Generic Splitting of Reductive Groups

(to appear in the Tôhoku Mathematical Journal)

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In memory of Ernst Witt

Abstract. Generic conditions for the occurrence of a parabolic subgroup of given type in a reductive algebraic group are described. Especially the notion of a generic splitting field of a reductive algebraic group is investigated. The given theory generalizes and unifies other investigations of various authors for special algebraic structures such as Azumaya algebras and quadratic forms.

Introduction

The "degree of splitting" of a connected semisimple algebraic group G over a field k is essentially determined by the types of parabolic k-subgroups of G. For example, G is anisotropic if G itself is the only parabolic k-subgroup, G is quasi-split if it contains a Borel subgroup, and G is split if it contains parabolic subgroups of every possible type.

We now assume that G is a connected reductive linear algebraic group over k. One of our main goals is to describe generic conditions for a field extension K of k, which guarantee the existence of a parabolic subgroup of G_K of a prescribed type, where $G_K = G \times_k K$ denotes the algebraic group over K obtained from G by scalar extension.

Let $\bar{k}$ denote an algebraic closure of k. It is known that $G_{\bar{k}}$ splits and that the conjugacy classes of parabolic subgroups of $G_{\bar{k}}$ are in one-to-one correspondence with the subsets of the vertices Δ of the Dynkin diagram of $G_{\bar{k}}$. The subset $\Theta \subseteq \Delta$ corresponding to the class of a parabolic subgroup P of $G_{\bar{k}}$ is called the *type* of P. The set Δ itself is the type of $G_{\bar{k}}$ and the empty set $\emptyset$ is the type of a Borel subgroup of $G_{\bar{k}}$.

In §3 we show that the occurrence of parabolic subgroups of given type is preserved under k-specializations. More precisely, in 3.9 we prove, for any field extension L of k: If there is a parabolic subgroup of type Θ in G_L, then there is a parabolic subgroup of type Θ in $G_{k'}$ for every k-specialization k' of L, that is, for every field extension k' such that there is a k-place $L \to k' \cup \{\infty\}$.

This leads us to the definition of a generic Θ-splitting field of G for any subset $\Theta \subseteq \Delta$. A field K is called a Θ-*splitting field* of G if G_K contains a parabolic K-subgroup of type Θ, and a Θ-splitting field F of G is called *generic* if every Θ-splitting field of G is a k-specialization of F. Especially a (generic) *quasi-splitting field* of G is a (generic) $\emptyset$-splitting field of G.

In order to find a generic Θ-splitting field of G, we study, in §3, the quotient variety $V_\Theta := G_{\bar{k}}/P$ for a parabolic subgroup P of $G_{\bar{k}}$ of any type Θ. Since P is self-normalizing, V_Θ can be identified with the conjugacy class of P in $G_{\bar{k}}$. It is known that the variety $V_\emptyset$ is always defined over k, and we will see in 3.11 that its function field $k(V_\emptyset)$ is a generic quasi-splitting field of G. We show, more generally, that V_Θ is defined over a "small" finite and separable field extension k_Θ of k, which is the smallest extension of k such that the so called $*$-action of the Galois group $\mathrm{Gal}\,(k_s/k_\Theta)$ on Δ leaves Θ invariant. (By k_s we denote the separable closure of k contained in $\bar{k}$.) Hence $k_\Theta = k$ in most cases. Especially this is always true for groups of inner type. We will see in 3.16 that, for any Θ, the function field $k_\Theta(V_\Theta)$ is a generic Θ-splitting field of G. Any Θ-splitting field of G contains a copy of k_Θ, and the Θ-splitting fields K of G are – as field extensions of k_Θ – characterized by the condition that $K(V_\Theta)$ is a purely transcendental extension of K (cf. 3.10, 3.16).

A Θ-splitting field K of G splits G "partially" in the sense that the rank of G_K is greater than or equal to the rank of G, but is not necessarily equal to the maximal possible value, in which case K would be a splitting field of G (cf. 1.7 and 1.10 below). If K is a splitting field of G, then the semisimple part of G_K is a group of Chevalley type.

Another main goal of this paper is to exhibit subsets Θ of Δ such that a corresponding generic Θ-splitting field is a generic splitting field of G. Similarly as above, a splitting field F of G is called *generic* if every splitting field of G is a k-specialization of F.

* Supported by the Sonderforschungsbereich 343 "Diskrete Strukturen in der Mathematik" at Universität Bielefeld sponsored by the Deutsche Forschungsgemeinschaft.

1991 Mathematics Subject Classification. Primary 20G15; Secondary 16H05, 11E04, 14L35, 14M15.

Our theory of generic Θ-splitting fields of reductive groups unifies several other investigations of similar kind for different special algebraic structures.

The earliest example of a generic splitting field has been given by Witt [37, 1935], who constructed a generic splitting field for a quaternion skew field D over k. Here splitting of course means the splitting of D into a full 2×2 matrix ring. We will show that the generic splitting field constructed by Witt is precisely the function field $k(V_\emptyset)$ for the algebraic k-group $G = \mathrm{SL}_1(D)$ (cf. 3.20). Witt's result has been generalized to central simple k-algebras by Amitsur [2, 1955]. The varieties which occur in this context are the Severi-Brauer varieties over k which can be described as the k-forms of projective space. A different approach to this construction making use of non-abelian Galois cohomology has been given by Roquette [23, 1963], [24, 1964]. These results occur as particular cases in our discussion of the partial generic splitting of the algebraic group $G = \mathrm{SL}_{r+1}(D)$ for a finite dimensional central skew field D over k and $r \geq 0$ (cf. 4.9 below). Moreover, Roquette proved [23, Th. 4, p. 413] that the function field of the Severi-Brauer variety related to the full matrix ring $M_{r+1}(D)$ over D is a purely transcendental extension of that of the Severi-Brauer variety of D. Translated into our theory, this becomes a particular case of the fact (cf. 3.18, 3.19 below) that the generic Θ-splitting field $k(V_\Theta)$ of G is a purely transcendental extension of a certain corresponding Θ_{an}-splitting field of the semisimple anisotropic kernel G_{an} of G (cf. 1.8 below).

To the best of our knowledge the first who had the idea of studying partial generic splitting instead of just total generic splitting was Knebusch in the 70's. He investigated partial generic splitting of quadratic forms [16, 1976], [17, 1977], thereby introducing his generic splitting towers. Then his student Heuser [12, 1976] studied partial generic splitting of central simple algebras, using the function fields of generalized Severi-Brauer varieties of prescribed level. It turned out that the splitting behavior of central simple algebras is much more uniform than that of quadratic forms (cf. 4.8 ii) and 5.8 below). As we were told, Knebusch, puzzled by this phenomenon, suggested already then to study partial generic splitting of linear algebraic groups.

Recently, also Blanchet [4, 1991] and Schofield/v. d. Bergh [26, 1991], [27, 1991] studied the partial generic splitting of central simple algebras by means of generalized Severi-Brauer varieties. As we point out in §4 these generalized Severi-Brauer varieties are the quotients of G by arbitrary *maximal* proper parabolic subgroups of $G = \mathrm{SL}_{r+1}(D)$ (here Θ consists of Δ minus one element).

Similarly, the layers of a generic splitting tower of a quadratic form q are, in our theory, achieved by the function fields of the quotients of $G = SO(q)$ modulo its various maximal proper parabolic subgroups. This is discussed in §5.

Another uniformizing approach which establishes and generalizes the results of Amitsur and Knebusch above and which uses the techniques and terminologies of Jordan algebras and Jordan pairs has been given independently by Petersson [21, 1984] and by Jacobson [14, 1985] (for a survey, see [15, 1989]). Especially [14, §7, p. 591] gives results on generic splitting of involutorial simple associative algebras which can be transformed into special cases of our Theorem 6.1.

The authors want to express their gratitude to Manfred Knebusch who enthusiastically encouraged them to investigate partial generic splitting of algebraic groups.

We now briefly describe the contents of the various sections of this paper.

In §1 we collect some facts about varieties, splitting fields, reductive linear algebraic groups and their anisotropic kernels.

In §2 the generic splitting of algebraic tori is discussed.

In §3 we set up the framework of reductive groups and the rational theory of parabolic subgroups in order to prove the main results: Theorem 3.6 describes how to obtain a generic splitting field from a generic quasi-splitting field, Theorem 3.10 describes the fundamental properties of the varieties V_Θ (becoming rational exactly over specializations of their function fields). 3.11 – 3.17 prove the existence and describe the properties of generic Θ-splitting fields, 3.18 – 3.19 relate the general results to the respective anisotropic kernel. We conclude this chapter with the discussion of Witt's first example (3.20) of generic splitting, namely the generic splitting of quaternion algebras.

In §4 we describe the generic Θ-splitting of groups of type ${}^1A_n$. We show how several essential results on generic splitting fields obtained by Amitsur, Roquette, Heuser, Blanchet and Schofield/v. d. Bergh can be deduced from our theory by taking proper maximal subsets Θ of Δ.

In §§5 and 6 we assume char $(k) \neq 2$. In §5 we discuss the partial and total generic splitting of quadratic forms including its relations to the work of Knebusch. In §6 we describe the generic Θ-splitting of the classical groups of types ${}^2A_n$, B_n, C_n, ${}^1D_n$ and ${}^2D_n$.

In §7 we give, for all characteristics, generic splitting and quasi-splitting fields of arbitrary almost simple groups including the groups of exceptional types.

1. Some basic definitions and lemmas

Let k be field and $\bar{k}$ an algebraic closure of k.

In this paper, a k-variety V is an absolutely reduced quasi-projective scheme over k. Any algebraic k-group is supposed to have a k-variety structure in this sense. For any field extension k' of k, we denote the set of k'-rational points of V by $V(k') = \mathrm{Hom}_{k-\mathrm{scheme}}(\mathrm{Spec}\ k', V)$ and we write $V_{k'} := V \times_k k'$ for the k'-scheme obtained from V by base extension with k'.

1.1 Lemma. *If V is an absolutely irreducible k-variety, then k is algebraically closed in the function field $k(V)$ and $k(V)$ is separably generated over k.*

For the proof see [20, Chap. II, §4, Prop. 4, p. 142].

Recall that a finitely generated field extension k' of k is said to be *regular* if $\bar{k}$ and k' are linearly disjoint or, equivalently, if k is algebraically closed in k', and k' is separably generated over k, cf. [36, Chap. I.7, Th. 5, p. 18]. If V is an absolutely irreducible k-variety, then, by 1.1, for any field extension k' of k, the free composite $k'k(V)$ is uniquely determined up to k-isomorphism [13, Chap. IV, Cor. 1, p. 203, Th. 26, p. 209] and is isomorphic to $k'(V_{k'})$ [36, Chap. I.7, Th. 5, p. 18].

1.2 Definition. A field extension k' of k is a *k-specialization* of an extension L of k if there is a k-place $L \to k' \cup \{\infty\}$.

1.3 Lemma. *Let V be an absolutely irreducible projective k-variety. Let L, k' be field extensions of k such that k' is a k-specialization of L. Then $V(L) \neq \emptyset$ implies $V(k') \neq \emptyset$.*

Proof. There is a homogeneous ideal I in the polynomial ring $k[X_0, \ldots, X_n]$ for a suitable n together with a bijection of sets

$$V(K) \xrightarrow{\sim} \{(x_0 : \ldots : x_n) \in \mathbb{P}^n(K) \mid f(x_0, \ldots, x_n) = 0,\ \forall f \in I\}$$

for every field extension K of k.

Let $\varphi\colon L \to k' \cup \{\infty\}$ be the k-place describing k' as a k-specialization of L, and let $\mathcal{O}_\varphi$ denote the valuation ring of φ.

Let $x = (x_0 : \ldots : x_n) \in V(L)$. We choose $j \in \{0, \ldots, n\}$ such that the principal ideal $x_j\mathcal{O}_\varphi$ is maximal among the ideals $x_i\mathcal{O}_\varphi$. This is possible because $\mathcal{O}_\varphi$ is a valuation ring, cf. [9, Chap. VI, §1, No. 2, Th. 1d].

Clearly, we have $x_j \neq 0$, whence $x = (x'_0 : \ldots : x'_n)$ with $x'_i := x_i/x_j \in \mathcal{O}_\varphi$ for all $i = 0, \ldots, n$ and $x'_j = 1$, and it follows that $(\varphi(x'_0) : \ldots : \varphi(x'_n)) \in V(k')$. □

Let G be a connected affine algebraic k-group. This implies that $G_{\bar{k}}$ is $\bar{k}$-connected [11 I, Exp. VI_2, Prop. 2.1.1, p. 296].

1.4 Theorem. *G has a maximal k-torus.*

For the proof see [11 II, Exp. XIV, Th. 1.1, p. 296] or [6, Th. 18.2, p. 218].

We assume for the rest of this paragraph that G is reductive.

1.5 Definition. Let K be a field extension of k.

i) K is a *splitting field* of G if G_K has a maximal K-torus which splits over K.

ii) K is a *quasi-splitting field* of G if G_K has a Borel subgroup defined over K.

iii) A splitting field (resp. quasi-splitting field) K of G is said to be *generic* if every splitting field (resp. quasi-splitting field) of G is a k-specialization of K.

For the notion of a *split* connected reductive affine algebraic group compare [6, 18.6, 18.7, p. 220ff] and [7, 8.1, 8.2, p. 481ff].

1.6 Remark. Obviously two generic splitting fields of G are k-equivalent to each other in the sense that they are k-specializations of each other. We shall show in 3.9 iii) that every k-specialization of a splitting field of G is also a splitting field of G. So we have the following result: If K and K' are k-equivalent field extensions of k and if one of them is a generic splitting field of G, so is the other. In particular, K is a generic splitting field of G if this is true for some purely transcendental extension $K(\{x_i\}_{i \in I})$ of K.

1.7 Remark.

i) A field extension K of k is a splitting field of G if and only if $\operatorname{rank}(G_K) = \operatorname{rank}(G_{\bar{k}})$ holds. We write *rank*(G) for the k-rank of the k-group G, that is, the dimension of a maximal k-split k-torus of G (cf. [8, 4.21, p. 93]).

ii) If k is finite then G is quasi-split (cf. [6, 16.6, p. 211]), and the semi-simple groups over k are completely classified (cf. [30], [31]). Therefore we will always assume that the base field k is infinite except in §§1 and 2.

iii) It is known that G has a splitting field which is finite and separable over k. This follows from 1.4 and the fact that any k-torus has such a splitting field [6, 8.11, p. 117]. However, if K is a generic quasi-splitting field of G, then K does not split any nontrivial anisotropic k-torus of G (cf. Cor. 3.12 below).

1.8 Definition.

i) G is *isotropic* if it contains a non-trivial k-split k-torus and is *anisotropic* if $\operatorname{rank}(G) = 0$.

ii) If S is a maximal k-split k-torus of G and $\mathcal{Z}(S)$ its centralizer in G, then the derived group $\mathcal{DZ}(S)$ is called a *semisimple anisotropic kernel of* G. If Z_{an} is the maximal anisotropic k-subtorus of the center of $\mathcal{Z}(S)$, then $\mathcal{DZ}(S) \cdot Z_{\text{an}}$ is called a *reductive anisotropic kernel of* G.

Our notion of an (an-)isotropic group seems to be standard now, as it is used in [5, 6.4, p. 13], [32, 2.2, p. 39], [8, 4.23, p. 93] and [6, 20.1, p. 224]. It differs, however, from the definition in [29, 6.5, p. 476], where G is said to be anisotropic if its split k-subtori all are central.

1.9 Proposition.

i) The semisimple anisotropic kernels of G are precisely the subgroups occurring as derived groups of Levi k-subgroups of minimal parabolic k-subgroups of G. Any two such are conjugate under $G(k)$.

ii) The anisotropic kernels of G are anisotropic k-groups.

iii) G is quasi-split if and only if its semisimple anisotropic kernel is trivial.

Proof. i) If S is a maximal k-split k-subtorus of G, then $\mathcal{Z}(S)$ is a Levi k-subgroup of a minimal parabolic k-subgroup P of G by [8, 4.15, 4.16, p. 91]. Conversely, if P is a minimal parabolic k-subgroup, then the Levi k-subgroups of P are the centralizers of maximal k-split k-tori of G (contained in the radical of P), [8, 4.16, p. 91]. This proves the first assertion of i). The second follows from the fact that all maximal k-split k-tori are conjugate over k by [8, 4.21, p. 93] or [6, 20.9, p. 228].

ii) It suffices to prove the statement for semisimple anisotropic kernels. Let S be a maximal k-split k-torus of G. Being reductive, $\mathcal{Z}(S)$ is an almost direct product of its maximal semisimple subgroup $\mathcal{DZ}(S)$ and the identity component of its center which contains S (cf. [8, 2.2, p. 64]). The maximality of S now asserts that $\mathcal{DZ}(S)$ does not contain any nontrivial k-split k-torus.

iii) By definition, G is quasi-split if and only if it contains a Borel k-subgroup. Hence the statement follows from i). □

1.10 Corollary (cf. [8, 4.17, p. 92])**.** *G contains a non-central k-split k-torus if and only if it contains a proper parabolic k-subgroup.*

2. Generic splitting of algebraic tori

Let k be a field and let T be an algebraic k-torus (i.e., there is a field extension K of k such that $T_K \cong \mathbb{G}_m \times_K \cdots \times_K \mathbb{G}_m$, where the multiplicative K-group $\mathbb{G}_m$ is defined by $\mathbb{G}_m(K) = K^*$).

2.1 Lemma. *Let L be a field extension of k which splits T. Then the subfield $\widetilde{k}$ of L of elements which are separable algebraic over k also splits T.*

Proof. Let $\varrho\colon T \to \mathrm{GL}(W)$ be a faithful k-rational representation on a finite dimensional linear k-space W. By assumption there is an L-basis of $W \otimes_k L$ such that every $t \in T(L)$ is described by a diagonal matrix with respect to this basis [6, 8.2 Prop. (d), p. 112]. Hence, for every $t \in T(\widetilde{k})$, the minimal polynomial $m_t(X) \in \widetilde{k}[X]$ decomposes into pairwise distinct linear factors

$$m_t(X) = \prod_i (X - \alpha_i^{(t)}), \text{ with } \alpha_i^{(t)} \in L.$$

It follows that the $\alpha_i^{(t)}$ are separable over $\widetilde{k}$, hence over k. Therefore every $t \in T(\widetilde{k})$ is diagonizable over $\widetilde{k}$. Since $T(\widetilde{k})$ is commutative, there is a $\widetilde{k}$-basis of $W \otimes_k \widetilde{k}$ which diagonalizes $T(\widetilde{k})$. □

2.2 Proposition. *Let L be a splitting field of T. Then any k-specialization k' of L is a splitting field of T.*

Proof. Let $\varphi: L \to k' \cup \{\infty\}$ be the place describing k' as a k-specialization of L. By 2.1, the subfield $\widetilde{k}$ of separable algebraic elements of L over k splits T. By [38, Cor. 1, p. 13], the restriction of φ to $\widetilde{k}$ is injective, hence k' splits T. □

2.3 Theorem. *An algebraic k-torus T has an algebraic generic splitting field, say F, which is unique up to k-isomorphism and is a finite Galois extension of k. Every splitting field of T contains a subfield isomorphic to F.*

Proof. Let $\bar{k}$ denote an algebraic closure of k, and define

$$F := \bigcap \{\ L \mid k \subseteq L \subseteq \bar{k}, L \text{ splitting field of } T\}.$$

Then F is a finite separable field extension of k because T has a finite separable splitting field (cf. [6, 8.11, p. 117], [7, Cor. 8.3, p. 482]).

Let K be a splitting field of T. By 2.1, the subfield $\widetilde{k} \subseteq K$ of elements separable over k splits T. Then $\widetilde{k}$ contains a subfield isomorphic to F by the definition of F.

We have to show that F splits T. Then it will follow from the above that F is a generic splitting field of T. Let $A = k[T]$ be the affine coordinate ring of T. Let k_1 be any extension of k. Then the set $\mathcal{X}(T_{k_1})$ of characters of T defined over k_1 are the k_1-group homomorphisms $T_{k_1} \to \mathbb{G}_m$. These are in one-to-one correspondence with the set of k_1-algebra homomorphisms $k_1[X, X^{-1}] \to A \otimes_k k_1$ with an indeterminate X or equivalently, using restrictions, to the k-algebra homomorphisms $k[X, X^{-1}] \to A \otimes_k k_1$. Let now k_1, k_2 be extensions of k both contained in a field k_3. Then any character defined over k_3 which, by restriction, gives a character defined over both k_1 and k_2, will also give a character over $k_1 \cap k_2$, as its associated k-homomorphisms $k[X, X^{-1}] \to A \otimes_k k_i$ will map X into both $A \otimes_k k_i$ for $i = 1, 2$ and hence into $A \otimes_k (k_1 \cap k_2)$.

Now since any extension $L \subseteq \bar{k}$ of k is a splitting field of T if and only if $\mathcal{X}(T_L) = \mathcal{X}(T_{\bar{k}})$ [6, 8.2, Cor., p. 112], it follows that the intersection of two splitting fields of T which are contained in $\bar{k}$ is also a splitting field of T. Hence F splits T. Since the same then is true for all the conjugates of F it follows from the definition of F that it is a Galois extension of k. □

2.4 Example. Let $\mathrm{char}(k) \neq 2$ and $a \in k^*$. Define a k-torus by

$$T(k) := \left\{ \begin{pmatrix} \alpha & \beta \\ a\beta & \alpha \end{pmatrix} \in \mathrm{M}_2(k) \mid \alpha^2 - a\beta^2 = 1 \right\}.$$

Then it is easily checked that T splits over some field extension K of k if and only if $a \in (K^*)^2$ and that $k(\sqrt{a})$ is a splitting field of T. In fact, it is just the generic splitting field F of T described in 2.3.

Let G be a connected affine algebraic k-group.

2.5 Corollary. *Suppose G is reductive and quasi-split, and T is a maximal k-torus of G contained in a Borel k-subgroup B of G. Then the generic splitting field F of T is a generic splitting field of G, and every splitting field of G contains a field isomorphic to F.*

Proof. Clearly F is a splitting field of G, since it splits one of its maximal tori. Let k' be a splitting field of G. Then $G_{k'}$ contains a maximal k'-torus which splits. This is contained in some Borel k'-subgroup of $G_{k'}$ and since this is conjugate in $G_{k'}$ to $B_{k'}$, the torus $T_{k'}$ splits. Hence k' is a splitting field of T and therefore contains F, by 2.3. This clearly implies that F is a generic splitting field of G. □

3. Parabolic subgroups

In this section k is an infinite field, k_s is the separable closure in an algebraic closure $\bar{k}$ of k, and G is a connected reductive affine algebraic k-group.

Let K be a splitting field of G (for example $K = \bar{k}$). Choose a maximal K-torus T of G_K which splits over K (cf. Definition 1.5). We denote by $\mathcal{X}(T)$ the character group $\mathrm{Hom}\,(T, \mathbb{G}_m)$.

Let $\Phi_K = \Phi(G_K, T) \subseteq \mathcal{X}(T)$ be the set of roots of G_K with respect to T. For every $\alpha \in \Phi_K$ there is a connected unipotent subgroup U_α of G_K such that $TU_\alpha = U_\alpha T$. Also, there is a K-isomorphism $x_\alpha: \mathbb{G}_a \to U_\alpha$, where the additive K-group $\mathbb{G}_a$ is defined by $\mathbb{G}_a(K) = K^+)$, such that

$$t\, x_\alpha(u)\, t^{-1} = x_\alpha(t^\alpha\, u) \quad (\forall u \in K,\ t \in T(K))$$

(cf. [8, 2.3, p. 64] or [6, 18.6, p. 221]).

We choose an ordering of Φ_K, denote the set of positive roots by Φ_K^+, and let $\Delta_K \subset \Phi_K^+$ be the basis (or the set of simple roots) of Φ_K with respect to that ordering.

For every subset $\Theta \subseteq \Delta_K$ we have the so-called standard parabolic subgroup P_Θ of G_K (with respect to T) defined by

$$P_\Theta := \langle T, U_\alpha \mid \alpha \in \Delta_K \text{ or } -\alpha \in \Theta \rangle.$$

It is known that the standard parabolic subgroups are in one-to-one correspondence with the conjugacy classes of parabolic subgroups of G_K [8, 4.6, p. 87]. Obviously we have $P_{\Delta_K} = G_K$, and $B := P_\emptyset$ is the standard Borel subgroup of G_K. More generally, there is the following description of P_Θ, cf. [8, 4.2, p. 85f] or [6, Prop. 14.18, p. 197].

3.1 Remark. We denote by H° the connected component of the identity element in an algebraic group H. Let $T_\Theta = \left(\bigcap_{\alpha\in\Theta} \operatorname{Ker}(\alpha)\right)^\circ$, let $\mathcal{Z}(T_\Theta)$ be its centralizer in G_K and $U_\Theta = \langle U_\alpha \mid \alpha \in u_\Theta \rangle$ where u_Θ is the set of all $\alpha \in \Phi_K^+$ which are not linear combinations of elements of Θ. Then $P_\Theta = \mathcal{Z}(T_\Theta)U_\Theta$ is the Levi decomposition of P_Θ with reductive part $\mathcal{Z}(T_\Theta)$ and unipotent radical $\mathcal{R}_u(P_\Theta) = U_\Theta$. If $U_\Theta^- = \langle U_\alpha \mid \alpha \in u_\Theta^- \rangle$ where u_Θ^- is the set of all $\alpha \in \Phi_K \setminus \Phi_K^+$ which are not linear combinations of elements of Θ, then $P_\Theta^- = \mathcal{Z}(T_\Theta)U_\Theta^-$ is, analogously, the Levi decomposition of the parabolic subgroup of G_K containig T which is opposite to P_Θ.

3.2 Lemma. *Let P be a parabolic k-subgroup of G. Then the unipotent radical $\mathcal{R}_u(P)$ is, as a k-variety, isomorphic to an affine k-space, and G/P is a rational variety over k. If P_K is conjugate to P_Θ for $\Theta \subseteq \Delta_K$, then the dimension of G/P equals that of $\mathcal{R}_u(P)$ which is given by the number of elements of u_Θ.*

Proof. Let P^- be a parabolic k-subgroup of G which is opposite to P. From [6, 14.21 (iii), p. 198f] we deduce that the product map in G induces a $\bar{k}$-rational map $\mathcal{R}_u(P^-)_{\bar{k}} \times_{\bar{k}} P_{\bar{k}}$ onto a $\bar{k}$-open subvariety of $G_{\bar{k}}$. By [6, 21.11, p. 233f and 21.20, p. 240] we find that $\mathcal{R}_u(P)$ and $\mathcal{R}_u(P^-)$ are affine k-spaces, and that G/P is a rational k-variety. It follows that dim $G/P =$ dim $\mathcal{R}_u(P^-)$ which equals the cardinality of u_Θ^- and hence of u_Θ. □

Let Δ denote the set of vertices of the Dynkin diagram of $G_{\bar{k}}$ and let $\iota = \iota_K : \Delta \to \Delta_K$ denote the natural one-to-one correspondence.

3.3 Definition. Let k' be a field extension of k contained in K and $\Theta \subseteq \Delta$. A parabolic subgroup P of $G_{k'}$ is said to be of *type* Θ if P_K is conjugate to $P_{\iota(\Theta)}$ in G_K.

3.4 Remark.

i) The type of a parabolic subgroup is independent of the choice of the splitting field K. To see this, let K_1 be another splitting field of G which contains k'. Then any free composite $\widetilde{K}$ of K, K_1 over k' is a splitting field of G as well. Hence, if T_1 is a maximal split K_1-torus of G_{K_1}, then $T_{\widetilde{K}}$ and $(T_1)_{\widetilde{K}}$ are conjugate over $\widetilde{K}$ by [8, Th. 4.21, p. 93]. This conjugation induces an isomorphism $i: \mathcal{X}(T_K) \to \mathcal{X}((T_1)_{K_1})$. Hence we obtain an ordered root system $\Phi(G_{K_1}, T_1)$ as the image of Φ_K under i, with basis $i(\Delta_K)$ as a set of simple roots of G_{K_1} with respect to T_1, and we have $\iota_{K_1} = i \circ \iota_K$.

ii) For $i = 1, 2$, let k_i be two field extensions of k, and let P_i be parabolic subgroups of G_{k_i}. Then P_1, P_2 are of the same type if and only if they are conjugate over some free composite of k_1, k_2. This follows from i) by using splitting field extensions $K_i \supseteq k_i$ of G and from [8, Th. 4.13 c), p. 90].

iii) Because of i), we will henceforth identify Δ_K with Δ. By ii), there is a one-to-one correspondence of the subsets $\Theta \subseteq \Delta$ and the conjugacy classes of parabolic subgroups of G_K for any splitting field K.

iv) Following [32, 2.3, p. 39] we define the $*$-action of the Galois group $\Gamma = \operatorname{Gal}(k_s/k)$ on $\Delta = \Delta_{k_s}$ as follows. As G splits over k_s, parabolic subgroups of every type are defined over k_s and hence Γ operates on the set of their conjugacy classes. Via iii) we get an induced action on Δ, if we restrict this operation to the conjugacy classes of maximal parabolic subgroups of G_{k_s} which are in obvious one-to-one correspondence with the elements of Δ: The element corresponding to P_Θ is the unique one in $\Delta \setminus \Theta$. This gives the wanted $*$-action. The permutation of Δ corresponding to $\gamma \in \Gamma$ will be denoted by γ^*. The group G is of *inner type* if the $*$-action is trivial on Δ and of *outer type* otherwise.

v) Let S denote a maximal split k-subtorus of G contained in a maximal k-torus T of G. By i) we may assume that $\Phi = \Phi(G_{k_s}, T_{k_s})$. The set of roots of Δ which vanish on S is usually denoted by

Δ_0, and the set of nontrivial restrictions of elements of Δ to S is ${}_k\Delta$. Hence we have the restriction map $\mathrm{res}_k : \Delta \to {}_k\Delta \cup \{0\}$ with $\mathrm{res}_k^{-1}({}_k\Delta) = \Delta \setminus \Delta_0$. The set Δ_0 is the set of simple roots of $\mathcal{DZ}(S)$. By 1.9 iii) G is quasi-split if and only if $\mathcal{DZ}(S)$ is trivial, which obviously is equivalent to $\Delta_0 = \emptyset$. On the other hand, the pre-images of single elements of ${}_k\Delta$ under res_k are precisely the equivalence classes of elements of $\Delta \setminus \Delta_0$ under the $*$-action of Γ [32, 2.5.1, p. 40]. Hence we can conclude: If G is of inner type, then the map res_k, restricted to $\Delta \setminus \Delta_0$, is injective. Moreover, if G is of inner type and quasi-split, $\mathrm{res}_k : \Delta \to {}_k\Delta$ is a bijection, hence the derived group $\mathcal{D}(G)$ of G is split.

3.5 Lemma. *There is a finite Galois extension k_{inn} of k which is unique up to k-isomorphism with the following properties:*

i) The group $G_{k_{\mathrm{inn}}}$ is of inner type.

ii) Every field extension k' of k such that $G_{k'}$ is of inner type contains a subfield isomorphic to k_{inn}.

Proof. Clearly the subgroup $\Gamma' = \{ \gamma \in \Gamma \mid \gamma^* = \mathrm{id} \}$ is normal of finite index in $\Gamma = \mathrm{Gal}\,(k_s/k)$. Hence its fixed field k_{inn} is a finite Galois extension of k such that $G_{k_{\mathrm{inn}}}$ is of inner type. Let k'_s be a separable closure of k' containing k_s. If $G_{k'}$ is of inner type, then the $*$-action of the Galois group $\mathrm{Gal}\,(k'_s/k')$ on Δ is trivial. Hence (cf. 3.4 iii), iv)) the $*$-action of $\mathrm{Gal}\,(k'k_s/k') \cong \mathrm{Gal}\,(k_s/(k' \cap k_s))$ is trivial as well, which implies $k_{\mathrm{inn}} \subset k' \cap k_s$. □

3.6 Theorem. *Let k_{alg} be the composite in k_s of k_{inn} and the generic splitting field of the maximal central torus of G (cf. 2.3).*

i) The free composite of k_{alg} and a generic quasi-splitting field of G is a generic splitting field of G.

ii) Any splitting field of G contains a subfield k-isomorphic to k_{alg}.

Proof. Let F be a field obtained from an generic quasi-splitting field L of G as in i). Since $G_{k_{\mathrm{alg}}}$ is, by 3.5, of inner type and since it has a split maximal central torus, it follows from 3.4 v) that F is a splitting field of $G_{k_{\mathrm{alg}}}$ and hence of G. Let k' be a splitting field of G. Then there is a k- place $\varphi : L \to k' \cup \{\infty\}$. Since k_{alg} is algebraic over k and since k_{alg} is contained in k' by 3.5 ii) and 2.3 we have a trivial k_{alg}-place $k_{\mathrm{alg}} \to k' \cup \{\infty\}$ [38, Chap. VI.4, p. 13]. Thus φ can be extended to a k_{alg}-place $F = L \cdot k_{\mathrm{alg}} \to k' \cup \{\infty\}$. This implies that F is generic and also proves ii). □

3.7 Lemma.

(1) If $\Theta \subseteq \Delta$ is $*$-invariant, then there is a unique projective k-variety V_Θ with the following property: For any field extension k' of k and any parabolic subgroup P' of $G_{k'}$ of type Θ one has $V_\Theta \times_k k' \cong G_{k'}/P'$.

(2) For arbitrary $\Theta \subseteq \Delta$ the following conditions i) – iii) are equivalent.

i) There is a parabolic subgroup of G of type Θ.

ii) Θ is $*$-invariant and $V_\Theta(k) \neq \emptyset$.

iii) Θ is $*$-invariant and $\Theta \supseteq \Delta_0 = \{\alpha \in \Delta \mid \mathrm{res}_k(\alpha) = 0\}$ (cf. 3.4 v)).

Proof. (1) Let V denote the k_s-variety given by the conjugacy class of parabolic subgroups of G_{k_s} of type Θ. By [8, 6.2 (3), p. 104], Θ is $*$-invariant if and only if $V(k_s)$ is Γ-stable. By [7, 8.4, p. 482], the Γ-stability of $V(k_s)$ implies that V is defined over k. For any field extension k' of k, let $\bar{k'}$ denote an algebraic closure of k'. Then, by [7, 7.2 (b), (i), p. 474], the set $M = V(\bar{k'})$ is a homogeneous (G,k)-set represented by a k-variety V_Θ . Hence M is a homogeneous $(G_{k'},k')$-set represented by the k'-variety $V_\Theta \times_k k'$ (cf. [7, 7.3, p. 475]). Let now P' be a parabolic subgroup of $G_{k'}$ of type Θ. Then $V_\Theta \times_k k' \cong G_{k'}/P'$, since parabolic subgroups are self-normalizing. The uniqueness of V_Θ now follows from [7, 7.5 (i), p. 475].

(2) If i) holds then clearly Θ is $*$-invariant. Hence to prove the equivalence of i), ii), iii) we may assume the $*$-invariance of Θ. For any field extension k' of k, the set $V_\Theta(k')$ is the set of parabolic subgroups in $G_{k'}$ of type Θ. This follows from [7, Prop. 7.6, p. 476] applied to the homogeneous (G,k)-set M above. Therefore i) is equivalent to ii). The equivalence of ii) and iii) follows from [8, 6.3 (1), p. 105] and [8, 6.8, p. 107]. □

3.8 Corollary. *G is quasi-split if and only if G contains a parabolic k-subgroup of type Θ for every $*$-invariant subset Θ of Δ, and G is split if and only if it contains parabolic subgroups of every type and its maximal central torus splits.*

Proof. We recall from 3.4 v) that G is quasi-split if and only if $\Delta_0 = \emptyset$. Hence the equivalence of i) and iii) in 3.7 says that every parabolic subgroup of $*$-invariant type occurs in the quasi-split case. The converse is trivial, as $\emptyset$ is $*$-invariant and the type of a Borel subgroup.

If G is split, then its maximal central torus splits and the $*$-action is trivial. It follows by the above that G has parabolic subgroups of every type. Conversely, if this is true, then certainly the $*$-action is trivial, G is quasi-split and therefore split by 3.4 v) if its maximal central torus splits. □

Remark. Let $\Theta \subseteq \Delta$ be $*$-invariant. For a parabolic subgroup P of G_K of type Θ the quotient G_K/P is a projective irreducible K-variety which defines, by 3.7, a k-variety V_Θ such that $V_\Theta \times_k K \cong G_K/P$. We will say that G_K/P is defined over k in spite of the fact that P is not necessarily defined over k. Note that V_Θ does not depend on the choice of the splitting field K.

3.9 Corollary. *Let k' and L be two field extensions of k such that k' is a k-specialization of L. If P is a parabolic subgroup of G_L, then there is a parabolic subgroup of $G_{k'}$ of the same type as P. Moreover we have the following:*

i) $\operatorname{rank}(G_{k'}) \geq \operatorname{rank}(G_L)$.

ii) Anisotropic reductive k-groups remain anisotropic under purely transcendental extensions of k.

iii) Every k-specialization of a splitting field of G is a splitting field of G.

Proof. By assumption, we have a k-place $\varphi : L \to k' \cup \{\infty\}$, and $V(L) \neq \emptyset$ with V being the quotient L-variety G_L/P. Clearly the type Θ of P is $*$-invariant with respect to the action of $\operatorname{Gal}(L_s/L)$. Since $\operatorname{Gal}(Lk_{\text{inn}}/L) \cong \operatorname{Gal}(k_{\text{inn}}/(L \cap k_{\text{inn}}))$, we find that Θ is $*$-invariant with respect to $\operatorname{Gal}(k_s/k_1)$, where $k_1 := L \cap k_{\text{inn}}$. Therefore V is defined over k_1 by 3.7. The field k_1 is finite separable over k and consequently it is mapped isomorphically by φ onto a subfield of k'. Identifying this subfield with k_1 we obtain that φ is a k_1-place, hence k' is a k_1-specialization of L. By 1.3 we obtain that $V(k') \neq \emptyset$, hence $G_{k'}$ contains a parabolic subgroup of type Θ.

We now prove i). Since G is an almost direct product of its maximal semisimple subgroup $\mathcal{D}(G)$ and a torus [8, 2.2, p. 63], it suffices to prove that assertion for tori and for semisimple groups. The statement for tori follows from 2.2 by induction on the dimension. If G is semisimple and P a minimal parabolic subgroup of G_L we find by the above that $G_{k'}$ contains a parabolic subgroup of the type of P and hence a k'-split k'-torus whose rank equals $\operatorname{rank}(G_L)$ (cf. [6, 20.6, p. 225]) which proves the inequality.

ii) and iii) are immediate consequences of i). □

3.10 Theorem. *Let $\Theta \subseteq \Delta$ be $*$-invariant and let $V := V_\Theta$ denote the corresponding k-variety as described in 3.7. Then the function field $k(V)$ is a regular extension of k, and for every field extension k' of k the following statements are equivalent:*

i) $V(k') \neq \emptyset$.

ii) The free composite of k' and $k(V)$ is a purely transcendental extension of k'.

ii′) There is a k-linear embedding $k(V) \hookrightarrow k'(X_1, \ldots, X_m)$ of $k(V)$ into a finitely generated purely transcendental extension of k'.

iii) k' is a k-specialization of $k(V)$.

Proof. It follows from 1.1 that $k(V)$ is regular.

"i) ⇒ ii)": If $V(k') \neq \emptyset$, then there is a parabolic subgroup Q of $G_{k'}$ of type Θ by 3.7. By 3.2 the k'-variety $V_{k'} \cong G_{k'}/Q$ is rational over k' which implies ii).

Clearly ii) implies ii′).

"ii′) ⇒ iii)": There is a k'-place $k'(X_1, \ldots, X_m) \to k' \cup \{\infty\}$, whose restriction to $k(V)$ gives a k-place $k(V) \to k' \cup \{\infty\}$, hence iii).

"iii) ⇒ i)": $k(V)$ is the residue field at the generic point of V and hence $V(k(V)) \neq \emptyset$ (cf. [20, Chap. II, §6, p. 161]). The assertion i) now follows from 1.3. □

3.11 Theorem. *The function field $F := k(V_\emptyset)$ is a generic quasi-splitting field of G. If G is semisimple of inner type, then F is a generic splitting field of G.*

Proof. Clearly $\emptyset \subseteq \Delta$ is $*$-invariant, and $\emptyset$ is the type of a Borel subgroup B of G_K. Hence $V_\emptyset$ is a k-variety such that $G_K/B \cong V_\emptyset \times_k K$ by 3.7. Since $V_\emptyset(F) \neq \emptyset$, the field F is a quasi-splitting field of G. If k' is a quasi-splitting field of G, then $V_\emptyset(k') \neq \emptyset$ (by 1.5 ii)), hence k' is a k-specialization of F by 3.10. Consequently F is a generic quasi-splitting field of G. By 3.4 v), every quasi-splitting field of an inner type semisimple group is a splitting field of that group. □

3.12 Corollary. *Let L be a generic quasi-splitting field of G. Then k is algebraically closed in L, and L does not split any nontrivial anisotropic k-torus of G.*

Proof. Let F be the generic quasi-splitting field of G as defined in 3.11. If L is a generic quasi-splitting of G, then there is a k-place $L \to F \cup \{\infty\}$ by Definition 1.5 iii). Since any algebraic extension of k in

L possesses only trivial k-places (cf. [38, Chap. VI, §4, p. 13]) and since k is algebraically closed in F by 1.1, it follows that k is algebraically closed in L. If L splits some k-torus T of G, then, by 2.1, the k-algebraic elements of L already form a splitting field of T. □

We now generalize the notion of a quasi-splitting field.

3.13 Definition. Let F be a field extension of k and let $\Theta \subseteq \Delta$.

i) F is a Θ-*splitting field* of G if G_F contains a parabolic subgroup of type Θ.

ii) A Θ-splitting field F of G is said to be *generic*, if every Θ-splitting field of G is a k-specialization of F.

3.14 Remark. It follows from 3.9, that every k-specialization of a Θ-splitting field of G is a Θ-splitting field of G.

3.15 Examples.

i) A (generic) quasi-splitting field is a (generic) $\emptyset$-splitting field.

ii) The field k is a generic Δ-splitting field of G.

iii) For any $*$-invariant subset $\Theta \subseteq \Delta$ the function field $k(V_\Theta)$ with V_Θ as in 3.7 is a generic Θ-splitting field of G as follows from 3.10.

3.16 Theorem. *Let $\Theta \subseteq \Delta$ be any subset. Then there is a finite separable field extension k_Θ of k, contained in the field k_{inn} of 3.5, with the following properties:*

i) Every Θ-splitting field of G contains a subfield isomorphic to k_Θ.

ii) Θ is invariant with respect to the $*$-action of the Galois group $\mathrm{Gal}\,(k_s/k_\Theta)$.

iii) If V_Θ denotes the k_Θ-variety defined in 3.7, then the function field $F_\Theta := k_\Theta(V_\Theta)$ is a generic Θ-splitting field of G.

iv) The field F_Θ is regular over k if and only if $k = k_\Theta$, hence if and only if Θ is invariant with respect to the $*$-action of $\mathrm{Gal}\,(k_s/k)$.

Proof. Let $\Gamma_\Theta = \{\gamma \in \mathrm{Gal}\,(k_s/k) \mid \gamma^*(\Theta) = \Theta\}$ and let k_Θ be its fixed field. Since k_{inn} is the fixed field of $\Gamma' = \{\gamma \in \mathrm{Gal}\,(k_s/k) \mid \gamma^* = \mathrm{id}\}$ and $\Gamma' \subseteq \Gamma_\Theta$ it follows that $k_\Theta \subseteq k_{\mathrm{inn}}$.

i) Let k' be a Θ-splitting field of G. Let k'_s be a separable closure of k' containing k_s. By assumption $\mathrm{Gal}\,(k'_s/k')$ leaves Θ $*$-invariant. Hence $\mathrm{Gal}\,(k'k_s/k') \cong \mathrm{Gal}\,(k_s/(k' \cap k_s))$ leaves Θ $*$-invariant, which implies $k_\Theta \subseteq k' \cap k_s$.

ii) This follows from the construction of k_Θ.

iii) Since $V_\Theta(F_\Theta) \neq \emptyset$, Lemma 3.7 implies that F_Θ is a Θ-splitting field of G_{k_Θ} and hence of G. Let k' be a Θ-splitting field of G. By i) we may assume that k' is a field extension of k_Θ. Thus 3.10, "i) $\Rightarrow$ iii)", implies that k' is a k_Θ-specialization of F_Θ, hence also a k-specialization of F_Θ. This proves iii).

iv) By 1.1, the field F_Θ is regular over k_Θ. Since k_Θ is algebraic over k, the first statement follows. If $k = k_\Theta$, then Θ is invariant with respect to the $*$-action of $\mathrm{Gal}\,(k_s/k)$ by ii). If the latter is true, then, by 3.7, the variety V_Θ is defined over k, and the function field $k(V_\Theta)$ is a Θ-splitting field of G. By i) it contains k_Θ, which implies $k = k_\Theta$, since $k(V_\Theta)$ is regular over k. □

The following corollary illustrates the functorial behavior of the map $\Theta \mapsto F_\Theta$.

3.17 Corollary. *Let $\Theta' \subseteq \Theta$ be a pair of subsets of Δ. Then the following is true.*

i) If k' is a Θ'-splitting field of G, then $k_\Theta k'$ is a k_Θ-specialization of F_Θ.

ii) If F_Θ is a Θ'-splitting field of G and if $k_\Theta \subseteq k_{\Theta'}$, then F_Θ is a generic Θ'-splitting field of G.

iii) If Θ is $*$-invariant and if F_Θ is a quasi-splitting field of G, then it is a generic quasi-splitting field of G.

iv) If F_Θ is a splitting field of G, then it is a generic splitting field of G.

Remark. The assumption that Θ is $*$-invariant in 3.17 iii) is necessary. See the example after 5.4.

Proof of 3.17. i) After replacing k by k_Θ we may assume that Θ is $*$-invariant, hence $k_\Theta = k$. By 3.7, "i) $\Rightarrow$ iii)", we have $\Delta_0 := \{\alpha \in \Delta \mid \mathrm{res}_{k'}(\alpha) = 0\} \subseteq \Theta'$. Since $\Theta' \subseteq \Theta$, it follows that k' is a Θ-splitting field of G by 3.7, "iii) $\Rightarrow$ i)". Thus k' is a k-specialization of F_Θ by 3.15 iii).

ii) Let k' be a Θ'-splitting field of G. Then $k_\Theta \subseteq k_{\Theta'} \subseteq k'$ by assumption and 3.16 i). So i) implies that k' is a k-specialization of F_Θ.

iii) This follows from ii) and 3.16 iv) for $\Theta' = \emptyset$.

iv) Let k' be a splitting field of G. By 3.6 ii), k' and especially F_Θ both contain a copy of the Galois extension k_{alg} of k. Replacing k by k_{alg} we may assume that G is of inner type by 3.5. Hence we may apply iii) to find that k' is a k-specialization of F_Θ which yields iv). □

It seems to be natural to expect that Θ-splitting of any group G can be achieved by the corresponding Θ_{an}-splitting of the anisotropic kernel G_{an} of G, where Θ_{an} is the appropriate set of vertices of the Dynkin diagram of $(G_{\mathrm{an}})_{\bar{k}}$. The precise meaning of this statement is given in 3.18, 3.19. It essentially is reflected by the fact that the generic Θ-splitting field of G is a purely transcendental extension of the corresponding Θ_{an}-splitting field of G_{an}, hence these two fields are obviously equivalent in the sense that they are k-specializations of each other. The reason for this is that G_{an} is given – up to a torus part, cf. 1.9 – by a Levi-subgroup of a minimal parabolic subgroup Q of G and that G/Q is a rational k-variety. This explains and generalizes an observation made by Roquette [23, Th. 4, p. 413], which will be discussed in 4.10 below.

3.18 Theorem. *Let Q be a parabolic k-subgroup of G of type $\Delta' \subseteq \Delta$ and let $\mathcal{L}$ be a Levi k-subgroup of Q. Consider Δ' as a root basis of $\mathcal{L}_{k_s}$. Let Θ be a $*$-invariant subset of Δ. Let V (resp. V') denote the projective k-varieties associated to Θ (with respect to G) (resp. to Θ' (with respect to $\mathcal{L}$)) according to 3.7. Then $k(V)$ is isomorphic to a purely transcendental extension of $k(V')$.*

Proof. In $\mathcal{L}$ we choose a maximal k-torus T of G containing a maximal k-split k-torus S of G, such that $\mathcal{Z}(S) \subseteq \mathcal{L}$ (cf. [6, 20.6, p. 225]). In G_{k_s} we choose a Borel subgroup B such that $T_{k_s} \subset B \subseteq Q_{k_s}$ and a parabolic subgroup $P \subseteq G_{k_s}$ of type Θ with $B \subseteq P$. We identify Δ with the basis of a root system $\Phi(G_{k_s}, T_{k_s})$ such that the parabolic subgroups of G_{k_s} containing B are those which are in standard position. Hence especially Q_{k_s} and P are standard parabolic subgroups of G_{k_s}. By [6, 21.13, p. 235], the subgroup $B_{\mathcal{L}} := B \cap \mathcal{L}_{k_s}$ is a Borel subgroup of $\mathcal{L}_{k_s}$ which we will use to define the standard position of parabolic subgroups of $\mathcal{L}_{k_s}$.

Since Q is defined over k, its type Δ' and hence also Θ' is $*$-invariant. $P \cap \mathcal{L}_{k_s}$ is a parabolic subgroup of $\mathcal{L}_{k_s}$ by [6, 21.13, p. 235]. It is obviously in standard position. From [8, 5.20, p. 102] it follows that its type is Θ'. We have $V_{k_s} := V \times_k k_s \cong G_{k_s}/P$ and $V'_{k_s} := V' \times_k k_s \cong \mathcal{L}_{k_s}/(P \cap \mathcal{L}_{k_s}) \cong Q_{k_s}/(P \cap Q_{k_s})$. The k-embedding $\mathcal{L} \to G$ induces a k_s-embedding $\iota : V'_{k_s} \to V_{k_s}$ by $g(P \cap \mathcal{L}_{k_s}) \mapsto gP$ for $g \in \mathcal{L}(k_s)$. By construction, $P = P_\Theta$ in G_{k_s} and $P \cap \mathcal{L}_{k_s} = P_{\Theta'}$ in $\mathcal{L}_{k_s}$.

We show that ι is $\mathrm{Gal}\,(k_s/k)$-equivariant: For $\sigma \in \mathrm{Gal}\,(k_s/k)$ there is a unique w_σ in the Weyl group of $\Phi(G_{k_s}, T_{k_s})$ such that $w_\sigma \sigma(\Delta) = \Delta$. For any root $\alpha \in \Delta$ we then have $w_\sigma \sigma(\alpha) = \sigma^*(\alpha)$ (cf. [32, 2.3, p. 39]). Let n_σ be a representative of w_σ in the normalizer of $T(k_s)$ in $G(k_s)$. The orders defined on $\mathcal{X}(T_{k_s})$ by Δ and by $\sigma(\Delta)$ induce the same order on $\mathcal{X}(S_{k_s})$. Therefore it follows from [8, 6.6, p. 107] that $n_\sigma \in \mathcal{Z}(S)(k_s) \subset \mathcal{L}(k_s)$. Thus ${}^\sigma P_\Theta = n_\sigma P_{\sigma^*(\Theta)} n_\sigma^{-1}$ and ${}^\sigma P_{\Theta'} = n_\sigma P_{\sigma^*(\Theta')} n_\sigma^{-1}$. Since the conjugacy class gPg^{-1} identifies with the coset gP in V and similarly for V' we obtain by the $*$-invariance of Θ and Θ' that $\iota({}^\sigma(gP_{\Theta'})) = \iota({}^\sigma g\, n_\sigma P_{\Theta'}) = {}^\sigma g\, n_\sigma P_\Theta = {}^\sigma g\, {}^\sigma P_\Theta = {}^\sigma(gP_\Theta) = {}^\sigma(\iota(gP_{\Theta'}))$ for any $g \in \mathcal{L}(k_s)$ which proves the $\mathrm{Gal}\,(k_s/k)$-equivariance of ι.

Hence ι is defined over k [6, AG.14.3, p. 31], i. e., it is obtained by base extension from a k-embedding $\iota : V' \to V$. Let Q^- and P^- denote parabolic subgroups of G (resp. G_{k_s}) which are opposite to Q and P.

By [6, 14.21, p. 198], the product maps

$$\mathcal{R}_u(P^-) \times_{k_s} P \to G_{k_s}, \qquad (\mathcal{R}_u(P^-) \cap \mathcal{L}_{k_s}) \times_{k_s} (P \cap \mathcal{L}_{k_s}) \to \mathcal{L}_{k_s}$$

induce k_s-isomorphisms of their pre-images onto open dense subvarieties of G_{k_s} and $\mathcal{L}_{k_s}$. Hence we obtain morphisms of k_s-varieties

$$\mathcal{R}_u(P^-) \to V_{k_s}, \qquad \mathcal{R}_u(P^-) \cap \mathcal{L}_{k_s} \to V'_{k_s}$$

which are k_s-isomorphisms of their pre-images onto open dense subvarieties of V_{k_s} and V'_{k_s}. Also the product map

$$(\mathcal{R}_u(P^-) \cap \mathcal{R}_u(Q^-_{k_s})) \times_{k_s} (\mathcal{R}_u(P^-) \cap \mathcal{L}_{k_s}) \to \mathcal{R}_u(P^-)$$

is an isomorphism of k_s-varieties which can be seen as follows: By 3.1, we have

$$\begin{aligned} \mathcal{R}_u(P^-) &= \langle\, U_\alpha \mid \alpha \in u_\Theta^- \,\rangle, \\ \mathcal{R}_u(P^-) \cap \mathcal{L}_{k_s} &= \langle\, U_\alpha \mid \alpha \in u_\Theta^- \cap \langle \Delta' \rangle \,\rangle, \\ \mathcal{R}_u(P^-) \cap \mathcal{R}_u(Q^-_{k_s}) &= \langle\, U_\alpha \mid \alpha \in u_\Theta^- \setminus \langle \Delta' \rangle \,\rangle, \end{aligned}$$

where u_Θ^- is the set of all negative roots which are not linear combinations of elements of Θ, and where $\langle \Delta' \rangle$ is the set of roots which are linear combinations of elements of Δ'. It follows from [6, 21.9 (ii), p. 232] that the three groups above are the direct span of their respectively generating groups U_α since each of their underlying sets of roots α is closed in the sense that it contains the sum of each two of its elements if this sum itself is a root. Therefore, as varieties, each of the above three groups is k_s-isomorphic to an affine space (cf. [6, 21.20 (i), p. 240]) and the product map induces a k_s-isomorphism.

Hence we obtain the following commutative diagram of k_s-morphisms, each of which is an isomorphism onto an open dense subvariety.

$$\begin{array}{ccccc}(\mathcal{R}_u(P^-)\cap\mathcal{R}_u(Q^-_{k_s})) & \times_{k_s} & (\mathcal{R}_u(P^-)\cap\mathcal{L}_{k_s}) & \longrightarrow & \mathcal{R}_u(P^-)\\ & & \downarrow & & \downarrow\\ (\mathcal{R}_u(P^-)\cap\mathcal{R}_u(Q^-_{k_s})) & \times_{k_s} & V'_{k_s} & \longrightarrow & V_{k_s}\end{array}$$

Here the latter horizontal map is just given by $(g, h(P\cap\mathcal{L}_{k_s}))\mapsto ghP$.

Let $\psi = u^-_\Theta\setminus\langle\Delta'\rangle$, so that $\mathcal{R}_u(P^-)\cap\mathcal{R}_u(Q^-_{k_s}) = \langle U_\alpha \mid \alpha\in\psi\rangle$ as above. As Θ and Δ' are both $*$-invariant, $\mathcal{R}_u(P^-)\cap\mathcal{R}_u(Q^-_{k_s})$ is $\mathrm{Gal}\,(k_s/k)$-invariant and hence defined over k (cf. [6, AG.14.4, p. 32]). Therefore we have a k-subvariety U of $\mathcal{R}_u(Q^-)$ such that $U\times_k k_s = \mathcal{R}_u(P^-)\cap\mathcal{R}_u(Q^-_{k_s})$. The image of ψ under res_k (cf. 3.4 v)) is a closed set of roots of G over k. Hence we conclude using [6, 21.20 (i), p. 240] that U is isomorphic, as a k-variety, to an affine k-space. Thus we obtain a morphism of k-varieties $U\times_k V'\to V$ which is an isomorphism onto an open and dense k-subvariety. Therefore $k(V)\cong k(U)\otimes_k k(V')$, and since U is isomorphic to an affine k-space, the theorem is proved. □

Let Q be a minimal parabolic k-subgroup of G, with Levi subgroup $\mathcal{L}$ and type $\Delta_0\subseteq\Delta$ (cf. 3.4 v)). Then, by 1.9, the derived group G_{an} of $\mathcal{L}$ is a semisimple anisotropic kernel of G. Let $\Theta\subseteq\Delta$ be $*$-invariant. As above, $\Theta_{\mathrm{an}} := \Theta\cap\Delta_0$ is $*$-invariant and can be considered as a set of roots of $\mathcal{L}_{k_s}$ and of $(G_{\mathrm{an}})_{k_s}$. If P is a parabolic subgroup of G_{k_s} of type Θ then $P_{\mathcal{L}} := P\cap\mathcal{L}_{k_s}$ and $P_{\mathrm{an}} := P\cap(G_{\mathrm{an}})_{k_s}$ are parabolic subgroups of type Θ_{an} of $\mathcal{L}_{k_s}$ (resp. $(G_{\mathrm{an}})_{k_s}$). Consequently, the associated quotient varieties G_{k_s}/P, $\mathcal{L}_{k_s}/P_{\mathcal{L}}$, $(G_{\mathrm{an}})_{k_s}/P_{\mathrm{an}}$ are defined over k by 3.7. We denote the respective k-structures by V_Θ, $V_{\mathcal{L},\Theta_{\mathrm{an}}}$, $V_{\Theta_{\mathrm{an}}}$. Since $\mathcal{L}$ is the product of its maximal central torus and G_{an}, the natural k-embedding $G_{\mathrm{an}}\to\mathcal{L}$ defines a k_s-isomorphism $(G_{\mathrm{an}})_{k_s}/P_{\mathrm{an}}\to\mathcal{L}_{k_s}/P_{\mathcal{L}}$ which is $\mathrm{Gal}\,(k_s/k)$-equivariant. Therefore it induces a k-isomorphism of the k-varieties $V_{\Theta_{\mathrm{an}}}\to V_{\mathcal{L},\Theta_{\mathrm{an}}}$ (cf. [6, AG.14.3, p. 31]). Hence $k(V_{\Theta_{\mathrm{an}}})$ is naturally isomorphic to $k(V_{\mathcal{L},\Theta_{\mathrm{an}}})$. By 3.18 we find that $k(V_\Theta)$ is purely transcendental over $k(V_{\mathcal{L},\Theta_{\mathrm{an}}})$. Hence we conclude:

3.19 Corollary. *For any $*$-invariant $\Theta\subseteq\Delta$, the generic Θ-splitting field $k(V_\Theta)$ of G is a purely transcendental extension of the corresponding induced generic Θ_{an}-splitting field $k(V_{\Theta_{\mathrm{an}}})$ of the semisimple anisotropic kernel G_{an} of G.* □

3.20 Example (Witt [37]). We first consider the case $\mathrm{char}(k)\neq 2$ which has been investigated by Witt and which is the origin of the theory of generic splitting.

Let $a, b\in k^*$ be such that $D = (a, b)$ is a quaternion algebra over k, that is, an Azumaya algebra over k of k-dimension 4. One can choose a k-basis $\{1, u, v, uv\}$ of D such that the multiplication in D is given by $u^2 = a, v^2 = b, vu = -uv$. Let $G = \mathrm{SL}_1(D)$ be the kernel of the reduced norm N_{red} of D over k restricted to the group $\mathrm{GL}_1(D)$ of invertible elements. G is an anisotropic k-form of $(\mathrm{SL}_2)_k$ if D is non-split. The Dynkin diagram of $G_{\bar k}$ consists of a single vertex only, hence the only conjugacy class of proper parabolic subgroups is given by the class of Borel subgroups, which can be represented, over $\bar k$, by the $\bar k$-group B of upper triangular matrices of determinant 1.

Now G operates k-morphically and k-linearly on the affine k-space D by conjugation. This operation gives an operation on the projective k-space $\mathbb{P}(D)_k\cong\mathbb{P}^3_k$. Let V denote the $\bar k$-subvariety of nilpotent lines of $\mathbb{P}(D)_{\bar k}$. It is easily checked that $G(\bar k)$ operates transitively on $V(\bar k)$ and that B is the stabilizer subgroup of the nilpotent line of $V(\bar k)$ represented by the matrix

$$\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}\in\mathrm{M}_2(\bar k)\cong D\otimes_k\bar k.$$

Hence $V\cong G_{\bar k}/B$. By 3.7 we know that V is defined over k, but we here will give an elementary argument for this fact which will give us the equation with coefficients in k defining the complete curve V. If $x = \xi+\eta_1 u+\eta_2 v+\eta_3 uv\in D$, then the reduced norm N_{red} and the reduced trace S_{red} are given by the formulae $N_{\mathrm{red}}(x) = \xi^2-\eta_1^2 a-\eta_2^2 b+\eta_3^2 ab$ and $S_{\mathrm{red}}(x) = 2\xi$. The variety of nilpotent elements of $D_{\bar k}$ is defined by the equations $N_{\mathrm{red}}(x) = S_{\mathrm{red}}(x) = 0$. These are equivalent to $\xi = 0$ and $\eta_1^2 a+\eta_2^2 b = \eta_3^2 ab$.

Hence a defining equation for $k(V)$ is given by $X^2a + Y^2b = ab$. This is the function field associated to $D = (a, b)$ by Witt [37, p. 464].

The field $k(V)$ is isomorphic to $F := k(Z)(y)$ with an indeterminate Z and $y = \sqrt{aZ^2 + b}$ (with $X = b/y,\ \ Y = aZ/y$, and $y = b/X,\ \ Z = bY/aX$). If D splits over k, then clearly F is a splitting field of G. So let D be non-split. Since $D_{k(Z)} \cong (-a/b, aZ^2 + b)$, we may assume that F is a maximal commutative subfield of $D_{k(Z)}$. We then obtain a maximal anisotropic $k(Z)$-torus T in $G_{k(Z)}$ defined as the kernel of $\mathrm{Norm}_{F/k(Z)}$ (restricted to the invertible elements) (cf. 2.4). The torus T is *not* defined over k. It splits over F and hence F is a splitting field of G.

Every splitting field of G is a splitting field of D and vice versa. If now L is such a splitting field, then, over L, the element b is a norm from the L-algebra $L[X]/(X^2 - a)$ (cf. [19, Th. 15.7, p. 149]), hence there are elements $\xi_1, \xi_2 \in L$ such that $\xi_1^2 = a\xi_2^2 + b$. We then have a k-place $\varphi\colon F \to L \cup \{\infty\}$ with $\varphi(Z) = \xi_2$ and $\varphi(y) = \xi_1$. Hence F is a generic splitting field of G.

We now assume $\mathrm{char}(k)$ to be arbitrary. Then for $a, b \in k$ with $b \neq 0$ we obtain a quaternion k-algebra D with k-basis $\{1, u, v, uv\}$ and multiplication defined by $u^2 = u + a,\ v^2 = b,\ vu = (1 - u)v$ which is a full 2×2-matrix ring over k if and only if the equation $b = \xi^2 + \xi\eta - a\eta^2$ has a solution $\xi, \eta \in k$ [1, Th. 26, p. 146], or equivalently, if b is a norm from the separable extension $k[X]/(X^2 - X - a)$.

It is easily checked that the reduced norm and trace of D for $x = \xi + \eta_1 u + \eta_2 v + \eta_3 uv \in D$ is given by the formulae $N_{\mathrm{red}}(x) = \xi^2 + \xi\eta_1 - \eta_1^2 a - (\eta_2^2 + \eta_2\eta_3 - \eta_3^2 a)b$ and $S_{\mathrm{red}}(x) = 2\xi + \eta_1$. As above, we get the variety $V \cong G_{\bar{k}}/B$ of nilpotent lines of $D_{\bar{k}}$ by the equations $N_{\mathrm{red}}(x) = S_{\mathrm{red}}(x) = 0$.

If $\mathrm{char}(k) = 2$, these equations are equivalent to the k-equation $\xi^2 b + \eta_2^2 + \eta_2\eta_3 + \eta_3^2 a = 0$. By [28, XIV, §5, Example, p. 221] this is the homogeneous equation defining the Severi-Brauer variety associated to D.

4. Generic splitting of Azumaya algebras over fields

Let A be an Azumaya algebra over an infinite field k, that is, A is a finite dimensional central simple k-algebra, and, by Wedderburn's theorem, there is a unique integer $r \geq 0$ and a central division algebra D over k which is unique up to k-isomorphism such that $A \cong \mathrm{M}_{r+1}(D)$. Let $d = \mathrm{ind}\,(A)$ denote the index of A (that is, $\dim_k D = d^2$), and let n be defined by

$$n + 1 = d(r + 1).$$

Then the semisimple k-group $G := \mathrm{SL}_{r+1}(D)$ has the k-rank r and the absolute rank n (cf. [6, 23.2, p. 254f]).

Let K be a splitting field of G. We then have $G_K \cong \mathrm{SL}_{n+1,K}$, and a maximal K-split torus T of G_K is given by the set of diagonal matrices

$$t = \mathrm{diag}\,(t_1, t_2, \ldots, t_{n+1}) := \begin{pmatrix} t_1 & 0 & \ldots & 0 \\ 0 & t_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & t_{n+1} \end{pmatrix} \ \text{with } \det(t) = 1\ .$$

A basis Δ of the root system $\Phi(G_K, T)$ is given by the K-rational characters $\alpha_i(t) := t_i t_{i+1}^{-1}$, for $i = 1, \ldots, n$, and its Dynkin diagram is given by

$$\underset{\alpha_1}{\circ} \!-\!\!-\!\!-\! \underset{\alpha_2}{\circ} \!-\!\!-\! \cdots \!-\!\!-\! \underset{\alpha_n}{\circ}\ .$$

Since G is of inner type ${}^1A_n$ we have a k-variety V_Θ for any subset $\Theta \subseteq \Delta$ by 3.7. The function field $F_\Theta = k(V_\Theta)$ has the properties described in 3.10 and is a generic Θ-splitting field of G by 3.16.

4.1 Theorem. *Let $\{\alpha_{i_1}, \ldots, \alpha_{i_l}\} = \Delta \setminus \Theta$ for $\Theta \subseteq \Delta$. Then, for every field extension k' of k, we have $V_\Theta(k') \neq \emptyset$ if and only if* $\mathrm{ind}\,(A \otimes_k k')$ *divides* $\gcd\,(d, i_1, \ldots, i_l)$.

Proof. It suffices to show the equivalence for the case $k' = k$. Using the description of the relative Dynkin diagram of ${}^1A_n$ as given in [32, Table II, p. 55] we see that $\Delta \setminus \Delta_0 = \{\alpha_d, \alpha_{2d}, \ldots, \alpha_{rd}\}$ with Δ_0 as in 3.4 v). It follows that $d \mid \gcd\,(d, i_1, \ldots, i_l)$ if and only if $\Delta \setminus \Theta \subseteq \Delta \setminus \Delta_0$. However, the latter condition is equivalent to $V_\Theta(k) \neq \emptyset$ by 3.7, "iii) $\Leftrightarrow$ ii)". $\square$

4.2 Corollary.

i) Let L and k' be two field extensions of k such that k' is a k-specialization of L. Then $\mathrm{ind}\,(A \otimes_k k')$ *divides* $\mathrm{ind}\,(A \otimes_k L)$.

ii) For $d = \mathrm{ind}\,(A)$ *and* $i_1, \ldots, i_l$ *as in 4.1 we have* $\mathrm{ind}\,(A \otimes_k F_\Theta) = \gcd\,(d, i_1, \ldots, i_l)$.

iii) Let i divide d. Then there is a parabolic subgroup P of G_{k_s} such that $V = G_{k_s}/P$ is defined over k and that $\mathrm{ind}\,(A \otimes_k k(V)) = i$. *Every field extension k' of k with* $\mathrm{ind}\,(A \otimes_k k')$ *dividing i is a k-specialization of $k(V)$. A possible choice is $P = P_{\Delta_i}$ for $\Delta_i = \Delta \setminus \{\alpha_i\}$.*

Proof. i) For $i = 1, \ldots, n$ let $V_i := V_{\Delta_i}$ with $\Delta_i = \Delta \setminus \{\alpha_i\}$, and let $F_i := k(V_i)$. It follows from 4.1 that, for any extension k_1 of k, the set of all $j \in \{1, \ldots, n\}$ with $V_j(k_1) \neq \emptyset$ consists precisely of the multiples of $\mathrm{ind}\,(A \otimes_k k_1)$. Applying this to the fields L, k' yields i), since $V_j(L) \neq \emptyset$ implies $V_j(k') \neq \emptyset$ by 1.3.

ii) We have $V_\Theta(F_\Theta) \neq \emptyset$ which implies $\mathrm{ind}\,(A \otimes_k F_\Theta) \mid g := \gcd\,(d, i_1, \ldots, i_l)$ by 4.1. Let p be a prime dividing g and p^s the highest power of p which divides g. It suffices to show that p^s divides $\mathrm{ind}\,(A \otimes_k F_\Theta)$. There is finite separable field extension k' of k such that $p^s = \mathrm{ind}\,(A \otimes_k k')$ (cf. [22, 14.4, Lemma b, p. 260]). Thus 4.1 and 3.10 yield that k' is a k-specialization of F_Θ. Now i) implies that p^s divides $\mathrm{ind}\,(A \otimes_k F_\Theta)$.

iii) For $P = P_{\Delta_i}$ the first statement follows immediately from ii). The second statement follows from 4.1 and 3.10. □

The generic splitting field $k(V_\emptyset)$ of G (cf. 3.11) is of transcendence degree $n(n+1)/2$ (apply 4.4 below with $\Theta = \emptyset$). If $n > 1$, then there are generic splitting fields of G of smaller transcendence degree, as follows from 4.3 and 4.4.

4.3 Corollary. *If the greatest common divisor of $i_1, \ldots, i_l$ and d is 1 then the function field $F_\Theta := k(V_\Theta)$ is a generic splitting field of $G = \mathrm{SL}_{r+1}(D)$.*

Proof. By 4.2 ii) we find that F_Θ is a splitting field of A and hence of G. Cor. 3.17 iv) now implies that F_Θ is a generic splitting field of G. □

4.4 Proposition. *For $\Theta \subset \Delta$, let V_Θ be the corresponding k-variety as defined in 3.7. If $\{\alpha_{i_1}, \ldots, \alpha_{i_l}\} = \Delta \setminus \Theta$ with $i_1 < \cdots < i_l$ and if $i_0 := 0$, then*

$$\dim\ V_\Theta = \sum_{j=1}^{l} (i_j - i_{j-1})(n + 1 - i_j).$$

Moreover, $V_\Theta \times_k K$ is isomorphic, as a K-variety, to the projective variety $\mathrm{Flag}_\Theta(K^{n+1})$ of flags of subspaces of the $(n+1)$-dimensional affine K-space $\{0\} = U_0 \subset U_1 \subset \cdots \subset U_l$ with $\dim_K U_j = i_j$ for $j = 0, \ldots, l$.

Proof. Let $\{e_1, \ldots, e_{n+1}\}$ denote the standard basis of K^{n+1}. The group $G_K \cong (\mathrm{SL}_{n+1})_K$ operates K-morphically and transitively on $\mathrm{Flag}_\Theta(K^{n+1})$, and the stabilizer subgroup P of the flag

$$\{0\} = U_0 \subset U_1 \subset \cdots \subset U_l, \quad U_j = Ke_1 \oplus \cdots \oplus Ke_{i_j} \subseteq K^{n+1}, \text{ for } j = 1, \ldots, l$$

is defined by the matrices in $\mathrm{SL}_{n+1}(K)$ of shape $(A_{jj'})_{j,j'=1,\ldots,l+1}$. Here $A_{jj'}$ is an $(i_j - i_{j-1}) \times (i_{j'} - i_{j'-1})$-matrix for $j, j' = 1, \ldots, l+1$, where we define $i_{l+1} := n+1$, and $A_{jj'} = 0$ for $j > j'$. Since P contains the Borel subgroup B of G_K defined by the upper triangular matrices, it is a parabolic subgroup of G_K. The dimension of its unipotent radical $\mathcal{R}_u(P)$ is equal to the sum of the number of entries of all matrices $A_{jj'}$ for $j < j'$. By 3.2, this yields the right hand side of the formula for the dimension of V_Θ.

By 3.1, the reductive part of P_Θ is the centralizer $\mathcal{Z}(T_\Theta)$ in G_K of the K-torus

$$T_\Theta := \left(\bigcap_{\nu=1, \nu \neq i_1, \ldots, i_l}^{n} \mathrm{Ker}\ (\alpha_\nu) \right)^\circ .$$

Hence $T_\Theta(K)$ consists of diagonal matrices

$$t = \mathrm{diag}\,(\underbrace{t_{i_1}, \ldots, t_{i_1}}_{i_1 \text{ times}}, \underbrace{t_{i_2}, \ldots, t_{i_2}}_{i_2 - i_1 \text{ times}}, \ldots, \underbrace{t_{i_{l+1}}, \ldots, t_{i_{l+1}}}_{i_{l+1} - i_l \text{ times}})$$

with $\det(t) = 1$ and $i_{l+1} = n+1$. Now it is easily checked that the Levi subgroup of P is the centralizer $\mathcal{Z}(T_\Theta)$ of T_Θ in G_K. Therefore we obtain

$$P = \mathcal{Z}(T_\Theta)\mathcal{R}_u(P) = \mathcal{Z}(T_\Theta) \cdot B = \mathcal{Z}(T_\Theta)\mathcal{R}_u(P_\Theta) = P_\Theta$$

by 3.1. Hence $P = P_\Theta$, which proves 4.4, since $G_K/P \cong \mathrm{Flag}_\Theta(K^{n+1})$. □

We now restrict our attention to proper maximal subsets of Δ. Set

$$\Theta := \Delta_i := \Delta \setminus \{\alpha_i\}, \quad V_i = V_{\Delta_i}, \quad F_i := k(V_i)$$

for $i \in \{1, \dots, n\}$. The following corollary is a direct consequence of 4.4.

4.5 Corollary. *For $i = 1, \dots, n$ we have* $\dim\ V_i = i(n+1-i)$*, and V_i is, as a K-variety, isomorphic to the Graßmann variety* $\mathrm{Grass}_i(K^{n+1})$.

4.6 Corollary. *For $i = 1, \dots, n$ the equality* $\mathrm{ind}\,(A \otimes_k F_i) = \gcd(d, i)$ *holds, and for every field extension k' of k we have $V_i(k') \neq \emptyset$ if and only if* $\mathrm{ind}\,(A \otimes_k k')$ *divides i. In particular, F_{jd} is a purely transcendental extension of k for $j = 1, \dots, r$.*

Proof. The equality holds by 4.2, the rest of 4.6 follows from 4.1 and 3.10. □

As it was mentioned in [4, p. 103], the generalized Severi-Brauer varieties described there are precisely the k-forms of the Graßmann varieties from 4.5, except in the case $2 \mid (n+1)$ and $i = (n+1)/2$, where also an outer form of $\mathrm{Grass}_i(k^{n+1})$ exists. This will naturally occur in the theory of the generic splitting of special unitary groups of type ${}^2A_n$ which will be discussed in §6 (cf. 6.5).

4.7 Corollary. *For every field extension L of k, the following statements are equivalent:*

i) L is a splitting field of G.

ii) $V_1 \times_k L \cong \mathbb{P}^n_L$.

iii) $V_1(L) \neq \emptyset$.

The statements i), ii), iii) remain equivalent if V_1 is replaced by V_n in ii) and iii).

Proof. The statement i) implies ii) by 4.4. Obviously, ii) implies iii). If iii) holds, then $d_L := \mathrm{ind}\,(A \otimes_k L) \mid 1$ (resp. n) by 4.6. Since $d_L \mid (n+1)$, it follows in both cases that L is a splitting field of A and hence of G. □

4.8 Remark. i) Since the separable closure k_s is a splitting field of G (cf. 1.7 iii)) it follows from 4.7 that $V_i \times_k k_s \cong \mathbb{P}^n_{k_s}$ for $i = 1, n$, hence V_1 and V_n are n-dimensional Severi-Brauer varieties over k [28, Chap. X, §6, p. 168]. More generally, all V_i for $i = 1, \dots, n$ are isomorphic to "generalized Severi-Brauer varieties" introduced in 1976 by Heuser (for $i \mid (n+1)$), [12, p.30, 46], and later (1991) for all i by Blanchet [4, p. 100, 102] and Schofield/v. d. Bergh [26]. The generalized Severi-Brauer varieties are the varieties W_i of rank i left ideals of A in [12] and [26] (and right ideals in [4]). Using the isomorphism $A \otimes_k K \cong \mathrm{M}_{n+1}(K)$, one verifies similarly as in the proof of 4.4 that G_K operates transitively on W_i and that P_{Δ_i} stabilizes a rank i left ideal of $\mathrm{M}_{n+1}(K)$ under left multiplication. Hence $W_i \cong G_K/P_{\Delta_i} = V_i$ for $i = 1, \dots, n$. (Note that P_{Δ_i} are precisely the proper *maximal* parabolic subgroups of G_K.) It follows that the fields F_i are the generic partial splitting fields of A, introduced by Heuser [12, Def. 7, p. 22 and p. 63], Blanchet [4, Def. 3 and Th. 2, p. 103] and Schofield/v. d. Bergh [26, Sec. 3]. The statement $\mathrm{ind}\,(A \otimes_k F_i) = \gcd(d, i)$ and the equivalence of 4.6 was proved by Heuser [12, p. 73, p. 43], Blanchet [4, Th. 3, p. 104, Prop. 3, p. 103] and Schofield/v. d. Bergh [27, Th. 2.5]. Blanchet and Schofield/v. d. Bergh also proved the equivalence of "i)" and "ii)" of 3.10 for the special case $V = V_i$.

ii) The assertion $\mathrm{ind}\,(A \otimes_k F_i) = \gcd(d, i)$ with $F_i = k(V_i)$ in 4.6 shows a significant difference in the behavior of the generic splitting of Azumaya algebras and that of quadratic forms as discussed in the next paragraph (cf. 5.8).

4.9 Remark. Taking $i = 1$ (or $i = n$) we obtain from 4.6, 4.7 and 3.10 the results of Amitsur [2, 9.1, p. 26] (see also [3, Th. 2, p. 1]), which were later proved by Roquette [23, Th. 2, p. 413] with methods from Galois cohomology and which generalize the result of Witt on quaternion algebras (cf. Example 3.20).

Amitsur also showed that the automorphism group of F over k is isomorphic to A^*/k^*. This can be shown in the following way: The automorphism group of V_1 is certainly a k-form of the group $(\mathrm{PGL}_{n+1})_k$ which is isomorphic to $(\mathrm{GL}_{n+1})_k$ modulo its center $\mathcal{C}((\mathrm{GL}_{n+1})_k)$. Obviously it contains the group $\mathrm{GL}_1(A)/\mathcal{C}(\mathrm{GL}_1(A)) \cong \mathrm{GL}_{r+1}(D)/\mathcal{C}(\mathrm{GL}_{r+1}(D))$ which is a k-form of the group above. For dimension reasons, this is already the full automorphism group of V_1. But its k-rational points are just given by $A^*/\mathcal{C}(A^*) \cong A^*/k^*$.

The corollary in [3, p. 3] characterizes splitting fields K of A by the condition that $k(V_1)$ is contained in a purely transcendental extension of K. This condition is, by 3.10, equivalent to $V_1(K) \neq \emptyset$, hence the assertion of the corollary follows from 4.6.

4.10 Remark. In [23, p. 424f], Roquette associates to every Galois-2-cocycle $\gamma \in H^2(\mathrm{Gal}\,(K/k), K^*)$ (where K is a finite Galois extension of k), and every multiple m of its Schur index d a "Brauer field" $F_m(\gamma)$ of transcendence degree $m-1$ over k.

In our terminology, the cocycle γ defines a central k-division algebra D of index d, the multiple m of d is just $n+1=(r+1)d$. These data define the semisimple group $G = \mathrm{SL}_{r+1}(D)$, and the Brauer field $F_m(\gamma)$ defined by Roquette is precisely the function field $k(V_1)$, where V_1 is a Severi-Brauer variety satisfying $G_K/P_{\Delta_1} = V_1 \times_k K$. Clearly we hereby obtain an infinite series of generic splitting fields of D as r ranges over all non-negative integers. It can easily be deduced from 3.18 that, for $m' \le m$, the field $F_m(\gamma)$ is a purely transcendental extension of the field $F_{m'}(\gamma)$, which is the content of [23, Th. 4, p. 413]. In particular, all the fields $F_m(\gamma)$ are purely transcendental over the smallest one, $F_1(\gamma)$, which is isomorphic to the generic splitting field of G_{an} as constructed in 3.19, since the semisimple anisotropic kernel of $\mathrm{SL}_{r+1}(D)$ is a direct product of $r+1$ copies of $SL_1(D)$.

5. Generic splitting of quadratic forms

Let k be an infinite field with $\mathrm{char}(k) \neq 2$ and let (M,q) be a regular quadratic k-space of dimension m, that is, M is an m-dimensional k-vector space and q is a quadratic form with nondegenerate associated bilinear form $(\ ,\ )$ such that $q(x+y) = q(x) + q(y) + (x,y)$ holds for all $x, y \in M$. The discriminant $d(M)$ of (M,q) is defined to be the square class $(-1)^{[m/2]} \det((u_i,u_j)_{i,j=1,\dots,m}) k^{*2} \in k^*/k^{*2}$. (Here $\{u_1,\dots,u_m\}$ denotes a k-basis of M.) We have a *Witt decomposition* of M into mutually orthogonal subspaces

$$M = \left(\mathop{\perp}_{i=1}^{r} H_i \right) \perp M_{\mathrm{an}}$$

where H_i is a hyperbolic plane for $i = 1,\dots,r$ and $(M_{\mathrm{an}}, q_{\mathrm{an}})$ with $q_{\mathrm{an}} := q|_{M_{\mathrm{an}}}$ is a maximal anisotropic subspace of (M,q) which is unique up to k-isometry and is called an *anisotropic kernel* of the quadratic space (M,q). The integer $r \ge 0$ is the *Witt index* of (M,q), and we have $m = 2r + \dim_k M_{\mathrm{an}}$ and $d(M) = d(M_{\mathrm{an}})$. It is convenient to choose a k-basis $\{e_1,\dots,e_m\}$ of M as follows. For $i = 1,\dots,r$, let $\{e_i, e_{m-i+1}\} \subset M$ be a basis of H_i such that $q(e_i) = q(e_{m-i+1}) = 0, (e_i, e_{m-i+1}) = 1$, and let $\{e_i \mid i = r+1,\dots,m-r\}$ be any basis of $(M_{\mathrm{an}}, q_{\mathrm{an}})$. A basis like this we will call a *Witt basis* of (M,q). We mention that $\{e_1,\dots,e_r\}$ (as well as $\{e_{m-r+1},\dots,e_m\}$) generate a maximal totally isotropic subspace of (M,q).

Let $G := \mathrm{SO}(q)$ be the special orthogonal group of (M,q). If $m = 2$, then G is a k-torus and its generic splitting field is described in 2.3 and 2.5. Hence we now assume $m \ge 3$. This implies that G is semisimple. The following proposition is obtained from [6, 23.4, p. 256f] and Definition 1.8.

5.1 Proposition. *Let (M,q) be a regular quadratic k-space of Witt index r. Then r is the rank of $G = \mathrm{SO}(q)$, and a maximal k-split k-torus S of G is given, with respect to a Witt basis $\{e_i\}$ of (M,q), by the diagonal matrices*

$$s = \mathrm{diag}\,(s_1,\dots,s_r,1,\dots,1,s_r^{-1},\dots,s_1^{-1}) \in \mathrm{GL}(M),\ \ s_1,\dots,s_r \in k^*.$$

A reductive anisotropic kernel of G is given by $G_{\mathrm{an}} = \mathrm{SO}(q_{\mathrm{an}})$, where $(M_{\mathrm{an}}, q_{\mathrm{an}})$ is the anisotropic kernel of (M,q). More precisely, we have $\mathcal{Z}(S) = S \times_k G_{\mathrm{an}}$, and G_{an} is semisimple if and only if $\dim_k M_{\mathrm{an}} \ge 3$, and is an anisotropic k-torus of rank 1 if and only if $\dim_k M_{\mathrm{an}} = 2$, in which case G is quasi-split but not split. G is split over k if and only if $\dim_k M_{\mathrm{an}} \le 1$.

Let K be any splitting field of G. Then the rank of G_K is $n := [m/2]$. We modify the Witt basis given above over k into one over K by setting $e'_i = e_i$ for $i \notin \{r+1,\dots,m-r\}$ and by replacing the basis e_i of M_{an} for $i = r+1,\dots,m-r$ by a Witt basis e'_i of $(M_{\mathrm{an}} \otimes_k K, q_{\mathrm{an}} \otimes_k K)$ such that $q(e'_i) = q(e'_{m-i+1}) = 0, (e'_i, e'_{m-i+1}) = 1$ for $i = r+1,\dots,n$. Let T be the K-torus of G_K which is given with respect to the new basis by the diagonal matrices

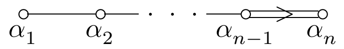
$$t = \mathrm{diag}\,(t_1,\dots,t_n,\hat{1},t_n^{-1},\dots,t_1^{-1}) \in \mathrm{GL}(M \otimes_k K),\ \ t_1,\dots,t_n \in K^*.$$

(Here the symbol $\hat{1}$ means that the component 1 occurs (resp. does not occur) in the middle according as m being odd (resp. even).) Then, by the above, T is a maximal torus of G_K which splits completely and contains S_K .

A basis Δ of the root system $\Phi = \Phi(G_K, T)$ is given by the K-rational characters $\alpha_i(t) := t_i\, t_{i+1}^{-1}$, for $i = 1, \dots, n-1$, and, in addition,

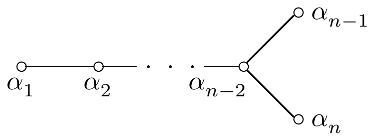
$$\alpha_n(t) = \begin{cases} t_n, & \text{if } m \text{ is odd, i. e., } G \text{ is of type } B_n; \\ t_{n-1}t_n, & \text{if } m \text{ is even, i. e., } G \text{ is of type } D_n. \end{cases}$$

The Dynkin diagram of G_K is, respectively, given by



in case B_n

and



in case D_n.

If m is odd or $d(M) = 1$, then G is of inner type, while G is of outer type ${}^2D_n$ for m even and $d(M) \neq 1$. [32, 2.3, p. 39 and Table II, p. 56f].

For $i = 1, \dots, n$, define standard parabolic subgroups of G_K by

$$P_i = P_{\Delta_i}, \text{ where } \Delta_i := \begin{cases} \Delta \setminus \{\alpha_i\} & \text{if } G \text{ is of inner type or } i \leq n-2; \\ \Delta \setminus \{\alpha_{n-1}, \alpha_n\} & \text{if } G \text{ is of outer type and } i = n-1. \end{cases}$$

(Intentionally, we leave P_n undefined in the outer type case.) Then P_i is, for every i, a proper parabolic subgroup of G_K such that $G_K/P_i \cong V_i \times_k K$ for some k-variety V_i and such that P_i is maximal with this property. This follows by 3.7, since in the outer type case, the subset $\{\alpha_{n-1}, \alpha_n\} \subset \Delta$ is the only equivalence class under the $*$-action which contains more than one element [32, Table II, p. 57]. Also we have $P_i \supset B = P_\emptyset = \langle T, U_\alpha \mid \alpha \in \Delta \rangle$, and B is the stabilizer of the complete isotropic flag of K-spaces given by $U_i = Ke'_1 \oplus \cdots \oplus Ke'_i$ for $i = 1, \dots, n$.

5.2 Lemma. *Let $i \in \{1, \dots, n\}$ if G is of inner type, and $i \in \{1, \dots, n-1\}$ otherwise.*

i) Let the Witt index of (M,q) be at least i. Then P_i is a k-parabolic subgroup of G. Moreover, the k-variety I_i of i-dimensional totally isotropic k-subspaces of M is equal to V_i except when G is of inner type D_n and $i \geq n-1$, in which case M splits totally and I_n equals $V_{n-1} \cup V_n$.

ii) Conversely, if there is a parabolic k-subgroup of type Δ_i, then there is a totally isotropic subspace U of M of dimension greater than or equal to i, hence the Witt index of (M, q) is at least i.

Proof. i) By Witt's cancellation theorem on quadratic forms, the operation of G on I_i is transitive unless we are in the exceptional case where I_n has two orbits.

For the Witt basis $\{e_1, \dots, e_m\}$, the k-subspace $U = ke_1 \oplus \cdots \oplus ke_i$ of M is totally isotropic and $U \otimes_k K = U_i$, hence the stabilizer subgroup P_U of U is a k-subgroup of G such that $P_{U,K} \supset B$. Therefore it is a parabolic k-subgroup of G.

The group $P_U(k)$ consists of matrices $(A_{jj'})_{j,j'=1,2,3}$ where A_{11}, A_{33} are $i \times i$-matrices, while A_{22} is a $(m-2i) \times (m-2i)$-matrix and $A_{jj'} = 0$ for $j > j'$. From the definition of the Witt basis, we conclude the identities $A_{33} = IA_{11}^{-t}I$ and $Q = A_{22}^t Q A_{22}$. Here I denotes the $i \times i$-matrix with 1's in the antidiagonal and zero elsewhere, while Q is the matrix describing the bilinear form on the subspace M_0 generated by $e_{i+1}, \dots, e_{m-i}$. This implies that A_{33} is uniquely determined by $A_{11} \in \mathrm{GL}_i(k)$ with $\det(A_{33}) = \det(A_{11})^{-1}$ and that $A_{22} \in G_0(k)$, where $G_0 := \mathrm{SO}(M_0, q|_{M_0})$. Hence P_U has a Levi k-subgroup isomorphic to $\mathrm{GL}_i \times G_0$ given by the matrices

$$\begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & IA_{11}^{-t}I \end{pmatrix}.$$

The torus $T_{\Delta_i} = \left(\bigcap_{\alpha \in \Delta_i} \mathrm{Ker}\,(\alpha)\right)^\circ$ is a k-torus and given by the diagonal matrices

$$\mathrm{diag}\,(t_1, \dots, t_i, 1, \dots, 1, t_i^{-1}, \dots, t_1^{-1})$$

with $t_1 = \cdots = t_i$ except when G is of type D_n and $i \geq n-1$, in which case T_{Δ_i} is given by the matrices

$$\mathrm{diag}\,(t_1, \dots, t_{n-1}, *, *, t_{n-1}^{-1}, \dots, t_1^{-1})$$

with $t_1 = \cdots = t_{n-1}$, where $*, *$ denotes entries which show up as diagonal elements only over a splitting field of G and which, otherwise, have to be replaced by 2×2 matrices which represent an anisotropic torus.

Now it is easily checked that, in any case, the Levi subgroup of P_U is the centralizer of T_{Δ_i} in G. Hence $P_{U,K} = (\mathcal{Z}(T_{\Delta_i})\mathcal{R}_u(P_U))_K = \mathcal{Z}(T_{\Delta_i})_K \cdot B = \mathcal{Z}(T_{\Delta_i})_K \cdot \mathcal{R}_u(P_i) = P_i$ by 3.1. This proves i).

ii) Applying the arguments above to the pair K, K instead of k, K we find that the existence of a parabolic k-subgroup of type Δ_i implies the existence of a totally isotropic k-subspace U of dimension i in M. □

By 3.15 iii) the function field $F_i = k(V_i)$ is a generic Δ_i-splitting field of G.

5.3 Theorem. *Let $i \in \{1, \ldots, n\}$ if G is of inner type, and $i \in \{1, \ldots, n-1\}$ otherwise. The field $F_i = k(V_i)$ is a generic field for splitting off at least i hyperbolic planes from the underlying quadratic space. Namely, $(M \otimes_k F_i, q \otimes_k F_i)$ has the Witt index $\geq i$, and for every field extension L of k the quadratic space $(M \otimes_k L, q \otimes_k L)$ has the Witt index $\geq i$ if and only if L is a k-specialization of F_i.*

Proof. Since $V_i(F_i) \neq \emptyset$ it follows from 5.2 ii) that $(M \otimes_k F_i, q \otimes_k F_i)$ has the Witt index $\geq i$. By 5.2 and 3.10 we have: $(M \otimes_k L, q \otimes_k L)$ has the Witt index $\geq i$ if and only if $V_i(L) \neq \emptyset$, and the latter holds if and only if L is a k-specialization of F_i. □

If G is of outer type, there is no regular extension of k which splits (M,q) totally (that is, gives the maximal Witt index); this follows from 3.5.

5.4 Corollary. *If G is of inner type, then $k(V_\emptyset)$ and F_n are generic splitting fields of G. If G is of inner type and m is even, then also F_{n-1} is a generic splitting field of G. If G is of outer type, then $k(V_\emptyset)$ and F_{n-1} are generic quasi-splitting fields of G, and the fields $k(\sqrt{d(M)}) \cdot F_{n-1}$ and $k(\sqrt{d(M)}) \cdot k(V_\emptyset)$ are generic splitting fields of G.*

Proof. It follows from 3.11 that $k(V_\emptyset)$ is a generic quasi-splitting field of G and that this is a generic splitting field if G is of inner type. In that case also F_n is a generic splitting field which is obvious from 5.1, 5.3, and 3.17 iv). If G is of inner type and m is even, then the discriminant of (M, q) is 1. Hence if L is a field extension of k such that the Witt index of $(M \otimes_k L, q \otimes_k L)$ is $\geq n-1$, then $M \otimes_k L \cong H \perp M'$ with a hyperbolic L-space H and a regular L-space M' of dimension 2 and discriminant 1, which therefore is a hyperbolic plane. Hence G_L splits. The last statement follows from 3.6 in combination with 3.17, since for $n \geq 3$ the field k_{alg} of 3.6 coincides with $k(\sqrt{d(M)})$. □

Example. To clarify the situation in the case of a non-$*$-invariant Θ we take m even and $\Theta = \{\alpha_1, \ldots, \alpha_{n-1}\}$. This is, in the outer type case, not $*$-invariant, and $F_\Theta \supseteq k_\Theta = k(\sqrt{d(M)})$ (cf. 3.16). Clearly G_{k_Θ} is of inner type, F_Θ is a splitting field of G_{k_Θ} by 5.4 and hence also of G. Of course F_Θ is then a fortiori a quasi-splitting field of G. However, it is not a generic quasi-splitting field of G by 3.12, since k is not algebraically closed in F_Θ.

5.5 Corollary. *The transcendence degree of F_i is given by the following formulae:*

$$\operatorname{trdeg}\ F_i = \begin{cases} i(4n-3i+1)/2 & \text{if } m \text{ is odd, } 1 \leq i \leq n; \\ i(4n-3i-1)/2 & \text{if } m \text{ is even, } 1 \leq i \leq n-2; \\ n(n-1)/2 & \text{if } m \text{ is even, } G \text{ is of inner type, } n-1 \leq i \leq n; \\ (n+2)(n-1)/2 & \text{if } m \text{ is even, } G \text{ is of outer type, } i = n-1; \end{cases}$$

$$\operatorname{trdeg}\ k(V_\emptyset) = \begin{cases} n^2 & \text{if } m \text{ is odd;} \\ n(n-1) & \text{if } m \text{ is even.} \end{cases}$$

There is an increasing sequence of (non-canonical) k-linear embeddings

$$F_1 \hookrightarrow \cdots \hookrightarrow F_i \hookrightarrow F_{i+1} \hookrightarrow \cdots \hookrightarrow F_{n'} \hookrightarrow k(V_\emptyset)$$

with $n' = n$ or $n-1$ according as G is inner or not.

Proof. By 3.2, the dimension of V_i is the cardinality of u_{Δ_i}. This can be computed by using the explicit descriptions of the root systems of types B_n and D_n as given in [10, p. 252 and p. 256] or [33, p. 30, p. 35]. (Note: The description of positive roots of D_n in [10, p. 256] is erroneous. For a correct description cf. [33, p. 35].)

By 5.3, F_{i+1} splits off at least $i+1$ hyperbolic planes of (M,q). Hence $V_i(F_{i+1}) \neq \emptyset$ by 5.2. Thus, by 3.10, there is a k-linear embedding of F_i into a purely transcendental extension of F_{i+1}. By the above we have trdeg $F_{i+1} \geq$ trdeg F_i, hence it follows from [24, Lemma 1, p. 209] that there is a k-linear embedding $F_i \hookrightarrow F_{i+1}$. A similar argument gives the k-linear embedding $F_{n'} \hookrightarrow k(V_\emptyset)$. Alternatively, we here can use the natural map induced by the inclusion $B \hookrightarrow P_{n'}$ for a proper choice of a Borel group B. □

5.6 Corollary. *We have $F_1 = k(V_1) \cong k(q)_0$, where $k(q)_0$ is a generic zero field as defined by Knebusch [16, 3.2, p. 69, and p. 71].*

Proof. V_1 is, by 5.2, the variety of the isotropic lines in M, which can be defined by the equation $q(x) = 0$ for $x \in M$. □

5.7 Corollary. *Let $i = 1, \dots, n'$ where $n' = [m/2]$ or $[m/2]-1$ according as G is of inner or outer type. Assume L is an arbitrary field extension of k. Then $(M \otimes_k L, q \otimes_k L)$ is of index $\geq i$ if and only if the free composite $F_i \cdot L$ is purely transcendental over L. In particular, (M,q) is of index $\geq i$ if and only if F_i is a purely transcendental extension of k.*

This follows from 5.3 and 3.10. Corollary 5.7 was obtained by Knebusch for $i = 1$ [16. 3.8 and 3.10, p. 72].

5.8 Remark. It is easily seen that a suitable subsequence $\{F_{i_j}\}$ of the sequence $\{F_i\}$ in 5.3 is a so-called generic splitting tower as originated by Knebusch [16, §5, p. 78]: Let $n' = n$ if G is of inner type and let $n' = n-1$ otherwise. We define i_j inductively. Let $i_0 = 0$ and $F_{i_0} := k$. If $i_j \leq n'$ is defined let $i_{j+1} \in \{1, \dots, n'\}$ be the smallest number such that the Witt index of $(M \otimes_k F_{i_{j+1}}, q \otimes_k F_{i_{j+1}})$ is bigger than that of $(M \otimes_k F_{i_j}, q \otimes_k F_{i_j})$. In the inner case the sequence $F_{i_j}, j \geq 1$, is a generic splitting tower. If G is of outer type and $F := F_{i_{j'}}$ is the last element of this sequence, then the anisotropic kernel of $(M \otimes_k F, q \otimes_k F)$ is a binary form, hence its special orthogonal group is an anisotropic F-torus which is generically split by the field $F(\sqrt{d(M)})$ (cf. 2.4). In this case we define $F_{i_{j'+1}} := F(\sqrt{d(M)})$ as the last element of the sequence.

Knebusch gives in [16, Example 5.7, p. 80] an example of an anisotropic form of arbitrary dimension together with a generic splitting tower $\{K_i\}$ such that every layer splits off precisely one hyperbolic plane. Clearly, for such a form, the sequence $\{F_i\}$ is also a generic splitting tower. We have $K_1 = F_1$, however, for $i > 1$, the transcendence degree of K_i exceeds that of F_i by $i(i-1)/2$ if m is odd or $i \leq n-2$, and if m is even and $i = n-1$, by $(n-1)n/2$ in the inner case and by $(n-1)(n-2)/2$ in the outer case.

On the other hand it is easy to see that there are forms for which the sequence $\{F_i\}$ degenerates completely in the sense that F_1 already is a generic splitting field of $\mathrm{SO}(\varphi)$. For example, any Pfister form φ has the property that it is hyperbolic already if it is isotropic [25, 4, Cor. 1.5, p. 144]. This implies that all the associated fields F_i are k-specializations of each other. Since the special orthogonal group $\mathrm{SO}(\varphi)$ is of inner type if the dimension of φ is ≥ 4 (the discriminant of a Pfister form of dimension ≥ 4 is 1), it follows from 3.17 iii) that the F_i are all generic splitting fields of $\mathrm{SO}(\varphi)$.

As has also been observed by Knebusch, a generic zero field of any orthogonal summand ψ of a Pfister form φ with dim $\psi = (\dim\ \varphi)/2 + 1$ is a generic splitting field of $\mathrm{SO}(\varphi)$.

This seems to indicate that in general it might be difficult to find a generic splitting field with minimal transcendence degree for an arbitrary reductive group.

As a consequence of Theorem 5.3 we obtain a corollary which can be also derived from [16, Th. 3.3, p. 69] by using a generic splitting tower of Knebusch (cf. [25, 4. Cor. 6.10, p. 160]).

5.9 Corollary. *Let L be an arbitrary field extension of k. If $i > 0$ is the Witt index of $(M \otimes_k L, q \otimes_k L)$, then i is the Witt index of $(M \otimes_k F_i, q \otimes_k F_i)$.*

Proof. The field L is a k-specialization of F_i by 5.3. However, L is not a k-specialization of F_{i+1}, for otherwise the Witt index of $(M \otimes_k L, q \otimes_k L)$ would be at least $i+1$ by 5.3. Thus the result follows from 5.3. □

6. Generic splitting of the classical groups

In §§4 and 5 we studied the generic splitting of groups of type ${}^1A_n$, B_n, and certain cases of type ${}^1D_n$ and ${}^2D_n$ (namely, those for which the underlying central k-division algebra is k itself).

In this section we investigate the generic Θ-splitting of G for arbitrary $*$-invariant subsets Θ of Δ and G of types ${}^2A_n$, B_n, C_n, ${}^1D_n$, ${}^2D_n$ in a uniform manner. This is possible because all these groups are special unitary groups of certain (skew-) Hermitian forms over some finite dimensional division algebras over k.

Let k be an infinite field of $\mathrm{char}(k) \neq 2$. Suppose E is a field extension of degree 1 or 2 over k and D is a central division E-algebra of degree d over E. Let $\sigma : D \to D, a \mapsto a^\sigma$, be an involution on D, so that σ is E-linear, of order ≤ 2 and $(ab)^\sigma = b^\sigma a^\sigma$ for all $a, b \in D$. Assume that M is an m-dimensional right D-vector space and that $h : M \times M \to D$ is a non-degenerate ϵ-σ-Hermitian form on M with $\epsilon = \pm 1$. In particular, we have $h(xa, yb) = a^\sigma h(x,y)b,\ \ h(y,x) = \epsilon h(x,y)^\sigma$ for $x, y \in M,\ a, b \in D$. The pair (M, h) is called an ϵ-σ-Hermitian space.

Let now $G := \mathrm{SU}(h)$ be the special unitary group of (M, h). Then the index r of (M, h) is the k-rank of G (cf. [6, 23.9, p. 266]).

If G is of type ${}^2A_n$, then the involution on D is of second type, hence E is separable of degree 2 over k. In this case we let $n \geq 1$. If G is of type B_n, then $d = 1$ and $D = E = k$ and we may assume $n \geq 2$. If G is of type C_n or D_n, we have $E = k$. We may then assume $n \geq 3$.

Let K be a splitting field of G (for example, K is a separable closure of k). The group G_K is isomorphic to $(\mathrm{SL}_{n+1})_K$ (resp. $(\mathrm{SO}_{2n+1})_K$, $(\mathrm{Sp}_{2n})_K$, $(\mathrm{SO}_{2n})_K$) if G is of type ${}^2A_n$ (resp. B_n, C_n, D_n).

Then the absolute rank n of G is the rank of G_K and is given by the formulae $n+1 = md$ in case ${}^2A_n$ and $n = [md/2]$ in the other cases.

In the case ${}^2A_n$ we take the maximal K-split K-torus T given by

$$t = \mathrm{diag}\,(t_1, \ldots, t_{n+1}) \in G(K), \qquad t_1, \ldots, t_{n+1} \in K^*$$

and the basis Δ from §4 for the root system $\Phi = \Phi(G_K, T)$, which is given by $\alpha_i(t) := t_i t_{i+1}^{-1}$, for $i = 1, \ldots, n$.

In the cases B_n, C_n, D_n we proceed as follows. Similarly as in §5 we can use a Witt basis of the underlying bilinear K-space K^{md} to embed G_K into $(\mathrm{SL}_{md})_K$. Then a maximal K-split K-subtorus T of G_K is defined by the set of diagonal matrices (cf. [6, 23.9, p. 266])

$$t = \mathrm{diag}\,(t_1, \ldots, t_n, \hat{1}, t_n^{-1}, \ldots, t_1^{-1}) \in G(K), \quad t_1, \ldots, t_n \in K^*.$$

(Here the symbol $\hat{1}$ means that the component 1 occurs (resp. does not occur) in the middle according as G is of type B_n or not.) A basis $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ of the root system $\Phi = \Phi(G_K, T)$ is given by $\alpha_i(t) := t_i t_{i+1}^{-1}$, for $i = 1, \ldots, n-1$, and, in addition,

$$\alpha_n(t) = \begin{cases} t_n, & \text{if } G \text{ is of type } B_n; \\ t_n^2, & \text{if } G \text{ is of type } C_n; \\ t_{n-1}t_n, & \text{if } G \text{ is of type } D_n \end{cases}$$

(cf. [33, p. 30, 32, 35] and [10, p. 252, 254, 256]).

If $\deg\,(E/k) = 2$, then for a field extension k' of k, the k-algebra $D \otimes_k k'$ is an Azumaya $E \otimes_k k'$-algebra if $E \otimes_k k'$ is a field, or it is a direct sum of two copies of an Azumaya k'-algebra A' if $E \otimes_k k' \cong k' \oplus k'$. In Theorem 6.1 below we use the following notation:

$$\mathrm{ind}\,(D \otimes_k k') := \begin{cases} \mathrm{ind}_{\,E\otimes_k k'}(D \otimes_k k') & \text{if } E \otimes_k k' \text{ is a field;} \\ \mathrm{ind}_{\,k'}(A') & \text{otherwise.} \end{cases}$$

We now prove a theorem corresponding to 4.1 for special unitary groups G. If G is of outer type, we have to replace the set $\Delta \setminus \Theta$ occurring in 4.1 by a suitable set of representatives in $\Delta \setminus \Theta$ of $*$-orbits. The function field $F_\Theta := k_\Theta(V_\Theta)$ is a generic Θ-splitting field of G by 3.16. It has the equivalent properties listed in 3.10 if Θ is $*$-invariant.

6.1 Theorem. *Let G be of type ${}^2A_n, B_n, C_n, {}^1D_n$ or ${}^2D_n$. For each $*$-invariant subset $\Theta \subset \Delta$ let $\{\alpha_{i_1}, \ldots, \alpha_{i_l}\}$ be the set of representatives of $*$-orbits of $\Delta \setminus \Theta$ such that $i_\nu \in \{1, \ldots, n-1\}$ in case ${}^2D_n$ and $i_\nu \in \{1, \ldots, [(n+1)/2]\}$ in case ${}^2A_n$, and let $\{\alpha_{i_1}, \ldots, \alpha_{i_l}\} = \Delta \setminus \Theta$ otherwise. If V_Θ is the k-variety* *associated to* Θ *according to 3.7, then for every field extension* k' *of* k *we have* $V_\Theta(k') \neq \emptyset$ *if and only if* $d' := \operatorname{ind}(D \otimes_k k')$ *divides* $\gcd(d, i_1, \ldots, i_l)$ *and* $\max(i_1, \ldots, i_l) \leq d' \cdot \operatorname{rank}(G_{k'})$.

Proof. If $G_{k'}$ is of inner type ${}^1A_n$ the equivalence follows from 4.1, since the last condition in 4.1 implies the condition $\max(i_1, \ldots, i_l) \leq d' \cdot \operatorname{rank}(G_{k'})$. Therefore it suffices to show the equivalence for the case $k' = k$. Namely, we have to show: The index $d = \operatorname{ind}(D)$ divides $\gcd(d, i_1, \ldots, i_l)$ and $\max(i_1, \ldots, i_l) \leq dr$ if and only if $V_\Theta(k) \neq \emptyset$.

A maximal k-split torus S of G is given, with respect to a Witt basis $\{e_i\}$ of (M, h), by the diagonal matrices $s = \operatorname{diag}(s_1, \ldots, s_r, 1, \ldots, 1, s_r^{-1}, \ldots, s_1^{-1}) \in \mathrm{M}_m(D)$ with $s_j \in k^*$ for $j = 1, \ldots, r$ (cf. [6, 23.9, p. 266]). Let k_s be a separable closure of k.

Using a Witt basis over k_s we may obtain an embedding $G_{k_s} \hookrightarrow (\mathrm{SL}_{md})_{k_s}$ such that a maximal k_s-torus T_{k_s} of G_{k_s} is, respectively, described by matrices $\operatorname{diag}(t_1, \ldots, t_{n+1})$ in case ${}^2A_n$ and $\operatorname{diag}(t_1, \ldots, t_n, \hat{1}, t_n^{-1}, \ldots, t_1^{-1})$ otherwise with $t_1, \ldots, t_{n+1} \in k_s^*$, and S_{k_s}, as a subtorus of T_{k_s}, is given by the following matrices

$$s = \operatorname{diag}(\underbrace{s_1, \ldots, s_1}_{d \text{ times}}, \ldots, \underbrace{s_r, \ldots, s_r}_{d \text{ times}}, \underbrace{1, \ldots, 1}_{(m-2r)d \text{ times}}, \underbrace{s_r^{-1}, \ldots, s_r^{-1}}_{d \text{ times}}, \ldots, \underbrace{s_1^{-1}, \ldots, s_1^{-1}}_{d \text{ times}})$$

with $s_1, \ldots, s_r$ as above.

We first evaluate the cases B_n, C_n, D_n. If G is of type B_n, then $d = 1$ and we find, for $i \in \{1, \ldots, n\}$,

$$\alpha_i(s) = \begin{cases} s_r & \text{if } i = r \leq n \\ s_i s_{i+1}^{-1} & \text{if } 1 \leq i < r \\ 1 & \text{if } i > r. \end{cases}$$

If G is of type D_n and $n = rd + 1$, then $d \leq 2$ and $\alpha_{n-1}(s) = \alpha_n(s) = s_r$. Hence G cannot be of inner type in this case and is necessarily of type ${}^2D_n$.

We find in case $d > 1$

$$\alpha_i(s) = \begin{cases} s_r & \text{if } i = rd < n \\ s_r^2 & \text{if } i = rd = n \\ s_j s_{j+1}^{-1} & \text{if } i = jd, \quad 1 \leq j < r \\ 1 & \text{if } d \not| \; i \text{ or } i > rd \end{cases}$$

for $i \in \{1, \ldots, n-1\}$ if G is of type ${}^2D_n$ and $i \in \{1, \ldots, n\}$ otherwise.

If $d = 1$ and G is of type D_n we have the same formula with the exception that $\alpha_i(s) = s_{n-1}s_n$, if $i = r = n$.

If $d = 1$ and G is of type C_n, then $r = n$ (cf. [35, §91, p. 31]), and we find that $\alpha_i|_S$ is not trivial for all $i = 1, \ldots, r = n$.

In the case ${}^2A_n$ we obtain similarly, for $i = 1, \ldots, [(n+1)/2]$,

$$\alpha_i(s) = \alpha_{n+1-i}(s) = \begin{cases} s_r & \text{if } i = rd < (n+1)/2 \\ s_r^2 & \text{if } i = rd = (n+1)/2 \\ s_j s_{j+1}^{-1} & \text{if } i = jd, \quad 1 \leq j < r \\ 1 & \text{if } d \not| \; i \text{ or } i > rd. \end{cases}$$

Using the notation of 3.4 v) we now see that

$$\Delta \setminus \Delta_0 = \begin{cases} \{\alpha_{jd}, \alpha_{n+1-jd} \mid j = 1, \ldots, r\} & \text{in case } {}^2A_n \\ \{\alpha_{jd} \mid j = 1, \ldots, r\} \cup \{\alpha_n\} & \text{in case } {}^2D_n, \text{ if } d \leq 2 \text{ and } n = rd + 1 \\ \{\alpha_{jd} \mid j = 1, \ldots, r\} & \text{otherwise} \end{cases}$$

for $d \geq 1$. It follows that $d \mid \gcd(d, i_1, \ldots, i_l)$ and $\max(i_1, \ldots, i_l) \leq rd$ if and only if $\Delta \setminus \Theta \subseteq \Delta \setminus \Delta_0$. The latter condition is equivalent to $V_\Theta(k) \neq \emptyset$ by 3.7, "iii) $\Leftrightarrow$ ii)". □

For any $\alpha_i \in \Delta$ let Δ_i be the maximal $*$-invariant subset of Δ which does not contain α_i. It follows from [34, Table II, p. 55ff] or also from the above calculations that all those sets can be described as follows.

$$\Delta_i = \begin{cases} \Delta \setminus \{\alpha_i, \alpha_{n+1-i}\} & \text{if } G \text{ is of type } {}^2A_n \text{ and } i \in \{1, \ldots, [(n+1)/2]\}; \\ \Delta \setminus \{\alpha_i\} & \text{if } G \text{ is of type } B_n, C_n \text{ or } {}^1D_n \text{ or } i < n-1; \\ \Delta \setminus \{\alpha_{n-1}, \alpha_n\} & \text{if } G \text{ is of type } {}^2D_n \text{ and } i = n-1. \end{cases}$$

We emphasize that, in case ${}^2A_n$, the set Δ_i is always of order $n-2$ except if $2 \mid (n+1)$ and $i = (n+1)/2$, in which case it is of order $n-1$ since $\alpha_i = \alpha_{n+1-i}$.

In all cases, let $V_i := V_{\Delta_i}$ be the k-variety associated to Δ_i according to 3.7, and let $F_i = k(V_i)$ be its function field. Applying 6.1 for $\Theta = \Delta_i$ we obtain the following corollary.

6.2 Corollary. *Let $i \in \{1,\dots,[(n+1)/2]\}$ if G is of type ${}^2A_n$, let $i \in \{1,\dots,n\}$ if G is of type B_n, C_n or ${}^1D_n$, and $i \in \{1,\dots,n-1\}$ if G is of type ${}^2D_n$. Then for every field extension k' of k we have $V_i(k') \neq \emptyset$ if and only if $d' := \operatorname{ind}(D \otimes_k k')$ divides i and $1 \le i/d' \le \operatorname{rank}(G_{k'})$.*

We now list generic (quasi-)splitting fields of G with low transcendence degrees. Most, but not all of them, are defined by maximal proper $*$-invariant subsets Θ of Δ. For G of type B_n this is discussed in 5.4.

6.3 Corollary. *Let the notation be as in 6.2.*

i) Let G be of type ${}^2A_n$ and let $n' = [(n+1)/2]$. If $\gcd(n',d) = 1$, then $F_{n'}$ is a generic quasi-splitting field of G. More generally, let $n'' \in \{1,\dots,n'\}$ be some integer such that $\gcd(n',n'',d) = 1$ and let $\Theta := \Delta \setminus \{\alpha_{n'}, \alpha_{n''}, \alpha_{n+1-n'}, \alpha_{n+1-n''}\}$. Then $k(V_\Theta)$ is also a generic quasi-splitting field of G. Moreover, the fields $E \cdot F_{n'}$ and $E \cdot k(V_\Theta)$ are generic splitting fields of G, respectively.

ii) Let G be of type C_n. Then every (generic) splitting field of D is a (generic) splitting field of G. For every odd $i \in \{1,\dots,n\}$ the field $F_i = k(V_i)$ is a generic splitting field of G.

iii) Let G be of type ${}^1D_n$. Then F_{n-1} is a generic splitting field of G. If n is odd or $d = 1$, then F_n is also a generic splitting field of G.

iv) Let G be of type ${}^2D_n$. If n is even or if $d = 1$, then F_{n-1} is a generic quasi-splitting field of G. If n is odd and $\Theta = \Delta \setminus \{\alpha_{n-2}, \alpha_{n-1}, \alpha_n\}$, then $k(V_\Theta)$ is a generic quasi-splitting field. If $d(M)$ denotes the discriminant of the Hermitian space (M,h), then a generic splitting field is given by $k(\sqrt{d(M)}) \cdot F_{n-1}$ if n is even or $d = 1$ and by $k(\sqrt{d(M)}) \cdot k(V_\Theta)$ if n is odd.

Proof. We recall once and for all that, by 3.17 iv) (resp. iii)), some field F_Θ is a generic splitting field (resp. quasi-splitting field) of G if it is a splitting field (resp. quasi-splitting field) of G.

i) Let k_1 be either $F_{n'}$ or $k(V_\Theta)$. Then it follows from 6.1 or 6.2 that $\operatorname{rank}(G_{k_1}) \ge n'$. Therefore G_{k_1} is a special unitary group $(\mathrm{SU}_{n+1})_{k_1}$ of maximal rank and hence quasi-split. Now it follows from 3.6 that $E \cdot k_1$ is a generic splitting field, since E is isomorphic to the field k_{alg}.

ii) If $d = 1$, then G splits [35, §91, p. 31]. Hence the first assertion follows. Since $V_i(F_i) \neq \emptyset$ and d is a power of 2, it follows from 6.1 that $\operatorname{ind}(D \otimes_k F_i) = 1$ for odd i. Thus G_{F_i} splits.

iii) Since $V_i(F_i) \neq \emptyset$ we have $d_i := \operatorname{ind}(D \otimes_k F_i) \mid \gcd(d,i)$ by 6.1. If n is even, then $d_{n-1} = 1$, and if n is odd, then $d_n = 1$, since d is a power of 2. Hence if n is odd we obtain $n \le \operatorname{rank}(G_{F_n})$ from 6.1, therefore G_{F_n} splits.

We have $(n-1)/d_{n-1} \le \operatorname{rank}(G_{F_{n-1}})$ by 6.1. Therefore, if $d_{n-1} = 1$, then $G_{F_{n-1}}$ is the orthogonal group of a quadratic form of dimension $2n$ with discriminant 1 and of Witt index $\ge n-1$. Hence the form is hyperbolic, which implies that $G_{F_{n-1}}$ splits.

It remains to show that $d_{n-1} = 1$ for odd n. If n is odd, then $d \le 2$, because $d \mid 2n$ and d is a power of 2. Hence $d_{n-1} \mid 2$. Assume $d_{n-1} = 2$. Then the rank r_{n-1} of $G_{F_{n-1}}$ is at least $(n-1)/2$ which implies $n = r_{n-1}d_{n-1} + 1$. This is impossible since G is of inner type ${}^1D_n$ (cf. [32. Table II, p. 56]). Hence necessarily $d_{n-1} = 1$.

iv) If n is even or $d = 1$, then, as above, $d_i := \operatorname{ind}(D \otimes_k F_i) = 1$, hence $G_{F_{n-1}}$ is the orthogonal group of a quadratic form of dimension $2n$ over F_{n-1} with Witt index $n-1$ and discriminant $d(M) \neq 1$, since F_{n-1} is regular over k. Hence $G_{F_{n-1}}$ is quasi-split. If n is odd, then $d \le 2$ (cf. the proof of iii)). Obviously Θ is $*$-invariant. Hence we may apply 6.1 to find that $d' = \operatorname{ind}(D \otimes_k F_\Theta) = 1$ and $n - 1 \le \operatorname{rank}(G_{F_\Theta})$. Therefore G_{F_Θ} is of rank $n-1$ which means that it is quasi-split. Since G is semisimple, we have $k(\sqrt{d(M)}) = k_{\mathrm{alg}}$, where k_{alg} is given by 3.6. The rest of the statement follows from 3.6. □

6.4 Remark. For groups of outer type there are other non-regular generic splitting fields with possibly lower transcendence degrees: Let $\Theta \subset \Delta$ be any not necessarily $*$-invariant subset. By 3.16 there is a generic Θ-splitting field $F_\Theta = k_\Theta(V_\Theta)$ of G. Applying the results of §4 (resp. §5) to G_{k_Θ} for special proper maximal subsets Θ we obtain for example:

i) In case ${}^2A_n$: Let $\Theta = \Delta \setminus \{\alpha_1\}$. Then $k_\Theta = E$ and V_Θ is an n-dimensional Severi-Brauer variety over E and the function field $E(V_\Theta)$ is a generic splitting field of G_{E_Θ}, and hence of G.

ii) In case ${}^2D_n$: Let $\Theta = \Delta \setminus \{\alpha_{n-1}\}$. Then $k_\Theta = k(\sqrt{d(M)})$ and V_Θ is an $n(n-1)/2$-dimensional variety (cf. 5.5) over k_Θ and the function field $k_\Theta(V_\Theta)$ is a generic splitting field of G_{k_Θ} by 6.3 iii), and hence of G.

6.5 Remark. In case ${}^2A_n$ and $2 \mid (n+1)$ we find an outer form of a generalized Severi-Brauer variety as discussed in §4: Let $i := (n+1)/2$ and $\Delta_i = \Delta \setminus \{\alpha_i\}$. Then Δ_i is $*$-invariant, hence the associated variety V_{Δ_i} is defined over k (cf. 3.7). This is the outer form of $\mathrm{Grass}_i(k^{n+1})$ mentioned after 4.6.

7. Generic splitting of almost simple groups

In this paragraph we will give generic splitting and quasi-splitting fields of the absolutely almost simple k-groups including the exceptional groups and groups over fields of characteristic 2 which have been excluded in §§5 and 6. We emphazise that the notions of quasi-splitting field and splitting field coincide in the case of semi-simple groups of inner type, as it follows from the last statement of 3.4 v). In the outer type case, one may obtain generic splitting fields out of generic quasi-splitting fields by applying 3.6. Therefore, in this case we only give quasi-splitting fields. But the same method can also be used to construct generic splitting fields directly.

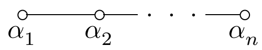
Let G be any almost simple k-group. Let T be a maximal k-torus of G. Let $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ be the set of simple roots of $G_{\bar{k}}$ with respect to $T_{\bar{k}}$ and some ordering of the root system. We will give generic splitting or quasi-splitting fields of G in terms of its Dynkin diagram (over a splitting field) by describing the maximal subsets Θ of Δ such that F_Θ is a generic splitting or quasi-splitting field. We will use the abbreviations $\Delta_i = \Delta \setminus \{\alpha_i\}$ and $F_i = F_{\Delta_i}$.

7.1 Lemma. *Let $\Theta \subset \Delta$. Then $\mathrm{res}_{F_\Theta}(\alpha) \neq 0$ for all $\alpha \in \Delta \setminus \Theta$.*

Proof. As F_Θ contains k_Θ, we may assume that $k = k_\Theta$. Hence we may assume that Θ is $*$-invariant. Then our claim will follow from 3.7,"i) $\Rightarrow$ iii)", applied to F_Θ instead of k. □

In the following, we will indicate how to use 7.1 together with 3.17 iii) or iv) and the information encoded in the index of G as described in [32, §2, p. 38ff] just for the particular case ${}^1A_n$, since the considerations in all the other cases are quite similar.

Dynkin diagram of type A_n:



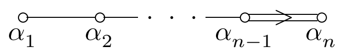
Generic splitting fields in case ${}^1A_n$ are given by F_i for any i which is coprime to the index d of the underlying central k-division algebra D (cf. 4.3). In order to see this, let G denote an almost simple k-group of type ${}^1A_n$ and let k' be a field extension of k.

We will verify that, for any $i \in \{1, \ldots, n\}$ coprime to d, the condition $\mathrm{res}_{k'}(\alpha_i) \neq 0$ implies that $G_{k'}$ is split. It follows from the description of the index of $G_{k'}$ in the sense of [32, §2, p. 38ff] that $\mathrm{res}_{k'}(\alpha_i) \neq 0$ if and only if i is a multiple of $d' := \mathrm{ind}\,(D \otimes_k k')$. As d' divides both d and i, and since $\gcd(d, i) = 1$ by assumption, we find $d' = 1$, which implies that $G_{k'}$ is split.

In particular, for $k' = F_i$, it follows from 7.1 that $\mathrm{res}_{F_i}(\alpha_i) \neq 0$. By the above, G_{F_i} splits, and 3.17 iii) or iv) proves that F_i is a generic splitting field of G.

For ${}^2A_n$, we use the notation of [33, p.55] or of §4 and let $\varrho = [(n+1)/2]$. Generic quasi-splitting fields in case ${}^2A_n$ are given by $\Delta \setminus \{\alpha_\varrho, \alpha_{n+1-\varrho}\}$ if $\gcd(\varrho, d) = 1$ and by $\Delta \setminus \{\alpha_\varrho, \alpha_{n+1-\varrho}, \alpha_{\varrho-1}, \alpha_{n+2-\varrho}\}$ if $\gcd(\varrho, d) \neq 1$ (cf. 6.3 i)).

Dynkin diagram of type B_n:

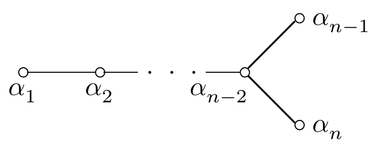


Generic splitting fields are given by Δ_n (cf. 5.4).

Dynkin diagram of type C_n:

Generic splitting fields are given by Δ_i for any i which is coprime to the index of the underlying division algebra. As this is a power of two, i just has to be odd in this case (cf. 6.3 ii)).

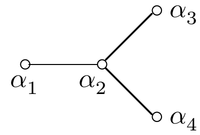
Dynkin diagram of type D_n:



Generic splitting fields for ${}^1D_n$ are given by Δ_{n-1} (cf. 6.3 iii)). For the outer case we again use the notation of [33, p.57] which is consistent with §§5 and 6.

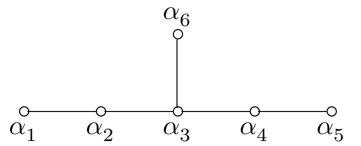
Generic quasi-splitting fields for ^2D_n are given by $\Delta \setminus \{\alpha_{n-1}, \alpha_n\}$ if $d = 1$ or $2 \mid n$ and by $\Delta \setminus \{\alpha_{n-2}, \alpha_{n-1}, \alpha_n\}$ if n is odd (which implies that $d \le 2$ since d is a power of 2 and divides $2n$) and $d = 2$ (cf. 6.3 iv)).

Dynkin diagram of type $^3D_4, {}^6D_4$:



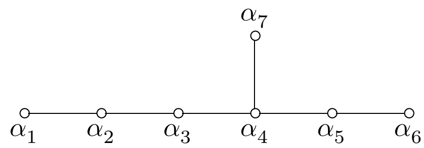
A generic quasi-splitting field is given by $\Theta = \{\alpha_2\}$.

Dynkin diagram of type E_6:



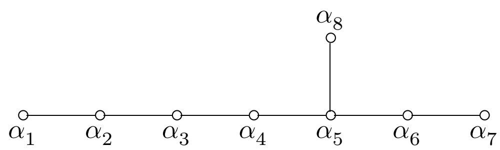
Generic quasi-splitting fields are given in case $^1E_6$ by Δ_2, Δ_4, in case $^2E_6$ by $\Delta \setminus \{\alpha_2, \alpha_4\}$.

Dynkin diagram of type E_7:



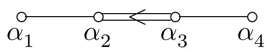
Generic splitting fields are given by $\Delta_3, \Delta_5, \Delta_7$.

Dynkin diagram of type E_8:



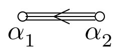
Generic splitting fields are given by $\Delta_4, \Delta_5, \Delta_6, \Delta_8$.

Dynkin diagram of type F_4:



Generic splitting fields are given by $\Delta_2, \Delta_3, \Delta_4$.

Dynkin diagram of type G_2:



Generic splitting fields are given by Δ_1, Δ_2.

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