which are short and concentrated, as well as longer, more diffuse pieces, in which both the mathematics and the problem solving ebb and flow. In general, the more we work to develop plot and characterization, the more diffuse or uneven is the mathematics. However, since the audience prefers richer plot and narrative structure, thus we need these elements for their appeal.

Consider a specific example. One constantly hears of the importance of conveying the idea that mathematics is a useful and powerful tool, applicable to a variety of situations in the "real world". However, exactly how to do this remains a major problem, unresolved as far as we are concerned. In our first season we attempted a number of 3- to 4-minute segments that showed real people (that is, non-actors) engaged in occupations in which mathematics was used with varying degrees of explicitness. These included a piece on a tugboat captain who uses mathematics in piloting her boat around New York harbor; a professional basketball player concerned with bouncing angles; and various people in the wood-products industry. The mathematics in these segments was of a fairly high density, but they were not successful on any grounds: formative research showed that viewers did not like the pieces, content department opinion judged them to be ineffective, and the expert judgment of the producers rated them below standard for the series. We have tried other approaches, including a much lower-density exhortation to learn mathematics because of its usefulness in some specific occupation. We are not satisfied with this approach either, and we continue to look for effective ways to convey the ideas.

Square One TV returns to the air earlier in September 1989. New shows will appear in January 1990. The prospects are good for funding for more production. We expect to apply the lessons of the past several years to improve our technique and effectiveness. The challenge continues to be to attract and to hold our audience while conveying valuable messages that will expand and enrich their conceptions of mathematics.

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Mathematics in Prime-Time Television: The Story of Fun and Games

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Yorkshire Television, one of the larger independent television companies, is well known for programmes popularising science. In October 1986, the science department came up with the idea to try to popularise mathematics in a similar manner. As Duncan Dallas, the head of the science department, wrote "As a kamikaze notion it could hardly be better ... (Mathematics) is the least popular or accessible of the sciences. It does not sell magazines in the same way as computing, it is not the subject of dinner party conversations as is ecology, nor is it a trendy part of our life style like technology. Indeed, it is universally acceptable to trumpet our ignorance whenever the subject is mentioned. Clearly mathematics is important enough to command our attention but on list of programme ideas rated by popularity it will probably become bottom" (Dallas 1988). The crucial question was to find a format - a way into the mathematical perspective. After considerable discussion, Yorkshire Television came up with the idea to base the programme around puzzles and games. The rationale was to capitalize on people's interest in puzzles, an interest which goes back for many years, and to use these puzzles as a vehicle to think about the embedded mathematical ideas - after all recreational mathematics has been the source of a great deal of mainstream mathematics. The programme was called Fun and Games and it was to be transmitted at prime-time, seven o'clock in the evening.

The next step was for Yorkshire Television to decide on the method of presentation of the games and puzzles - another crucial question. The thought was that members of the studio audience would be participant puzzlers, there would be two presenters, one a TV 'personality' and the other a mathematical 'expert' to 'explain' what mathematics had gone on. Eventually, I was asked to be the mathematical 'expert' and to present the programme with Johnny Ball, a TV comedian, well-known for his television programmes concerned with children's mathematics. When I was first approached I was most excited by the prospect of doing the programmes, indeed, the audacity of the idea! I have long been aware of the pervasive negative image of mathematics. Indeed, in my inaugural lecture I stated that "The majority of pupils are anxious about mathematics, alienated from it, or simply bored by it. They rarely become involved in the subject, treating it as something to be got through rather than understood. Even 'successful' pupils have been found to harbour incorrect ideas at the most fundamental level. Mathematics urgently needs, therefore, to be made more accessible and meaningful for pupils" (Hoyles 1985). Schooling seems to turn people off mathematics and the majority of the population do not see any way that the subject can be interesting and intriguing, let alone fun. Their view of mathematics is of rows of sums or meaningless rule-following. Mathematics does not relate to people's intuitions nor does it appear to have any coherent structure in itself. Formal representations are imposed 'from above' in school and these formal representations thus have little meaning and less interest.

I therefore was delighted to join the programmes and to try a new approach to the problem of popularisation of mathematics. I suppose I had a vision that people would, around dinner tables or in the pub, try one of our puzzles themselves, think about the mathematics and, perhaps more importantly, talk about it. Thus the 'real' nature of mathematics could be glimpsed by more than the present tiny élite minority. I also acknowledged the need for a female in the role of mathematician given the enormous literature on gender stereotyping in mathematics (see for example Fennema 1985). The format for the first series was as originally conceived, but without the competitive element that was at first planned. The first programmes were transmitted in July 1987. Much to everybody's surprise, and, indeed, delight, the programmes were a great success and the series achieved audiences of between eight and nine million during high summer. In fact the programme reached the ITV's Top Ten and was widely heralded as "streaking ahead of its great Wednesday night rival, Terry Wogan!" (Daily Mirror August 1987).

I think it is true to say that Fun and Games is the first television programme to try to use mathematics as a focus in a prime-time slot. Sometimes the mathematics is simple, sometimes quite hard but we always try to highlight the mathematical principles employed in the puzzles. We want to challenge people's long-held conceptions of what mathematics is all about by getting their involvement in the subject in an enjoyable setting and bringing in the surprise factor - the 'Aha!' when people suddenly see a new way through or when they look at a problem differently - that is, mathematically! So to recapitulate, what were our aims? To show that mathematics is not just 'boring old numbers'; to show how mathematics can be the key to solving puzzles; to show that mathematics can be fun, intriguing and perhaps exciting; and, finally, to get mathematics into the general popular culture as something which people can think about and participate in.

There was another series of Fun and Games in the summer of 1988 where I was joined by a new co-presenter, Dr Rob Buckman and a third series in the spring of 1989. The audience figures and loyalty seem to be holding up. Some things about the programme have not changed. There is still no competition, no extrinsic incentives such as rewards or prizes. We also have continued to have two presenters, myself and Rob Buckman. Rob's role is crucial in providing a clear exposition of the puzzles and in keeping the atmosphere of the programme effervescent and lively. My role is to give the mathematical perspective - but how this is done has changed over the three years of the programme. In 1986 the puzzlers solved the puzzle, with or without assistance, and then I was brought in to explain the mathematics that they thought they had used! This sometimes worked but sometimes it was very evident that they had not used that mathematics or, indeed, having finished the puzzle, saw no relevance in my presence. Now we aim that my role is to give to a hint part way through the action: to try to highlight a particular part of the puzzle which is significant; to bring in a new perspective. I suppose I try to give the puzzlers and the audience a mathematical lens, through which they can look at the puzzle and then, with this new view, see ways to solution. We want to try to capture the real thrill of suddenly 'seeing it'. My hint has four aspects.

* What is difficult about the puzzle ? What might be the problems for solution?
* What mathematics do I know that would seem relevant to the difficulty? What abstract thoughts or relationships or theories might it be possible to bring to bear?
* What do I know about these abstract thoughts? What deductive or formal reasoning can I use in this mathematical plane?
* How far does this help when applied back to the puzzle?

So to summarise, my role is to try to pin-point what the puzzlers might have found difficult, to try and introduce mathematical ideas, work with them a little in the discourse of mathematics and finally apply what I have done to the puzzle. In the 1989 series we have also introduced another new feature where sometimes, if the puzzlers have finished the puzzles successfully, I just comment on it very briefly and then offer an extension that people might like to think about at home.

Apart from the viewing figures the response of some press reports are of interest. In the Daily Mail for example, Friday 28 April 1989, Philip Purser said "Beneath its relentless high spirits Fun and Games (ITV) is intellectually unique among games shows and quizzes. It demands answers actually worked out by mental agility rather than by recall. In theory, a bright five year old and a law lord stand an equal chance of winning since acquired knowledge and wisdom play no part". This may not be true but it is interesting to read! Some reporters expressed surprise at the success of Fun and Games, for example Northants Evening Telegraph under the heading "Calling all Clever Cloggs" wrote on 27 April 1989, "Television doctor and clever cloggs Celia Hoyles team up again tonight to present another programme packed with puzzles, conundrums, brain-teasers and games. Fun and Games is the show that aims to shake the cobwebs out of mathematics. But probably the biggest puzzle is how it has managed to capture the imagination of the public". Obviously, although the majority of press reports have been complimentary, some have not. Most of the criticisms are interesting since they probably arise from a stereotypical view of mathematics. For example, in the Sunday Express April 30 1989, it was suggested that I was "expected to recite twenty things you never knew about an Isosceles triangle!" Some reviewers also objected to the fact that at prime-time television you were expected to think a little! This point of view was actually contradicted by Yorkshire Post in a piece on Friday April 28 1989. Eric Roberts wrote that he felt, in the past, television had been "guilty in underestimating the concentration level of viewers". He noted that people were joining in the problem-solving in Fun and Games, both during the programme and at home. They loved the challenge of the puzzles. The same reviewer also wrote that there was a welcome absence in Fun and Games of "brass bands and the off-screen shouting which characterised and ruined so many other shows". What I think Mr Roberts is hinting at is that Fun and Games, in a way, tries to tap and build upon the intuitive talents of the population, in, we hope, a way that is not patronising. The audience is <u>not</u> rehearsed and the puzzles are <u>not</u> rehearsed by the puzzlers. There are some gloriously funny moments as well as some gloriously stimulating mathematical insights.

So what are *the major constraints* of putting mathematics on prime-time television? The *first* that immediately comes to mind is <u>the stringent time constraint</u>. Each Fun and Games programme is twenty five minutes five seconds long and, on an average, we have five puzzles per show. So, roughly, each puzzle takes about four minutes. In that short space of time, the puzzle has to be stated clearly, the mathematical hint has to be given, and hopefully the puzzlers must bring the conundrum to a satisfactory conclusion. That is just the mathematics - there are jokes and interactions between presenters and puzzlers which all bring the show alive. So there is enormous time pressure. So why is all this pace necessary? From my perspective, I would prefer to spend far longer on a fewer number of puzzles but I am assured we would lose our audience. We cannot expect viewers to grapple at length <u>there and then</u> - we are not teaching. All we can hope for is to leave the viewers with just a few ideas, a few puzzles that they can struggle with later at their leisure. In fact we do have a lot of telephone calls and letters from people who do just that - from mathematicians to members of the public who boast their previous loathing of mathematics. Some letters sadly do not refer to mathematics and just say, for example, that they like my clothes but the majority do give 'new' solutions or extensions to our puzzles. One example of a puzzle that has triggered off quite a large correspondence amongst mathematicians was the Riffle-Shuffle Puzzle which was transmitted in 1988. Here we had a croupier who was doing 'perfect' riffle-shuffles, that is cutting the pack in half and then shuffling the cards so they went left, right, left, right, in perfect order. The puzzle was: how many riffle-shuffles were needed to bring the cards back to the same order that they were at the beginning. People wrote computer programmes, people generalized from 52 cards to n cards, children undertook GCSE investigations all around this puzzle!

The *second constraint* on the mathematical presentation in a programme like Fun and Games is that we have to make a very conscious effort <u>not to appear to be teaching mathematics</u> or to use any medium that calls up school mathematics. Sometimes it would seem to me to be obvious to have a diagram or, dare I say, a blackboard but we have to avoid this. We want people to break free of their past school experiences. A *third constraint* is on the selection of puzzles. We try to use a variety of types of puzzle, some topological, some number manipulation, some more geometric, some games. All the puzzles though have to involve the puzzlers in <u>doing something active</u>. It is very bad television to see people just standing thinking! This obviously is a constraint because mathematics requires reflective thought. However, we do not expect people to become instant mathematicians! We just hope that we trigger an idea in their minds upon which they can reflect later. The *final constraint* is that we do not use any formal mathematical language. This naturally restricts potential generality and power - although it is quite interesting that applying formal symbolisation to the solution of a puzzle frequently stops one from adopting an intuitive approach - as I have found out for myself!

In conclusion, I believe that Fun and Games has achieved at least some of its objectives. I believe that people are getting involved in mathematics although how far they are thinking mathematically we will never know. I would hope that at least some people have changed their view of mathematics and mathematicians as a result of following the series. Is it possible to imagine a party when somebody says, in answer to the question "What do you do?", "I'm a mathematician" and instead of the usual "Oh, I could never do mathematics", or "I always hated mathematics", or deadly silence because nothing can be said, the answer comes up 'Oh, how interesting, do you know this puzzle: 'if you've got a milk crate with 25 places in it, how many ways can you arrange 10 bottles so that there is never more than 2 bottles in any one line (horizontal, vertical or diagonal)?' " Wouldn't that be wonderful? May I end by saying that mathematics is obviously much more than fun and games. Mathematics is a discipline with its own language and structures. Learning mathematics is also much more than watching Fun and Games or even doing the puzzles. Learning mathematics requires hard study, considerable thought and persistence. But if we want to popularise mathematics and open up the mathematical culture to a wider range of participants, first of all surely we must open people's eyes to what mathematics could be all about?

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Television, Parents and Primary Mathematics

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Since the 1960s parental interest has been recognised as an important aspect of children's education. The recent government document *Better Schools* (1985) emphasises the value of a shared commitment between schools and parents. Despite this, it is not always easy for parents to find out the kind of educational support that is needed. What do you do if a child seems to be making slow progress? Should help be given at home? Will wrong teaching methods cause even more confusion? Mathematics suffers more than most subjects in this respect. Parents are often afraid to interfere when it comes to maths and the reasons why are not hard to find. Many adults fear and dislike the subject. It may be that they themselves under-achieved at school and feel insecure through a lack of basic understanding. Nevertheless, many parents are concerned about their children's progress in mathematics and recognise its future importance. They often hope their children will be better at mathematics than they themselves were.

Moves are now afoot in the educational world to ease this situation. It has long been recognised that parents can play an important part in helping their children to read but only recently has it been acknowledged that this help can be extended to learning mathematics. There are, however, problems which must be faced. Methods of teaching mathematics have changed: parents need reassurance, they need to know the sort of activities they can share with their children and how these help learning. Should children learn their tables by rote? Does a calculator rot the brain or can it aid mathematical thinking? What has happened to pencil and paper arithmetic? Why is the use of concrete apparatus and discussion so important? The most significant message to get across is that teachers, parents and children can all work together and support each other in the learning process. No one needs to be a mathematical genius to help primary-school age children; there are many everyday tasks such as shopping, gardening, cooking and going on a journey which lend themselves to mathematical activity. Even pre-school children are familiar with simple number, shape and sorting procedures. Many a child asks for and realises the potential of 'two' long before they can say the word 'biscuit', though I daresay, for obvious reasons, few of us would want to extend this method to learning three, four, five, six...We all use numbers and given the opportunity children quickly become familiar and adept with their use in a home context.

The PrIME Project (Primary Initiatives in Mathematics Education Project) is working with Local Education Authorities and Primary Schools to involve parents in their children's mathematical education. Events which are school-based seek to persuade parents to do more mathematics themselves through activities with their children. Scheduled events often include workshops, games evenings and maths trails (a maths trail is similar to a nature trail with the focus on mathematical observations rather than natural history). Many schools also organise displays of children's work, run a maths games library, and encourage parents to work in school with groups of children.

It is in conjunction with PrIME Project teacher co-ordinators that the BBC Continuing Education Department has commissioned Palace Gate Productions to produce the *Help Your Child with Maths* series in the Autumn. Its aim is to encourage parents to share maths with their children through games and activities in the home. In doing so it is hoped that adults will come to understand today's teaching methods, including those concerned with calculators and computers and the move towards discussion rather than pencil and paper methods for younger children. It is also hoped that the series will contribute to parents' own mathematical understanding and confidence. Building confidence and persuading people to take the necessary action makes certain demands when such a series is produced. The style of presentation must not conflict with the way families think of themselves. If it does, then the task may appear to be out of reach. It is necessary to show ordinary families doing ordinary, everyday things, leaving an 'I could do that' impression. Parents must feel confident to handle the tasks set. Tasks for adults and children need to be carefully chosen, not so easy that they are uninteresting but not so hard that they are beyond comprehension. A strategy game such as NIM drops easily into this category:

> With twenty matches each player takes it in turn to take one, two or three matches. The loser is the person to take the last match or matches.

Whilst younger children may concentrate on subtraction and calculating the number left, older children and adults will no doubt be working out winning strategies.

The *Help Your Child with Maths* series is scheduled for Sunday mornings in the winter months, so that there is immediate opportunity for parents and children to play games, try out problems, make models or just talk about what has happened. Since the likelihood of response decreases with time, it is essential that the series is broadcast when families are free to interact and do things together. An objective of the series is to provoke action. The response which television characteristically produces is to leave people sitting down. The programmes themselves introduce activities so that families can get started straight away. In the programme *Celebrations and Festivals* for example, the following problem arises:

> Write down the date of your birthday or the date of a celebration. Use the digits to make the numbers from 10 to 20 or 10 to 100, if you want to try something harder.
> e.g. Christmas Day will be 25.12.88
>
> So $10=2+5+1+2$
>
> $11=12+5+2-8$
>
> What about 12, 13, 14 . . . up to 20?

This type of problem is irresistible and special to each of us if birthdays are used, yet within range of many people's mathematical experience.

There are also two support publications; a games pack which will be sent free to parents on request and a book on sale in high-street bookshops. The pack is intended as an introduction, whilst the book contains over one hundred mathematical games, puzzles and tasks for families to try. Both explain the mathematical concepts and educational practice underlying each activity.

Last but not least, parents need encouragement. Enjoying family activity is a fairly natural process but parents need to be convinced that they can help their children learn mathematics and that it is not such a mysterious experience after all. Television provides a powerful medium to show that tasks shared as a family are not only interesting and good fun but that they also contribute to a child's mathematical education.

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"NOTHING WITHOUT HARD WORK"

From a cross-cultural perspective, the arguments for the popularization of mathematics are compelling enough. The world is becoming more quantitative. Mathematical models are being used to inform decisions which effect us all -- technological, environmental, social, legal. In order to understand our world, we need a sense of what mathematics can and cannot tell us, otherwise we are in increasing danger of institutionalizing a split between a mathematically and technologically literate elite and a lay public cut off from the decision making process.

From a U.S. perspective,popularizing mathematics has become an absolute necessity. Huge percentages of our population not only don't understnad mathematics, but dislike, distrust and abhor it. These attitudes are quickly passed from generation to generation. creating an environment where it is socially acceptable for a student to take as few math courses as possible and to do poorly in those he/she does choose. It is long past time for this cycle to be broken. It is simply not acceptable. Mathematics has become too important to the quality of life in our world.

And this, I believe, must be our starting point. We know that mathematics is not simply a collection of random algorithms, or the union of subfields called algebra, geometry, analysis, etc. -- our students, and hence the general public do not. We know of the universal utility of our subject -- models of decision making, weather predictions, epidemics and cryptography, image processing, the environment -- the public does not. This is both our fault and our responsibility. It is no longer a comfort to feel smug in our superior knowledge and understanding. We must act. We have two arenas available to us - one familiar and one of virtually unchartered waters. First, when we speak of educating the public, we must remember that essentially every member of the public is in a math course at one time in their lives. If we want them to understand the power and utility of mathematics as they mature, then we must change our curricula to insure that they see these ideas in class.

Rather than pretend that there is some magic linear sequence of math courses from birth to the PH.D. where each course is prerequisite for the next; we must treat each course as though it were a prerequisite for life. In the U.S. this has, in fact, become true for half our students each year. Recent statistics show that from the ninth grade on we lose approximately half of the U.S. mathematics students taking math courses each year. Half-life = 1-Year! We must insure that when students stop taking courses they are left, not with a collection of half-remembered algorithms, but with life skills - positive attitudes towards mathematics and a sense of both the subject and its place in their lives. To do this will take a massive overhaul of our present curricula, focusing on end goals -- knowledge and skills we hope will still be there ten and twenty years out from school.

And that's the easy part. For all the changes implied above, the world of school and curricula is a world we understand and know how to effect. The second area is much more difficult - the world of popular culture - magazines, newspapers, radio, television. This is a world in which we are not the experts. It is time to learn and there is much to learn.

I have recently completed work on two 26 half-hour television series in mathematics, and am working on a third. When I began this work, my naivety was staggering. I wanted to put mathematics on television. I wanted to show the public that all mathematics was not 3,000 years old and spoken in Greek. I wanted to show people how mathematics was being used in ways which directly effected them by people who looked like them and not some caricature of a mad scientist. To do that COMAP brought together a distinguished group of mathematicians and math educators to design the content of the programs and write an accompanying book. Oh yes, and we hired a television production company to produce the series.

I have learned more from the people in that TV production company than I would ever have thought possible. This medium (and I suspect all such media) is a subject in itself. It has its own language, axioms, and theorems. Many of us are fond of saying that one doesn't truly understand a subject until one is forced to teach it. Let me say, that one doesn't truly understand a subject until one is forced to design a television script to explain it. In part this is become of the visual aspects which must be designed, but more directly, the TV format forces you to condense and compress information. What can you say about a deep subject or important application in a half-hour or ten minutes or sometimes sixty seconds?

This is terribly important. In class we have the luxury of time. We can show the complexity of an idea and/or let students discover the complexities for themselves. In the media we must choose our words extremely carefully - we have so few. We must decide on the level (and vocabulary) of our intended audiences. Being absolutely correct is not as important as conveying a general understanding.

In this setting mathematics education becomes like sex education . When a child first asks where a baby comes from, you do not take out an advanced biology text. In a similar way, when we explain mathematical ideas on TV we must refrain from telling all we know. Rather, we must use words and ideas familiar to our audience. It is a difficult task. We wish to present real ideas and not insult our audiences' intelligence, nor leave them hopelessy confused. As I said - a difficult task, but clearly necessary.

In effect, we are explaining to people what we do and why we do it. It may have been easier to say "you wouldn't undestand this" or "you'd have to study for several years before I could really explain it to you." No more. We were really cheating anyway. If we are agreed that the popularization of mathematics is a necessary and positive good, then we must discipline ourselves to the task of explaining our subject in terms that non-specialists can appreciate.

While many colleges have mottos such as "in truth there is beauty" my undergraduate motto was "nil signe magna labore" - nothing without hard work. This is appropriate for our task of popularizing mathematics. We not only have to decide which ideas we hope to communicate, but we must learn a new vocabulary, new rules - in fact, a new subject. From experience, I can tell you it is both a humbling and extremely gratifying experience.

The FAMILY MATH Program: It's Role in Popularizing Mathematics

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How many of us have been in social situations when the question: "And what kind of work do you do?" has been met with complete silence when we answer mathematics and/or mathematics education, followed by an tirade of how terrible our new acquaintance was in school mathematics, how they can't do any of it now, and how much trouble their children are having with their school math.

It is socially acceptable to discuss and even brag about such matters in the U.S. and many European countries. In contrast, we never meet anyone who will tell us: "Oh, I never learned to read, it was just too confusing to understand." In fact, those adults who do not read spend a fair amount of energy working out ways to hide the fact. What accounts for these two antithetical situations? What have we been doing in mathematics education that causes the majority of the population to leave school disliking mathematics, feeling that they are incapable of doing math, and yet feeling comfortable bragging about it? If we want to change these attitudes, we will need to make changes in school instruction as well as promoting a better image of the mathematical sciences outside of school, for children, their parents, and the community at large.

How can we kindle interest in mathematics for adults who are so completely turned off? One approach is based in the common desire we hear expressed so often. Parents want to help their children with math, but don't know where to begin. The FAMILY MATH program provides materials and assistance to interested families around the world. In achieving its goal of involving parents in their children's mathematics education, the project benefits many adults by developing their confidence and appreciation for the subject of mathematics.

A FAMILY MATH course taught by a parent, teacher, or community worker in a school, church, or home, gives parents, together with their children, opportunities to develop their problem-solving skills and to understand mathematical concepts in areas that reinforce and supplement the school mathematics curriculum. The classes provide a setting for families to enjoy doing mathematics together. In addition, the families have the opportunity to meet women and men from the community who are working in math-based fields and to learn about the importance of mathematics to future education and work.

Background

FAMILY MATH might well be called a daughter of the EQUALS project, which is based at the Lawrence Hall of Science. For the past twelve years, this project has presented inservice workshops for K-12 teachers addressing the issue: How can we encourage *all* students to continue with mathematics when it becomes optional in high school? In particular, how can we reach and encourage those students who have been traditionally the first to drop out--Black, Hispanic, and Native American students and females of all races (in the U.S.)?

The EQUALS teachers reminded us of the important role that parents can play in encouraging their children to continue with mathematics, and wanted a way to involve parents in their children's mathematics education. We felt that if we could immerse both parents and their children in learning mathematics, provide a supportive environment, and model a teaching style that encourages persistence in problem-solving, we could make a positive impact on those students who are particularly "at risk."

Pilot courses for FAMILY MATH began in low-income neighborhoods in Richmond, California, in January 1982. They continued in similar neighborhoods in Richmond and Oakland, California. with funding from the U.S. Department of Education, during the 1982-1983 academic year. Parents attended classes with their children in grade-level spans (K-1, 2-3, 4-6, 7-9) to learn "hands-on" activities that were designed to be repeated at home, with inexpensive materials.

Classes met one and a half to two hours a week for six weeks. The curriculum included measurement, geometry, probability, statistics, estimation, the use of calculators, logical thinking, and arithmetic. Parents were given overviews of the mathematics topics covered at their children's grade levels and explanations of how the FAMILY MATH activities related to these topics. In addition, one evening, or parts of several evenings, were devoted to doing activities that provide information about the mathematics requirements for the workplace and tertiary education The older students and their parents also heard a role model panel of university students from their community speak about their studies in math-based fields.

What We Learned

Out of these successful pilot programs came our goals for FAMILY MATH: to provide parents with activities to help their children with mathematics at home; to build awareness that mathematics consists of more than arithmetic and rote computations; to develop problem-solving skills and the ability to talk about mathematics; to foster positive attitudes toward mathematics; to provide parents with information about the importance of mathematics in future schooling and work; to inform families about the equity issues around mathematics and that mathematics is important for *all* students; to help parents feel that they can and do make a difference in their children's mathematics education; and to provide an opportunity for all members of the family to enjoy doing mathematics.

Follow-up questionnaires from the parents who attended FAMILY MATH classes in our first year showed that parents did continue math-related activities at home. Between 80 and 90% of the respondents said that they had played math games with their children, helped with math homework, talked to their child's teacher about mathematics, and/or talked about the math they encountered in everyday life. Fifty percent of the respondents talked to their children about careers in relationship with math and the same percent bought a math book or puzzle for their child.

Parents also took actions for themselves. Fifty percent bought a math book or puzzle, 27% bought a refresher book and 18% enrolled in a math class. One-third of the respondents reported that they had learned some new mathematics from the course.

Reaching a Larger Audience
To ensure that FAMILY MATH classes would continue beyond the ones that the EQUALS staff could teach, we developed, in the fall of 1983, a two-day inservice workshop to prepare teachers, parents, and community members to instruct FAMILY MATH courses in their own communities. The first workshops drew teachers in the EQUALS network, who live in the San Francisco Bay Area. Interest and enthusiasm for the program spread rapidly, so that there are now a total of 31 FAMILY MATH sites presenting the two-day inservices throughout California, across the U.S., and in several other countries (Australia, New Zealand, Canada, Sweden, Costa Rica, Puerto Rico and Ireland). To date the program has reached over 50,000 families and 100,000 parents and children.

We published the FAMILY MATH curriculum guide in 1986 with funding from the Carnegie Foundation of New York. The book, which has been translated into Spanish (1987) and Swedish (1988), has greatly facilitated the dissemination of the program.

Although, we were reaching some families from groups who are underrepresented in math-based fields in the U.S., we felt that there was more to do. In 1984, we received a grant from the National Science Foundation (NSF) to work with community-based organizations to disseminate the program to their constituents. With this funding, FAMILY MATH programs have been established in tribal Indian, Hispanic, and urban Black communities. Participating organizations are the National Urban Coalition in Washington, D.C., the Indianapolis Urban League, the Oregon Indian Education Association and two Hispanic groups, Alba in San Diego, California, and Valle del Sol in Phoenix, Arizona.

The People of FAMILY MATH--The Program's Influence
The Families--Parents and Children
The families of FAMILY MATH come from all walks of life. We know that the program has been successful across the U.S. in inner-city settings among Black parents, in Hispanic communities, in tribal Indian settings, as well as in predominantly White areas. The community-based organizations mentioned above used their NSF funds to extend their outreach to the hardest-to-get segments of their

constituents--the poorest, the most remote, the most in need. In some communities, classes have successfully brought poor and well-off families together.

FAMILY MATH is also working in Toronto, Canada, noted for its large immigrant population, across Australia, throughout New Zealand, in northern and southern Sweden, and in Puerto Rico and Costa Rica. And there is a pilot project beginning in southern South Africa, for Afrikaans and Black families.

We have found, especially when classes are held at a school site, that it is easier to recruit parents from the middle and upper income groups. However, this does not mean that they come with any more confidence in their mathematics knowledge or love of the subject. Limited education, poor experiences in learning mathematics themselves, and confusion about the "new" math have made parents from *all* backgrounds feel incapable of helping their children.

Many parents come to FAMILY MATH classes expecting to sit back and watch their children learn the multiplication tables. They are very surprised when they find themselves enjoying the class and learning along with their children. One parent wrote after the first class: *"You know I came to this class because I thought I had to; but I hated the idea of spending an evening learning math. Now I can't wait for the next class. This is a blast."* Both the parents and class instructors are often surprised at how much math the the adults learn. Many go on to study more math for themselves.

Parents appreciate the opportunity to observe their child learning, and are frequently surprised at the leadership roles the children take in group work. For many, especially single parents, the most valuable aspect is the quality time spent with their child without interruptions.

Parents who have experienced FAMILY MATH and have seen the positive response of their children to this approach become more supportive of new curriculum topics and methods (an important need as we disseminate the NCTM Standards in the U.S. and make similar curricular changes in other countries). Part of the popularization of mathematics will be convincing parents and the community to support the needed changes in mathematics education.

The children attending FAMILY MATH also represent a cross-section of the population. Many of them convince their parents to attend the classes in the first place. Others attend at their parents urging and may come with feelings that math can't be that interesting. However, after the first class, they all are eager to come back. We have heard many tales of children who, too ill to attend school, suddenly recover in time to attend FAMILY MATH class.

Teachers note: *"The biggest change you see in kids is that they are finally willing to take a chance. They become risk takers. They are more motivated and excited about learning math in the classroom."*

The Instructors

We expected that the people that we would recruit for our first FAMILY MATH inservices would be teachers with good math backgrounds and extra experience with the subject. Although, in our first inservice, we did attract participants with such a background, this has not held generally. We attracte people who often had unpleasant experiences with mathematics themselves, but who want to help parents become involved in their children's mathematics education in a positive manner. Despite their personal discomfort with math, they are willing to work to help others. Gradually parents became interested in teaming with a teacher to present classes or instructing classes on their own. In several cases, it was parents who brought the program to a school or district.

An immediate question that might come to mind is: Can such people possible lead a FAMILY MATH course successfully? The answer is *yes*. In many cases, there is advantage for the instructor who can say: *"Math was never easy for me, but this works."*

A parent who is not a credentialed teacher credits FAMILY MATH with converting her from a mathophobe to an inservice math instructor. She writes: *"When I started teaching FAMILY MATH, I felt very insecure about my own math understanding, and I had never worked with kids. I perceived the role of the FAMILY MATH instructor as facilitator rather than lecturer. I perceived those as very very different skills... My feeling is very strong that anybody can provide FAMILY MATH classes. You don't have to be an educator, but you do have to have commitment to help others become adept at math."*

The Indianapolis Urban League invited parents from their target community to become FAMILY MATH instructors. Many of these adults had dropped out of high school as young teenagers. The program not only increased their math skills, but enhanced their self-esteem.

One observer writes: *"The FAMILY MATH workshops [inservices] can function as a "leveler" for parents and teachers. Parents often come to the workshops with minimal educational backgrounds and few have done any kind of teaching. Taking a workshop with teachers is a potentially daunting experience. For many parents, a key outcome of their participation is a new sense of self-confidence. ...parents spoke of their fears and hesitancy, followed by amazement at their ability to participate on a par with many of the teachers."* (Others have noted the same effect with children and their parents.)

What Makes the Program Work?

There is a need and demand for these materials. Parents know how to help their children with reading: reading bedtime stories, looking at picture books, going to the library--all very pleasant and amiable. On the other hand, how do most parents help with math? By using flash cards for basic facts and insisting that homework be done before play begins. Somehow these activities do not create the same friendly feelings in families. FAMILY MATH provides parents with a way to help their children with mathematics so that everyone can learn in a pleasurable way.

The program provides a **supportive and nonthreatening environment** in which parents and children feel comfortable doing mathematics. Such an environment encourages the experimentation and taking risks necessary to develop problem-solving skills. It builds self-confidence for both the parents and the children.

FAMILY MATH has a **sound mathematical curriculum** and it is presented in a manner so that both parents and children can begin to see how mathematical concepts are interrelated. We start with number-based activities that are familiar to adults as elementary school math and gradually introduce the broader range of topics included in today's curriculum. Most of the activities are of interest across age-groups so that much of the time there is an opportunity for the parents to learn along with their children.

The materials are also of interest across ethnic and socio-economic lines. We are collecting and developing activities that add more cultural aspects to the program, but the basic curriculum has proven successful with few or no changes with the many different groups who have seen it.

We model a **teaching style** that provides success for learners. This helps math-anxious parents feel comfortable and giving them an approach to use when they continue activities at home.

We make the **link between mathematics and future schooling and work**. Activities and role models provide information about math requirements and needs on the job. Thus families can begin to see the relevance of learning mathematics.

The program **creates excellent school/community public relations.** School principals and administrators are blissful to find a roomful of families happily doing mathematics together in their schools. Newspaper reporters find such scenes the source of fascinating feature stories.

When looking at the question of why has the program spread so rapidly, the answers subsume those above, but also include some of the following.

First, **it's a simple and popular idea** that is easy to explain. No one can complain about a program that will help parents help their children with math and have fun at the same time.

The EQUALS experience in presenting staff development insured the development of an **strong two-day inservice**. Participants receive the opportunity to actually do the activities in the same manner that they will teach later. They have a chance to discuss the logistics of setting up a series of classes and time to plan.

The program allows instructors **ownership**. The inservice and book serve as resources for instructors to use as they plan classes. Instructors are encouraged to supplement and adjust these ideas to fit the particular needs of their own situations.

The **FAMILY MATH book** itself is an excellent resource and documentation of how to set up classes. We know that some people have been able to initiate their own program of classes after reading the appendix on Organizing a FAMILY MATH class.

Relationship to the Popularization of Mathematics

There is no question that FAMILY MATH has helped improve attitudes toward mathematics--both children and adults. There are some general characteristics about the program that can carry over to other work in this area.

One essential key to the success of FAMILY MATH is what motivates parents to attend: the desire to be able to help their children do better in mathematics. Many math-anxious adults who would otherwise avoid anything to do with the subject become involved.

The nonthreatening and supportive atmosphere combined with a respect for the families keep them coming back. We help parents help their children. They are learning mathematics with their children in an appropriate and worthwhile manner--while enjoying themselves.

We measure our success in what others might call small steps. We consider it a big breakthrough when parents understand that math is more than just arithmetic. We feel we are making progress if we can change the attitudes that contribute to the creation of strereotypes about math. We rejoice when a parent writes: *"I hadn't looked at math that way before. I didn't realize how much I was influencing my child by my comments that I was no good at math as a kid."*

We are happy when we receive reports that: Indians and Mexican-Americans meet regularly in the unincorporated town of Guadalupe, Arizona, to do mathematics with their children; the basement of a Baptist church in northeast Washington, D.C. is filled with inner-city families doing math; the schools in the border town of San Ysidro, California are open late to accommodate the FAMILY MATH classes taught by parents with the assistance of a teacher; tribal members are teaching adults and children math activities in a small Oregon costal town; parents and teachers who have not met before are sitting together in a large room at the Broadway Methodist Church in Indianapolis, Indiana, learning mathematics.

In summary, the FAMILY MATH project has served as a vehicle for the popularization of mathematics, although that was not its initial primary goal. Its successes depend on the program's inherent interest to potential participants, the nonthreatening manner in which the materials are presented, and their relevance to the partcipants. These three characteristics will be essential in all popularization efforts.

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POPULARIZING GEOMETRICAL CONCEPTS: THE CASE OF THE KALEIDOSCOPE

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> "La date de l'année 1823 était pourtant indiquée par les deux objets à la mode alors dans la classe bourgeoise qui étaient sur une table, savoir un kaléidoscope et une lampe de fer-blanc moiré."
>
> Victor Hugo, *Les Misérables.* (II, III, I)

Among devices based on a substantial mathematical phenomenon having enjoyed a wide dissemination through popular enthusiasm, the kaleidoscope stands as a rather unique case. Ever since its invention more than 170 years ago, it has been an inexhaustible source of fascination for people of all ages filled with wonder at the infinite richness of the pictures created by the interplay of mirrors. The French writer André Gide has aptly described the intense delight he took in the kaleidoscope as a six-year-old child:

> "Un autre jeu dont je raffolais, c'est cet instrument de merveilles qu'on appelle kaléidoscope: une sorte de lorgnette qui, dans l'extrémité opposée à celle de l'oeil, propose au regard une toujours changeante rosace, formée de mobiles verres de couleur emprisonnés entre deux vitres translucides. L'intérieur de la lorgnette est tapissée de miroirs où se multiplie symétriquement la fantasmagorie des verres, que déplace entre les deux vitres le moindre mouvement de l'appareil. Le changement d'aspect des rosaces me plongeait dans un ravissement indicible. (...) Bref, je passais des heures et des jours à ce jeu."
>
> André Gide, *Si le grain ne meurt.* (I, I)

The curiosity and interest aroused by such an instrument can serve as a motivation for the need of a clear understanding of the underlying principles. It is thus instructive to look more closely at the kaleidoscope from the point of view of popularization of mathematics. This shows how some elementary concepts of geometry

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† Part of this work was done while the first author was visiting the Centre de recherches mathématiques de l'Université de Montréal.

can be brought into play to create an attractive context suitable for significant mathematical observations. Although the original kaleidoscope was simply conceived from basic properties of mirrors, new technologies, and in particular computer graphics, now allow for a more thorough exploration of the kaleidoscopic phenomenon.

1. HISTORICAL NOTES ON THE KALEIDOSCOPE

From the misfortune of Narcissus, enamoured of his own image seen in the water of a fountain, to the adventures of Alice exploring the Looking-Glass House, the mirror has always been a source of mystery and fascination. (A study of various myths surrounding mirrors and interpretation of their effects can be found in the essay [Bal].)

While reflection in a single plane surface generates an identical image, different and more complex patterns are produced by the use of multiple mirrors. The effects of combining mirrors so as to produce a multiplication of images have been described by early writers on optics, such as Giambattista Della Porta (1535-1615) or Athanasius Kircher (1601-1680): the former presented in his *Magia naturalis* (1558) experiments done with two rectangular mirrors joined by one of their sides so they could be opened or shut like a book, while the latter stressed in his *Ars magna lucis et umbrae* (1645) the connection between the angle of the mirrors and the number of images formed. Applications of such combinations of mirrors have been described by the botanist Richard Bradley (1688-1732) in his book *New Improvements in Planting and Gardening, Both Philosophical and Practical* (1717), where sets of mirrors are used for preparing symmetrical designs for formal gardens.

In 1817, the Scotch physicist Sir David Brewster (1781-1868) patented a "philosophical toy" which he called the kaleidoscope (a name Brewster derived from the Greek words «kalos», *beautiful*, «eidos», *aspect*, and «skopein», *to see*). This optical instrument, which Brewster "invented for creating and exhibiting beautiful forms" [Br2, p. 443], was a direct result of his studies of the theory of polarization of light by multiple reflections. In the Supplement to the 6th edition of *The Encyclopædia Britannica* (1824), the kaleidoscope is described as

> "an optical instrument, invented by Sir David Brewster, which, by a particular arrangement of mirrors, or reflecting surfaces, presents to the eye, placed in a certain position, symmetrical combinations of images, remarkable for their beauty and the infinite variations of which they are susceptible" [EB, p. 163].

(This article on the kaleidoscope, written by P.M. Roget, of *Thesaurus* fame, runs for more than nine pages, including pictures, and appears also in the 7th edition (1842); the kaleidoscope occupies less than two pages in the 9th edition (1880) of *Britannica* and a mere half a column in the 15th edition (1985).) Figures 1 and 2, taken from [EB], show respectively the arrangement of mirrors in a basic kaleidoscope and a typical kaleidoscopic rosace (or "rose-pattern") obtained by the coalescence of the images of a basic motif reflected in the mirrors AC and BC.

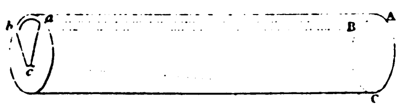

Figure 1

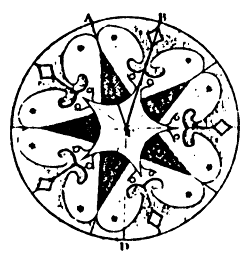

Figure 2

It has been reported in many places that the kaleidoscope was phenomenally popular as soon as it started being manufactured. It is said that no fewer than two hundred thousand instruments were sold in London and Paris in the space of three months [Br1, p. 7]. The article in *Britannica* comments that

> "the sensation it excited in London throughout all ranks of people was astonishing. Kaleidoscopes were manufactured in immense numbers, and were sold as rapidly as they could be made. The instrument was in every body's hands, and people were every where seen, even at the corners of streets, looking through the kaleidoscope. It afforded delight to the poor as well as the rich; to the old as well as the young. Large cargoes of them were sent abroad, particularly to the East Indies. They very soon became known throughout Europe, and have been met with by travellers even in the most obscure and retired villages in Switzerland." [EB, p. 163]

And Brewster himself states in a letter to his wife from London in May 1818:

> "You can form no conception of the effect which the instrument excited in London; all that you have heard falls infinitely short of the reality. No book and no instrument in the memory of man ever produced such a singular effect." (Quoted in [Wad, p. 196])

It should be mentionned, for historical record, that Brewster was rather bitter about what happened in relation to the kaleidoscope. Not only was there a controversy about the originality of his invention (a full chapter of [Br1] is devoted to the defence of his priority -- a similar controversy took place over another of his inventions, the stereoscope, as is reported in [Wad]), but also his patent seems to have been mishandled, resulting in an important loss of revenue at a time when his financial position was not secure (see [Wad]). Moreover Brewster complains that out of the immense number of instruments produced, there are very few "constructed upon scientific principles, and capable of giving any thing like a correct idea of the power of the kaleidoscope" [Br1, p. 7].

The immense popularity that the kaleidoscope enjoyed at that time is not without reminder of a mathematical "toy" of a more recent vintage, the Rubik's cube. It should be noted however that both instruments, once the objects of a craze, came to a more proper popular status. After the fad of the early 1980s, the Rubik's cube lost some of its fascination, and many cubes (and analog offsprings) are now probably sleeping in the back of drawers. Something similar seems to have happened to the kaleidoscope. A chronicler of the last century even reported the following, probably with a certain exaggeration:

> "Vers 1820, le kaléidoscope fit fureur à Paris; on en trouvait alors dans tous les salons, sur les tables où les *Albums* et les romans du jour les ont remplacés depuis; et il y a longtemps qu'il est tombé dans l'oubli. C'est tout au plus si quelque marchand forain se hasarde à en offrir à ces chalands comme prix à gagner à l'un de ces jeux *où à tout coup l'on gagne*." [En]

Despite the fact that the kaleidoscope is no more a source of wild excitement as it was at the time it was invented, it is nevertheless an extremely attractive instrument which is still commercially offered, after more than 170 years, in many versions at various prices. And it infallibly fascinates its users, as can be easily judged by letting people actually handle it. The kaleidoscope, as well indeed as Rubik's cube, offers a rich occasion to involve a wide audience in some nice mathematics. Even if both the kaleidoscope and the cube can be looked at from a group-theoretic point of view (see for example [Co2] and [Ban] respectively), they are essentially geometrical devices, the former being concerned mainly with reflectional symmetry while the inventor of the latter, Ernö Rubik, was aiming at the development of spatial intuition through his articulated cube.

2. UNDERSTANDING THE KALEIDOSCOPE

What advantage is to be gained from an instrument, like the kaleidoscope, built from given mathematical principles? On the one hand, it can make easily accessible phenomena otherwise difficult even to conceive; Charles Wheatstone (1802-1875), a contemporary and at times opponent of Brewster, spoke in the following terms of one of his inventions, the kaleidophone (having no similarity but the name with Brewster's invention and serving to illustrate acoustical phenomena): it "renders obvious to the common observer what has hitherto been confined to the calculations of the mathematician" (quoted in [Wad, p. 205]). But conversely it can also attract the observer and induce him or her to investigate the causes of the phenomena with sustained interest; it can provoke involvement. In his treatise on the stereoscope [Br3], Brewster opens a chapter on the "Applications of the stereoscope to purposes of amusement" with the following comments, surely valid with respect to the kaleidoscope:

> "Every experiment in science, and every instrument depending on scientific principles, when employed for the purpose of amusement, must necessarily be instructive. 'Philosophy in sport' never fails to become 'Science in earnest'. The toy which amuses the child will instruct the sage, and many an eminent discoverer and inventor can trace the pursuits which immortalize them to some experiment or instrument which amused them at school. The soap bubble, the kite, the balloon, the water wheel, the sun-dial, the burning-glass, the magnet, &c., have all been valuable incentives to the study of the sciences." [Br3, p. 204]

In the case of the kaleidoscope, the astounding effect produced by the interplay of mirrors leads to a need for a better understanding of the way mirror-images are being produced. One is thus introduced to the mathematical model of geometrical reflection, the mathematical equivalent of the mirror effect. This need for a careful study was described as follows in *Britannica*:

> "This circular arrangement of the images, however legitimately it may have been deduced from the simplest law of optics, appears to be so extraordinary an illusion of the sense, as to call for somewhat further examination before we can feel perfectly assured that it is a necessary consequence of that law. Perhaps the most satisfactory method of prosecuting their examination is to investigate separately the mode in which each of the images results from the successive reflections by the two mirrors." [EB, p. 164]

But mirrors being omnipresent in our daily lives, we all feel quite confortable about the way reflection acts. Or do we really? Even with a single mirror, some confusing situations can occur. In his charming book *The Ambidextrous Universe* [Ga], M. Gardner asks for example the following naïve question: "Why does a mirror reverse only the left and right sides of things, and not up and down?" [Ga, p. 6] And if this seems to be a rather simple-minded or even silly question, Gardner then suggests the following experiment with two mirrors: we are not surprised by the fact that rotation of a single mirror by a quarter turn clockwise will not turn the image of our face upside down; but if two mirrors are joined at a right angle by one of their sides (thus becoming a 90°-kaleidoscope), they produce an image which is not reversed, and a quarter turn rotation will then turn this image upside down (as illustrated in Figure 3, showing two pictures taken from [Ga]).

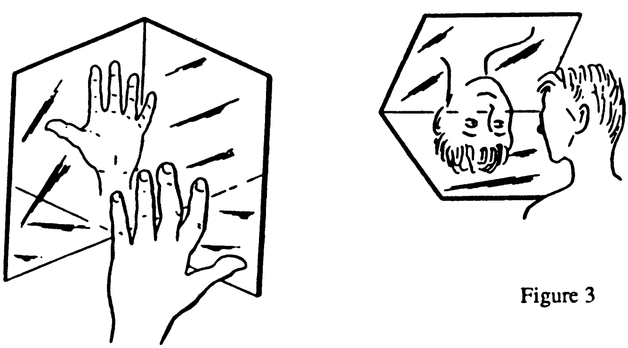

Figure 3

Giving a clear explanation of such phenomena is not self-evident and involves some good mathematical thinking. As Gardner says, adults "take mirror reflections for granted without attempting to get clear in their mind exactly what a mirror does" [Ga, p. 7]. Understanding how two mirrors will act upon the images produced by each of them calls for a sound knowledge of the mathematical notion of reflection. Basic manipulations of mirrors are thus excellent starting points in the exploration of certain fundamental concepts of geometry, as is so well illustrated by the "Mirror Cards" and other works of Marion Walter about mirror geometry (see for instance [Wal]). Such exploration then leads to the identification of pertinent mathematical principles and the building of a good mathematical model. (A discussion of the left-right but not up-down reversal is presented in [Ga, p. 23].)

Once the interplay of the images produced by each of the mirrors is understood, it is a direct matter to derive general rules about the generation of images in a kaleidoscope. Although any angle between the two mirrors can result in interesting reflected patterns, consideration of various possibilities leads to the identification of the principles that were so dear to Brewster for the production of "perfectly beautiful and symmetrical forms": he held that, in a "true" kaleidoscope,

> "the reflectors should be placed at an angle, which was an *even* or an *odd* aliquot part of a circle, when the object was regular, and similarly situated with respect to both the mirrors; or the *even* aliquot part of a circle when the object was irregular, and had any position whatever." [Br1, pp. 4-5]

And these principles, clearly emerging from actual manipulations, can be fully justified on a theoretical level: properties of geometrical reflection show the inconsistency in orientation occurring with an odd aliquot part.

Besides the typical kaleidoscope, in which the objects being looked at are fragments of coloured glass placed in a case that can be revolved at one end of the mirrors, Brewster also constructed "telescopic" kaleidoscopes [EB, p. 170], having a convex lens in place of the case so that any object can occupy the field of vision, as well as "polyangular" kaleidoscopes [EB, p. 171], in which the angle between the mirrors can be altered at pleasure. Other variations include "polycentral kaleidoscopes" [EB, p. 167], built from a greater number of mirrors and thus producing groups of images around several centres spreading in all directions. Configurations suitable for the production of symmetrical combination of images are either the square (or rectangular) polycentral kaleidoscope or the three cases of triangular polycentral kaleidoscopes built respectively on triangles of 60° - 60° - 60°, of 90° - 45° - 45° and of 90° - 60° - 30°. Figure 4, taken again from [EB], illustrates typical patterns generated by such arrangements of mirrors.

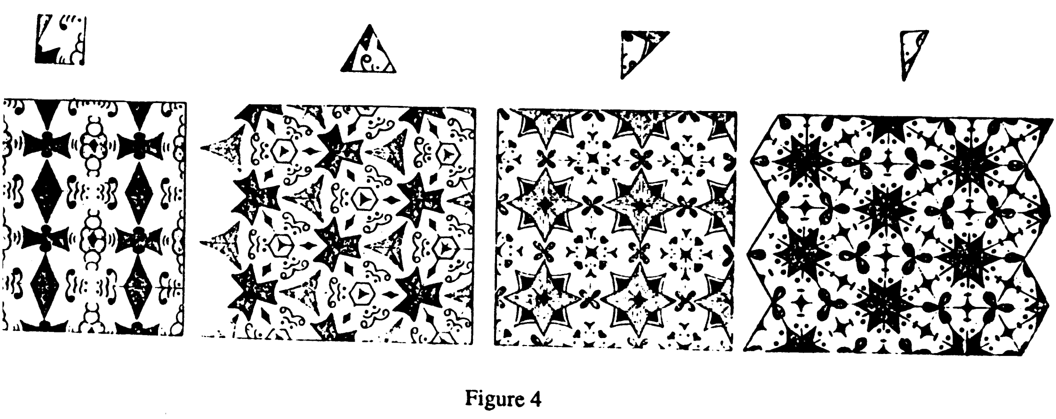

Figure 4

Early expositions on the kaleidoscope, like [Br1], [EB] or [EE], took great care to give an abundance of technical details about the construction of instruments producing the best possible effects. Such considerations concern mostly the practical use of the kaleidoscope and can be of great importance in applications, for example to the ornamental arts, a case strongly supported by Brewster. But with regard to an appreciation of the theoretical principles underlying the kaleidoscope, these concerns can become a distraction from the mathematical content. This might explain why Brewster's contribution seems to have found little echo in the mathematical literature, possibly being considered as of strictly physical interest. A notable exception in this respect is Coxeter's remarkable study [Co1], where proper credit is given to Brewster for his work on the kaleidoscope and "mirror geometry".

3. OBSERVING THE KALEIDOSCOPIC PHENOMENON

Attempts at popularizing mathematical concepts using a given instrument greatly depend on the attractiveness and the availability of the instrument. The kaleidoscope, in this respect, can serve in many different contexts.

First of all, as is testified to by the early works of Della Porta, Kircher or Bradley, the kaleidoscope is easily made from standard mirrors (although possibly not quite correctly built according to Brewster's stipulations!). In fact, we frequently encounter kaleidoscopic phenomena in our daily lives, be it with hinged mirrors in bathrooms or with the limiting case of parallel mirrors at the tailor or the hairdresser. Kaleidoscopes are thus there around us; it belongs to the mathematician to provide a context where a fruitful and informative exploration can be made. Devices suitable for studying the properties of reflection in multiple mirrors can easily and inexpensively be built. Appropriate manipulations can then be made, even with young children in the classroom context, and the basic rules gradually identified (see for instance [Ho] for such an exploration of the "kaleidoscope geometry").

It should be mentionned that at the other extreme, the kaleidoscope can become a very sophisticated and expensive device, a collector's delight found only in specialized stores. Illustrations of such instruments and their visual effects are given in the book *Through the Kaleidoscope* by C. Baker [Bak]. The author also presents there a variety of information on the kaleidoscope both from an arts and crafts and from a popular point of view. (Information can be obtained from C. Baker about the so-called Brewster Society, a "kaleido-club for designers, collectors and lovers of kaleidoscopes".)

Although all that is needed for basic exploration is a simple kaleidoscope made of two small mirrors, it is interesting to note that various modifications using modern technology are possible, thus revamping the kaleidoscope into an up-dated instrument for a high-tech society. For example, the paper [AS1] introduces some "present-day" kaleidoscopes in which polarizing filters and light-emitting diodes are used to produce spectacular effects. But such modernizations are to a certain extent superfluous: encountering the effects of plain mirrors is still the crux of the matter and no fancy sophistications, however interesting they might be, can prompt one to develop a sound understanding of geometrical reflection better than basic experiments with mirrors. The paper [AS2] illustrates this point very well, using as a motivation the bewilderment provoked by mirror mazes. The feelings aroused by actual involvement in these explorations can also be nicely conveyed through animated films: the geometrical properties of mirror interplay suit very well to the potentialities of such a medium, as is so well illustrated by the films *Dihedral Kaleidoscopes* or *Symmetries of the Cube* [UM], produced under the guidance of H.M.S. Coxeter. Animated films like these can be an excellent way of having a large number of people experiment a succinct and systematized exploration of a given mathematical phenomenon with a minimum of material support.

Among such "modern" approaches to the kaleidoscopic phenomenon, there is one which stands out as particularly interesting in that it provides simultaneously new stimulation and deeper understanding. Graphic capabilities of computers, and even of standard microcomputers, allow for effective simulation on the screen of mirror reflections and of iterations of these reflections. A complex kaleidoscopic rosace can then be generated either all at once, as seen in real mirrors, or in a step by step manner, leaving control of the generation of each of the images to the user. The next section presents an implementation of such an approach (see [Gr1], [Gr2]) based on the LOGO language.

4. THE KALEIDOSCOPE WITH THE COMPUTER

a. Background from didactics

As explained before, the different types of kaleidoscopes are powerful means to start investigations of mathematical problems, geometrical ones as well as analytical ones. The intention is always to arrive at some small or larger formal theory on iterated axial or rotational symmetries, on regular polygons (from kaleidoscopes with an angle dividing 360°), on regular stars (angle not dividing 360°), on parquetting or tiling a plane (by applying special "polycentral" kaleidoscopes), etc.

In mathematics education, before reaching this final formal level of description and problem solving, different media are used for investigation. Learning is first supported in enactive phases of action, then the learner goes on to iconic ones and finally he or she ends up with symbolic ones. Very often the iconic means are rather poor or limited and so the learning step to the formal level is rather big. In this section we want to point out that computer graphics is an excellent tool to fill this gap.

The mathematical problems connected with kaleidoscopes can be worked on the following five levels:

1. looking through the real kaleidoscope;
2. reducing the kaleidoscope to a model with two or more real mirrors placed on a sheet of paper carrying some figure;
3. abstracting the mirrors and their effects to reflections in straight lines (axial reflections), constructed with ruler and compasses;
4. transferring these constructions to a computer graphics display;
5. using formal methods to describe the phenomena, like calculus of different complexity up to analytical geometry or linear algebra.

There should always be a very careful examination of advantages for learning before the computer is used in some field of mathematical education. There is no use in transferring manual or mental activities (like constructions with ruler and compasses) to the computer, unless this brings about more efficiency in learning. Another good reason exists, if the computer allows activities which the students cannot achieve with their hands or brains. Then the computer is like an additional tool, increasing the traditional abilities of the students. We think that our kaleidoscope software is a good example for such a kind of tool. It offers additional help in exploring mathematical problems. It allows

- a great variety of investigations with little effort;
- easy experimenting;
- high quality of complex geometrical constructions turning up when studying many iterations of reflections or complicated figures to reflect;
- doing manually impossible constructions like pointwise (pseudo-) simultaneous construction of two or more figures, e.g. one original object and its two images under two axial reflections;

- studying "fiction kaleidoscopes" which have no real material equivalent, namely mathematical models of kaleidoscopes using central reflections instead of axial reflections;
- introducing and exercising simple methods of CAD, a technique which has replaced manual technical drawing to some extent.

Whereas in the first years of computer application in mathematics education the numerical power of computers was dominating, we are now in a position to use its graphical power wherever and whenever helpful from a mathematical or methodical point of view.

b. Software developed and examples

The development of our software started in a course at Freie Universität Berlin for teacher students and in-service-teachers on "computers in mathematics education", using LOGO as a programming language. We selected LOGO as an adequate tool for the programming of graphics without caring too much about a general philosophy of LOGO. This language allowed us to construct a very transparent package of educational software, open to comfortable adaptation by each teacher using it in his or her class. Our software has been developed for an IBM compatible PC with graphics card and display, not necessarily with colour (which is helpful, however).

Besides the correct control of the different kaleidoscope constructions, we took great efforts to get safe interaction between the user and the computer. The dialogue is controlled by menus which allow very flexible investigations. When looking at the pictures, the user can easily switch between full graphics screen and mixed screen. There is a choice of colours for the different objects in the kaleidoscope, which can also be changed easily via the main menu. Moreover, this menu allows to switch between manual control of the growing of the images in the kaleidoscope and automatic generation.

The main menu, which is shown in Figure 5, offers a choice of four different types of kaleidoscopes (all of them with two mirrors only; we are at present working on the general case with three or more mirrors). *Mode 1* leads into a dialogue about putting up a kaleidoscope with an arbitrary angle. The user gives the positions of the axis and then the position of the object selected from another menu (square, triangle, point, cross, line, etc.; see Figure 6) to be reflected in the kaleidoscope. After this setting the computer displays the two axis on the screen and shows the original object. It then constructs and displays one reflection after the other until the pattern is complete. This can be done with a pause after each image, allowing local studying, or in an automatic mode. It is also possible to fill the kaleidoscope with several objects of different colours. When controlling the construction manually, the user can always interrupt and start a new construction.

```
SIMULATION EINES KALEIDOSKOPS
- - - - - - - - - - - - - - -

1 ... ACHSEN FREI WAEHLBAR,
..... SPLITTER FREI WAEHLBAR

2 ... ACHSEN 45 60 72 90 120 GRAD,
..... SPLITTERAUSWAHL EINGESCHRAENKT

3 ... HODGSON - BILDER

4 ... DEMO VON 3 NACHEINANDER
..... AUSGEFUEHRTEN SPIEGELUNGEN



T ... TASTENSTEUERUNG NEU FESTLEGEN
F ... FARBENTSCHEIDUNG NEU TREFFEN
# ... ENDE DER ARBEIT

DEINE WAHL :
```

Figure 5

```
WAS FÜR EIN SPLITTER?

1 - PUNKT
2 - KREUZ
3 - GLEICHSEIT. DREIECK
4 - RHOMBUS
5 - QUADRAT
6 - HAKEN
7 - BRÜCKE
8 - STRECKE ( ACHSEN 60 GRAD )
9 - STRECKE ( ACHSEN 45 GRAD )
10 - STRECKE ( ACHSEN 90 GRAD )
11 - SCHRÄGSTRECKE
12 - KREUZUNG
13 - WINKEL ( ACHSEN 90 GRAD )
14 - WINKEL ( ACHSEN 60 GRAD )
15 - ECKE
16 - GLEICHSCH. DREIECK ( 72 )
17 - GLEICHSCH. DREIECK ( 120 )
18 - STREIFEN
19 - BAND
20 - WOLKEN

DEINE WAHL :
```

Figure 6

Mode 2 allows to select a kaleidoscope with angle 45°, 60°, 72°, 90° or 120° and then proceeds as above. Figure 7 shows some steps in the development of a 60° pattern in Mode 2 and Figure 8 shows the same process for a 70° pattern in Mode 1.

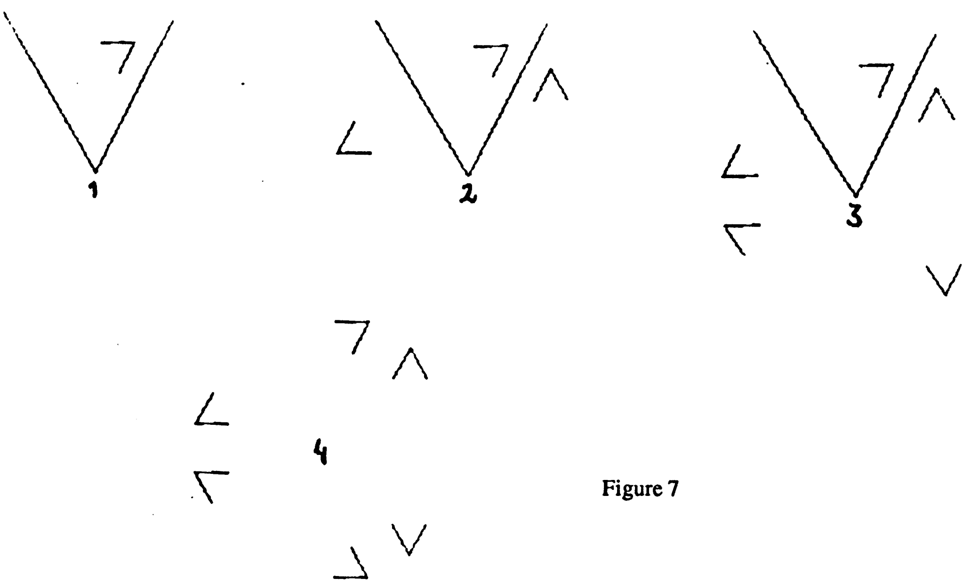

Figure 7

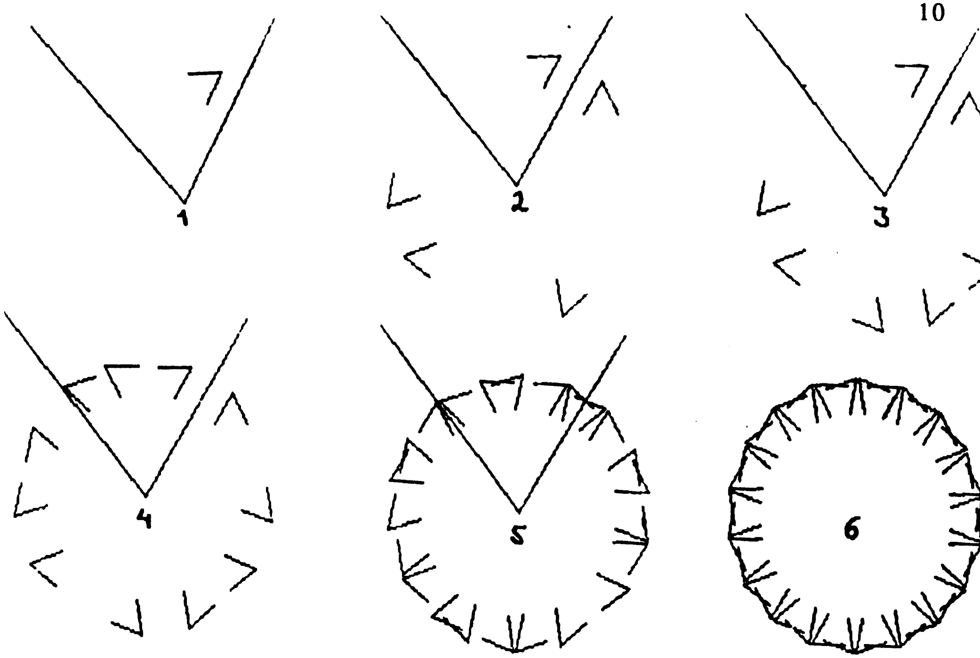

Figure 8

It becomes quite clear that repeated reflections run through a circle for several times and of course the question arises: How many times before the pattern begins to repeat itself? *Mode 3* produces the images corresponding to the situations used in the exploratory approach suggested in [Ho] to answer this question. Figure 9 gives two examples from this mode.

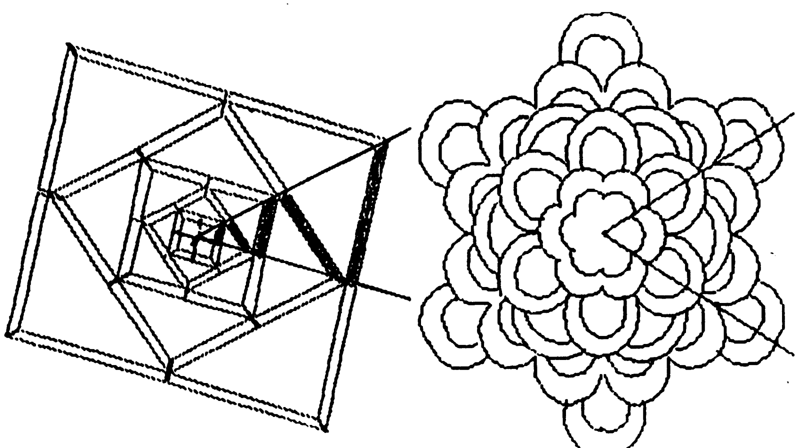

Figure 9

The last mode, *Mode 4*, generates an example of the image obtained under 60° with three different objects and three different colours. Finally, we have shown in Figure 10 some more examples with decorative and geometrical properties; such images have proved to provide an excellent context for provoking mathematical discussions.

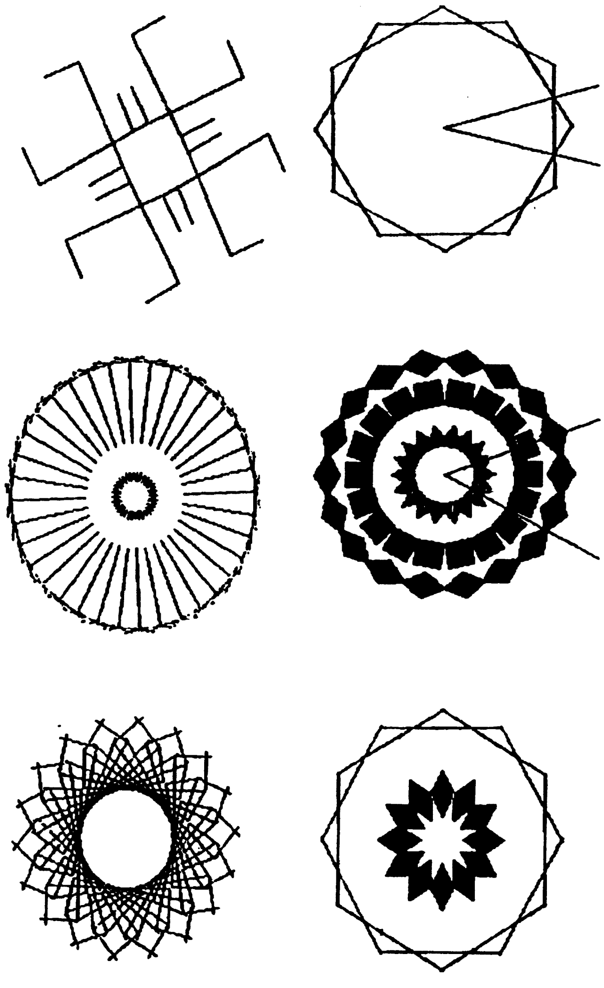

Figure 10

5. CONCLUSION

Part of the previous discussion is related to material developed or used in the context of teacher training activities. Such a link between the mathematical preparation of teachers and the popularization of mathematics is to a certain extent natural, since the two fields have many points in common. In both cases, the main objective is to develop an awareness for the omnipresence and usefulness of mathematics as well as an appreciation of its intrinsic beauty. Mathematics teachers, especially at the elementary level, should themselves be extremely sensitive to the importance of a "mathematical vision" of the world which, although not replacing other essential visions like the historical, political, religious or artistic ones, has nonetheless a crucial rôle to play in our society. In order to make the general public comfortable with various mathematical paradigms, early and various contacts with bona fide mathematical ideas are necessary. In this respect, encountering and truly mastering a phenomenon like the kaleidoscopic multiplication of images can be an extremely important experience. Firm knowledge about such a situation can illustrate the way mathematics works and prepare citizens to appreciate the significance of the numerous mathematical models so commonly used nowadays. The satisfaction gained through a full understanding of the mathematical principles behind a phenomenon could help develop a positive attitude towards mathematics and could even eventually attract some new advocates to the field. Such concerns are essential ingredients of the task of a teacher of mathematics.

The kaleidoscope represents a truly interesting case in the popularization of mathematical concepts. While the simplicity of its construction contrasts with the richness of patterns it can generate, it is an instrument remarkably suitable for a first contact with a non trivial mathematical phenomenon. Considering the fascination infallibly aroused by the interaction of mirrors, it is then no surprise the kaleidoscope has been present for such a long time.

Acknowledgement

The kaleidoscope software presented here was programmed and put in its professional interactive shape by Eva Pilz, assistant in mathematics and computer science education at Freie Universität Berlin.

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DEUX EXEMPLES DE CONCEPTS IMPORTANTS EN MATHEMATIQUES SINGULARITE ET BIFURCATION

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Le projet de populariser les mathématiques ne peut atteindre ses objectifs si le public n'est pas lui-même profondément convaincu de l'intérêt de ce projet, c'est-à-dire, en définitive, de l'intérêt véritable des mathématiques. On peut se demander si la mise en avant des seuls aspects ludiques, spectaculaires et esthétiques des mathématiques, par lesquels on tente de séduire les esprits les plus rebelles, est réellement suffisant pour convaincre ces derniers de la nécessité de s'adonner à notre discipline, de lui conserver la place centrale qu'elle occupe encore au sein de nos systèmes d'enseignement, d'en financer hardiment le développement.

Il est certain que si l'on parvenait à mieux définir et cerner le rôle des mathématiques dans la formation de la pensée, à présenter de manière plus précise et détaillée l'importance de l'outil mathématique dans les différentes branches du savoir, à montrer davantage son caractère indispensable dans la conception des nouveaux produits qui apparaissent sur les marchés, dans l'organisation même de notre univers industriel, économique et social, alors pourraient s'abaisser certaines barrières psychologiques qui peuvent faire obstacle à l'accès aux mathématiques.

Parce qu'il transcende en partie les aspects précédents, je voudrais insister ici sur la possibilité de tirer parti de l'un des caractères les profonds des mathématiques. Celles-ci ne sont pas un simple jeu de l'esprit. Partant des données de l'environnement, leur construction consiste d'abord en une épuration des propriétés physiques secondaires pour ne retenir que les propriétés les plus universelles et les intrinsèques. La création de l'univers mathématiques'accompagne ainsi de la mise en lumière de données essentielles faits ou comportements, dont l'extrême importance et généralité avait souvent échappé à l'attention de nos prédécesseurs, et qui renouvellent notre compréhension du monde, en même temps qu'elles la rendent à la fois plus vaste et plus intime. En faisant connaissance, par une illustration concréte et immédiatement accessible, même de manière approchée, avec les concepts mathématiques parmi les plus profonds et essentiels, le néophyte peut non seulement avoir le sentiment de pénétrer en plein coeur des mathématiques, mais il peut aussi se convaincre du bien-fondé de cette science, éprouver la nécessité de développer le bourgeon de savoir qui s'est formé, d'enrichir son acquis en faits mathématiques, de comprendre la manière dont ils sont établis.

Celui qui serait tenté d'entreprendre une popularisation des mathématiques dans l'optique qui vient d'être proposée, devrait naturellement opérer un choix justifié parmi les concepts très variés et importants existant en mathématiques, et qui relèvent parfois de philosophies différentes mais complémentaires.

Pour ma part, je suis davantage attiré par un choix qu'on pourrait qualifier de géométrique, ou mieux de topologique. Naturellement, des considérations d'ordre subjective se mêlent à des considérations d'ordre objectif pour guider ce choix. De manière objective, celui-ci repose sur l'observation que tous les objets et phénomènes que nous observons sont localisés dans l'espace et dans le temps. Il en résulte la nécessité de commencer par bien comprendre ce qu'est un espace, comment est-il structuré, quelles peuvent en être les propriétés. Je n'entrerai pas à l'intérieur de cette problématique qui touche autant à la métaphysique qu'à la mathématique.

Je me contenterai ici de rappeler que la définition d'un espace fait d'abord appel à des considérations d'ordre topologique: on veut dire par là que l'espace est avant tout caractérisé par la disposition des lieux, par les rapports structurels que les différents lieux entretiennent entre eux, par la manière dont s'organise l'espace autour de chaque lieu particulier.

Ce sont, d'une manière tant locale que globale, d'une part les voisinages de points, d'autre part la manière (relativement simple et standard) dont s'organisent entre eux les voisinages qui consituent les données premières qui, pour le mathématicien et faute de mieux, caractérisent les espaces. Les relations métriques se surajoutent en général à ces notions proprement topologiques. Une bonne manière de faire comprendre ces notions et leurs principales propriétés est de s'en référer aux "espaces humains", à leurs caractéristiques à la fois géographiques et sociales.

La notion de voisinage fermé (ou ouvert) est plus abstraite, i.e. plus idéale, et, pour la faire comprendre, on définira en premier lieu la notion de boule (orange) et de sphère (la peau, l'écorce infiniment mince de l'orange); on pourra alors concevoir les notions de boule fermée (l'orange avec son écorce) puis ouverte (l'orange démunie de son écorce idéale), puis revenir à la présentation de ces mêmes notions sur la droite numérique, abusivement d'ailleurs présentée de manière essentiellement rectiligne.

Ces espaces sont peuplés d'objets, souvent des sous-variétés plongées, qu'on peut parfois définir géométriquement comme graphes d'applications, présentant en général à l'observation des singularités, où adviennent des bifurcations, de forme, de comportement, de structure.

Singularité et bifurcation, le second se situant dans le prolongement dynamique du premier, sont des concepts d'une extrême importance - ce qui justifie d'ailleurs le nombre impressionnant de publications relatives à leur sujet. Pour chacun de ces concepts, je me propose, brièvement dans ce texte, de montrer sur des exemples pratiques et mathématiques les propriétés essentielles.

1. LA NOTION DE SINGULARITE

1.1 Exemples ordinaires

1.1.1 Exemples géographiques:

- Le sommet d'une aiguille se détache dans le ciel: ce sommet est un point singulier.
- Le fond de ce lac de montagne possède un point plus bas que tous les autres: ce point est un point singulier.

1.1.2 Exemples sociologiques:

- On parle beaucoup en ce moment de ce chef d'Etat, de ce président de Société, de cet acteur célèbre: ce sont des singularités de la vie politique, économique, culturelle.

- Dans de nombreux pays, le 1er Mai, le 25 Décembre sont jours de fête: ce sont des jours singuliers de l'année.

1.1.3 Exemple biologique:

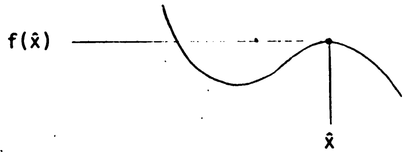
Dans ce tissus embryonnaire apparaît une minuscule zone, pratiquement réduite à un point, où se produit une différenciation du tissus: ce point est singulier.

1.1.4 Exemples physiques:

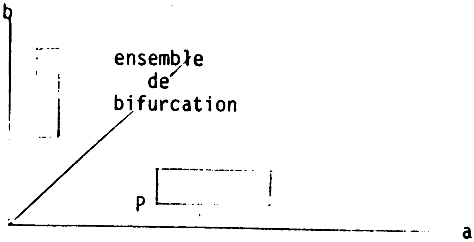
- Sur un pan de ciel, seule une étoile brille, la nuit: ce point lumineux est un point singulier.
- A la pression ordinaire $\hat{p}$ = 760 mmHg, et à la température habituelle $\hat{t}$ = 0°C, se produit la transition glace-eau. Ce point de l'espace (p,t) pression-température où se produit ce changement de phase est un point singulier.
- Un potier fait tourner un vase. Le centre du fond de la cuvette est point singulier de l'ensemble des points du vase qui sont en mouvement.

1.1.5 Exemples mathématiques

- Soit $(\hat{x},f(\hat{x}))$ un extrémum local du graphe de la fonction réelle à une variable f: ce point est un point singulier de ce graphe.



- Soit l'ensemble des rectangles dont un sommet P(a,b) est situé dans le premier quadrant Q^+ de R^2. Parmi ces rectangles, certains sont du type H (a supérieur à b), d'autres du type V (a inférieur à b), d'autres enfin du type C (a = b). Conformément à l'exemple précédent, les extrémums finis de la fonction f(p,q) = p/q où (p,q) Q^+ sont les poins singuliers de f: ils sont atteints lorsque p = q (f = 1): les rectangles correspondants sont du type C, en d'autres termes, la forme carrée est une forme singulière de l'univers des formes rectangles.
- Dans Q^+, la demi-bissectrice p = q forme un ensemble singulier de points.



1.2 Propriétés principales des éléments singuliers

1.2.1 Les points et éléments singuliers ne sauraient former des ensembles denses; bien au contraire, ils sont rares, et forment des ensembles de mesure nulle. Ainsi, dans le dernier exemple précédent, les aires des demi-quadrants de Q^+ ne sont pas nulles alors que l'aire de la demi-bissectrice qui les sépare est nulle.

Ce caractère de rareté des éléments singuliers est un fait patent. On peut lire sous la plume de G.Condé (Le Monde du 19 Avril 1989) le texte suivant: "Testament par le sujet, somme par la musique d'une densité constante (richesse de l'orchestration, du contrepoint, inventions rythmiques et harmoniques), *Doktor Faust*est une oeuvre singulière, résolument. Une de celles, assez rares, dont l'unité manifeste ne peut se réduire à une formule et dont on a jamais fini de faire le tour."

Il est clair que l'oeuvre musicale de Busoni dont parle le critique est également singulière au sens mathématique du terme, elle présente de plus des caractères d'extrémalité soulignés par G.Condé.

1.2.2 Physiquement, la présence d'une singularité marque la limite d'une transformation interne accompagnant un déplacement le long d'un chemin tracé sur un objet géométrique.

Prenons par exemple le sommet d'un cône. Dessinons sur ce cône une trajectoire qui feuillette la surface du cône et vient mourir en son sommet: une vitesse, une dérivée donc s'y annule.

La singularité est ainsi un lieu de dégénérescence d'une structure interne, ou, tout au contraire, le lieu de naissance d'une nouvelle organisation. La singularité correspond donc à ce qu'on appelle un centre organisateur en biologie, et, en sciences humaines, un centre de pouvoir aussi bien qu'un lieu de perdition ou un foyer séditieux.

Supposons par exemple que l'on considère la singularité constituée par un extrémum local d'une fonction f définie sur R^n et à valeurs dans R. La croissance ou la décroissance de f dégénère en cette singularité, ce qui se traduit par le fait que toutes les dérivées partielles f_{x_i} de f par rapport aux x_i (i = 1,2,...,n) s'annulent: la forme linéaire df = (f_{x_1}, ..., f_{x_n}) est donc de rang nul en $\hat{x}$, alors que son corang est 1.

Ce point de vue analytique n'est que la traduction du fait géométrique bien connu suivant. L'analogue géométrique de la fonction f est son graphe dans R^nxR, c'est-à-dire le dessin, la forme dans cet espace que réalise l'ensemble des couples de points (x,f(x)). Réciproquement, une telle figure étant donnée, on lui associe une application f admettant cette figure pour graphe. Au point singulier "d'abcisse " $\hat{x}$ de la figure, l'espace qui lui est tangent et que l'on suppose ici exister, est parallèle à l'espace des "abcisses" R^n.

Plus généralement, si f = (f_1, ..., f_p) est une application de R^n dans R^p (f_i(x) = y_i est la i-ème composante dans R^p de l'image y = f(x) de x), dont la représentation géométrique est le graphe de f dans R^nxR^p admettant un point singulier ($\hat{x}$,f($\hat{x}$)), alors l'espace tangent à ce graphe au point singulier indiqué est tel qu'un au moins de ses sous-espaces de dimension n est parallèle à l'espace des "abcisses" R^n. La traduction en termes d'algèbre linéaire de ce fait géométrique est la suivante: le corang de f, défini comme (min(n,p) - rang de f) est positif.

1.2.3 Un point singulier peut être présenté comme un extrémum local: en tant qu'extrémum, il est visible, souvent voyant, il attire l'attention.

En tant que centre organisateur, ses caractères propres déterminent, en son voisinage, l'organisation de l'objet dont il est un élément singulier. Cette propriété est vraie tant dans la réalité quotidienne qu'en mathématique (l'objet géométrique est localement homéomorphe à un bouquet de sphères).

1.2.4 Il résulte de la remarque faite à l'instant, qu'un objet peut être entièrement décrit par la donnée de ses singularités et de leurs propriétés. Il existe ainsi une dualité profonde entre les singularités de l'objet et l'objet lui-même. Cette dualité est remarquable puisque la réalisation dans l'objet des singularités (que l'on peut donc définir dans un espace dual) forme un sous-ensemble de points de l'objet ayant une mesure nulle.

Cette dualité permet de réaliser un gain très appréciable dans la description, la codification, la mémorisation de l'objet. On ne retient de celui-ci que ce que j'ai proposé d'appeler "le squelette perceptif", l'ensemble des traits pertinents.

Notons au passage que la notion de singularité permet ainsi de donner une représentation fonctionnelle et géométrique à une notion sémantique, le concept de trait pertinent.

A contrario, on obtient une représentation fonctionnelle et géométrique de propriétés ou de faits presque partout présents ou vrais, alors qualifiés de génériques. Voici en pratique comment l'on procède.

Prenons d'abord l'exemple de deux droites du plan, d'une part la droite d'équation $y = 0$ appelée V_0, d'autre part la droite d'équation $y = ax + b$ appelée $V_{a,b}$. Pour presque tout couple (a,b) de R^2, ces deux droites se coupent selon un angle non nul, on dit qu'elles sont transversales. Dans le cas singulier où $a = 0$, $b = 0$, elles sont en fait tangentes (et donc ici confondues) elles ne sont plus transversales.

Etant donnée alors une propriété P, on la géométrise plus ou moins grossièrement par une sous-variété $V_{a,b}$: cette propriété est générique si lorsqu'on modifie les modalités (a,b) de sa réalisation effective, $V_{a,b}$ reste transversale, c'est-à-dire non tangente, à une sous-variété V_0 fixée à l'avance.

1.2.5 Avant d'aborder des aspects plus dynamiques liés à la notion de singularité, il convient d'examiner la manière dont on définit la singularité en analyse fonctionnelle: qu'entend-on par application singulière ?

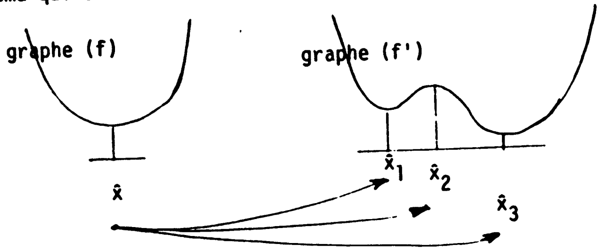
Comme on l'a vu précédemment, on représente géométriquement une application f de R^n dans R^p par son graphe dans $R^n x R^p$. Celui-ci est caractérisé par ses singularités.

Si donc f' est une autre application de R^n dans R^p qui possède en commun avec f la singularité $\hat{x}$, et à laquelle est associée un graphe G', elle ne pourra être dans un "bon" voisinage de f que si d'une part, au voisinage de la singularité les graphes G et G' sont "voisins", et si d'autre part f et f' ont même nombre potentiel de singularités.

Par exemple, si au voisinage de la singularité, le graphe de f est localement donné par une forme polynômiale de degré k - on dit que f est localement k-déterminée, alors tout ajout à f de formes polynômiales de degré inférieur ou égal à k ne changera pas le nombre de singularités de l'application obtenue f'.

Pour que l'application f soit singulière, notons-la alors f, il faut qu'elle présente des propriétés d'extrémalité par rapport aux applications voisines f'. Une manière d'exprimer cette extrêmalité est de dire que le nombre de paramètres et de termes que possède f est minimal pour la propriété de conservation du nombre potentiel de singularités (i.e. le nombre de racines possible de l'équation $f^{(1)}(x) = 0$, où $f^{(1)}$ désigne ici la "dérivée" de f). Dans ce cas, f est telle qu'elle ne possède qu'une seule singularité apparente $\hat{x}$, alors que dans le voisinage de f peuvent figurer des f' possédant plusieurs singularités issues de la singularité orginelle.

Un cas typique est celui par exemple de la fonction f de R dans R définie par $f(x) = x^4/4$. Elle possède à l'origine une singularité d'ordre 3, 0 est en effet racine triple de la dérivée $f^{(1)}(x) = x^3$. L'addition de termes supplémentaires comme ux^2 + vx transforme f en $f'(x) = f(x) + ux^2 + vx$: f et f' ont même nombre de singularités potentielles. Mais pour des valeurs convenables de u et de v, celles-ci peuvent se matérialiser comme il apparait dans le schéma qui suit:



Toutes les singularités ponctuelles de la singularité fonctionnelle f étaient concentrées à l'origine. Sur cet exemple, on assiste au déploiement de cette singularité ($\hat{x} = 0$, $f(\hat{x}) = 0$) en la gerbe maximale de singularités ($\hat{x}_1$, $f'(\hat{x}_1)$), ($\hat{x}_2$, $f'(\hat{x}_2)$), ($\hat{x}_3$, $f'(\hat{x}_3)$).

Par extension, on dit également que f' est un déploiement de f.

Puisque, encore une fois, la détermination des singularités d'un objet permet de le reconnaître, le travail a été entrepris de trouver les singularités (fonctionnelles) des objets définis en tant qu'espaces d'applications. Ce travail de recherche et de classification a permis de mettre en évidence la structure stratifiée de ces espaces.

Comme problème à résoudre où les singularités fonctionnelles sont impliquées, je poserai celui-ci: un problème d'équations aux dérivées partielles bien posé étant donné, à quelles conditions admet-il pour solution une application singulière; s'il admet une telle solution, à quelles conditions des déploiements de l'application singulière sont elles également des solutions; comment déformer le problème initial pour que ces déploiements soient également solutions, dans le cas où la question à la seconde question serait négative ?

1.2.6 Nous avons, jusqu'à présent, principalement caractérisé les singularités en termes d'extrémalité fonctionnelle.

Si l'on souhaite des caractéristiques plus intrinsèques des singularités, l'examen de leurs groupes de symétrie sera du plus grand intérêt. On obtiendra alors des caractérisations en termes d'extrémalité structurelle.

Une famille d'objets étant donnée, par exemple celle des ellispses obtenus par déformation régulière des rectangles, un objet de cette famille sera singulier s'il accumule le plus grand nombre de symétries.

Le carré ou le cercle sont des objets singuliers car les rectangles ou les ellipses ne possèdent pas à un même degré la symétrie métrique. De même, le couple $\hat{x}_1 = 0$, $\hat{x}_2 = 0$, solution de $x^2 = 0$ a davantage de symétrie que le couple de solutions $\hat{x}'_1=-a$, $\hat{x}'_2 = +a$ de l'équation $x^2 - a^2 = 0$, car $\hat{x}'_1$ et $\hat{x}'_2$ diffèrent par leur orientation, révélée ici par leur signe.

Une telle recherche de caractérisations structurelles des singularités n'est encore qu'en projet. Elle est parfois partiellement menée lorsque les physiciens traitent de problèmes de groupes liés aux brisures de symétrie, ou lorsqu'on étudie les phénomènes de bifurcation en présence de symétries.

1.2.7 Les singularités qui, par leur caractère de rareté, peuvent échapper totalement à l'analyse statistique, et c'est là le point faible de ce type d'approche, jouent un rôle naturellement attractant dans la dynamique mentale des être animés.

On peut alors, ce qui non plus n'a pas encore été entrepris consciemment, se livrer à une étude linguistique sur la place dans la phrase du terme désignant la singularité, sur son rôle dans l'organisation du discours.

Voici l'une des rares évocations de cette problématique trouvée dans la littérature, dans un texte sur la métaphore "De même, suivant Tversky, on dira "une ellipse est semblable à un cercle" et non "un cercle est semblable à une ellipse", car la plus typique des coniques fermées, c'est le cercle et non l'ellipse."

L'explication ici avancée l'auteur des lignes précédentes, C.Boisson [1] , nous laisse sur notre faim, mais du moins a-t-elle le mérite de dégager le caractère original de la singularité, à laquelle est accordée une vertu paradigmatique. Ce point n'est-il pas justifié dans la mesure où justement la singularité est le lieu où s'équilibrent toutes les tendances opposées qui peuvent engendrer le déploiement ?

1.2.8 Quelle que soit la manière dont on retourne la question, il apparaît qu'en définitive, ce sont des propriétés d'extrémalité qui confèrent à l'élément privilégié son caractère singulier. Le concept d'extrémalité, dont j'ai souligné depuis longtemps l'importance en philosophienaturelle ([2] ,[3]),mérite assurément d'avoir le pas sur le concept de singularité. Faut-il alors voir dans l'extrémalité un principe organisateur ontologique ? Oui et non. Oui, si l'on s'en tient à une modélisation approchée et rapide des phénomènes, ce que nous sommes obligés de faire la plupart du temps, compte tenu de nos possibilités d'analyse des phénomènes. Non, si l'on se rend compte que l'extrémalité n'est que la conséquence des contraintes d'inter-relations qui pèsent sur les modes de construction des organisations, et sur les mouvements internes d'autoévolution qui limitent leur développement. Ces limites ne sont pas les mêmes pour tous, d'où résultent les positions d'extrémalité relative.

On retiendra ici la nécessité de lier extrémalité et singularité, et la possibilité offerte au pédagogue de faire d'une pierre deux coups en présentant simultanément ces deux notions essentielles.

1.2.9 Il est nécessaire, pour terminer ce paragraphe, de présenter un dernier visage de la notion de singularité, celle que l'on rencontre en dynamique.

Dans un monde à la Parménide, peuplé d'objets presque toujours en mouvement, ceux d'entre eux qui restent immobiles apparaissent inévitablement comme singuliers. Si, par exemple, chaque point $x = (x_1, \ldots, x_n)$ d'un espace R^n est en mouvement, la vitesse de chaque composante x_i de x étant analytiquement donnée par une fonction f_i:

$$\frac{dx_i}{dt} = f_i(x,a,t)$$

où a est un vecteur de paramètres, les points singuliers seront ceux de vitesse nulle, vérifiant donc $f(\hat{x},a,) = 0$.

Cette notion d'élément invariant au cours du temps s'étend en celle d'ensemble de points globalement invariants au cours du temps, et qu'on appelle des attracteurs.

L'étude de la variation des morphologies des attracteurs sous l'influence de perturbations sur le vecteur a occupe une place centrale en dynamique qualitative, et relève de la théorie de la bifurcation.

2. LA NOTION DE BIFURCATION

2.1 Introduction

Dans un univers figé, la singularité apparaît, tel un sommet étincelant et immuable. Dans un monde en évolution, on observe des singularités quasi inertes et figées dans leur morphologie et d'autres où adviennent de spectaculaires transformations.

Ces transformations sont appelées en mathématiques des bifurcations. On les rencontre bien sûr dans tous les règnes de la nature. On les appelle alors par exemple naissances, morts, brisures de symétrie, changements de phase, parfois cassures, catastrophes, flambages, ou bien dédifférenciation, déstructuration, restructuration, différenciation, voire métamorphoses. L'étude des bifurcations permet également de comprendre les naissances et les disparitions de comportements périodiques, comme par exemple en hydrodynamique les cellules de convection, ou comme ceux que l'on rencontre en chimie et en biochimie. On a de bonnes raisons de penser que la modélisation complète et dynamique des bio-rythmes, ou du phénomène du rire, fera apparaître des bifurcations classiques entraînant des mouvements internes oscillants. Il est également bien connu que certains phénomènes turbulents peuvent être interprétés en termes de bifurcation. Par ailleurs, les développements de morphologies ramifiées comme celles qu'on observe dans le règne végétal pourraient fort bien résulter de processus de bifurcation. Dans un tout autre domaine, l'économie, où l'on étudie le bouillonnement de certaines activités humaines liées aux modalités de l'organisation sociale, l'existence de certains phénomènes ou comportements stables peut être traduite en termes de présence de singularités, dont le maintien à un niveau donné ou le suivi de leur évolution relève de la théorie de la bifurcation.

Les phénomènes de bifurcation sont donc parmi les plus passionnants à étudier dans la mesure ils nous permettent de fournir des explications ou des métaphores explicatives de la genèse et de la fin des choses.

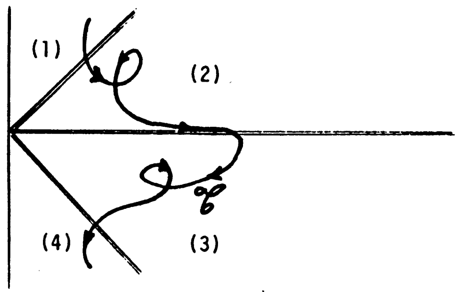
On n'en finirait pas de relever les exemples de bifurcation et la diversité de leurs modalités. La brièveté imposée à l'exposé ne permet guère d'entrer dans quelque détail. On me pardonnera sans doute de m'en tenir ici à la présentation rapide d'exemples simples quoique essentiels, et à faire quelques considérations d'ordre général.

La suivante est d'ordre technique. La théorie des singularités fonctionnelles s'appuie fondamentalement sur la théorie des groupes (groupes de transformations de contact, algèbres et espaces vectoriels divers). Il en sera de même des théories de la bifurcation, basées sur la perturbation des singularités fonctionnelles. En fait, toutes les théories de la bifurcation reposent sur l'action de groupes de transformations, certains éventuellement induits pas la perturbation de paramètres extérieurs - ainsi, la perturbation des paramètres d'une équation polynômiale influe sur le groupe de Galois de ses racines. Par suite, la détermination des divers comportements possibles résulte de la classification de sous-groupes appropriés des groupes considérés.

2.2 Exemples élémentaires

2.2.1 Exemple géométrique

On étend ici le second exemple présenté au paragraphe 1.1.5. Considérons dans l'espace ordinaire à trois dimensions un objet dont le contour apparent C supposé plan est défini par une équation de la forme $ax^2 + by^2 = 1$, où a est un réel positif, b un réel quelconque. On appelle le plan (a,b) le plan des variables de forme, de contrôle ou de bifurcation.

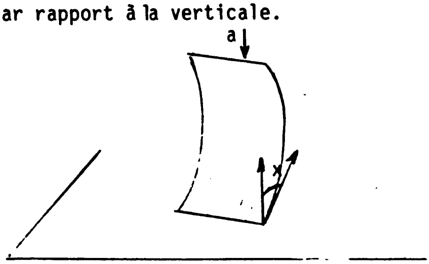


Dans le domaine (1), les objets sont du type EV (ellipses allongées verticalement); dans le domaine (2), les objets sont du type EH (ellipses allongées horzontalement); dans le domaine (3) (resp.(4))les objets sont du type HO (hyperboles "ouvertes") (resp. HF (hyperboles fermées)). Supposons qu'on déplace l'objet de sorte qu'au cours du temps, a et b se modifient. Dessinons dans le plan (a,b) la trajectoire $\mathcal{C}$(a(t),b(t)) de ce couple de paramètres de forme associés à O. La forme persiste quand le point P(t) = (a(t),b(t)) est à l'intérieur de chacun des domaines que l'on vient de spécifier. Lorsque P franchit la demi-droite (1-2), l'apparence change brusquement. : la forme initialement étirée dans le sens vertical devient soudain étirée dans le sens horizontal. La forme cercle, intermédiaire n'est réalisée que ponctuellement: elle ne peut véritablement apparaître. Un phénomène analogue se produit lorsqu'on examine la transition (3-4). Dans ces deux transitions, on remarquera que la trajectoire est transversale aux demi-droites dites de bifurcation (1-2) et (3-4). Le passage de (2) à (3) est le plus spectaculaire: on passe d'ine forme ellipse à une forme hyperbole. La transition (2-3) n'est pas moins étonnante puisqu'elle se compose de deux droites. Dans le cas présent, la trajectoire n'est pas transversale à la demi-droite de bifurcation (2-3), la forme transitoire persiste quelque temps avant de disparaître.

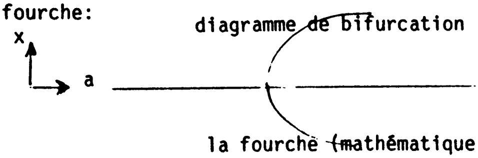
Les demi-droites où adviennent les bifurcations forment l'ensemble de bifurcation: leur mesure (celle de l'aire du domaine qu'elles forment) est nulle dans l'espace des variables de bifurcation

2.2.2 Exemple mécanique

C'est l'exemple d'un rectangle en carton que l'on fixe verticalement sur la table le long de l'un des côtés de plus petite longueur. On exerce avec le doigt une force a sur le bord opposé au bord fixé, et l'on étudie les positions d'équilibre du carton. Pour cela, on calcule la fonction d'énergie potentielle de ce système mécanique. Les positions d'équilibre sont des extrémums de ce potentiel, par conséquent des zéros de la dérivée de cette fonction par rapport à la variable d'état x qui mesure l'inclinaison du bord du carton par rapport à la verticale.



Autour de son point singulier, la fonction dérivée est équivalente à un déploiement $F(x,a) = x^3 - (a - 1/2)x$ de la fonction x^3, de sorte qu'il y a, localement, correspondance biunivoque entre les zéros de F et ceux de la fonction dérivée. Les zéros de F forment dans la variété (espace des variables d'état)x(espace des variables de bifurcation), ici le plan (x,a), une figure appelée diagramme de bifurcation. Dans le cas présent, ce diagramme est appelé la fourche:



Si a est inférieur à 1/2, on a qu'une seule position d'équilibre. Si a est supérieur à 1/2, il en existe trois a priori.

Une étude complémentaire de stabilité élimine la solution x = 0.

Dans cet exemple, R forme l'ensemble des valeurs possibles du paramètre de bifurcation a, et le point a_c = 1/2 constitue à lui seul l'ensemble de bifurcation.

In fine, une équation a été résolue, on a discuté, selon la valeur de ses coefficients, du nombre de ses racines. On peut concevoir toute équation à résoudre comme une équation associée à un problème de bifurcation. Traiter le problème de bifurcation revient à discuter du nombre de racines de l'équation en fonction des valeurs de ses coefficients. On peut ainsi donner une présentation tout à fait moderne du problème de la résolution des équations, abordé dans les lycées avec l'étude des équations polynômiales à une et deux inconnues.

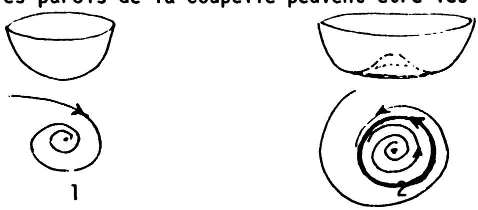
Dans le même ordre d'idées, étudier le signe d'une dérivée au voisinage d'un point singulier n'est autre qu'étudier la changement morphologique du graphe de f au voisinage de la singularité: c'est encore un problème de bifurcation.

On constate ici la possibilité d'introduire très tôt le concept de bifurcation, en changeant tout simplement la manière de concevoir, de présenter les problèmes.

2.2.3 <u>Exemple dynamique</u>

Comme tout système dynamique plan est somme d'un système gradient et d'un système hamiltonien, on peut réaliser toutes les dynamiques du plan en considérant des écoulements de fluides incompressibles sur des surfaces géographiques, éventuellement animées de mouvements de rotation. Les trajectoires correspondent aux lignes d'écoulement du fluide.

Supposons alors donnée une coupelle déformable, en rotation autour de son axe de symétrie, et dont une section verticale passant par l'axe de symétrie est une portion convenable du graphe des fonctions f et f' du paragraphe 1.2.5. Par projection sur le plan sur lequel repose la coupelle, les trajectoires d'un fluide s'écoulant sur les parois de la coupelle peuvent être les suivantes:



Dans la situation 2, la trajectoire fermée autour de laquelle viennent s'enrouler les autres trajectoires s'appelle un cycle. Cette trajectoire fermée est associée à une évolution périodique du genre $x_1 = a \cos wt$, $x_2 = b \sin wt$. Le passage de 1 à 2 s'appelle la bifurcation de Poincaré-Hopf

Un modèle mathématique du fonctionnement de l'oscillateur électronique du type push-pull a été proposé dans les années 30 par Van der Pol. L'étude de ce modèle dynamique et de son extension par Liénard montre l'existence d'une bifurcation de Poincaré-Hopf. Tant qu'on reste en deçà de certaines valeurs des coefficients évaluant une résistance, une capacité et une self-induction, le système électrique ne connaît pas d'oscillation: un point singulier représente son état électrique (tension, intensité). Au delà de ces valeurs critiques, tension et intensité varient de manière périodique, un cycle représente leur évolution.

Dans les études de bifurcation en dynamique, en se plaçant au voisinage d'un point singulier tel qu'il a été défini en 1.2.9, on considère l'approximation linéaire du système à étudier. Si $dx_i/dt = f_i(x,a)$ admet 0 pour point singulier ($f_i(0,a) = 0$), alors on examine les propriétés du système linéaire $dy_i/dt = f_i^{(1)}(0,a).y$, où $f_i^{(1)}(0,a)$ est la valeur en 0 de la forme linéaire des dérivées partielles premières de f_i par rapport aux composantes de x. Ecrivons ce système sous la forme $dy/dt = M(a).y$.

A la manière de ce que nous avons vu en 1.2.2, ce système linéaire dépendant de a présentera une singularité pour la valeur a_c lorsque, par simple changement éventuel de répère, M(a) devenant alors M'(a), on pourra écrire le système sous une forme telle que l'une au moins des équations $dy_i/dt = m_i'(a_c).y$ soit nulle: pour la valeur de bifurcation a_c, le rang de M'(a) baisse d'au moins une unité, fait qui a son pendant sur les racines d'un polynôme attaché à M(a) et qu'on appelle son polynôme caractéristique.

Nous retiendrons encore ici le fait qu'un système dynamique présente un caractère singulier lorsque se produit une dégénérescence interne: pour la valeur critique a_c, le système perd au une variable d'état.

2.3 Les deux propriétés principales d'une singularité de bifurcation

Reprenons ici ce que nous avons observé:

2.3.1 Les bifurcations apparaissent pour des valeurs des variables de bifurcation formant des ensembles de mesure nulle dans l'espace des variables de bifurcation.

On n'observe donc pas des bifurcations à tout instant, mais quand on les observe, on s'en souvient !

2.3.2 Pour ces valeurs singulières, les objets mathématiques présentent des formes de dégérescence.

Elles correspondent à des situations physiques et réelles et parfaitement compréhensibles. Pour que la structure d'un objet puisse être modifiée, il faut qu'il passe au préalable par une phase où la structure initiale puisse être partiellement dissoute.

Cette phase ne peut être que transitoire, car l'objet perd des qualités de stabilité immédiates.

Il acquiert par contre des potentialités supplémentaires: une variable y_i devient momentanément muette, elle ne pèse d'aucun poids sur la structure de l'objet. Elle n'est pas moins potentiellement présente, et peut retrouver un rôle structurel nouveau au sein d'une architecture différente.

On pressent ici l'existence d'un invariant global pour tout objet, liant trois concepts entéléchiques et dont seul le premier, pour l'instant, fait l'objet d'une quantification: l'énergie, les degrés de potentialité, les degrés de stabilité.

Si, durant la phase de stabilisation, un agent extérieur ou endogène pèse davantage que les autres sur l'évolution du système, cet agent va conduire rapidement cette évolution vers une nouvelle structure qui peut être beaucoup plus stable, et mathématiquement représentée par un nouvel attracteur de la dynamique du système.

3. CONCLUSION GENERALE

Il est de tradition et très naturel, dans nos démarches intellectuelles, de commencer par percevoir et par analyser les situations statiques, les plus longtemps visibles, avant d'étudier les mêmes données dans le cadre dynamique, plus vaste et plus complet, au travers duquel on peut réellement comprendre les raisons de nos observations.

S'il est raisonnable de commencer par étudier les singularités, c'est la notion de bifurcation qu'il faut viser à introduire.

On peut s'y employer, dans des limites naturellement modestes, dès le secondaire et dès les premières années de l'enseignement post-secondaire.

J'avancerais même qu'on devrait s'y employer. On n'exerce ra jamais assez tôt l'esprit humain à s'orienter vers l'essentiel, à apprécier l'importance des voies nouvelles qui permettent l'accès à une compréhension plus approfondie et plus ample des réalités de notre univers.

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