Splitting Patterns of Quadratic Forms

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Introduction.

Let q be an anisotropic quadratic form of dimension $n \geq 2$ over a field k of characteristic $\neq 2$. The splitting pattern of q is made up from all distinct Witt indices of q which occur over field extensions of k. The entries of the splitting pattern of q are called the higher Witt indices of q.

For excellent quadratic forms q, the splitting pattern depends only on the dimension n of q, and is known explicitly for every n. This goes back to the originating work of M. Knebusch [10, 11] and has been studied further in [5]. However, surprisingly little is known about the splitting pattern of quadratic forms q, in general, as soon as there are at least 2 non-zero Witt indices in the splitting pattern of q.

In section 1 we investigate general constraints on splitting patterns. In theorem 1.6 we describe in which way the splitting pattern of a quadratic form is influenced by its Clifford algebra. For this we use the index reduction formulas of A. S. Merkurjev [14] as the crucial tool. It turns out that the splitting of the Clifford algebra determines the "high end" of the splitting pattern.

On its "low end", we find that the first higher Witt index $i_1(q)$ of q always is less than or equal to the first higher Witt index of the anisotropic excellent forms of the same dimension as q. In particular, $i_1(q)$ is less than or equal to the excess of $\dim q$ over the biggest 2-power which is strictly smaller than $\dim q$. If equality holds, then q is a "stable Pfister neighbor"; that is, there is a field extension K of k such that q_K is an anisotropic Pfister neighbor.

Similar results on $i_1(q)$ have earlier been obtained by D. W. Hoffmann [4].

Here we give a short proof based on Theorem 1.7, which describes a useful criterion to determine when an anisotropic form q is an orthogonal summand of an anisotropic Pfister form π. Corollary 1.8 establishes the existence of a unique field extension K of k in a generic splitting tower of $\pi \perp -q$ such that π_K is anisotropic and has the anisotropic kernel $(q_K)_{\mathrm{an}}$ of q as an orthogonal summand.

In section 2 we study the "algebraic" splitting pattern; that is, we look at the possible Witt indices which can occur over algebraic extensions of k. In general, the algebraic splitting pattern is different from the splitting pattern. However, we show that for given dimension n there is a number t such that, for every form q of dimension n over any field k, the splitting pattern of q is equal to the algebraic splitting pattern of q_K, where K is a purely transcendental field extension of k of degree t.

In section 3 we approach the wide open case of quadratic forms of height 2; that is, with exactly 2 non zero Witt indices in the splitting pattern of q; see, in particular, theorems 3.4 and 3.11 for so-called good forms of degree 3 and forms of degree 2 that are not good, respectively. The general theorem 3.9 makes explicit the splitting pattern of good forms of arbitrary degree, assuming that the conjectures by B. Jacob

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and M. Rost [6] on higher cohomological invariants are true. Under this assumption those forms have always twice the dimension of their leading form, and over a suitable field extension they are (still anisotropic) a product of a four dimensional form and a Pfister form. This also answers a question of M. Knebusch [11, 10.6, p. 30].

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0. Setup.

Let q be a regular quadratic form of index $\mathrm{i}(q)$ and let

$$i_0 := \mathrm{i}(q) < i_1 < \ldots < i_{h-1} < i_h = [n/2]$$

denote all distinct Witt indices which occur for the quadratic forms $q_L := q \otimes L$ with L running through all field extensions of k.

The natural number h is called the *height* of q. The $(h+1)$-tuple $(i_0, i_1, \ldots, i_h)$ of strictly increasing integers is called the *splitting pattern of* q (cf. [5, 1.1]). If necessary, we will write $i_\nu(q)$ and $h(q)$ instead of i_ν and h. We now illustrate the extreme cases $h = 1$ and $h = m := [n/2]$.

0.1 Example (Height 1; cf. [10, Thm. 5.8, p. 81]). *An anisotropic quadratic form q of dimension ≥ 2 over k has splitting pattern $(0, m)$ for some $m \in \mathbb{N}$ if and only if q is similar to an orthogonal summand of codimension ≤ 1 of a Pfister form over k.*

0.2 Example (Height m; cf. [10, Ex. 5.7, p. 80]). *Let $q = \langle X_1, X_2, \ldots, X_n \rangle$ with indeterminates X_i over $k = F(X_1, \ldots, X_n)$ for any field F. Then q has splitting pattern*

$$(0, 1, 2, \ldots, m-2, m-1, m).$$

Given $m \in \mathbb{N}$, there are 2^{m-1} different tuples $(0, \ldots, m)$ of strictly increasing integers. What about realizing them as splitting patterns? By example 0.1, there is *no* quadratic form q with splitting pattern $(0, 5)$ or $(0, 6)$, say, since there do not exist Pfister forms of appropriate dimensions.

The basic question of our paper is: What are the splitting patterns of quadratic forms?

The above observation concerning splitting patterns of quadratic forms of height 1 yields immediately that a tuple $(0, \ldots, m-a, m)$ can be realized to be a splitting pattern of some quadratic form of arbitrary height *only if* a or $a-1$ is a 2-power.

We will look for further restrictions on splitting patterns and will comment on the next-to-extreme cases $h = 2$ and $h = m-1$.

By the disciminant $d(q)$ of any quadratic form q we always mean its *signed* discriminant. The Witt invariant is defined as in [13, Ch. V.3, p. 120]. We recall the standard notation for Pfister forms: For $n > 0$ and $a_1, \ldots, a_n \in k^*$ the 2^n-dimensional quadratic form $\langle\langle a_1, \ldots, a_n \rangle\rangle := \langle 1, a_1 \rangle \otimes \cdots \otimes \langle 1, a_n \rangle$ is called an n-fold Pfister form. Sometimes it makes sense to call the form $\langle 1 \rangle$ a 0-fold Pfister form.

For the definition of the *leading field* and the *leading form* of a quadratic form we refer to [10, 5.4, p. 79 and 5.9, p. 82]. The *degree* of a form q is defined in [10, §6, p. 88].

1. General constraints on splitting patterns

We will present an analog to example 0.2. Consider the rational function field $k = F(X_1, \ldots, X_{n-1})$ with indeterminates $X_1, \ldots X_{n-1}$ for any field F of characteristic $\neq 2$, let $c \in F^*$.

1.1 Lemma. *The n-dimensional quadratic form*

$$q = \langle X_1, X_2, \ldots, X_{n-1}, cX_1X_2 \cdots X_{n-1} \rangle$$

is anisotropic over k for any $n \geq 3$.

Proof. Assume that q is isotropic over k. Then there are relatively prime polynomials $p_1, \ldots, p_n$ in $F[X_1, \ldots X_{n-1}]$ with

$$(*) \qquad X_1p_1^2 + X_2p_2^2 + \cdots + X_{n-1}p_{n-1}^2 = -cX_1X_2 \cdots X_{n-1}p_n^2.$$

As the form $\langle X_{i_1} \rangle \perp \ldots \perp \langle X_{i_r} \rangle$ is anisotropic over k for every subset $\{i_1, \ldots, i_r\}$ of $\{1, \ldots, n-1\}$ we conclude, using the substitution $X_i = 0$, that X_i divides p_j for every $j \in \{1, \ldots, n-1\}, j \neq i$. Thus $X_jp_j^2 = X_j(\prod_{j \neq i} X_i^2)\widetilde{p}_j^2$ for some $\widetilde{p}_j \in F[X_1, \ldots, X_{n-1}]$ for all $j = 1, \ldots, n-1$. So every summand in $(*)$ is divisible by $X := X_1 \cdots X_{n-1}$. Dividing $(*)$ by X yields

$$(**) \qquad \hat{X}_1\widetilde{p}_1^2 + \ldots + \hat{X}_{n-1}\widetilde{p}_{n-1}^2 = -cp_n^2$$

where $\hat{X}_i := X/X_i = \Pi_{j \neq i} X_j$. From $(**)$ it follows that X_{n-1} divides p_n if and only if it divides $\widetilde{p}_{n-1}$, but this cannot happen since the p_i are assumed to be relatively prime. By substituting $X_{n-1} = 0$ we obtain an equation $\hat{X}_{n-1}\widetilde{\widetilde{p}}_{n-1}^{\,2} = -c\widetilde{p}_n^{\,2}$ with non-zero $\widetilde{\widetilde{p}}_{n-1}, \widetilde{p}_n \in F[X_1, \ldots, X_{n-2}]$. Now this implies that $-\hat{X}_{n-1}/c$ is a square in k. But for $n \geq 3$ this contradicts the fact that $F[X_1, \ldots, X_{n-1}]$ is a unique factorization domain. □

Let us apply this lemma to the case $n = 2m$ and $c = (-1)^m$ to obtain:

1.2 Proposition. *Let $k = F(X_1, \ldots, X_{2m-1})$ with indeterminates X_i over F and $2 \leq m$. Then $q = \langle X_1, X_2, \ldots, X_{2m-1}, (-1)^m X_1X_2 \cdots X_{2m-1} \rangle$ has height $m-1$ and splitting pattern $(0, 1, 2, \ldots, m-2, m)$.*
Moreover, the Clifford algebra $C(q)$ of q is isomorphic to $\mathrm{M}_2(D)$, with a central division algebra D that is a tensor product of $m-1$ quaternion algebras.

Proof. The $2m$-dmional quadratic form q is anisotropic over k by Lemma 1.1. We have arranged for q to have determinant $(-1)^m$, which means that its discriminant is 1. Thus if, for some extension L of k, the form q_L splits off at least $m-1$ hyperbolic planes, then q_L splits completely. That is, $m-1$ does not appear as a Witt index over any extension of k.

Hence we are done if $m = 2$, obtaining splitting pattern $(0, 2)$. We proceed by induction. We write $q_m := q$ for $m \geq 3$ and assume that the proposition has been proved for all $q_{m'}$ with $m' < m$. Then over $L = k(\sqrt{-X_{2m-2}X_{2m-1}})$ the form q_m is isometric to $q_{m-1} \perp \langle X_{2m-2}, -X_{2m-2} \rangle$. Thus it splits off exactly one hyperbolic plane, since by Lemma 1.1 the first summand is still anisotropic over L. Using the induction hypothesis we get our first claim.

To prove our statement about $C(q)$ we write $a_m = (-1)^m X_1 X_2 \cdots X_{2m-1}$, $Y = X_{2m-2}/a_{m-1}$, $Z = X_{2m-1}/a_{m-1}$ and observe that

$$q_{m-1} \perp (-a_{m-1})\langle\langle -Y, -Z\rangle\rangle \cong q_m \perp \langle 1, -1\rangle,$$

which yields $C(q_{m-1}) \otimes \mathrm{M}_2(Q) \cong \mathrm{M}_2(C(q_m))$ for some quaternion skew field Q over k [13, Ch. V, 2.7, p. 113 and 3.3, p. 116]. Clearly the index of $C(q_m)$ is twice the index of $C(q_{m-1})$, and the assertion follows by induction on m. □

Contrasting Proposition 1.2, the reader notices that no odd dimensional form has splitting pattern $(0, 1, 2, \ldots, m-2, m)$ since no 5-dimensional form has height 1.

More generally, we can see that, for $a \geq 1$, no odd dimensional form q can have a splitting pattern $(0, 1, 2, \ldots, m-2^a, m)$ since this would require the existence of an anisotropic form of dimension $2^{a+1}+1$ of height 1 contradicting 0.1.

An even dimensional form with splitting pattern $(0, 1, 2, \ldots, m-2^a, m)$ and $a \geq 2$ would require the existence of an anisotropic form q of discriminant 1, Witt invariant 1 (by 1.6 below), height 2 and splitting pattern $(0, 1, 2^a+1)$, hence of dimension $2^{a+1}+2$. This seems us to be unlikely, especially because of our results in §3. We can exclude the case $a = 2$ since then $\dim q = 10$, in which case it follows from [15, Proof of Satz 14, No. 5, p. 123] that a form q having invariants as mentioned above is necessarily isotropic.

1.3 Question. Given $a \geq 3$, is there an even dimensional form q over some field k with splitting pattern $(0, 1, 2^a+1)$? If the answer is no as we expect, is there a general lower bound on $i_1(q)$ (independent of the field k) for even dimensional anisotropic forms q of height 2 as an increasing function of the degree? (3.9 below yields this under certain assumptions for excellent resp. "good" forms of height 2.)

Next we will exhibit constraints on the first non-zero entry of the splitting pattern of a quadratic form.

We begin with excellent forms as introduced by Knebusch [11, 7.7, p. 3]. For those forms, the splitting patterns depend only on the dimension of the form. Let $i_1(n)$ and $h(n)$ stand for the first non-zero Witt index and the height of all n-dimensional (anisotropic) excellent forms of dimension n over any field k (of characteristic $\neq 2$).

Given $j \in \mathbb{N}$, what is the best possible lower bound for $i_1(n)$, where n ranges over all dimensions with $h(n) = j$? An answer is given by the following proposition which sharpens 3.1 in [5].

1.4 Proposition. *Let $h \geq 1$. Then the natural number*

$$n_h := \begin{cases} (2^{h+2}-2)/3 & \text{if } h \text{ is odd,} \\ (2^{h+2}-1)/3 & \text{if } h \text{ is even} \end{cases}$$

is the minimal dimension of an excellent form of height h. Moreover we have $i_1(n) \geq n_{h(n)-2}$ for any $n \in \mathbb{N}$ with $h(n) \geq 3$, with equality for excellent forms of dimension $n = n_h$.

Proof. We prove the first statement by induction on h. Obviously $n_1 = 2$, and 5 is the lowest dimension of an excellent form of height 2, hence $n_2 = 5$ and we are done for

$h \leq 2$. We now assume that $h \geq 3$, and that the first statement has been shown for excellent forms of height $< h$. We write $h = 2u + \varepsilon$ with $\varepsilon = 0$ or $\varepsilon = 1$ and find

$$(1) \qquad n_h = 2^\varepsilon(4^{u+1} - 1)/3 = 2^\varepsilon(\sum_{i=0}^{u} 4^i) = 2^\varepsilon 4^u + n_{h-2} = 2^h + n_{h-2}.$$

If q is an excellent form of height $h \geq 2$ and if q_2 denotes its second complementary form it follows from [5, 2.6 and 2.10] that

$$(2) \qquad \dim q = 2^a - 2^b + \dim q_2$$

with $a > b \geq h$. Clearly q_2 has height $h - 2$. Hence, using the induction hypothesis and (1), we obtain from (2) that $\dim q \geq 2^{b+1} - 2^b + \dim q_2 \geq 2^h + n_{h-2} = n_h$. Thus the first statement is proved.

It follows from [5, 1.8] that the dimension of the first complementary form q_1 of q is given – in terms of (2) – by $2^b - \dim q_2$, and hence we obtain

$$(3) \qquad \begin{aligned} i_1(n) &= (\dim q - \dim q_1)/2 = (2^a - 2^b + \dim q_2 - 2^b + \dim q_2)/2 \\ &= 2^{a-1} - 2^b + \dim q_2 \geq \dim q_2 \geq n_{h(n)-2}. \end{aligned}$$

If $n = n_h$, then a comparison of (1) and (2) together with the equality $\dim q_2 = n_{h-2}$ shows that $2^a - 2^b = 2^h$, hence (3) holds with an equality everywhere. □

1.5 Remark. In particular, we have $i_1(n) > 1$ for all n with $h(n) \geq 3$.

We now study the influence of the splitting behavior of the Clifford algebra $C(q)$ on the splitting pattern of an arbitrary regular form q.

Let us remark that the function field $k(q)$ of a quadratic form q is a purely transcendental extension followed by a quadratic extension, hence if a central simple algebra over k is tensored by $k(q)$, then its index goes down by a factor of at most 2.

It follows from [13, Ch. V, Thm. 2.5, p. 111] that in case $\dim q > 0$ even and $d(q) = 1$ the Clifford algebra of q is of the form $C(q) \cong \mathrm{M}_2(E(q))$ for some central simple k-algebra $E(q)$. In the other cases we define $E(q)$ as follows. If $\dim q$ is even and $d(q) \neq 1$, then $E(q) := C(q)$. If $\dim q$ is odd, then $E(q) := C_0(q)$, the even part of the Clifford algebra.

If $q \cong \langle 1, -1 \rangle \perp q'$ and $q' \neq 0$, then $E(q) \cong \mathrm{M}_2(k) \otimes E(q') \cong \mathrm{M}_2(E(q'))$ by [13, Ch. V, 2.7, 2.9, p. 113f].

Hence if $E(q)$ is a skew field $\neq k$, then q is anisotropic.

1.6 Theorem. *Let q be anisotropic and let $E(q) \cong \mathrm{M}_{2^r}(D)$ for some skew field $D \not\cong k$ and some $r \geq 0$.*
i) *Let $r > 0$. Then $D_{k(q)}$ is a skew field, $i_1(q) \leq r$, and there is some ν such that $i_\nu(q) = r$.*
ii) *Let $r = 0$. Then $E(q)$ is a skew field, and this also holds for all higher anisotropic kernels of q. If $\dim q$ is odd or $d(q) \neq 1$, then the splitting pattern of q is $(0, 1, 2, \ldots, m-1, m)$ and $E(q)$ is a product of m quaternion skew fields. If $\dim q$ even and $d(q) = 1$, then the splitting pattern of q is $(0, 1, 2, \ldots, m-2, m)$ and $E(q)$ is a product of $m-1$ quaternion skew fields. (Here $m = [\dim q/2]$.)*

Proof. For $i = i_1(q)$ and $q' = (q_{k(q)})_{\mathrm{an}}$ we have $q_{k(q)} \cong \langle 1, -1 \rangle^i \perp q'$, hence by [13, Ch. V, 2.7, 2.9, p. 113f]

$$(*) \qquad \mathrm{M}_{2^r}(D_{k(q)}) \cong E(q_{k(q)}) \cong \mathrm{M}_{2^i}(k(q)) \otimes_{k(q)} E(q') \cong \mathrm{M}_{2^i}(E(q')) \text{ for } q' \neq 0.$$

i) If $r > 0$, then it follows from Merkurjev's index reduction theorems [14, Thm. 1, 2, 3, p. 218] that $D_{k(q)}$ is a skew field. Therefore $i_1(q) = i \le r$ by $(*)$. Repeating this procedure yields our assertion.

ii) If $r = 0$, then it follows from $(*)$ that $D_{k(q)}$ is not a skew field, but then its index is exactly half of the index of $D \cong E(q)$. Hence i must be 1 unless $d(q) = 1$ and $\dim q = 4$, in which case $i = 2$. An induction proves the result. □

1.7 Theorem. *Let π be an anisotropic Pfister form, let q be any regular form with $\dim q < \dim \pi$, and let $\widetilde{q} := \pi \perp -q$. Then $\mathrm{i}(\widetilde{q}) + \mathrm{i}(q) \le \dim q$, and the following are equivalent.*

i) $\mathrm{i}(\widetilde{q}) + \mathrm{i}(q) = \dim q$.

ii) $\pi_{k(\widetilde{q}_{\mathrm{an}})}$ *is isotropic.*

iii) q_{an} *is an orthogonal summand of* π.

Proof. We have $\pi \perp (-q)_{\mathrm{an}} \cong \widetilde{q}_{\mathrm{an}} \perp \mathbb{H}^\delta$ with $\delta := \mathrm{i}(\widetilde{q}) - \mathrm{i}(q)$. Since π is anisotropic this yields $\delta \le \dim q_{\mathrm{an}}$, which is equivalent to $\mathrm{i}(\widetilde{q}) + \mathrm{i}(q) \le \dim q$. There are regular subforms $\pi' \subseteq \pi$ and $q' \subseteq q_{\mathrm{an}}$ each of codimension δ such that $\pi' \perp (-q') = \widetilde{q}_{\mathrm{an}}$.

If i) holds, then $\delta = \dim q_{\mathrm{an}}$, hence $q' = 0$ and $\widetilde{q}_{\mathrm{an}}$ must be an orthogonal summand of π which implies ii).

We show that ii) implies iii). Since $\dim q < \dim \pi$ we have $\pi' \neq 0$, hence there is some $s \in k^*$ represented by π and by $\widetilde{q}_{\mathrm{an}}$. Let $k' := k(\widetilde{q}_{\mathrm{an}})$. If $\pi_{k'}$ is isotropic, then it is hyperbolic. By [12, 7.4, p. 22], the form $\widetilde{q}_{\mathrm{an}} = s^2\widetilde{q}_{\mathrm{an}}$ is an orthogonal summand of π, hence there is a regular form q_1 such that $\pi = \widetilde{q}_{\mathrm{an}} \perp q_1$. By Witt cancellation we get $q_1 \perp (-q)_{\mathrm{an}} = \mathbb{H}^\delta$ and thus $\delta = \dim q_{\mathrm{an}}$. This implies $q' = 0$ and therefore $\widetilde{q}_{\mathrm{an}} = \pi' \subseteq \pi$, hence the orthogonal complement of π' in π is isomorphic to q_{an}.

If iii) holds, then $\delta = \dim q_{\mathrm{an}}$ which yields i). □

The notion of a generic splitting tower of a quadratic form has been introduced by Knebusch in [10, p. 78]. However, we here will use the slightly more general version of [5, 1.1, p. 184].

1.8 Corollary. *In any generic splitting tower of $\widetilde{q}$ there is a (unique) field extension K of k such that π_K is anisotropic and $(q_K)_{\mathrm{an}}$ is an orthogonal summand of π_K.*

Any two such field extensions are k-equivalent, i. e. they are k-specializations of each other.

Proof. A generic splitting tower $\{K_\nu\}$ of $\widetilde{q}$ can be constructed inductively by $K_0 := k$, $K_{\nu+1} := K_\nu((\widetilde{q}_{K_\nu})_{\mathrm{an}})$ (cf. [10, p. 78]).

Take ν maximal such that π_{K_ν} is anisotropic. Then $\pi_{K_{\nu+1}}$ is isotropic and we can apply 1.7 with k replaced by $K := K_\nu$ to obtain that $(q_K)_{\mathrm{an}}$ is an orthogonal summand of π_K.

If now K and K' are two fields from generic splitting towers of $\widetilde{q}$ such that π_K and π'_K are anisotropic and $(q_K)_{\mathrm{an}}$ (resp. $(q_{K'})_{\mathrm{an}}$) is an orthogonal summand of π_K (resp. $\pi_{K'}$), then, by 1.7, we obtain $\mathrm{i}(\widetilde{q}_K) + \mathrm{i}(q_K) = \dim q = \mathrm{i}(\widetilde{q}_{K'}) + \mathrm{i}(q_{K'})$. We may assume that $\mathrm{i}(\widetilde{q}_K) \ge \mathrm{i}(\widetilde{q}_{K'})$, hence there is a k-specialization $K' \to K \cup \{\infty\}$ which implies $\mathrm{i}(q_K) \ge \mathrm{i}(q_{K'})$. Therefore we get $\mathrm{i}(\widetilde{q}_K) = \dim q - \mathrm{i}(q_K) \le \dim q - \mathrm{i}(q_{K'}) = \mathrm{i}(\widetilde{q}_{K'})$ and hence $\mathrm{i}(\widetilde{q}_K) = \mathrm{i}(\widetilde{q}_{K'})$, obtaining a k-specialization $K \to K' \cup \{\infty\}$. □

1.9 Corollary. *Let $\dim \pi = 2^{r+1}$ and K be as in Corollary 1.8.*

i) *If $\dim q - \mathrm{i}(q_K) \le 2^r$, then $K(\pi_K)$ is k-equivalent to $k(\pi)$.*

ii) *Assume $q_{k(\pi)}$ is anisotropic. If either* $\dim q \le 2^r$ *or* $\dim q = 2^r + i_1(q)$*, then q_K is an orthogonal summand of the anisotropic Pfister form π_K.*

Proof. By 1.7, 1.8 we have $\dim q - \mathrm{i}(q_K) = \mathrm{i}(\widetilde{q}_K)$. Let V denote the variety of $(\min\{2^r, \mathrm{i}(\widetilde{q}_K)\})$-dimensional totally isotropic subspaces of $\widetilde{q}$. Then V has a rational point over $k(\pi)$, since $\mathrm{i}(\widetilde{q}_{k(\pi)}) \ge 2^r$. Hence it follows from [9, Thm. 3.10, p. 46, and 5.2, p. 58] that $k(\pi)(V_{k(\pi)}) \cong k(V)(\pi_{k(V)})$ is purely transcendental over $k(\pi)$, and that there is a k-specialization $k(V) \to K \cup \{\infty\}$. Hence $\mathrm{i}(\widetilde{q}_{k(V)}) \le \mathrm{i}(\widetilde{q}_K)$ with equality if and only if $k(V)$ and K are k-equivalent.

i) We have $\mathrm{i}(\widetilde{q}_{k(V)}) = \mathrm{i}(\widetilde{q}_K)$, hence the fields $K(\pi_K)$ and $k(V)(\pi_{k(V)})$ are k-equivalent. Thus i) follows since $k(V)(\pi_{k(V)})$ is purely transcendental over $k(\pi)$.

ii) By 1.8, we have to show that $\mathrm{i}(q_K) = 0$. If K and $k(V)$ are k-equivalent, then $\mathrm{i}(q_K) = \mathrm{i}(q_{k(V)}) = 0$, since $q_{k(\pi)}$ is anisotropic and $k(V)(\pi_{k(V)})$ is purely transcendental over $k(\pi)$. Otherwise $2^r \le \mathrm{i}(\widetilde{q}_{k(V)}) < \mathrm{i}(\widetilde{q}_K) \le \mathrm{i}(\widetilde{q}_K) + \mathrm{i}(q_K) = \dim q = 2^r + i_1(q)$ which implies $\mathrm{i}(q_K) \le i_1(q) - 1$, hence $\mathrm{i}(q_K) = 0$. □

1.10 Lemma. *Let $k' := k(X_0, \ldots, X_r)$ and $\pi := \langle\langle X_0, \ldots, X_r \rangle\rangle$. Then π is anisotropic, and $k'(\pi)$ is unirational over k. If q is a regular form over k, then q and $q_{k'(\pi)}$ have the same splitting pattern.*

Proof. Since π is isotropic over $k'' := k'(\langle 1, X_0 \rangle)$ we find that $k''(\pi)$ is purely transcendental over k''. As k'' is purely transcendental over k, it follows that $k'(\pi) \subset k''(\pi)$ is unirational over k.

If L is a field extension of k, then the free composite $L' := Lk'$ is isomorphic to the rational function field $L(X_0, \ldots, X_r)$ and, by the above, $L_1 := L'(\pi_{L'})$ is unirational over L and contains $k'(\pi)$. Hence $\mathrm{i}(q_L) = \mathrm{i}(q_{L_1})$. This shows that every Witt index which occurs over an extension of k also occurs over an extension of $k'(\pi)$. □

Except of the statements on splitting patterns, the following three corollaries 1.11, 1.12, 1.13 are contained in [4, Thm. 2, Cor. 3 and Thm. 1]. We here give shorter proofs based on Thm. 1.7.

1.11 Corollary. *Let q be an anisotropic form with either* $\dim q \le 2^r$ *or* $\dim q = 2^r + i_1(q) < 2^{r+1}$*. Then there is a field extension K of k and an anisotropic $(r+1)$-fold Pfister form π over K such that $q_K \subseteq \pi$. If* $\dim q \le 2^r$*, then K and π can be chosen such that, for every regular form q' over k, the forms q' and $q'_{K(\pi)}$ have the same splitting pattern.*

Proof. We apply Corollary 1.9, with k replaced by k' from Lemma 1.10 and with the particular π defined there and with q replaced by $q_{k'}$. The unirationality of the field $k'(\pi)$ over k as stated in Lemma 1.10 gives the anisotropy of $q_{k'(\pi)}$ required by 1.9 ii). □

1.12 Corollary (Hoffmann, cf. [4, Thm. 1]). *Let q, q' be regular forms such that q' is anisotropic and, for some natural number r, we have* $\dim q' \le 2^r < \dim q$*. Then $q'_{k(q)}$ is anisotropic. More generally, q' and $q'_{k(q)}$ have the same splitting pattern.*

Proof. If q is isotropic, then $k(q)$ is a purely transcendental field extension of k [10, 3.9, p. 72] and there is nothing to show. Otherwise take K, π as in Corollary 1.11, but for the form q'. If $q'_{k(q)}$ is isotropic, then $\pi_{K(q_K)}$ splits, hence q_K is a subform of π by [10, 4.5, p. 75]. As $q_{K(\pi)}$ is anisotropic this implies $\dim q \le 2^r$.

Now if L is any field extension of k, then $L(q_L)$ is a field extension of $k(q)$, and $(q'_L)_{\mathrm{an}}$ stays anisotropic over $L(q_L)$ by what we just showed applied to $(q'_L)_{\mathrm{an}}$ and q_L. Hence the splitting patterns of q' and $q'_{k(q)}$ are the same. □

1.13 Corollary. *Let q be an anisotropic quadratic form of dimension n. Then we have $i_1(q) \leq i_1(n)$. If equality holds, then there is a field extension K of k such that q_K is an anisotropic Pfister neighbor.*
Conversely, if q is a Pfister neighbor with complementary form q', then $-q'_{k(q)}$ is the anisotropic kernel of $q_{k(q)}$ and $i_1(q) = i_1(n)$ holds.

Proof. By [5, 2.8, p. 187], $i_1(n)$ is the difference of n and the greatest 2-power which is strictly smaller than n. Hence the first part of the theorem follows, since, by 1.12, no subform of q of codimension $i_1(n)$ becomes isotropic over $k(q)$. If equality holds, we apply 1.11 if $\dim q$ is not a power of 2, otherwise we apply 0.1.

If q is a Pfister neighbor with complementary form q' we set $d = \dim q - 2^r$. Then $\dim q' = 2^r - d$, hence $q'_{k(q)}$ is anisotropic by 1.12. It then follows from [5, 2.9 (i), p. 187] that $-q'_{k(q)}$ is the anisotropic kernel of $q_{k(q)}$. But this means that $i_1(q) = d = i_1(n)$. □

1.14 Remark. Let q be anisotropic of dimension n. If q_K becomes an anisotropic Pfister neighbor for some field extension K of k, then it is not necessarily true that $i_1(q) = i_1(n)$ as can be seen by some combination of 1.18 and, say, 0.2. In fact 1.18 implies that there are fields k such that every anisotropic quadratic form becomes an anisotropic Pfister neighbor over some field extension of k. As B. Kahn has mentioned to us, $i_1(q) = i_1(n)$ if and only if there exists a field extension K of k such that q_K is an anisotropic Pfister neighbor and $h(q_K) = h(q)$.

The case $l = 1$ of the following statement has also been observed by D. W. Hoffmann.

1.15 Corollary. *Let q be anisotropic and* $\dim q < 2^r$. *Then, for any natural number l, there is a field extension K of k and an anisotropic form $\widetilde{q}$ over K with $\widetilde{n} := \dim \widetilde{q} < 2^{r+l}$ which is a q_K-extension of order l in the sense of [5, 2.2, p. 185]; that is, there is a sequence $\widetilde{q} = q_0, q_1, \ldots, q_l = q_K$ of anisotropic forms over K such that q_j is a Pfister neighbor with complementary form q_{j+1} for all $j < l$.*
If the splitting pattern of q is $(0, i_1, \ldots, i_h)$, then the splitting pattern of $\widetilde{q}$ is given by $(0, i_1(\widetilde{n}), \ldots, i_l(\widetilde{n}), i_1 + i_l(\widetilde{n}), \ldots, i_h + i_l(\widetilde{n}))$. Every (generic) splitting field of q_K is a (generic) splitting field of $\widetilde{q}$.

Proof. Assume $l = 1$. Corollary 1.11 implies the existence of a field extension K of k and an $r+1$-fold anisotropic Pfister form π over K such that q_K is a subform of π, hence it is the complementary form of some Pfister neighbor $\widetilde{q}$ of π. Since $\dim q < 2^r$ it follows from 1.11 that K can be choosen such that q and q_K have the same splitting pattern. Clearly $\dim \widetilde{q} > 2^r$, hence q and $q_{K(\widetilde{q})}$ have the same splitting pattern by 1.12. Together with the second half of 1.13 this now yields the result on the splitting pattern. The statement about the (generic) splitting field follows from [5, 2.4, 2.5, p. 186]. A straightforward induction now proves the case of arbitrary l. □

1.16 Example. The splitting patterns of anisotropic forms q with $\dim q = n \leq 9$ are given by

$n = 2$	$(0,1)$
$n = 3$	$(0,1)$
$n = 4$	$(0,2)$, $(0,1,2)$

$n = 5$	$(0, 1, 2)$
$n = 6$	$(0, 2, 3),\ (0, 1, 3),\ (0, 1, 2, 3)$
$n = 7$	$(0, 3),\ (0, 1, 2, 3)$
$n = 8$	$(0, 4),\ (0, 2, 4),\ (0, 1, 2, 4),\ (0, 1, 3, 4),\ (0, 1, 2, 3, 4)$
	and possibly $(0, 2, 3, 4)$
$n = 9$	$(0, 1, 4),\ (0, 1, 2, 3, 4)$.

Explanation. The first entry in each line above is the excellent splitting pattern, as can easily be checked from [11, 7.10, p. 4] or [5, 2.8, p. 187].

Everything is clear for dimension ≤ 3. For $n = 4$ both patterns occur by examples 0.1 and 0.2. For $n = 5$, $(0, 2)$ fails to be a splitting pattern by 0.1. For $n = 6$, the pattern $(0, 1, 3)$ occurs by 1.2, $(0, 2, 3)$ is the excellent pattern, $(0, 1, 2, 3)$ is realizable by 0.2, but $(0, 3)$ is not by 0.1.

For $n = 7$, $(0, 3)$ and $(0, 1, 2, 3)$ occur by 0.1 and 0.2, respectively. The tuple $(0, 1, 3)$ fails to be a splitting pattern since $(0, 2)$ does not occur for $n = 5$. We show that the only remaining tuple $(0, 2, 3)$ also is not realizable. A corresponding form q cannot be excellent, hence its Witt invariant $c(q)$ is non trivial by [11, p. 11]. By 1.6 we conclude that $c(q)$ is given by a single quaternion skew field, hence the leading form of q is defined over k by [11, 9.8, p. 23]. But this contradicts the initial remark of [11, §10, p. 27].

For $n = 8$, $(0, 4)$ occurs by 0.1; for $(0, 2, 4)$ we refer to Remark 3.3 below, and for $(0, 1, 2, 4)$ to Prop. 1.2. Moreover, the tuple $(0, 1, 4)$ is not realizable since $(0, 3)$ is not realizable in dimension 6. It is left to determine the splitting patterns of forms of degree 1; that is, of discriminant $\neq 1$. We can exclude $(0, 3, 4)$ as a pattern by Remark 3.2. The tuple $(0, 1, 3, 4)$ is realizable by the form $\langle 1, 1, 1, 1, 1, 1, 1, X\rangle$ over $\mathbb{Q}(X)$. For the tuple $(0, 1, 2, 3, 4)$ we can refer to 0.2 or 1.6 ii).

For $n = 9$, $(0, 1, 4)$ is the excellent pattern, $(0, 1, 2, 3, 4)$ occurs by 0.2. All other six tuples can be excluded, namely: $(0, 4)$, $(0, 2, 4)$, $(0, 3, 4)$, and $(0, 2, 3, 4)$ are not realizable as splitting patterns since $i_1(9) = 1$ by Corollary 1.13. Finally, $(0, 1, 2, 4)$ does not occur since $(0, 2)$ is not realizable in dimension 5, and $(0, 1, 3, 4)$ does not occur since $(0, 2, 3)$ is not realizable in dimension 7.

1.17 Definition. A regular anisotropic form q over k is called *stably excellent* if there is a field extension K of k such that q_K is anisotropic and excellent.

1.18 Proposition.

i) *If k is a formally real field and if $q = \langle a_1, \ldots, a_n\rangle$ where all the elements a_i are positive under some real embedding of k, then q is stably excellent.*

ii) *An anisotropic form q is stably excellent if* $\dim q \leq 5$ *or if k is a number field.*

Proof. i) Let $K = k(\sqrt{a_1}, \ldots, \sqrt{a_n})$. Then $q_K \simeq n \times \langle 1\rangle$; hence q_K is anisotropic and excellent.

ii) For $\dim q \leq 3$ there is nothing to show since every such form is excellent. A form q of dimension 4 and of discriminant $d \neq 1$ is anisotropic if and only if $q_{k(\sqrt{d})}$ is anisotropic, but the latter has discriminant 1 and hence is similar to the norm form of a quaternion skew field, hence to a Pfister form and therefore it is excellent. If $\dim q = 5$, then 1.13 yields $i_1(q) = i_1(n) = 1$, and there is a field extension K of k such that q_K is an anisotropic Pfister neighbor. Then the first kernel form of q_K is of dimension 3 and hence excellent, therefore q_K is excellent. If k is a number field and $\dim q \geq 5$ then, by the theorem of Minkowski-Hasse, the anisotropy of q implies that there is a real embedding of k over which q is (positive or negative) definite and hence anisotropic. Now the assertion follows from i). □

1.19 Remark. The splitting pattern of an excellent 6-dimensional form is $(0, 2, 3)$. Hence no 6-dimensional form with pattern $(0, 1, 3)$ is stably excellent; compare Theorem 3.11 and Remark 3.13 below.

2. Algebraic splitting patterns

2.1 Definition. The *algebraic* splitting pattern of a regular quadratic form q over k is the subsequence of its splitting pattern whose entries can be obtained as the Witt indices of q_K for algebraic field extensions K of k.

2.2 Example. Let q denote the sum of 9 squares over $\mathbb{Q}$, which is clearly anisotropic, and let F be any number field. An application of the Minkowski-Hasse Theorem shows that if q_F is isotropic, then F has level ≤ 4 and q splits totally; that is, q has Witt index 4. But the transcendental extension $k(q)$ is a field extension of level 8 and hence the Witt index of $q_{k(q)}$ is just 1.

This yields that the algebraic splitting pattern of q is $(0, 4)$, whereas the splitting pattern is $(0, 1, 4)$. A complete determination of the algebraic splitting pattern of excellent forms over $\mathbb{Q}$ is given in [5, Thm. 3.5].

Hence the algebraic splitting pattern in general differs from the splitting pattern. However, there is always a purely transcendental field extension K of k (so that the Witt indices of q_K and q are the same) such that the algebraic splitting pattern of q_K is the same as the splitting pattern of q, as we will see below. The transcendence degree of K over k can be chosen in a fairly uniform manner depending just on $\dim q$ and *not* on the underlying field k or on the form q itself.

More precisely we have the following.

2.3 Theorem. *For any natural number n, there is a natural number $t = t(n)$ with the following property:*

Let k be any field and let q be any regular quadratic form over k of dimension n. Then if K is a purely transcendental extension of k of degree t, every Witt index which does occur for q over some extension of k is also obtained as the Witt index of $q_{K'}$ for a finite separable extension K' of K.

A possible choice for t is the dimension of the unipotent radical of a Borel subgroup of $\mathrm{SO}_q \times k_s$, where SO_q is the special orthogonal group of q, and k_s is some separable algebraic closure of k.

Proof. The number t as described in the last part of the theorem is exactly half of the number of the roots in a root system of $G := \mathrm{SO}_q \times k_s$ and depends on the dimension of q only, not on the choice of the field k or of q itself.

It follows from [9, 5.2 and 5.3] that a generic splitting tower $\{K_i\}_{i=0,\dots,h}$ of q can be obtained, except possibly for the last field K_h, as a subsequence of the function fields of the connected components V_ν of the k-varieties of ν-dimensional totally isotropic subspaces of q for $\nu = 0, \dots, [n/2]$. For K_h we obtain a similar description, however the corresponding variety is defined over $k(\sqrt{d(q)})$. Hence each K_i is finitely and separably generated over k. The variety $V_\nu \times_k k_s$ is isomorphic to a quotient of G by some maximal proper parabolic subgroup P_ν of G, and its dimension equals the dimension of the unipotent radical of P_ν which is contained in the unipotent radical of some Borel subgroup. Let t_i denote the transcendence degree of K_i. Then $t_i \leq t$, and hence, using a separating transcendence basis, we can embed K_i into a finite separable extension K_i' of K such that K_i' is a purely transcendental extension of K_i. Thus K_i and K_i' are k-specializations of each other and therefore the Witt indices of q_{K_i} and of $q_{K_i'}$ are the same. This proves the theorem. □

2.4 Remark. The theorem shows: Extending the field of definition of q by some purely transcendental field extension K of course does not increase the Witt index of q, but it does increase the magnitude of maximal subtori of SO_q to an extent which is required to produce a more subtle splitting behavior of q_K in the formation of separable extensions of K.

The theorem is not restricted to the case of characteristic $\neq 2$. The proof holds in general, except that one has to replace the field $k(\sqrt{d(q)})$ by the corresponding field extension obtained by the Arf invariant.

The given bound is not optimal. As the proof shows it suffices to take for t the maximum of the dimensions of the unipotent radicals of maximal proper parabolic subgroups which are computed in [9, Cor. 5.5].

A theorem analogous to 2.3 holds in general for reductive groups. This will be discussed elsewhere.

3. On forms of height two.

3.0 Definition ([3, §1, p. 340]). A quadratic form q over k is called *good* if its leading form is defined over k.

We now assume that q is anisotropic.

3.1 Remark.

i) It follows from [11, 9.2 iii), p. 19] that, for a good form q over k, the leading form of q is always defined by a unique Pfister form over k.

ii) Let q be a good form of height 2 and odd dimension. Then q is known to be excellent (cf. [11, §10, p. 27]). This implies that $\dim q = 2^r - 2^s + 1$ with $2 \leq s < r$, and the splitting pattern of q is given by $(0, 2^{r-1} - 2^s + 1, 2^{r-1} - 2^{s-1})$ [5, 2.8 p. 187]. We refer also to [3, 1.2, p. 340].

In case of degree 1 we have:

3.2 Remark. Let q be of even dimension. If $\deg q = 1$, then $d(q) \neq 1$ and the leading form of q is defined by $\langle\langle -d(q) \rangle\rangle$ (cf. [10, 5.10 ii), p. 82]), hence q is good. If in addition q is of height 2, then, by [11, 10.3, p. 28], we have the following two cases:

i) q is excellent. We then have $q \cong a\langle 1, -b\rangle \otimes \rho'$ with $a, b \in k^*$ where ρ' denotes the pure part of some Pfister form ρ; that is, ρ' is the orthogonal complement of the subform $\langle 1\rangle$ of ρ. In this case $\dim q = 2^r - 2$ with $r \geq 3$, and the splitting pattern of q is given by $(0, 2^{r-1} - 2, 2^{r-1} - 1)$.

ii) q is not excellent. Then we have $\dim q = 4$, and the splitting pattern of q is given by $(0, 1, 2)$.

We now assume that q is of height 2. In case of degree 2 we have for good forms:

3.3 Remark. Let q be good of height 2 and even dimension. If $\deg q = 2$, then by [3, Thm. 1.6, p. 342] either q is excellent with $\dim q = 2^r - 4$, $r \geq 4$ and splitting pattern

$(0, 2^{r-1}-4, 2^{r-1}-2)$, or q is not excellent, $q \cong q_1 \otimes \langle\langle a \rangle\rangle$ with $\dim q_1 = 4$, which implies $\dim q = 8$ and splitting pattern $(0, 2, 4)$.

Forms q of height and degree 2 that are not good are treated in 3.11. For good forms of degree 3 we will obtain, as a special case of Theorem 3.9 below, the following natural generalization of the above remarks.

3.4 Theorem. *Let q be good of height 2 with even dimension. If $\deg q = 3$, then either q is excellent, $\dim q = 2^r - 8$ with $r \geq 5$ and splitting pattern $(0, 2^{r-1}-8, 2^{r-1}-4)$, or q is not excellent, $q_K \cong q_1 \otimes \langle\langle a, b \rangle\rangle$ is anisotropic of height 2 over some field extension K of k with $\dim q_1 = 4$, hence $\dim q = 16$, and splitting pattern $(0, 4, 8)$.*

Remark. In fact in [8, Thm. 2.12] it is shown that, for the non-excellent case, $q \cong q_1 \otimes \langle\langle a, b \rangle\rangle$ holds already over k.

We denote by $I(k)$ the fundamental ideal of even dimensional form classes in the Witt ring of k.

3.5 Definition. Let $n \in \mathbb{N}$. We say that *property* E_n *holds* if and only if for every field k (of characteristic $\neq 2$) the following conditions i) and ii) are satisfied:
i) For every $i \in \{1, \ldots, n+1\}$ the map

$$\langle\langle -a_1, -a_2, \ldots, -a_i \rangle\rangle \mapsto (a_1) \cup (a_2) \cup \ldots \cup (a_i)$$

extends to a homomorphism $e_i : \mathrm{I}^i(k) \to \mathrm{H}^i(k, \mathbb{Z}/2\mathbb{Z})$ with kernel $\mathrm{I}^{i+1}(k)$.
ii) For every $i \in \{1, \ldots, n\}$ and for every i-fold Pfister form τ over k we have an equality

$$\ker(\mathrm{H}^{i+1}(k, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{H}^{i+1}(k(\tau), \mathbb{Z}/2\mathbb{Z})) = e_i(\tau) \cup \mathrm{H}^1(k, \mathbb{Z}/2\mathbb{Z}).$$

3.6 Remark. i) e_0 is the parity of the dimension, e_1 is given by the discriminant, e_2 by the Clifford invariant.
ii) Arason [1, 5.7, p. 490] resp. Jacob/Rost [6, Main Theorem, p. 552] have shown that the homomorphisms e_3 resp. e_4 exists and that properties E_2 resp. E_3 hold.

The content of the following lemma can essentially be found in [11, §10, p. 29-30], but, for convenience, we will give a proof here.

3.7 Lemma. *Let q be an anisotropic form with $\dim q = 2^N$ for some $N \geq 2$, $h(q) = 2$ and $\deg q = n < N$ and assume that q is good with leading form defined over k by τ. Then there is a field extension K of k such that q_K is anisotropic, $h(q_K) = 2$ and q_K and τ_K are divisible by an $(n-1)$-fold Pfister form over K.*

Proof. Let μ be a Pfister divisor of q of maximal degree r. Then if $r > 0$, by [11, 9.7, p. 23], τ splits over $k(\mu)$ and hence μ divides τ. By [11, 10.1, p. 28], τ does not divide q and hence $r \leq n-1$.

If $r = n-1$ we are done. Hence let us assume that $r < n-1$. After replacing q by some scalar multiple we may assume that q represents 1 over k. It follows from [11, 10.5, p. 29] that $q \perp (-\tau)$ has index $\dim \mu$, and that we can write

$$(*) \qquad q \perp (-\tau) \cong \mu \otimes \langle 1, -1 \rangle \perp \alpha$$

where α denotes the anisotropic kernel of $q \perp (-\tau)$.

Then $\dim q = 2^N < 2^N + 2^n - 2^{r+1} = \dim\alpha$, and it follows from 1.12 that $q_{k(\alpha)}$ is anisotropic and $h(q_{k(\alpha)}) = 2$. Clearly the leading form of $q_{k(\alpha)}$ is $\tau_{k(\alpha)}$.

Tensoring $(*)$ by $k(\alpha)$ we find that the index of $q_{k(\alpha)} \perp (-\tau_{k(\alpha)})$ is bigger than that of $q \perp (-\tau)$, hence we obtain from [11, 10.5, p. 29] that $q_{k(\alpha)}$ has a Pfister divisor of degree bigger than r, but $\leq n-1$. Repeating this procedure, if necessary, yields the lemma. □

3.8 Lemma. *For $n \geq 1$ let $q = \psi\otimes\varphi$ with an $(n-1)$-fold Pfister form φ and an even dimensional regular form ψ. Then the following holds.*

i) *$\psi \perp -\langle\langle -d\psi\rangle\rangle \in \mathrm{I}^2(k)$ and $q \in \mathrm{I}^n(k)$, and the forms $(\psi \perp -\langle\langle -d\psi\rangle\rangle)\otimes\varphi \in \mathrm{I}^{n+1}(k)$ and $\langle\langle -d\psi\rangle\rangle\otimes\varphi \in \mathrm{I}^n(k)$ depend on q only.*

ii) *Assume that $e_i : \mathrm{I}^i(k) \to \mathrm{H}^i(k, \mathbb{Z}/2\mathbb{Z})$ is defined for all $i = 0, \ldots, n+1$. Then the cohomology class*

$$\widetilde{e}_{n+1}(q) := e_{n+1}((\psi \perp -\langle\langle -d\psi\rangle\rangle)\otimes\varphi) \in \mathrm{H}^{n+1}(k, \mathbb{Z}/2\mathbb{Z})$$

depends on q only.
If $q \in \mathrm{I}^{n+1}(k)$, then $\widetilde{e}_{n+1}(q) = e_{n+1}(q)$.
If π is some n-fold Pfister form and $a \in k^*$ we have $\widetilde{e}_{n+1}(a\pi) = e_{n+1}(-\langle\langle -a\rangle\rangle\otimes\pi)$.

Proof. i) The form $\psi \perp -\langle\langle -d\psi\rangle\rangle$ has even dimension and discriminant 1. It is therefore in $\mathrm{I}^2(k)$ by [13, Ch. II, 2.2, p. 40]. After tensoring it with φ we obtain a form in $\mathrm{I}^{n+1}(k)$. Since in the Witt ring of k we have

$$q \perp (-\psi \perp \langle\langle -d\psi\rangle\rangle)\otimes\varphi \sim \langle\langle -d\psi\rangle\rangle\otimes\varphi,$$

this shows the congruence

$$(*) \qquad q \equiv \langle\langle -d\psi\rangle\rangle\otimes\varphi \bmod \mathrm{I}^{n+1}(k),$$

hence $\langle\langle -d\psi\rangle\rangle\otimes\varphi \in \mathrm{I}^n(k)$. If $q = \psi'\otimes\varphi'$ is another decomposition of q with an $(n-1)$-fold Pfister form φ', then ψ' is also even dimensional. By the above we obtain

$$\langle\langle -d\psi\rangle\rangle\otimes\varphi \perp -\langle\langle -d\psi'\rangle\rangle\otimes\varphi' \equiv q \perp -q \sim 0 \bmod \mathrm{I}^{n+1}(k).$$

Since the left hand side is a difference of two n-fold Pfister forms and hence isotropic of dimension 2^{n+1} and the right hand side is in $\mathrm{I}^{n+1}(k)$ it follows from the Arason-Pfister Hauptsatz [13, 3.1, p. 289] that the left hand side is trivial in the Witt ring, hence $\langle\langle -d\psi\rangle\rangle\otimes\varphi = \langle\langle -d\psi'\rangle\rangle\otimes\varphi'$. This proves i).

ii) Clearly $\widetilde{e}_{n+1}(q)$ depends only on q by i).

If $q \in \mathrm{I}^{n+1}(k)$ then, by $(*)$, it follows that $\langle\langle -d\psi\rangle\rangle\otimes\varphi \in \mathrm{I}^{n+1}(k)$, and this form must be split by the Arason-Pfister Hauptsatz. Hence $\widetilde{e}_{n+1}(q) = e_{n+1}(q)$.

We write the n-fold Pfister form $\pi = \langle\langle -b\rangle\rangle\otimes\varphi$ with some $(n-1)$-fold Pfister form φ and obtain from the above

$$\begin{aligned}\widetilde{e}_{n+1}(a\pi) &= e_{n+1}((a\langle\langle -b\rangle\rangle \perp -\langle\langle -b\rangle\rangle)\otimes\varphi)\\ &= e_{n+1}(-\langle\langle -a\rangle\rangle\otimes\pi).\end{aligned}$$

□

3.9 Theorem. *Assume that property* E_n *(from* 3.5*) holds. Let* q *be a good form of height* 2 *and even dimension. If* $\deg q = n$*, then either* q *is excellent,* $\dim q = 2^r - 2^n$ *with* $r \geq n+2$ *and splitting pattern* $(0, 2^{r-1} - 2^n, 2^{r-1} - 2^{n-1})$ *or* q *is not excellent,* $q_K \cong q_1 \otimes \langle\langle a_1, \ldots, a_{n-1}\rangle\rangle$ *is anisotropic over some field extension* K *of* k *with* $\dim q_1 =$ 4*, which implies* $\dim q = 2^{n+1}$*, and splitting pattern* $(0, 2^{n-1}, 2^n)$.

For $n = 2$, this theorem was communicated to us by B. Kahn [7, 8], see also [3, Thm. 1.6, p. 342].

In the special cases $n = 1, 2, 3$ the reader will recognize the relationship to our statements in 3.2, 3.3 and 3.4.

The standard non-excellent example for 3.9 which is in generic position is given by the form

$$q = \langle X_1, X_2, X_3, X_4\rangle \otimes \langle\langle Y_1, Y_2, \ldots Y_{n-1}\rangle\rangle$$

over $k = \mathbb{Q}(X_1, \ldots, X_4, Y_1, \ldots, Y_{n-1})$, say. The form q is anisotropic over k, since otherwise $\langle\langle 1,1\rangle\rangle \otimes \langle\langle 1, \ldots, 1\rangle\rangle$ would not be anisotropic over $\mathbb{Q}$. Over $K := k(\sqrt{-X_1X_2})$, we have

$$\begin{aligned} q_K &\cong (\langle 1, -1\rangle \perp \langle a, -aX_3X_4\rangle) \otimes \langle\langle Y_1, \ldots, Y_{n-1}\rangle\rangle \\ &\cong 2^{n-1} \times \langle 1, -1\rangle \perp a\langle 1, -X_3X_4\rangle \otimes \langle\langle Y_1, \ldots, Y_{n-1}\rangle\rangle \\ &\cong 2^{n-1} \times \langle 1, -1\rangle \perp a\langle\langle -X_3X_4, Y_1, \ldots, Y_{n-1}\rangle\rangle \end{aligned}$$

with the last orthogonal summand being anisotropic over K. Thus we have $i(q_K) = 2^{n-1}$. Hence $h(q) > 1$ and q is not similar to a Pfister form, so, q is *not excellent.*

If, for some field extension F of k, the form q_F is isotropic, then $q_F \cong q_1 \otimes \Theta$ with $\dim q_1 = 4, \Theta = \langle\langle Y_1, \ldots, Y_{n-1}\rangle\rangle$, and q_1 isotropic over F (cf. [2, Thm. 1.4, proof, p. 185]). So $q_1 \cong \langle 1, -1\rangle \perp \langle b, c\rangle$ and $q_F \cong 2^{n-1} \times \langle 1, -1\rangle \perp \langle b, c\rangle \otimes \Theta$ with $\langle b, c\rangle \otimes \Theta$ being similar to a Pfister form. If $\langle b, c\rangle \otimes \Theta$ is anisotropic over F we have $i(q_F) = 2^{n-1}$. Otherwise it will split completely and $i(q_F) = 2^n$. In any case we find $h(q) \leq 2$.

Therefore, $h(q) = 2$ with splitting pattern $(0, 2^{n-1}, 2^n)$.

Proof of Thm. 3.9. Everything is clear if q is excellent, since we know all dimensions and splitting patterns of excellent forms of any height and degree, by [5].

Thus we assume that q is not excellent and that the leading form of q is defined by an n-fold Pfister form τ over k.

Step 1. $q_{k(\tau)}$ is similar to a Pfister form of degree $\geq n+1$. This follows from [11, 10.1 ii), p. 28]. Hence we know that $\dim q = 2^r$ for some $r \geq n+1$ and we want to show that $r = n+1$. In other words, we know that $q_{k(\tau)} \in \mathrm{I}^{n+1}(k(\tau))$ and we want to show that $q_{k(\tau)} \notin \mathrm{I}^{n+2}(k(\tau))$.

Step 2. By 3.7, there is a field extension K of k such that q_K is anisotropic and divisible by an $(n-1)$-fold Pfister form, and q_K is still of height 2.

Thus we might as well assume that already over k we have $q = q_1 \otimes \Theta$ for some $(n-1)$-fold Pfister form Θ and some form q_1 whose dimension is an even power of 2.

Step 3. By 3.8 we obtain $\widetilde{e}_{n+1}(q) \in \mathrm{H}^{n+1}(k, \mathbb{Z}/2\mathbb{Z})$.

Assume that $q_{k(\tau)} \in \mathrm{I}^{n+2}(k(\tau))$, which means, by E_n (3.5 i)), that $e_{n+1}(q_{k(\tau)}) = 0$. By E_n (3.5 ii)) and 3.8 we obtain for some $a \in k^*$

$$\widetilde{e}_{n+1}(q) = e_n(\tau) \cup (a) = e_{n+1}(\langle\langle -a\rangle\rangle \otimes \tau). \tag{1}$$

Since $\tau_{k(q)}$ is the leading form of q, there is an $f \in k(q)^*$ such that $(q_{k(q)})_{\mathrm{an}} = f\tau_{k(q)}$, and we obtain using 3.8 ii)

$$\widetilde{e}_{n+1}(q_{k(q)}) = e_{n+1}(-\langle\langle -f\rangle\rangle \otimes \tau_{k(q)}). \tag{2}$$

Let $\rho := f\tau_{k(q)} \perp -a\tau_{k(q)}$. Then, in the Witt ring of $k(q)$, we have an equivalence

$$\rho \sim (-\langle\!\langle -f \rangle\!\rangle \perp \langle\!\langle -a \rangle\!\rangle_{k(q)}) \otimes \tau_{k(q)},$$

hence we conclude that $\rho \in \mathrm{I}^{n+1}(k(q))$, and equations (1), (2) yield $e_{n+1}(\rho) = 0$. It now follows from E_n (3.5 i)) that $\rho \in \mathrm{I}^{n+2}(k(q))$. Since $\dim \rho = 2^{n+1}$, the Arason-Pfister Hauptsatz [13, Ch. X, 3.1, p. 289] tells us that ρ splits; in other terms, $a\tau_{k(q)} \cong f\tau_{k(q)}$.

Since $a \in k^*$, we conclude that also $f\tau_{k(q)}$ is defined over k; that is, the anisotropic kernel of q over $k(q)$, and hence all higher kernels of q, are defined over k. This means that q is excellent [11, 7.14, p. 6], which is a contradiction.

So, indeed, $\dim q = 2^{n+1}$ and $\dim q_1 = 4$.

Step 4. It only remains to determine the splitting pattern of an anisotropic form $q = q_1 \otimes \Theta$ of height 2 with $\dim q_1 = 4$ and a Pfister form Θ.

It follows from [2, Thm. 1.4, p. 185] that if $q_1 \otimes \Theta$ is isotropic, then there is an isotropic form q_1' with $q_1 \otimes \Theta \cong q_1' \otimes \Theta$. Thus $i_1(q) \geq \dim \Theta$, which in our case yields $i_1(q) = 2^{n-1}$; that is, q has splitting pattern $(0, 2^{n-1}, 2^n)$ (cf. the standard example above). □

3.10 Corollary. *If* E_n *holds, then there are no anisotropic forms* q *of height* 2 *and degree* n *with leading form defined over the base field and* $\dim q = 2^N$ *with* $N \geq n+2$.

Hence if the condition E_n holds, then the answer to the question 10.6 posed by M. Knebusch [11, 10.6, p. 30] is negative. In particular, 3.6 ii) shows that the answer is negative for $n = 2, 3$.

Let us now discuss the situation when q is of height 2 but not good. Then q is not excellent, hence if $\dim q$ is odd, then $\dim q \geq 11$ by 1.16. If $\dim q$ is even, then $\deg q \geq 2$ by 3.2, in particular $\dim q \geq 6$.

3.11 Theorem. *Let* q *be an anisotropic form of height* 2 *which is of even dimension and not good. If* $\deg q = 2$, *then* q *is of dimension* 6 *and has splitting pattern* $(0, 1, 3)$.

This theorem is due to B. Kahn [8, 2.11]. We will give here a proof based on 1.6.

Proof. As in 1.6, we have $E(q) = \mathrm{M}_{2^r}(D)$ for some skew field D, and it follows from the assumption on the degree that $D \neq k$ and that $d(q) = 1$. If D were a quaternion skew field, then the leading form of q would be defined over k by [11, 9.8, p. 23]. Hence D is the product of at least two quaternion skew fields.

Since the height is 2 we are in case ii) of 1.6, hence $\dim q = 6$ and the splitting pattern is $(0, 1, 3)$. □

3.12 Remark. In [3, Prop. 2.8, p. 348] it was proved that forms as in 3.11 have a dimension ≤ 26.

3.13 Remark. The standard example of a (6-dimensional) quadratic form of height 2 and degree 2 that is not good is given by

$$q = \langle X_1, X_2, X_3, X_4, X_5, -X_1X_2X_3X_4X_5 \rangle$$

over $k = F(X_1, X_2, X_3, X_4, X_5)$; see Prop. 1.2.

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