Anisotropic Groups over Arbitrary Fields*
Ulf Rehmann

Why anisotropic groups?

1888/9 Killing classifies semisimple groups and introduces the types
\[ A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2 \]
of semisimple Lie groups.

1961 Chevalley shows: These groups have a \( \mathbb{Z} \)-structur, (“Chevalley groups”)
Killing’s Classification holds over algebraically closed fields.

1965ff Borel and Tits describe the internal structure of these groups
insomuch they contain unipotent elements,
that is, up to their “anisotropic kernel”.
This reduces the classification and structure theory to the investigation of anisotropic semisimple groups.

1984 T. A. Springer writes in a survey article over linear algebraic groups:

The most difficult part of a classification of reductive \( k \)-groups is the classification of semi-simple anisotropic \( k \)-groups . . .
A complete classification of all anisotropic \( k \)-groups seems out of reach.

(Perspectives in Math., Anniv. of Oberwolfach, 1984, p. 477)

* Colloquium Talk, Regensburg, June 6., 1997
Anisotropic groups have, by definition, no proper parabolic subgroups.

**Examples of anisotropic groups:**

i) Compact semisimple Lie groups (alg. groups over $\mathbb{R}$),

ii) $\text{SL}_1(D), D$: finite dim. central simple $k$-division algebra, Killing type $A_n$,

iii) $\text{SO}(q), \text{Spin}(q), q$: anisotropic quadratic form over $k$, Killing type $B_n, D_n$,

iv) All Killing types have anisotropic “twists” over suitable fields.

Some methods to investigate anisotropic groups:

1. Galois cohomological methods
2. Splitting Patterns (with J. Hurrelbrink.)
3. Excellence properties of algebraic groups (with I. Kersten.)
1. Galois cohomological methods

Notation:

\( k \) : an arbitrary field,

\( k_s \) : separable closure of \( k \),

\( \Gamma = \text{Gal}(k_s/k) \) : its Galois group.

\( G \) : semisimple linear alg. group über \( k \).

\( H^1(k, G) \) : poset of classes of 1-cocycles \( z : \Gamma \to G(k_s) \)

\( H^1(k, \text{Aut}(G)) \) classifies all \( k \)-twists of \( G \),
i.e., all groups \( G' \) over \( k \) with \( G'(k_s) \cong G(k_s) \).

A Galois cohomological invariant of \( G \) is a transformation

\[ H^1(k, G) \longrightarrow H^i(k, C), \]

functorial in field extensions of \( k \), with

\( C \) = Torsion-\( \Gamma \) module

\[ H^i(k, C) = H^i(\Gamma, C) \) (Galois cohomology)

This invariant is trivial if the cocycle of the split twist of \( G \) in
\( H^1(k, G) \) is mapped to \( 1 \in H^i(k, C) \).

(Def. needs to be modified in more complicated situations.)
Examples:

(For simiplicity: let tor $C$ and char $k$ be coprime.)

$i = 1$: $G$ adjoint : $C := \text{Aut}(\text{Dynkin}(G))$ (finite!)

$G \hookrightarrow \text{Aut}(G) \rightarrow C$ exact $\Rightarrow \exists \delta_G : H^1(k, \text{Aut}(G)) \rightarrow H^1(k, C)$
the discriminant invariant

$i = 2$: $G$ inner, adjoint : $\tilde{G}$ s.c. covering,

$C := \text{center}(\tilde{G}(k_s))$ (finite!)

$C \hookrightarrow \tilde{G} \rightarrow G$ exact $\Rightarrow \exists \delta^2_G : H^1(k, G) \rightarrow H^2(k, C)$

$x \in \text{Hom}(C, \mathbb{G}_m)$ gives $x_* : H^2(k, C) \rightarrow H^2(k, \mathbb{G}_m) = \text{Br}(k)$,
the Brauer invariant is obtained by:

$\beta_G : C^* := \text{Hom}(C, \mathbb{G}_m) \rightarrow \text{Br}(k), x \mapsto x_*\delta^2_G(c)$,

where $c \in H^1(k, G)$ is the class of the split twist of $G$.

$i = 3$: $G = \text{Spin}(q) : \exists \alpha_G : H^1(k, G) \rightarrow H^3(k, \mathbb{Z}/2\mathbb{Z})$
the Arason invariant

$i \geq 4$: $G = \text{Spin}(q) : \exists \nu_G^i : H^1(k, G) \rightarrow H^i(k, \mathbb{Z}/2\mathbb{Z})$

(according to Voevodsky’s proof of the Milnor conjecture)
these maps are only “partially defined”: $\nu^3_G = \alpha_G$ and
$\nu^i_G$ is defined on $\text{Ker} \nu^{i-1}_G$ for $i > 3$. 

4
**Brauer invariant of inner twists** (i.e. $\delta_G$ trivial):

$$C^* \cong \Lambda/\Lambda^r = \langle \text{dual roots} \rangle / \langle \text{roots} \rangle.$$

Group structure of $C^*$:

- $A_n$: $\omega \rightarrow 2\omega \rightarrow \cdots \rightarrow n\omega$  
  \[ C^* \cong \mathbb{Z}_{n+1} = \langle \omega \rangle \]

- $B_n$: $\omega \rightarrow \cdots \rightarrow \omega$  
  \[ C^* \cong \mathbb{Z}_2 = \langle \omega \rangle \]

- $C_n$: $\omega \rightarrow 2\omega \rightarrow \cdots \rightarrow (n-1)\omega \rightarrow n\omega$  
  \[ C^* \cong \mathbb{Z}_2 = \langle \omega \rangle \]

- $D_n$: $\omega_0 \rightarrow 2\omega_0 \rightarrow \cdots \rightarrow (n-2)\omega_0 \rightarrow \omega_0$  
  \[
  \begin{cases}
    2|n: & C^* \cong \mathbb{Z}_4 = \langle \omega' = -\omega \rangle \\
    2\n: & C^* \cong \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle \omega \rangle \times \langle \omega' \rangle \\
    \omega_0 = \omega - \omega' 
  \end{cases}
  \]

- $E_6$: $\omega \rightarrow 2\omega \rightarrow \omega \rightarrow 2\omega$  
  \[ C^* \cong \mathbb{Z}_3 = \langle \omega \rangle \]

- $E_7$: $\omega \rightarrow \omega \rightarrow \omega \rightarrow \omega \rightarrow \omega$  
  \[ C^* \cong \mathbb{Z}_2 = \langle \omega \rangle \]

$E_8$, $F_4$, $G_2$ have trivial $C^*$.

Meaning of $\beta_G$:

The irreducible representation $G_{k_s} \rightarrow \text{GL}_m(k_s)$ with highest weight $\omega_\alpha$ has $k$ structure $G \rightarrow \text{SL}_1(A)$, where $A$ is a central simple $k$ algebra of class $\beta_G(\omega_\alpha)$ in $\text{Br}(k)$.
Remarks:

1. $G$ is of \begin{align*}
\text{outer type}, & \text{ if } \delta_G \text{ not trivial} \\
\text{inner type}, & \text{ if } \delta_G \text{ trivial}
\end{align*}

2. Tits (1990) defines:

   $G$ is of strongly inner type (SI), if $\beta_G, \delta_G$ trivial.

   Discovers (among other things) (over $\mathbb{R}$) the series

   $B_{4m}, B_{4m+3}, D_{4m}$ of SI anisotropic groups.

3. complementary concept (UR):

   $G$ is of Brauer type (BT), if $G_L$ splits for every field extension $L/k$, for which $\beta_{G_L}$ is trivial.

4. Consequences:

   i) groups of inner type $A_n, C_n$ are always BT
   ii) over p-adic fields, inner groups are BT.
   iii) $G_2, F_4, E_8$ are always SI, $E_6, E_7$ mixed, and anisotropic SI types exist
   iv) $B_n, D_n$ are SI $\iff$ the Clifford algebra is a matrix ring.

5. Theorem (UR). For every $G$, there is a generic Brauer splitting field $K/k$ (i.e., $\beta_{G_K}$ generically trivial).

   Question: Under which conditions is $G_K$ anisotropic?

   ("Anisotropic splitting")
6. Let $\text{char } k \neq 2$ and $G$ of type $B_n, D_n$:

An outer group $G$ is said to be of \textit{discriminant} type, if $G_L$ splits over every field extension $L/k$, for which $\delta_{G_L}$ is trivial.

For orthogonal groups of excellent quadratic forms we have (UR, based on earlier work of M. Knebusch, I. Kersten/UR, J. Hurrelbrink/UR, B. Kahn):

**Theorem.** Let $G = \text{SO}_q$ for some anisotropic excellent quadratic form $q$ over $k$ with $\text{char } k \neq 2$.

i) Let $\dim q = 8m + \rho$, where $0 \leq \rho \leq 7$. Then one has the following table for the Killing and the twist type of $G$:

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>Killing type:</th>
<th>Twist type:</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$D_{4m}$</td>
<td>strongly inner</td>
</tr>
<tr>
<td>1</td>
<td>$B_{4m}$</td>
<td>strongly inner</td>
</tr>
<tr>
<td>2</td>
<td>$D_{4m+1}$</td>
<td>discriminant (outer)</td>
</tr>
<tr>
<td>3</td>
<td>$B_{4m+1}$</td>
<td>Brauer (inner)</td>
</tr>
<tr>
<td>4</td>
<td>$D_{4m+2}$</td>
<td>Brauer (inner)</td>
</tr>
<tr>
<td>5</td>
<td>$B_{4m+2}$</td>
<td>Brauer (inner)</td>
</tr>
<tr>
<td>6</td>
<td>$D_{4m+3}$</td>
<td>discriminant (outer)</td>
</tr>
<tr>
<td>7</td>
<td>$B_{4m+3}$</td>
<td>strongly inner.</td>
</tr>
</tbody>
</table>
ii) Let $\dim q = 8m + \rho$, where $0 \leq \rho \leq 15$. Then one has the following table for the Killing and the twist type of $G$:

<table>
<thead>
<tr>
<th>$\rho$</th>
<th>Killing type:</th>
<th>twist type:</th>
<th>l.c.i.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$D_{8m}$</td>
<td>$(\alpha_G$ trivial, strongly inner)</td>
<td>$v_2(16m)$</td>
</tr>
<tr>
<td>1</td>
<td>$B_{8m}$</td>
<td>$(\alpha_G$ trivial, strongly inner)</td>
<td>$v_2(16m)$</td>
</tr>
<tr>
<td>2</td>
<td>$D_{8m+1}$</td>
<td>discriminant (outer)</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>$B_{8m+1}$</td>
<td>Brauer (inner)</td>
<td>2</td>
</tr>
<tr>
<td>4</td>
<td>$D_{8m+2}$</td>
<td>Brauer (inner)</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>$B_{8m+2}$</td>
<td>Brauer (inner)</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>$D_{8m+3}$</td>
<td>discriminant (outer)</td>
<td>1</td>
</tr>
<tr>
<td>7</td>
<td>$B_{8m+3}$</td>
<td>Arason (strongly inner)</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>$D_{8m+4}$</td>
<td>Arason (strongly inner)</td>
<td>3</td>
</tr>
<tr>
<td>9</td>
<td>$B_{8m+4}$</td>
<td>Arason (strongly inner)</td>
<td>3</td>
</tr>
<tr>
<td>10</td>
<td>$D_{8m+5}$</td>
<td>discriminant (outer)</td>
<td>1</td>
</tr>
<tr>
<td>11</td>
<td>$B_{8m+5}$</td>
<td>Brauer (inner)</td>
<td>2</td>
</tr>
<tr>
<td>12</td>
<td>$D_{8m+6}$</td>
<td>Brauer (inner)</td>
<td>2</td>
</tr>
<tr>
<td>13</td>
<td>$B_{8m+6}$</td>
<td>Brauer (inner)</td>
<td>2</td>
</tr>
<tr>
<td>14</td>
<td>$D_{8m+7}$</td>
<td>discriminant (outer)</td>
<td>1</td>
</tr>
<tr>
<td>15</td>
<td>$B_{8m+7}$</td>
<td>$(\alpha_G$ trivial, strongly inner) $v_2(16(m + 1))$</td>
<td></td>
</tr>
</tbody>
</table>

$G$ is said here to be of Arason type, if $\beta_G, \delta_G$ is trivial, and if $G_L$ splits for every extension $L/k$, for which $\alpha_{G_L}$ is trivial. 

*l.c.i.* is the lowest dimension with non trivial cohomological invariant for $G$. 

8
With this technique, one can determine the Galois cohomology of spin groups of excellent quadratic forms.

**Example:** Let

\[ q(X_i) = \sum_{i=1}^{1000001} X_i^2. \]

Then, the following list describes the possible indices and dimensions of non trivial Galois cohomology for \( \text{Spin}(q) \) over any field \( k \):

<table>
<thead>
<tr>
<th>Indices</th>
<th>Dimensions of n.t.c.</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>6 7 9 10 14 15 16 20</td>
</tr>
<tr>
<td>475713</td>
<td>6 7 9 10 14 15 16</td>
</tr>
<tr>
<td>491520</td>
<td>6 7 9 10 14 15</td>
</tr>
<tr>
<td>492097</td>
<td>6 7 9 10 14 15</td>
</tr>
<tr>
<td>499712</td>
<td>6 7 9 10</td>
</tr>
<tr>
<td>499777</td>
<td>6 7 9</td>
</tr>
<tr>
<td>499968</td>
<td>6 7</td>
</tr>
<tr>
<td>499969</td>
<td>6</td>
</tr>
<tr>
<td>500000</td>
<td>–none–</td>
</tr>
</tbody>
</table>

For example, the index 475713 occurs if and only if

\[ 2^{16} \leq s(k) < 2^{20} \]

for the level \( s(k) \) of \( k \).

\( s(k) \) = lowest \( s \), such that \(-1\) is a sum of \( s \) squares in \( k \). If \( s(k) \) is finite, then it is a power of 2.)
Computation of the cohomological invariants for $G = \text{Spin}(q)$ and anisotropic excellent $q$:

Let $q = q_0, \ldots, q_h$, $\dim q_h \leq 1$

the “higher anisotropic kernels” of $q$.

Those are defined over $k$ and unique up to isometry (Knebusch).

Let $\tilde{q}_i = q_i \perp H$ with $H$ hyperbolic and $\dim \tilde{q}_i = \dim q$.

Then $\text{Spin}(\tilde{q}_i)$ is a twist of $\text{Spin}(q)$, and if $d(q) = d(q_i), c(q) = c(q_i)$, then $q_i$ defines $x_i \in H^1(k, \text{Spin}(q))$.

These invariants give canonically

$$\nu^{a_i}_G(x_i) \in H^{a_i}(k, \mathbb{Z}/2\mathbb{Z}), \ i = 1 \ldots, h.$$  

$a_i$ is the maximal number with $q - q_i \in \Gamma^{a_i}(k)$ for $i \in \{1, \ldots, h\}$.

With $\nu^{a_i}_G(x_i)$, the groups

Spin$(q), \ q$ excellent,

can be described uniquely up to isomorphy.

Conversely: All relations between the $\nu^{a_i}_G(x_i)$ in the cohomology ring $H^*(k, \mathbb{Z}/2\mathbb{Z})$ are known, i.e., if there is a set of these elements fulfilling the relations, then there is a group of the above type with these elements as invariants.


2. Splitting Patterns
(jointly with J. Hurrelbrink.)

Explained just by an example.

**Def.** The *Splitting Pattern* $\text{SP}_G$ of $G$ is the category of all Tits-Dynkin diagrams of $G_L$, where $L/k$ ranges over all field extensions of $k$; morphisms are the rank increasing $k$-specializations.

Splitting patterns allow the distinction of anisotropic groups:

Let, e.g., $G$ be of orthogonal type. Then, $\text{SP}_G$ is given by the sequence of Witt indices of the underlying quadratic forms, which may occur over extensions of $k$, morphisms are given by the relation $<$.

**Example.** Let $q$ be anisotropic, $\dim q = n$.

$q$ Pfister form: $\text{SP}_{\text{SO}(q)} = (0, n/2);$

$q$ “generic”: $\text{SP}_{\text{SO}(q)} = (0, 1, 2, \ldots, [n/2]).$

$h(q) = \#\text{SP}_q - 1$ is the *height* of $q$.

**Thm.** (Wadsworth/Knebusch) $h(q) = 1 \iff q$ is similar to an orthogonal summand of a Pfister form of codim $\leq 1$. 

Forms of height 2 are not completely known. However:

2.1 **Thm.** Let $q$ be a form of even dimension with $h(q) = 2$, whose “leading form” is defined over $k$. Then $q$ is of one of the following types.

i) $\dim q = 2^a - 2^b, a - 1 > b > 0$, $\text{SP}_q = (0, 2^{a-1} - 2^b, 2^{a-1} - 2^{b-1})$, and for every $K/k$ the anisotropic kernel of $q_K$ is defined over $k$, i.e., $q$ is excellent.

ii) $\dim q = 2^a$, $\text{SP}_q = (0, 2^{a-2}, 2^{a-1})$, $q$ not excellent. In this case there is $L/k$, such that $q_L$ is an anisotropic Pfister-Form. ("Anisotropic splitting")

**Remark.** Part i) was proved by Knebusch (1976), part ii) verifies a conjecture by Knebusch, who had proved ii) for the case $a \leq 3$.

Proved 1993 by H-R under a cohomological condition, which was proved 1996 by Voevodsky (Milnor-Vermutung). Also, in 1996, D. Hoffmann gave an elementary proof without using Voevodsky’s result.

**Recently** (Hurrelbrink-R): Description of all quadratic forms which are linear combinations of two Pfister forms (to appear in Crelle J.)
3. Excellence properties of algebraic groups

**Def.** $G$ is called *excellent*, if for every $L/k$ the anisotropic kernel of $G_L$ is defined over $k$.

There is an analogue notion for Azumaya algebras.

3.1 **Theorem** (Kersten-R.): For an Azumaya algebra $A$ over $k$ the following statements are equivalent:

i) $A$ is excellent.

ii) The index of $A$ is squarefree.

iii) Index and exponent of $A_L$ are equal for every $L/k$.

iv) $G = SL_1(A)$ is excellent.

3.2 **Theorem** (Kersten-R.): (char $k \neq 2$)

Let $q$ be a regular quadratic form over $k$. Then equivalent:

i) $G = SO(q)$ excellent

ii) For every extension $L/k$ there is a quadratic form $\tau$ over $k$ and $a \in L^*$ with $a\tau_L \cong (q_L)_{an}$.

Hence: Excellence of orthogonal groups is more general then excellence of quadratic forms.

(for $B_n$ equivalent, but not for $D_n$).

**Example:** if $q$ is anisotropic with dim $q = 4$ and $d(q) \neq 1$, then $SO(q)$ is excellent, but $q$ is not.

More general, forms from 2.1. ii) have this property.
Apparently, excellent groups have a very simple cohomological nature \((G = \text{SO}(q), q \text{ excellent}: \text{see above.})\).

\[ G := \text{SL}_1(D), \quad D/k \text{ a skew field, central, finite dimensional over } k: \]

\( n = \text{ind}(D) \) squarefree:

i) \( \exp(D) = \text{ind}(D) \) \quad \text{(Brauer)}

ii) \( \text{SK}_1(D) := \frac{G(k)}{[D^*, D^*]} = 1 \) \quad \text{(Wang)}

iii) \( H^1(k, G) \hookrightarrow H^3(k, \mu_n \otimes^2) \) \( (n, \text{char } k \text{ coprime}) \) \quad \text{(Merkurjev-Suslin)}

iv) \( G \) excellent \quad \text{(Kersten-R)}

\( n = \text{ind}(D) \text{ not squarefree:} \)

- \( \exists L/k \text{ with } \exp(D_L) \neq \text{ind}(D_L) \) \quad \text{(K-R)}
- \( \exists \text{ “anisotropic splitting: exp. reduction”} \) \quad \text{(K-R)}
- \( \exists D/k \text{ with } \text{SK}_1(D) \neq 1 \) \quad \text{(Platonov, Draxl)}
- \( H^1(k, G) \rightarrow H^3(k, \mu_n \otimes^2) \) i.g. not injective \quad \text{(Merkurjev)}
- \( G \) not excellent \quad \text{(K-R)}

Conjecture (Suslin, 1991):

\[ \text{SK}_1(D_{k(G)}) = 1 \iff \text{ind}(D) \text{ squarefree} \]

One knows (Merkurjev 1993):

\( G \) rational \( (\text{i.e., } k(G)/k \text{ purely transcendental.}) \Rightarrow \text{SK}_1(D_{k(G)}) = 1 \)

\( \text{char } k \neq 2 \text{ und } 4 \mid \text{ind}(D) \Rightarrow \text{SK}_1(D_{k(G)}) \neq 1. \)

Excellence of \( G \) is a sort of “rationality property” of \( G \).

By 3.1: \( G \) excellent \( \iff \text{ind}(D) \text{ squarefree.} \)

Question:

\( G \) rational \( \iff \text{SK}_1(D_{k(G)}) = 1 \iff G \) excellent

\( \iff H^1(k, G) \hookrightarrow H^3(k, \mu_n \otimes^2) \) for \( n, \text{char } k \text{ coprime} \)


I. Kersten, U. Rehmann, Excellent algebraic groups I, Prep. 96-120, SFB 343 (Bielefeld, 1996), to app. *J. Algebra.*


