Anisotropic Groups over Arbitrary Fields*

Ulf Rehmann

Why anisotropic groups ?

1888/9 **Killing** classifies semisimple groups
and introduces the types

$$\mathrm{A}_n,\ \mathrm{B}_n,\ \mathrm{C}_n,\ \mathrm{D}_n,\ \mathrm{E}_6,\ \mathrm{E}_7,\ \mathrm{E}_8,\ \mathrm{F}_4,\ \mathrm{G}_2$$

of semisimple Lie groups.

1961 **Chevalley** shows: These groups have a
$\mathbb{Z}$-structur, ("Chevalley groups")
Killing's Classification holds over
algebraically closed fields.

1965ff **Borel** and **Tits** describe the
internal structure of these groups
insomuch they contain unipotent elements,
that is, up to their "anisotropic kernel".
This reduces the classification and structure theory
to the investigation of anisotropic semisimple groups.

1984 **T. A. Springer** writes in a survey article over
linear algebraic groups:

The most difficult part of a classification of reductive k-groups is the classification of semi-simple anisotropic *k-groups ...*
A complete classification of all anisotropic k-groups seems out of reach.
(Perspectives in Math., Anniv. of Oberwolfach, 1984, p. 477)

---

* Colloquium Talk, Regensburg, June 6., 1997

Anisotropic groups have, by definition, no proper parabolic subgroups.

Examples of anisotropic groups:

i) Compact semisimple Lie groups (alg. groups over $\mathbb{R}$),

ii) $\mathrm{SL}_1(D)$, D: finite dim. central simple k-**division** algebra, Killing type A_n,

iii) $\mathrm{SO}(q)$, $\mathrm{Spin}(q)$, q: **anisotropic** quadratic form over k, Killing type B_n, D_n,

iv) All Killing types have anisotropic "twists" over suitable fields.

Some methods to investigate anisotropic groups:

1. Galois cohomological methods
2. Splitting Patterns (with J. Hurrelbrink.)
3. Excellence properties of algebraic groups (with I. Kersten.)

1. Galois cohomological methods

Notation:

k :	an arbitrary field,
k_s :	separable closure of k,
$\Gamma = \mathrm{Gal}(k_s/k)$:	its Galois group.
G :	semisimple linear alg. group über k.
$\mathrm{H}^1(k, G)$:	poset of classes of 1-cocycles $z : \Gamma \to G(k_s)$

$\mathrm{H}^1(k, \mathrm{Aut}(G))$ classifies all k-twists of G,

i.e., all groups G' over k with $G'(k_s) \cong G(k_s)$.

A *Galois cohomological invariant of* G is a transformation

$$\mathrm{H}^1(k, G) \longrightarrow \mathrm{H}^i(k, C),$$

functorial in field extensions of k, with

$C \quad = \text{Torsion-}\Gamma \text{ module}$

$\mathrm{H}^i(k, C) = \mathrm{H}^i(\Gamma, C)$ (Galois cohomology)

This invariant is *trivial* if the cocycle of the split twist of G in $\mathrm{H}^1(k, G)$ is mapped to $1 \in \mathrm{H}^i(k, C)$.

(Def. needs to be modified in more complicated situations.)

Examples:

(For simiplicity: let tor C and char k be coprime.)

$\underline{i=1}$: G adjoint : $C := \mathrm{Aut}(\mathrm{Dynkin}(G))$ (finite!)

$G \hookrightarrow \mathrm{Aut}(G) \twoheadrightarrow C$ exact $\Rightarrow \exists\ \delta_G : \mathrm{H}^1(k, \mathrm{Aut}(G)) \to \mathrm{H}^1(k, C)$

the *discriminant* invariant

$\underline{i=2}$: G inner, adjoint : $\tilde{G}$ s.c. covering,

$C := \mathrm{center}(\tilde{G}(k_s))$ (finite!)

$C \hookrightarrow \tilde{G} \twoheadrightarrow G$ exact $\Rightarrow \exists\ \delta^2_G : \mathrm{H}^1(k, G) \to \mathrm{H}^2(k, C)$

$x \in \mathrm{Hom}(C, \mathbb{G}_m)$ gives $x_* : \mathrm{H}^2(k, C) \to \mathrm{H}^2(k, \mathbb{G}_m) = \mathrm{Br}(k)$,

the *Brauer* invariant is obtained by:

$\beta_G : C^* := \mathrm{Hom}(C, \mathbb{G}_m) \to \mathrm{Br}(k), x \mapsto x_* \delta^2_G(c)$,

where $c \in \mathrm{H}^1(k, G)$ is the class of the split twist of G.

$\underline{i=3}$: $G = \mathrm{Spin}(q)$: $\exists\ \alpha_G : \mathrm{H}^1(k, G) \to \mathrm{H}^3(k, \mathrm{Z}/2\mathrm{Z})$

the *Arason* invariant

$\underline{i \geq 4}$: $G = \mathrm{Spin}(q)$: $\exists\ \nu^i_G : \mathrm{H}^1(k, G) \to \mathrm{H}^i(k, \mathrm{Z}/2\mathrm{Z})$

(according to Voevodsky's proof of the Milnor conjecture)

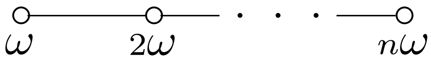
these maps are only "partially defined": $\nu^3_G = \alpha_G$ and

ν^i_G is defined on Ker ν^{i-1}_G for $i > 3$.

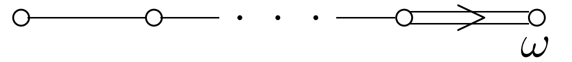
Brauer invariant of inner twists (i.e. δ_G trivial):

$C^* \cong \Lambda/\Lambda^r = \langle\text{dual roots}\rangle/\langle\text{roots}\rangle$.

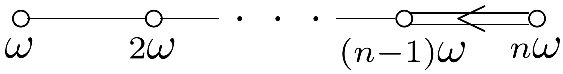
Group structure of C^*:

A_n: 

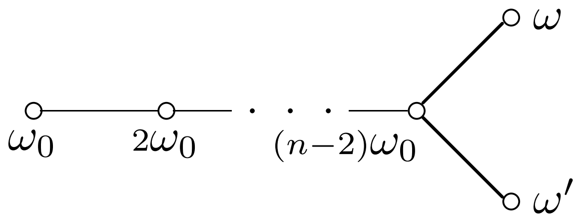
$C^* \cong \mathrm{Z}_{n+1} = \langle\omega\rangle$

B_n: 

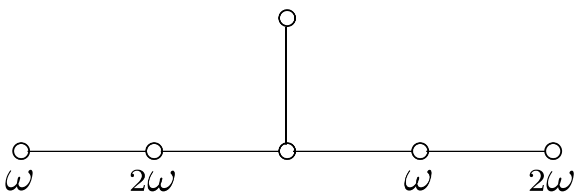
$C^* \cong \mathrm{Z}_2 = \langle\omega\rangle$

C_n: 

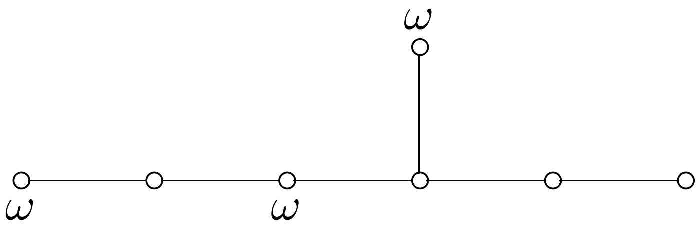
$C^* \cong \mathrm{Z}_2 = \langle\omega\rangle$

D_n: 

$$\begin{cases} 2\nmid n: & C^* \cong \mathrm{Z}_4 = \langle\omega' = -\omega\rangle \\ 2\mid n: & C^* \cong \mathrm{Z}_2 \times \mathrm{Z}_2 = \langle\omega\rangle \times \langle\omega'\rangle \\ \omega_0 = \omega - \omega' & \end{cases}$$

E_6: 

$C^* \cong \mathrm{Z}_3 = \langle\omega\rangle$

E_7: 

$C^* \cong \mathrm{Z}_2 = \langle\omega\rangle$

E_8, F_4, G_2 have trivial C^*.

Meaning of β_G:

The irreducible representation $G_{k_s} \to \mathrm{GL}_m(k_s)$ with highest weight ω_α has k structure $G \to SL_1(A)$, where A is a central simple k algebra of class $\beta_G(\omega_\alpha)$ in $\mathrm{Br}(k)$.

Remarks:

1. G is of $\begin{cases} \textit{outer type}, \text{ if } \delta_G \text{ not trivial} \\ \textit{inner type}, \text{ if } \delta_G \text{ trivial} \end{cases}$
2. Tits (1990) defines:

   G is of *strongly inner type* (SI), if β_G, δ_G trivial.

   Discovers (among other things) (over $\mathbb{R}$) the series

   B_{4m}, B_{4m+3}, D_{4m} of SI anisotropic groups.
3. **complementary concept** (UR):

   G is of *Brauer type* (BT), if G_L splits for every field extension L/k, for which β_{G_L} is trivial.
4. Consequences:
   - i) groups of inner type A_n, C_n are always BT
   - ii) over p-adic fields, inner groups are BT.
   - iii) $\mathrm{G}_2, \mathrm{F}_4, \mathrm{E}_8$ are always SI, $\mathrm{E}_6, \mathrm{E}_7$ mixed, and anisotropic SI types exist
   - iv) $\mathrm{B}_n, \mathrm{D}_n$ are SI $\Leftrightarrow$ the Clifford algebra is a matrix ring.
5. **Theorem** (UR). For every G, there is a *generic Brauer splitting field* K/k (i.e., β_{G_K} generically trivial).

   Question: Under which conditions is G_K anisotropic ?

   (*"Anisotropic splitting"*)

6. Let char $k \neq 2$ and G of type B_n, D_n:
An outer group G is said to be of *discriminant* type, if G_L splits over every field extension L/k, for which δ_{G_L} is trivial.

For orthogonal groups of excellent quadratic forms we have (UR, based on earlier work of M. Knebusch, I. Kersten/UR, J. Hurrelbrink/UR, B. Kahn):

Theorem. *Let $G = \mathrm{SO}_q$ for some anisotropic excellent quadratic form q over k with* char $k \neq 2$.

i) Let $\dim q = 8m + \rho$, where $0 \leq \rho \leq 7$. Then one has the following table for the Killing and the twist type of G:

ρ :	*Killing type:*	*Twist type:*
0	D_{4m}	strongly inner
1	B_{4m}	strongly inner
2	D_{4m+1}	discriminant (outer)
3	B_{4m+1}	Brauer (inner)
4	D_{4m+2}	Brauer (inner)
5	B_{4m+2}	Brauer (inner)
6	D_{4m+3}	discriminant (outer)
7	B_{4m+3}	strongly inner.

ii) Let $\dim q = 8m + \rho$, where $0 \le \rho \le 15$. Then one has the following table for the Killing and the twist type of G:

ρ :	*Killing type:*	*twist type:*	*l.c.i.*
0	D_{8m}	(α_G trivial, strongly inner)	$v_2(16m)$
1	B_{8m}	(α_G trivial, strongly inner)	$v_2(16m)$
2	D_{8m+1}	discriminant (outer)	1
3	B_{8m+1}	Brauer (inner)	2
4	D_{8m+2}	Brauer (inner)	2
5	B_{8m+2}	Brauer (inner)	2
6	D_{8m+3}	discriminant (outer)	1
7	B_{8m+3}	Arason (strongly inner)	3
8	D_{8m+4}	Arason (strongly inner)	3
9	B_{8m+4}	Arason (strongly inner)	3
10	D_{8m+5}	discriminant (outer)	1
11	B_{8m+5}	Brauer (inner)	2
12	D_{8m+6}	Brauer (inner)	2
13	B_{8m+6}	Brauer (inner)	2
14	D_{8m+7}	discriminant (outer)	1
15	B_{8m+7}	(α_G trivial, strongly inner)	$v_2(16(m+1))$

G is said here to be of *Arason type*, if β_G, δ_G is trivial, and if G_L splits for every extension L/k, for which α_{G_L} is trivial. *l.c.i.* is the lowest dimension with non trivial cohomological invariant for G.

With this technique, one can determine the Galois cohomology of spin groups of excellent quadratic forms.

Example: Let

$$q(X_i) = \sum_{i=1}^{1000001} X_i^2.$$

Then, the following list describes the possible indices and dimensions of non trivial Galois cohomology for Spin(q) *over any field* k:

Indices:	Dimensions of n.t.c.
0	6 7 9 10 14 15 16 20
475713	6 7 9 10 14 15 16
491520	6 7 9 10 14 15
492097	6 7 9 10 14
499712	6 7 9 10
499777	6 7 9
499968	6 7
499969	6
500000	–none–

For example, the index 475713 occurs if and only if

$$2^{16} \leq s(k) < 2^{20}$$

for the level $s(k)$ of k.

($s(k)$ = lowest s, such that -1 is a sum of s squares in k. If $s(k)$ is finite, then it is a power of 2.)

Computation of the cohomological invariants
for $G = \mathrm{Spin}(q)$ and anisotropic excellent q:
Let $q = q_0, \ldots, q_h,\ \dim q_h \leq 1$
the "higher anisotropic kernels" of q.
Those are defined over k and unique up to isometry
(Knebusch).

Let $\tilde{q}_i = q_i \perp \mathbb{H}$ with $\mathbb{H}$ hyperbolic and $\dim \tilde{q}_i = \dim q$.
Then $\mathrm{Spin}(\tilde{q}_i)$ is a twist of $\mathrm{Spin}(q)$,
and if $\mathrm{d}(q) = \mathrm{d}(q_i), c(q) = c(q_i)$,
then q_i defines $x_i \in \mathrm{H}^1(k, \mathrm{Spin}(q))$.

These invariants give canonically

$$\nu_G^{a_i}(x_i) \in \mathrm{H}^{a_i}(k, Z/2Z),\ i = 1 \ldots, h.$$

a_i is the maximal number with $q - q_i \in \mathrm{I}^{a_i}(k)$ for $i \in \{1, \ldots, h\}$.

With $\nu_G^{a_i}(x_i)$, the groups

$$\mathrm{Spin}(q),\ q \text{ excellent},$$

can be described uniquely up to isomorphy.

Conversely: All relations between the $\nu_G^{a_i}(x_i)$ in the cohomology ring $\mathrm{H}^*(k, Z/2Z)$ are known, i.e., if there is a set of these elements fulfilling the relations, then there is a group of the above type with these elements as invariants.

2. Splitting Patterns

(jointly with J. Hurrelbrink.)

Explained just by an example.

Def. The *Splitting Pattern* SP_G *of* G is the category of all Tits-Dynkin diagrams of G_L, where L/k ranges over all field extensions of k;
morphisms are the rank increasing k-specializations.

Splitting patterns allow the distiction of anisotropic groups:

Let, e.g., G be of orthogonal type. Then, SP_G is given by the sequence of Witt indices of the underlying quadratic forms, which may occur over extensions of k, morphisms are given by the relation $<$.

Example. Let q be anisotropic, $\dim q = n$.

q Pfister form: $\mathrm{SP}_{\mathrm{SO}(q)} = (0, n/2)$;
q "generic": $\mathrm{SP}_{\mathrm{SO}(q)} = (0, 1, 2, \ldots, [n/2])$.

$h(q) = \#\mathrm{SP}_q - 1$ is the *height* of q.

Thm. (Wadsworth/Knebusch) $h(q) = 1 \Leftrightarrow q$ is similar to an orthogonal summand of a Pfister form of codim ≤ 1.

Forms of height 2 are not completely known. However:

2.1 **Thm.** *Let q be a form of even dimension with $h(q) = 2$, whose "leading form" is defined over k. Then q is of one of the following types.*

i) $\dim q = 2^a - 2^b, a-1 > b > 0$, $\mathrm{SP}_q = (0, 2^{a-1} - 2^b, 2^{a-1} - 2^{b-1})$, *and for every K/k the anisotropic kernel of q_K is defined over k, i.e., q is excellent.*

ii) $\dim q = 2^a$, $\mathrm{SP}_q = (0, 2^{a-2}, 2^{a-1})$, *$q$ not excellent. In this case there is L/k, such that q_L is an anisotropic Pfister-Form. ("Anisotropic splitting")*

Remark. Part i) was proved by Knebusch (1976), part ii) verifies a conjecture by Knebusch, who had proved ii) for the case $a \leq 3$.

Proved 1993 by H-R under a cohomological condition, which was proved 1996 by Voevodsky (Milnor-Vermutung). Also, in 1996, D. Hoffmann gave an elementary proof without using Voevodsky's result.

Recently (Hurrelbrink-R): Description of all quadratic forms which are linear combinations of two Pfister forms (to appear in Crelle J.)

3. Excellence properties of algebraic groups

Def. G is called *excellent*, if for every L/k the anisotropic kernel of G_L is defined over k.

There is an analogue notion for Azumaya algebas.

3.1 **Theorem** (Kersten-R.): For an Azumaya algebra A over k the following statements are equivalent:

i) A is excellent.
ii) The index of A is squarefree.
iii) Index and exponent of A_L are equal for every L/k.
iv) $G = \mathrm{SL}_1(A)$ is excellent.

3.2 **Theorem** (Kersten-R.): (char $k \neq 2$)
Let q be a regular quadratic form over k. Then equivalent:

i) $G = \mathrm{SO}(q)$ excellent
ii) For every extension L/k there is a quadratic form τ over k and $a \in L^*$ with $a\tau_L \cong (q_L)_{\mathrm{an}}$.

Hence: Excellence of orthogonal groups is more general then excellence of quadratic forms.
(for B_n equivalent, but not for D_n).

Example: if q is anisotropic with dim $q = 4$ and d(q) $\neq 1$, then $\mathrm{SO}(q)$ is excellent, but q is not.
More general, forms from 2.1. ii) have this property.

Apparently, excellent groups have a very simple cohomological nature ($G = \mathrm{SO}(q), q$ excellent: see above.).

$G := \mathrm{SL}_1(D)$, D/k a skew field, central, finite dimensional over k:

$n = \mathrm{ind}(D)$ squarefree:

i)	$\exp(D) = \mathrm{ind}(D)$	(Brauer)
ii)	$\mathrm{SK}_1(D) := G(k)/[D^*, D^*] = 1$	(Wang)
iii)	$\mathrm{H}^1(k, G) \hookrightarrow \mathrm{H}^3(k, \mu_n^{\otimes 2})$ (n, char k coprime)	(Merkurjev-Suslin)
iv)	G excellent	(Kersten-R)

$n = \mathrm{ind}(D)$ **not** squarefree:

•	$\exists\ L/k$ with $\exp(D_L) \neq \mathrm{ind}(D_L)$	(K-R)
•	$\exists$ *"anisotropic splitting: exp. reduction"*	(K-R)
•	$\exists\ D/k$ with $\mathrm{SK}_1(D) \neq 1$	(Platonov, Draxl)
•	$\mathrm{H}^1(k, G) \to H^3(k, \mu_n^{\otimes 2})$ i.g. not injective	(Merkurjev)
•	G not excellent	(K-R)

Conjecture (Suslin, 1991):

$$\mathrm{SK}_1(D_{k(G)}) = 1 \iff \mathrm{ind}(D) \text{ squarefree}$$

One knows (Merkurjev 1993):

G rational (i.e., $k(G)/k$ purely transcendental.) $\Rightarrow \mathrm{SK}_1(D_{k(G)}) = 1$

char $k \neq 2$ und $4 \mid \mathrm{ind}(D)$ $\Rightarrow \mathrm{SK}_1(D_{k(G)}) \neq 1.$

Excellence of G is a sort of "rationality property" of G.

By 3.1: G excellent $\Leftrightarrow$ $\mathrm{ind}(D)$ squarefree.

Question :

$$G \text{ rational} \Leftrightarrow \mathrm{SK}_1(D_{k(G)}) = 1 \Leftrightarrow G \text{ excellent}$$
$$\Leftrightarrow \mathrm{H}^1(k, G) \hookrightarrow \mathrm{H}^3(k, \mu_n^{\otimes 2}) \text{ for } n, \text{ char } k \text{ coprime}$$

J. Hurrelbrink, U. Rehmann, Splitting patterns of excellent quadratic forms, *J. reine ang. Math.* **444** (1993), 183–192.

J. Hurrelbrink, U. Rehmann, Splitting patterns of quadratic forms, *Math. Nachr. 176* (1995), 111–127.

J. Hurrelbrink, U. Rehmann, Splitting patterns and linear combinations of two Pfister forms, preprint 1997, to appear in *J. reine ang. Math.* (1998).

A. S. Merkurjev, Generic element in SK_1 for simple algebras, *K-Theory* **7** (1993), 1–3.

A. S. Merkurjev, K-Theory of simple algebras, *Proc Symp. Pure Math.* **58.1** (1995), 65–83.

B. Kahn, Formes quadratique de hauteur et de degré 2, *Indag. Mathem., N.S.*, **7**(1), (1996) 47–66.

I. Kersten, U. Rehmann, Generic splitting of reductive groups, *Tôhoku Math. J.* **46** (1994) 35-70.

I. Kersten, U. Rehmann, Excellent algebraic groups I, Prep. 96-120, SFB 343 (Bielefeld, 1996), to app. *J. Algebra.*

J-P. Serre, Cohomology Galoisienne : Progrès et Problèmes *Sem. Bourbaki* **783** (1993-4) 229–257.

A. A. Suslin, SK_1 of division algebras and Galois cohomology, *Advances in Sov. Math.* **4** 1991) 75–99.

Tits, Jacques Strongly inner anisotropic forms of simple algebraic groups. J. Algebra 131 (1990), no. 2, 648–677.

S. Wang, On the commutator group of a simple algebra, *Amer. J. Math.* **72**, (1950) 323-334.