Splitting Patterns of Excellent Quadratic Forms

(to appear in Crelle's Journal)

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Dedicated to Martin Kneser on the occasion of his 65^{th} *birthday*

Abstract. The Witt indices which may occur after base field extension for a given anisotropic excellent quadratic form are uniquely determined by the dimension of the form and independent of the base field and of the particular form chosen. We generalize that result to a larger class of forms and give an easy method of computing these indices. In particular, this can be done for the form given by a sum of squares. This gives a natural generalization of Pfister's theorems on the level of fields. Another result is the existence of families of anisotropic special orthogonal groups of unbounded absolute rank which have a joint generic splitting field of small transcendence degree including examples of transcendence degree 0.

Introduction

The splitting pattern of a regular quadratic form q over a field k of characteristic different from 2 is the sequence of distinct Witt indices of q which may occur after base field extension. For a given regular form $\bar{q}$ we define a class of forms which we call $\bar{q}$-extensions. We will show that the anisotropic members q of this class all have a splitting pattern depending only on the dimension of q and on the splitting pattern of $\bar{q}$. This dependency is determined in §2. For example, Pfister forms are of this type (here we have $\bar{q} = 0$); their Witt index is either 0 or maximal. For every form $\bar{q}$ there are $\bar{q}$-extensions of arbitrarily high dimension. The $\bar{q}$-extensions with $\dim \bar{q} \leq 1$ are precisely the "excellent" forms introduced by Knebusch, who gave a recursive formula for their possible Witt indices in [3, 7.11]. In general, the determination of the splitting pattern of a given quadratic form seems to be very difficult, however, for excellent forms there is an easy way of describing their splitting patterns; this will be discussed in §2.

Our investigations have interesting consequences. For example, they generalize the results of Pfister that the level of fields is infinite or a power of 2, and that fields exist having a prescribed power of 2 as its level (cf. 3.4).

Another important consequence is the fact that families of anisotropic forms of arbitrarily high dimension exist which have a field of low transcendence degree as a joint generic splitting field (cf. 2.5, 2.14). This behavior is completely different from the splitting behavior of (finite-dimensional) central simple division algebras over k, since a theorem of Witt, Amitsur and Roquette says that a generic splitting field of such an algebra D splits precisely those division algebras which represent powers of D in the Brauer group. This is always a finite set of division algebras.

1. Generic splitting towers of quadratic forms

Let V be a vector space of dimension $n \geq 2$ over a field k of characteristic different from 2 and $q : V \to k$ be a regular quadratic form on V with discriminant $d(q) \in k^*/k^{*2}$. We are interested in the possible Witt indices

$$0 \leq i_0 < i_1 < \ldots < i_{h-1} < i_h = [n/2]$$

* The first-named author gratefully acknowledges support by the Sonderforschungsbereich 343 "Diskrete Strukturen in der Mathematik" during a stay at the University of Bielefeld in May 1992

which occur for the quadratic forms $q_L := q \otimes L$ with L running through all field extensions of k.

These Witt indices can be investigated by means of the notion of the so-called generic splitting tower of q considered by Knebusch [2, §5] (cf. also [4, Ch. 4, 6.9, p. 160]). However, our definition is slightly more general than that of [2].

For any quadratic form q, let $i(q)$ denote its Witt index.

1.1 Definition. A *generic splitting tower of* q is a sequence $k = K_0, K_1, \dots, K_h$ of field extensions of k with the following properties:

i) The associated sequence of Witt indices $i_j := i(q_{K_j})$ is strictly increasing.
ii) For every field extension L of k, there is a j such that $i(q_L) = i_j$.
iii) Every field extension L of k with $i(q_L) \geq i_j$ is a k-specialization of K_j (that is, there is a k-place $K_j \to L \cup \{\infty\}$).

The sequence $(i_0, i_1, \dots, i_h)$ is called the *splitting pattern* of q, the number h is called the *height* of q and will be denoted by $h(q)$.

1.2 Remark. i) and iii) imply the existence of the k-specializations $K_i \to K_j$ for $i < j$. Hence we obtain a sequence $k = K_0 \to K_1 \to \dots \to K_h$ of k-specializations. In [2] a sequence of embeddings was required. But our definition allows generic splitting towers $\{K_i\}$ with not necessarily non-decreasing transcendence degrees, which often exist, as can easily be seen from 2.5, 2.10, 2.14.

If $\{k = L_0, L_1, \dots, L_{h'}\}$ is a second generic splitting tower of q, then obviously $h' = h$ and for $j = 0, \dots, h$ the fields K_j and L_j are k-spezializations of each other.

A natural choice for a generic splitting tower of q is described in [1, §5]: For $i = 1, \dots, [n/2]$, the variety of i-dimensional totally isotropic subspaces of V is defined over k except if n is even, $i = n/2$ and $d(q) \neq 1$, in which case it is defined over $k(\sqrt{d(q)})$. (This variety is absolutely irreducible except when n is even and $i \geq n/2-1$. In this case it consists of two irreducible components, and for our purposes we just may take one of them, cf. [1, 5.2].) Its function field F_i is a generic field for the problem of splitting off at least i hyperbolic planes from V. That is, $i(q_{F_i}) \geq i$, and every field extension L of k with $i(q_L) \geq i$ is a k-specialization of F_i. Also, for every such L, we have $i(q_L) \geq i$ if and only if the free composite $F_i \cdot L$ is a purely transcendental extension of L [1, 5.3 and 5.7]. We get a generic splitting tower of q by taking the subsequence of $\{F_i\}$ obtained by avoiding repetitions of Witt indices; that is, we define $K_0 = k$, and, if K_j is already defined and $i_j < [n/2]$, then K_{j+1} is F_i where i is minimal with $i > i_j$. If $i_j = [n/2]$ we set $h = j$ (cf. [1, 5.8 and 5.9]). If necessary, we write $F_i(q)$ (resp. $K_i(q)$) instead of F_i (resp. K_i).

It is immediate from 1.1 ii) that the numbers i_j are the only possible Witt indices for q_L, where L runs through all field extensions of K.

For example, if q is a regular subform of codimension ≤ 1 of an anisotropic Pfister form, then the only possible Witt indices of q are $i_0 = 0$ and $i_1 = [n/2]$ since, for every such form q and every field extension L of k, we have: if q_L is isotropic, then q_L splits completely. For Pfister forms this follows from [4, Ch. 4, Cor. 1.5, p. 144]. If q is of codimension 1 in some Pfister form, say $\varphi = q \perp \langle a \rangle$ for some $a \in k^*$, and if q_L is isotropic, then φ_L splits totally and therefore has an $(n+1)/2$-dimensional totally isotropic subspace which intersects the space of q_L in a subspace of dimension $[n/2]$, hence q_L splits completely.

2. Excellent quadratic forms

2.1 Remark. Let $\bar{q} = \langle a_1, \ldots, a_n \rangle$ be a regular quadratic form over k. Then the Pfister form $\varphi := \langle 1, a_1^{-1} a_2 \rangle \otimes \ldots \otimes \langle 1, a_1^{-1} a_n \rangle$ is of dimension 2^{n-1} and has an orthogonal summand $\langle 1, a_1^{-1} a_2, \ldots, a_1^{-1} a_n \rangle$ which is similar to $\bar{q}$. Thus we have an orthogonal decomposition $a_1 \varphi \cong q \perp \bar{q}$ with some regular form q.

2.2 Definition. Let q be a regular quadratic form over k.

i) The *level* $\ell(q)$ of q is the smallest number l such that q is an orthogonal summand of a scalar multiple of an anisotropic l-dimensional Pfister form.

ii) q is a *Pfister neighbour* if there exists a Pfister form φ with $(\dim \varphi)/2 < \dim q$, some $a \in k^*$ and some form q' such that $a\varphi \cong q \perp q'$. The forms φ (resp. q') are called the *associated* Pfister form (resp. the *complementary* form) of q, and $\dim q'$ is called the *codimension* of q.

iii) Let $\bar{q}$ be a regular quadratic form over k. The form q is called a *$\bar{q}$-extension* if there is a sequence $q = q_0, q_1, \ldots, q_r = \bar{q}$ of quadratic forms over k of length $r \geq 0$ such that, for $j < r$, the form q_j is a Pfister neighbour with complementary form q_{j+1}. The number r is called the *order* of the *$\bar{q}$-extension* q and denoted by $\operatorname{ord}(q, \bar{q})$. An *excellent* form q is a $\bar{q}$-extension with $\dim \bar{q} \leq 1$, and $\operatorname{ord}(q)$ denotes its order.

2.3 Remarks and Examples.

i) The notions of a Pfister neighbour and of an excellent form have been introduced and studied by Knebusch [3, 7.4ff.]. It is proved in [3, l.c.] that φ and q' in 2.2 ii) are uniquely determined up to isomorphism by q, hence it follows that the sequence $q_0, \ldots, q_r$ in 2.2 iii) is uniquely determined up to isomorphism.

ii) Clearly the level of a form q is always a power of 2. By 2.1 we obtain $\ell(q) \leq 2^{(\dim q)-1}$. A form q is a Pfister neighbour if and only if $\ell(q) < 2 \dim q$.

iii) Let $n, t \in \mathbb{N}$ and let t be minimal such that $2^t \geq n$. Then the quadratic form $n \times \langle 1 \rangle$ is excellent of level $l = 2^t$, as is easily seen by induction. Hence there are (anisotropic) excellent forms of any dimension.

iv) If $\bar{q}$ is excellent (which is in particular true if $\dim \bar{q} \leq 3$), then every $\bar{q}$-extension is also excellent.

v) For excellent anisotropic q, it follows from [3, Prop. 7.9] that $h(q) = \operatorname{ord}(q)$. In 2.12 below we will show that, for an arbitrary anisotropic $\bar{q}$-extension q, the relation $h(q) \leq \operatorname{ord}(q, \bar{q}) + h(\bar{q})$ holds with equality for excellent forms, which shows that $\operatorname{ord}(q, \bar{q})$ plays the role of a 'relative' height for anisotropic forms.

2.4 Proposition. *Let q be a $\bar{q}$-extension of order r. If $\bar{q}$ is isotropic, then the anisotropic kernels of q and of $(-1)^r \bar{q}$ are isomorphic. If $\bar{q}$ is anisotropic, then the anisotropic kernel of q is isomorphic to one of its 'signed' complementary forms $(-1)^j q_j$ for some j with $0 \leq j \leq r$. In particular, the anisotropic kernel of an excellent form is excellent.*

Proof. We may assume that q is isotropic. Therefore $q \perp q'$ splits completely since it is similar to an isotropic Pfister form, and hence q defines the same Witt class as $-q'$, which gives the assertion. □

We give an example concerning generic splitting fields of special orthogonal groups. Recall that a generic splitting field of a reductive linear algebraic k-group G in the sense of [1] is a field extension F of k such that G_F is split and such that the field extensions

L of k for which G_L is split are the k-specializations of F. If $G = \mathrm{SO}(q)$ is the special orthogonal group of the regular quadratic form q of dimension n, then G_F is split if and only if $i(q_F) = [n/2]$. This condition means that q_F is totally split.

2.5 Theorem. *Let q be a $\bar{q}$-extension with $\dim \bar{q} \geq 2$, let $G = \mathrm{SO}(q)$ and $\bar{G} = \mathrm{SO}(\bar{q})$. A field extension F of k is a (generic) splitting field of G if and only if it is a (generic) splitting field of $\bar{G}$.*

Proof. It follows immediately from 2.4 and the preceding remark that, for $\dim \bar{q} \geq 2$, the field F is a splitting field of G if and only if it is a splitting field of $\bar{G}$.

Let F be a generic splitting field of $\bar{G}$ and L a splitting field of G. Then L is a splitting field of $\bar{G}$ and hence is a k-specialization of F. Therefore F is a generic splitting field of G. The converse follows similarly. □

2.6 Lemma. *Every natural number n can be written uniquely as an alternating sum of 2-powers*

$$n = 2^{a_h} - 2^{a_{h-1}} + 2^{a_{h-2}} - + \ldots + (-1)^{h-1} 2^{a_1} + (-1)^h \epsilon$$

with $\epsilon, h, a_1, \ldots, a_h \in \mathbb{N} \cup \{0\}$ satisfying $0 < a_1 < a_2 < \ldots < a_{h-1} < a_h$ and

$$\epsilon = \begin{cases} 0 \text{ and } a_1 < a_2 - 1 & \text{if } n \text{ is even,} \\ 1 \text{ and } 1 < a_1 & \text{if } n \text{ is odd.} \end{cases}$$

In particular, 2^{a_h} is the smallest 2-power greater than or equal to n.

Proof. Let $n = 2^g \epsilon_0 + 2^{g-1} \epsilon_1 + \ldots + 2\epsilon_{g-1} + \epsilon_g$ denote the dyadic expansion of n with $\epsilon_i \in \{0, 1\}$. Each subsequence of $\{\epsilon_i\}$ consisting of consecutive 1's only and being maximal with this property represents some subsummand of type $2^a - 2^b$ with $a > b$. If the last one is such that $a = b + 1$, then this can be replaced by 2^b. This proves the Lemma. □

2.7 Definition. For $n \in \mathbb{N}$, the expansion of 2.6 is called the *alternating 2-expansion of n*. For $j = 0, \ldots, h$, the numbers $2^{a_j} - 2^{a_{j-1}} + - \ldots \pm 2^{a_1} \mp \epsilon$ are called the *j-th alternating 2-partial sums of n* and denoted by $n^{(j)}$. The number h is called the *height* $h(n)$ of n.

2.8 Corollary. *Let q be a $\bar{q}$-extension of order r with $\dim q = n$, $\dim \bar{q} = n_0$, and let n be expanded as in 2.6. Then $h = h(n) = r + h(n_0)$, $\ell(q) = 2^{a_h}$ and $\dim q_j = n^{(h-j)}$ for any $j = 0, \ldots, r$.*
If $\bar{q}$ is anisotropic and j is such that $(-1)^j q_j$ is the anisotropic kernel of q, then $i(q)$ is given by

$$i_j(n) := (n - n^{(h-j)})/2 = \begin{cases} 2^{a_h - 1} - + \ldots - 2^{a_{h-j+1} - 1} & \text{if } j \text{ is even,} \\ 2^{a_h - 1} - + \ldots + 2^{a_{h-j+1} - 1} - \dim q_j & \text{if } j \text{ is odd.} \end{cases}$$

In particular, for $j = 1$, the smallest non-trivial Witt index is given by

$$\begin{aligned} i_1(n) &= 2^{a_h - 1} - 2^{a_{h-1}} + 2^{a_{h-2}} - + \ldots + (-1)^{h-1} 2^{a_1} + (-1)^h \epsilon \\ &= n - 2^{a_h - 1} = n - \ell(q)/2, \end{aligned}$$

which is the remainder of n modulo the highest power of 2 below n.

2.9 Proposition. *Let q be an anisotropic $\bar{q}$-extension with $r = \operatorname{ord}(q, \bar{q}) > 0$ and with complementary form q'.*

i) Let $r > 1$ or let $q'_{F_1(q)}$ be anisotropic. Then $-q'_{F_1(q)}$ is isomorphic to the anisotropic kernel of $q_{F_1(q)}$.

ii) If $q'_{F_1(q)}$ is isotropic, then the fields $F_1(q)$ and $F_1(q')$ are k-specializations of each other.

Remark. It is not known whether the condition in 2.9 ii) can hold for a Pfister neighbour q with complementary form q'. (Cf. [3, 8.3].)

Proof. i) By 2.4 we only have to show that $q'_{F_1(q)}$ is anisotropic if q' is a Pfister neighbour. Hence we may assume that $r > 1$. Let φ' be the Pfister form associated to q'. Remark 2.3.ii) implies $\dim q > \dim \varphi'$. If $q'_{F_1(q)}$ were isotropic, then $\varphi'_{F_1(q)}$ would split completely. Hence, by the Cassels-Pfister subform theorem [4, Ch. 4, 5.4 ii), p. 155], q would be a subform of φ', which is impossible.

ii) Clearly $F_1(q)$ is a k-specialization of $F_1(q')$. Conversely, $F = F_1(q')$ splits the Pfister form φ associated to q_F totally, and hence any of its maximal isotropic subspaces intersects the space of q_F non-trivially for dimension reasons, hence q_F is isotropic and therefore F is a k-specialization of $F_1(q)$. □

2.10 Corollary. *Let $1 \le i \le [n/2]$.*

i) For $j = 1, \dots, r-1$ the anisotropic kernel of $q_{F_i(q)}$ is isomorphic to $(-1)^j (q_j)_{F_i(q)}$ if and only if $i_{j-1}(n) + 1 \le i \le i_j(n)$. This is also true for $j = r$ if $\bar{q}_{F_1(q_{r-1})}$ is anisotropic.

ii) Let $K_0 = k$ and for $j = 1, \dots, r$ let $K_j = F_1(q_{j-1})$. Then the sequence $\{K_j\}$ is an initial sequence of a generic splitting tower of q. In particular, if q is excellent, then this is a generic splitting tower.

Proof. i) If $1 \le i \le i_1(n)$, then the free composite F of the fields $F_1(q)$ and $F_i(q)$ is a purely transcendental extension of $F_1(q)$ by [1, Th. 5.3]. Therefore $-q'$ is anisotropic over F, since it is so over $F_1(q)$ by 2.9, and hence it is anisotropic over $F_i(q)$. It follows that the anisotropic kernels of $q_{F_1(q)}$ and of $q_{F_i(q)}$ are both obtained from $-q'$ by base extension. On the other hand, if $i > i_1(n)$, then $i(q_{F_i(q)}) \ge i > i_1(n)$ and hence $q'_{F_i(q)}$ is isotropic. This proves the claim for $r = 1$. Since the complementary form of q_j is q_{j+1}, an induction on j gives the rest of the statement.

ii) This follows from 2.8, applied to the form q_{K_j} over K_j. □

2.11 Theorem. *Let q be an anisotropic $\bar{q}$-extension of order $r = \operatorname{ord}(q, \bar{q})$ with $n = \dim q$. Then the first r components of the splitting pattern of q are always given by $i_j = i_j(n)$ for $j = 0, \dots, r-1$. Its components for $j \ge r$ are given by*

$$i_j = i_r(n) + \begin{cases} j_{j-r} & \text{if } \bar{q}_{F_1(q_{r-1})} \text{ is anisotropic,} \\ j_{j-r+1} & \text{otherwise,} \end{cases}$$

where $(j_0, \dots, j_t)$ is the splitting pattern of $\bar{q}$.

Proof. The j^{th} component of the splitting pattern of q is the Witt index of q_{K_j}. By 2.8, applied to q_{K_j} instead of q, this is $i_j(n)$ for $j < r$, and the same argument, together with 2.9, works for $j = r$. For $j > r$ the statement follows from 2.4, applied to $q_{K_\nu(\bar{q})}$, where $K_\nu(\bar{q})$ runs through a generic splitting tower of $\bar{q}$. □

2.12 Corollary. *If q is an anisotropic $\bar{q}$-extension of order r, then $h(q) \leq \operatorname{ord}(q,\bar{q}) + h(\bar{q})$. Equality holds if $\bar{q}_{F_1(q_{r-1})}$ is anisotropic, and otherwise we have $h(q) = \operatorname{ord}(q,\bar{q}) + h(\bar{q}) - 1$. In particular, if q is excellent, then we have $h(q) = h(\dim q)$.*

2.13 Illustration. Let q be an anisotropic excellent form of dimension n over a field k. The following data hold independently of the field k and of the particular excellent form q. For example, they hold for the form given by the sum of n squares over any field with level $\geq n$.

i) $n = 12 = 2^4 - 2^2$, thus $h(q) = 2$ and
$\dim q_1 = 4, \dim q_2 = 0$.
Splitting pattern: $(0, 4, 6)$.

ii) $n = 123 = 2^7 - 2^3 + 2^2 - 1$, thus $h(q) = 3$ and
$\dim q_1 = 5, \dim q_2 = 3, \dim q_3 = 1$.
Splitting pattern: $(0, 59, 60, 61)$.

iii) $n = 1234 = 2^{11} - 2^{10} + 2^8 - 2^6 + 2^5 - 2^4 + 2$, thus $h(q) = 7$ and
$\dim q_1 = 814, \dim q_2 = 210, \dim q_3 = 46, \dim q_4 = 18$,
$\dim q_5 = 14, \dim q_6 = 2, \dim q_7 = 0$.
Splitting pattern: $(0, 210, 512, 594, 608, 610, 616, 617)$.

The anisotropic quadratic forms of height 1 are known to be the forms similar to an anisotropic subform of a Pfister form of codimension ≤ 1; that is, these are exactly the anisotropic excellent forms of dimension 2^{a_1} or $2^{a_1} - 1$ (cf. [2, Th. 5.8]).

The anisotropic excellent forms of height 2 are clearly the anisotropic excellent forms of dimension $2^{a_2} - 2^{a_1}$ with $a_1 \geq 1, a_2 > a_1 + 1$ or of dimension $2^{a_2} - 2^{a_1} + 1$ with $a_1 \geq 2$ and $a_2 > a_1$. Compare [3, §10] for results and questions about anisotropic forms of height 2 that are not excellent.

2.14 Corollary. *Let q be excellent of dimension n. Let n be expanded as in 2.6. Then $G = \mathrm{SO}(q)$ has a generic splitting field of transcendence degree $\leq 2^{a_1} - \epsilon - 2$. In particular, G has an algebraic generic splitting field of degree 2 – namely $k(\sqrt{d(q)})$ – if $n \equiv 2 \pmod 4$, and G has a generic splitting field which is the function field of a complete curve defined over k if $n \equiv 3, 4, 5 \pmod 8$. The curve can be taken to be the Severi-Brauer variety associated to the even Clifford algebra of the ternary form q_{h-1} in case $n \equiv 3, 5 \pmod 8$. Similarly, in case $n \equiv 4 \pmod 8$ the curve can be taken to be the Severi-Brauer variety associated to the quaternion algebra which is related to the quaternary form q_{h-1}.*

Proof. q is a q_{h-1}-extension for the (excellent) form q_{h-1} of dimension $n_1 = 2^{a_1} - \epsilon$. Hence, by 2.5, a generic splitting field of $\mathrm{SO}(q_{h-1})$ is one of G. By the preceding remark and by [1, 5.4], q_{h-1} is of height 1, hence the field $F_1(q_{h-1})$ (resp. $k(\sqrt{d(q_{h-1})})$) is a generic splitting field of $\mathrm{SO}(q_{h-1})$ if $n_1 \geq 3$ (resp. $n_1 = 2$). By [1, 5.5], this field has transcendence degree $n_1 - 2$. Let $n \equiv 3, 5 \pmod 8$. It follows from [4, Ch. 2, 14.3 (i)], that the ternary form q_{h-1} splits if and only if its even Clifford algebra $C = C(q_{h-1})$ splits. But C is a quaternion algebra, and it has been discovered by Witt [5] a long time ago that the generic splitting field of a quaternion algebra is the function field of its associated Severi-Brauer variety; that is, of the variety of its nilpotent elements (cf. [1, 3.20]).

The curve in case $n \equiv 4 \pmod 8$ is obtained as the Severi-Brauer variety of the quaternion algebra A associated to the Pfister form related to q_{h-1}. Namely, since in this case $d(q_{h-1}) = 1$, it follows from [4, Ch. 2, 14.3 (ii)] that q_{h-1} splits if and only if A splits. □

3. Witt indices, the level of fields, and applications to number fields.

Corollary 2.8 gives some interesting relations between the possible Witt indices and the height of excellent quadratic forms. For anisotropic excellent forms q of height 1 everything is clear: $i_1 = [n/2]$ with $n = \dim q$. In the case of height ≥ 2 we obtain:

3.1 Proposition. *Let q be an anisotropic excellent form of height $h \geq 2$. Then $i_1(n) \geq 2^{h-2}$ with strict inequality for $h \geq 4$.*

Proof. For $h = 2$ there is nothing to prove. Hence we may assume $h \geq 3$. Let n be given in alternating 2-expansion as in 2.6. It follows from 2.8 that

$$i_1(n) = 2^{a_h - 1} - 2^{a_{h-1}} + \dim q_{h-2} = 2^{a_{h-1}}(2^{a_h - a_{h-1} - 1} - 1) + \dim q_{h-2} \geq \dim q_{h-2}.$$

Hence we have $i_1(n) \geq \dim q_{h-2} > \ell(q_{h-2})/2 = 2^{a_{h-2}-1}$, where the inequality follows immediately from the definition of the level of a quadratic form, and since $a_j - 1 \geq j$ for every $j = 2, \ldots, h$, we obtain in fact $i_1(n) > 2^{h-2}$ for $h \geq 4$. □

The information about the dimensions of the anisotropic kernels q_j stated in 2.8 makes the parity of all Witt indices explicit for anisotropic excellent forms of dimension n. One concludes:

Let n be even; then all Witt indices are even, except – of course – possibly the last one, $i_h = [n/2]$.

Let n be odd; then the parity of the Witt indices is alternating; namely, $i_j \equiv j \pmod 2$ for $0 \leq j \leq h$.

In particular, this yields for odd dimensional forms:

3.2 Remark. An anisotropic excellent form of odd dimension n has an even height if and only if $n \equiv 1 \pmod 4$.

Proof. For anisotropic excellent forms of odd dimension n we have $n = 2\, i_h + 1$ with $i_h \equiv h \pmod 2$. □

An immediate consequence of the last statement in 2.8 is the following corollary.

3.3 Corollary. *An isotropic excellent form q is of Witt index 1 if and only if its (first) complementary form is anisotropic and $\dim q = 2^\kappa + 1$ for some $\kappa \in \mathbb{N} \cup \{0\}$.*

3.4 Remark. Of course 3.3 implies, in the case of sums of squares, Pfister's theorem that the level of any field k is either infinite or a power of 2 (cf. [4, Ch. 2, 10.8, p. 71]). Namely, the level $s = s(k)$ of k is the smallest number s such that $Q_s := s \times \langle 1 \rangle$ represents -1 over k, or, equivalently, the biggest s such that Q_s is anisotropic over k. It follows from 2.3 iii) and 3.3, applied to Q_{s+1}, that this number is infinite or a power of 2, which is Pfister's result. If $s(k)$ is finite, the anisotropic form $Q_{s(k)}$ is a Pfister form and hence its level is just the level of k.

The statements 2.9 and 3.3 also give a proof of the fact that fields of arbitrary level exist (cf. [4, Ch. 4, 4.3 , p. 152]): Let $q = (2^\kappa + 1) \times \langle 1 \rangle$ over some formally real field k. Then, by 2.9 and 3.3, the field $F_1(q)$ is of level 2^κ. In fact, if k is taken to be $\mathbb{Q}$, the field of rationals, then this gives a "generic construction" of such a field.

As another application, we describe the splitting behavior of excellent forms over $\mathbb{Q}$ after base extension by number fields F.

3.5 Theorem. *Let q be an anisotropic excellent form over $\mathbb{Q}$ of dimension n. Then, for every number field F, the only possible Witt indices of q_F occur among*

$$\begin{array}{lll} 0, [n/2] & \text{if } n \equiv 0, \pm 1 & \pmod 8, \\ 0, [n/2]-1, [n/2] & \text{if } n \equiv \pm 2, \pm 3 & \pmod 8, \\ 0, [n/2]-2, [n/2] & \text{if } n \equiv 4 & \pmod 8. \end{array}$$

Proof. For $n \leq 3$ our claim is obvious; for $n = 4$ it is true since the splitting pattern is given by $i_0 = 0, i_1 = 2$. Hence we may assume that q is of dimension $n \geq 5$. Let F be a number field such that q_F is isotropic.

Since every 5-dimensional indefinite form q over $\mathbb{Q}$ is isotropic, we know that q is definite; $q = \langle x_1, \ldots, x_n \rangle$ with $x_i \in \mathbb{Q}^*$, all x_i being positive or all x_i being negative. If the number field F had a real embedding, then q_F would still be anisotropic. So, F is totally imaginary and hence, by Minkowski-Hasse, the anisotropic kernel of q_F is of dimension ≤ 4. Thus, the Witt index $i = i(q_F)$ satisfies $i \geq (n-4)/2 \geq [n/2] - 2$. We expand n as in Lemma 2.6. It follows immediately from 2.6 that $a_2 \geq 3$ and hence

$$n \equiv (-1)^{h-1} n^{(1)} = (-1)^{h-1}(2^{a_1} - \epsilon) \pmod 8.$$

If $n \equiv 0, \pm 1 \pmod 8$, then $a_1 \geq 3$ and hence $i = [n/2]$.
If $n \equiv \pm 2 \pmod 8$, then $\epsilon = 0$, $a_1 = 1$ and hence $i = [n/2]$ or $[n/2] - 1$.
If $n \equiv \pm 3 \pmod 8$, then $\epsilon = 1$, $a_1 = 2$ and hence also $i = [n/2]$ or $[n/2] - 1$.
If $n \equiv 4 \pmod 8$, then $\epsilon = 0$, $a_1 = 2$ and hence $i = [n/2]$ or $[n/2] - 2$. □

It is straightforward to extract from the statement of 3.5 the well-known fact that the level of number fields, if finite, is 1,2 or 4.

3.6 Remark. The indices mentioned in 3.5 do occur over number fields. To see this, let F be a number field of level 4. Put $Q_n = n \times \langle 1 \rangle$. Then e.g. $i((Q_6)_F) = 2 = [6/2] - 1$ and $i((Q_{12})_F) = 4 = [12/2] - 2$ and similarly for other values of n.

Proof. Clearly $(Q_n)_F$ is a $(Q_j)_F$-extension for some $j \leq 4$. Now one applies 2.4. □

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