INTAS 93-2618: Final Report

for the period of June 1, 1995, till November 30, 1996

Initial remark: This report covers the total 18 months of the whole project. The project was very successful and is intended to be extended. The results of the project so far have been reported on 3 international conferences [C1,C2,C3], which were partially supported by the resources of this project, and the results will be published in at least 51 scientific publications.

1. Title

Linear Algebraic Groups, Algebraic K-Theory,
and related Structures in Algebra and Topology

2. Objectives

The research of this project is concentrated on the investigation of the splitting behavior of linear algebraic groups, the investigation of splitting properties of quadratic forms, the investigation of Lie algebras and their cohomology groups and their non-split and central extensions, in particular the computation of those groups, investigation of Azumaya algebras and Brauer groups of local hyper-elliptic curves, investigation of algebraic K-theories for various objects including homogeneous spaces (of algebraic groups) and special normed algebras over normed commutative rings.

3. Research Activities and Results Obtained

3.1 Description of the research activities

The main activities of the project were done by individual research in the above mentioned fields. Several interim results have been reported on the occasion of three international conferences mentioned below, also some individual visits of the NIS participants at various western universities. All fields mentioned in section 2. (Objectives) have been investigated intensively.

[C1] Euro-Conference on "Linear Algebraic Groups and Related Structures", organized by Ina Kersten (Bielefeld), Alexander Merkurjev (St. Petersburg) and Ulf Rehmann (Bielefeld), held at the Sonderforschungsbereich "Diskrete Strukturen in der Mathematik" at the University of Bielefeld (Germany) on July 3 - July 7, 1995.
International conference with 57 participants, sponsored mainly by the DFG (Deutsche Forschungsgemeinschaft), by the European community as a Euro-Conference, and by INTAS 93-2618 (Participation of 5 scientists from the former Soviet Union states was financed totally or in parts by INTAS.)

[C2] Conference on "Algebraic K-Theory", organized by Dan Grayson (Urbana,Illinois) and Ulf Rehmann at the "Mathematisches Forschungsinstitut" Oberwolfach (Germany), held on June 10 - 14, 1996. International conference with 44 participants.

[C3] Euro-Conference on "Linear Algebraic Groups and Related Structures", organized by Jean-Louis Colliot-Thélène (Paris), Eva Bayer-Fluckiger Besançon), Jean-Pierre Tignol (Louvain-la Neuve), Paul Van Praag, held on June 17 - 21, 1996, at Louvain-la-Neuve (Belgium). (Participation of 3 scientists from the former Soviet Union states and of one scientist from Western Europe was financed totally or in parts by INTAS.)

Besides these conferences there are various research activities of the single labs partially in interaction with each other , which are described separately for each participant of the project below.

3.2 Results

The results of the project are or will be documented by at least 51 publications mentioned below in section 4.1.1. Out of these publications, more than 40 have already been finished and most of them (at the time being: 29) already have appeared in or have been accepted by international scientific journals. Several others are are mentioned of being in preparation, and this indicates the expected further achievement of a project extension.
The publications all are on the field of linear algebraic groups and related structures and their content is described in detail in section 4.1.1, where the contributions of the individual participants are presented.

3.3 Application of Results

The results on splitting of algebraic groups and other linear structures will be applied to gain a better understanding of anisotropic groups and structures over arbitrary fields, which is a wide open area, and as such part of the research project on linear algebraic groups at the Sonderforschungsbereich 343 "Diskrete Structuren in der Mathematik" at Bielefeld. For example, in 1998 there will be another conference on "Linear algebraic groups and related structures" at Bielefeld (organized by I. Kersten and U. Rehmann) in the framework of this long running project. This conference will be a satellite conference to the International Congress of Mathematicians at Berlin in 1998.
The Sonderforschungsbereich will continue to integrate the NIS participants of this project into his work by short and long term invitations, and it would be good to have a continued INTAS project of this type to give these participants more flexibility in attending conferences and seminars at various places in Europe.

4. Management and Financial Information

4.1 Project management

4.1.1 Co-operation with reference to each contractor

Participant 1 – Bielefeld:

Activities: Organization of conferences [C1], [C2]; Research in linear algebraic groups and related structures, in collaboration with A. Merkurjev (St. Petersburg), who visits Bielefeld as a Humboldt Prize Winner from June 1995 till August 1996. In December 1996 U. Rehmann visits Besançon (supported by INTAS). I. Kersten and U. Rehmann participated in all or some of the conferences C1, C2, C3.
Publications:
[HR-I] J. Hurrelbrink and U. Rehmann, Splitting patterns of quadratic forms, Math. Nachr. (1995), 111-127.
Abstract: General constraints for splitting patterns of quadratic forms are discussed. It is investigated how the splitting of a quadratic form is influenced by the splitting pattern of its Clifford algebra. Criteria for a quadratic form being a "stable Pfister neighbor" are given, an alternative proof of a recent theorem of Hoffmann on the first higher Witt index is given. Also, properties of forms of height 2 are studied in terms of certain cohomological invariants.
[HR-II] J. Hurrelbrink and U. Rehmann, Splitting patterns of trace forms (preprint 1996), to appear in Canadian Mathematical Bulletin.
Abstract: The splitting pattern of a quadratic form q over a field k consists of all distinct Witt indices that occur for q over extension fields of k. In small dimensions, the complete list of splitting patterns of quadratic forms is known. It is shown that *all* splitting patterns of quadratic forms of dimension at most nine can be realized by trace forms.
[KR] I. Kersten and U. Rehmann, Excellence properties of semi-simple algebraic groups (preprint 1996).
Abstract: A semi-simple algebraic group over a field k is called excellent, when for every field extension K of k, the anisotropic kernel of G_K is defined over k. This parallels the definition of excellent quadratic forms as given by Knebusch, however, it is not true that a quadratic form is excellent if and only if its orthogonal group is excellent. The excellence of a form implies the excellence of its orthogonal group, but the converse is false. The excellent groups of inner type A_n are determined, in parallel an excellence condition for Azumaya algebras is formulated, and it is proved that exactly those Azumaya algebras are excellent which have square-free index.
[R] U. Rehmann, Semi-simple groups of Brauer type (in preparation).
Abstract: A semi-simple algebraic group over a field k is of Brauer type, when it splits over every field extension of k over which its second cohomological invariant (which has values in the Brauer group) becomes trivial. This notion is complementary to the notion of groups of strongly inner type introduced by Tits. Examples are discussed, and for groups of orthogonal type a new periodicity of twist types depending on the rank modulo 8 is given.

[MRY-I] G. Margolin, U. Rehmann, V. Yanchevskii, Quaternion generation of the 2-torsion part of the Brauer group of a local quintic (the type of splitting (1,1,1,1)), preprint 1996 (27 pages, see also Participant 4)
[MRY-II] G. Margolin, U. Rehmann, V. Yanchevskii, Quaternion generation of the 2-torsion part of the Brauer group of a local quintic (the type of splitting (1,1,1,2)), preprint 1996 (25 pages, see also Participant 4).
Abstract: The splitting behavior of the Brauer group for the field extension given by the function field K of a quintic over some local field k is determined by giving the elements of the kernel of the natural map of Brauer groups $\mathrm{Br}(k) \to \mathrm{Br}(K)$.

Participant 2 – Besançon:

Eva Bayer-Fluckiger, Laura Fainsilber, Detlev Hoffmann, Anne Queguiner from Besançon participated at the conferences C1 and/or C3 and contributed by giving talks. The travel of Bayer-Fluckiger from Besançon to Louvain-la-Neuve on occasion of C3 was paid from funds of this project as well as the travel for the visit of Bayer-Fluckiger in St. Petersburg in October 1996.
[1] E. Bayer-Fluckiger, J. Morales "Multiples of trace forms in number fields", *Proc. Symp. Pure Math.* **58.2** (1995), 73–81.

Soient K un corps de nombres, G un groupe fini et L une G-algèbre galoisienne sur K. Notons q_L la forme trace de L, et q_0 la forme trace de la G-algèbre déployée. Le but de cet article est de montrer que les G-formes $q_L \oplus q_L$ et $q_0 \oplus q_0$ sont isomorphes si et seulement si le discriminant de toute sous-algèbre quadratique de L est une somme de deux carrés.

[2] E. Bayer-Fluckiger, "Formes quadratiques devenant isotropes sur une extension", *Enseignement Math.*, **41** (1995), 111–122.

Soit k un corps de caractéristique $\neq 2$. Soit q une forme quadratique anisotrope sur k. Si E est une extension de k, on note q_E la forme quadratique obtenue par extension des scalaires à E. On dit que q *devient isotrope* sur E si la forme q_E est isotrope. Soit $k[X_1, ..., X_m]$ l'anneau des polynômes à m variables sur k. Soit $f \in k[X_1, ..., X_m]$ un polynôme irréductible, et notons $k(f)$ le corps des fractions de $k[X_1, ..., X_m]/(f)$. Le but de cette note est de présenter un critère nécessaire et suffisant pour que q devienne isotrope sur $k(f)$, et d'en donner quelques applications. Sous une forme légèrement différente, ce critère avait été obtenu par Witt.

[3] E. Bayer-Fluckiger, R. Parimala, "Galois cohomology of the classical groups over fields of cohomological dimension 2", *Invent. Math.*, **122** (1995), 195–229.

Soient K un corps parfait de caractéristique $\neq 2$, K_s une clôture séparable de K, et $G_K = \mathrm{Gal}(K_s/K)$. Pour tout G_K–module discret C et pour tout m entier ≥ 0, on note $H^m(K, C) = H^m(G_K, C)$. On dit que la *dimension cohomologique* de K est $\leq n$ si pour tout $m > n$ et pour tout G_K–module de torsion C, on a $H^m(K, C) = 0$. On écrit alors $\mathrm{cd}(K) \leq n$.

Soit G un groupe algébrique linéaire défini sur K. On définit le premier ensemble de cohomologie $H^1(K, G) = H^1(G_K, G(K_s))$.

En 1962, Serre a conjecturé que si K est un corps de dimension cohomologique ≤ 2, et si G est un groupe algébrique linéaire semi-simple et simplement connexe défini sur K, alors $H^1(K,G) = 0$. Cette conjecture est connue pour les corps p-adiques (Kneser), et pour les corps de nombres totalement imaginaires (Kneser, Harder, Chernousov).

Dans cet article, nous montrons que la conjecture est vraie pour les groupes classiques, ainsi que pour les groupes de type G_2 et F_4. La démonstration utilise le théorème de Merkurjev–Suslin, des résultats de Yanchevskii, de Merkurjev, de Parimala–Sridharan–Suresh, ainsi qu'un résultat commun avec Lenstra.

[4] E. Bayer-Fluckiger, "Systèmes de formes quadratiques d'algèbre donnée", *C.R. Acad. Sci. Paris* **321** (1995), 1291–1293.

Soit k un corps de caractéristique différente de 2. A tout système de formes quadratiques sur k est associée une algèbre à involution dont le groupe unitaire est le groupe des automorphismes du système. Le but de cette note est de montrer que toute algèbre à involution se réalise comme l'algèbre d'un système de trois formes quadratiques.

[5] E. Bayer-Fluckiger, L. Fainsilber "Non–unimodular hermitian forms", *Invent. Math.* (1996), à paraître.

Soit A un anneau, et soit M un A–module réflexif de type fini. Soit M^* le dual de M. On dit qu'une forme hermitienne $h : M \times M \to A$ est *unimodulaire* si h induit un isomorphisme $M \to M^*$. La théorie de Quebbemann, Scharlau et Schulte apporte de nombreux renseignements sur les formes unimodulaires. Par contre, les formes non unimodulaires ont été très peu étudiées, sauf dans des cas très particuliers. Cet article ramène (essentiellement) la classification des formes non unimodulaires sur A à celle de formes unimodulaires sur un autre anneau.

Une *algèbre p-adique* est une algèbre A de rang fini sur $\mathbf{Z}_p$. On dit que deux formes hermitiennes sont *stablement isomorphes* si elles deviennent isomorphes après addition (somme orthogonale) d'une même forme hermitienne. Un des problèmes de la théorie des formes hermitiennes est de déterminer si deux formes stablement isomorphes sont isomorphes. Si c'est le cas, on dit que le théorème de simplification de Witt vaut pour les formes hermitiennes sur A. Ce théorème n'était connu que dans peu de cas (certaines algèbres commutatives, ainsi que certaines formes hermitiennes sur les ordres héreditaires). Cet article établit le théorème de simplification de Witt en toute généralité. On obtient aussi une généralisation du théorème de Springer aux algèbres p-adiques, ainsi que des applications à la forme trace.

[6] E. Bayer-Fluckiger, R. Parimala, "Hasse principle for the classical groups over fields of virtual cohomological dimension 2", preprint 1995.

On dit qu'un corps k est de *dimension cohomologique virtuelle* n si $\mathrm{cd}(\mathrm{k}(\sqrt{-1}) = \mathrm{n}$. Notons Ω l'ensemble des ordres de k. Pour $v \in \Omega$, on note k_v la clôture réelle de k par rapport à v. Soit G un groupe semi–simple, simplement connexe de type classique, ou de type G_2 ou F_4. Dans cet article, on montre que l'application canonique

$$H^1(k,G) \to \prod_{v \in \Omega} H^1(k_v, G)$$

est injective.

Ceci généralise la conjecture II de Serre ainsi que le "principe de Hasse" classique pour les corps de nombres (du moins lorsque G est de l'un des types ci–dessus).

[7] E. Bayer-Fluckiger, "Lattices with automorphisms of given characteristic polynomial", en préparation.

Rappelons qu'un réseau est appelé f–réseau s'il a une isométrie de polynôme caractéristique f. Comme le groupe des isométries d'un réseau est fini, l'existence d'un f–réseau entraîne que f est un produit de polynômes cyclotomiques. Le but de cette recherche est de donner la liste complète des polynômes f pour lesquels il existe un f–réseau unimodulaire. Plus généralement, les méthodes de l'article permettent de caractériser les formes discriminant des f–réseaux.

[8] A. Quéguiner, Signature des involutions de deuxième espèce, *Arch. Math.* **65** (1995) 408-412.
Abstract: The signature of algebras with involution of the second kind is defined and investigated, and its impact on the decomposability of such algebras is studied.

[9] A. Quéguiner, Cohomological invariants of algebras with involutions, *J. Algebra*, to appear.
Abstract: The trace form of an algebra with involution gives rise to a cohomological invariant of dimension 2 of that algebra with involution. The interrelationship of this invariant with other "classical" invariants is investigated.

Participant 3 – St. Petersburg:

O. Izhboldin, A. Merkurjev, I. Panin, N. Nenashev from St. Petersburg participated in all or some of the conferences C1, C2, C3 and contributed by giving talks. The stay of Panin, Nenashev and Izhboldin at C1 and the stay of Panin and Izhboldin at C3 was supported by this project. They all including S. Yagonov contributed by writing several research papers (see below). A. Merkurjev from St. Petersburg visited Bielefeld for one year (ending in August 1996) as a Humboldt prize winner (upon suggestion by the project co-ordinator of this project) to do research in the area of linear algebraic groups and related structures. He co-organized the conference C1 at Bielefeld, and contributed to conferences C2, C3 by giving talks there.
i) Ivan Panin.
a) The K-theory of algebraic tori and toric varieties were computed. In particular it was shown that $K_0(T)$ for a tori T is a finitely generated Abelian group.
b) Index reduction formulas for inner forms of flag varieties were calculated. More precisely, inner forms of the flag varieties for classical groups were considered.
c) A variant of Gersten's Conjecture was proved for the K-theory of Azumaya algebras over a smooth local ring of geometric type. The corollary of this is the positive solution of Grothedieck's conjecture on principal homogeneous spaces for the special linear group $SL_{1,D}$.

d) Explicit and uncomplicated index reduction formulas for outer forms of twisted flag varieties were calculated. More precisely outer forms of twisted flag varieties were considered.

Publications:

[A] A. S. Merkurjev and I. A. Panin, *K-Theory of algebraic tori and toric varieties*, K-Theory, Vol. **10**, No.6 (1996), 517–569.
[B] A. S. Merkurjev, I. A. Panin, and A. R. Wadsworth, *Index reduction formulas for inner forms of flag varieties, I.* K-Theory, Vol. **10**, No.6 (1996), 517–569.
[C] I. A. Panin and A. A. Suslin, *On a Grothendieck's Conjecture for Azumaya algebras.* Preprint POMI, 1996, submitted to St.-Petersburg Math. J.
[D] A. S. Merkurjev, I. A. Panin, and A. R. Wadsworth, *Index reduction formulas for twisted flag varieties, II.* submitted to K-Theory.

ii) Oleg Izhboldin.
a) The cohomology groups $H^n_p(F((t)))$ and $H^n_p(F(t))$ were computed for any field F of characteristic $p > 0$, in terms of invariants of the field F. In particular the following theorem is proved: The homomorphism $H^n(F(t)) \to \coprod_v H^n(\hat{F}(t)_v)$ is injective, where v runs over all discrete valuations of $F(t)$, and $\hat{F}(t)_v$ denotes the corresponding completion of the field $F(t)$.
b) For any $n \geq 3$, a field F and an n-fold Pfister form ϕ were constructed such that the field extension $F(\phi)/F$ is not excellent. Moreover for any field and quadratic form ϕ over F the following theorem is proved: The field extension $F(\phi)/F$ is universally excellent if and only if ϕ is a Pfister neighbor of dimension ≤ 4.
c) Quadratic forms ϕ with maximal splitting were investigated under some condition on the dimension of ϕ. In particular we find a simple proof of the following fact: any quadratic form with maximal splitting of dimension $2^n - 5$ is necessarily a Pfister neighbor. d) For any biquaternion algebra A we construct an unirational field extension E/F such that $E(SB(A))/E$ is not excellent. Moreover we find a condition on a central simple algebra A so that $F(SB(A))/F$ is universally excellent. Namely, we prove that the field extension $F(SB(A))/F$ is universally excellent only in the following two cases: 1) the index of A is odd; 2) algebra A has the form $Q \otimes_F D$, where Q is a quaternion algebra and the index of D is odd.
e) We prove, that under certain conditions on the quadratic form ϕ ($\dim \phi \leq 8$, and ϕ contains no a quadratic form with trivial determinant) the isotropy of $\phi_{F(\psi)}$ implies that there exists a homomorphism of even parts of theirs Clifford algebras, $C_0(\psi) \to C_0(\phi)$.

Publications:

[A] *On the cohomology groups for the field of rational function*, Amer. Math. Soc. Transl. (2) Vol. **174**, 1996, 21–44.
[B] *On the non-excellence of field extension $F(\pi)/F$*, Doc. Math. J. DMV, **1**, 1996, pp. 127–136.

[C] *On the quadratic forms with maximal splitting*, to appear in St.-Petersburg Math. J., 1996. [D]
On the nonexcellence of the function fields of Severi–Brauer varieties, Preprint MPI, (1996).
[E] *On the isotropy of quadratic forms of small dimension over the function field of a quadric*, Preprint MPI, (1996)

iii) Alexander Nenashev.
a) We prove that for an arbitrary exact category A, every element of $K_1(A)$ can be represented by algebraic data called a double short exact sequence.
b) We establish relations between the images of double short exact sequences in $K_1(A)$.
c) We prove that the relations established in (b) provide a complete system of defining relations for $K_1(A)$. Thus we obtain a presentation of K_1 by generators and relations in terms of the underlying exact category.

Publications:
[A] *Double short exact sequences produce all elements of Quillen's* K_1, Algebraic K-Theory (Banaszak et al., Editors), Contemp. Math. Proc. Series, **199**, AMS Providence RI, (1996), 151–160.
[B] *Double short exact sequences and* K_1 *of an exact category*, Preprint MPI 95-118; submitted to K-Theory.
[C] K_1 *by generators and relations*, Preprint MPI, 96-156, 1996.

iv) Serge Yagunov.
We completely calculate the homology of bi-Grassmannian complexes with rational coefficients. As an immediate application we obtain a rational case of van der Kallen's stabilization theorem.

Publications:
[A] *Homology of bi-Grassmannian complexes*, Preprint 1996 (K-theory Preprint archive http://math.uiuc.edu/K-theory/0101), to appear in K-theory.

Participant 4 – Minsk:

A. Chernousov and V. Yanchevskii from Minsk participate in this project. Chernousov is visiting Bielefeld right now for several months, supported by Alexander v. Humboldt foundation, and working in the area of this project. Chernousov was supported by this project in order to attend a conference in Barcelona. Yanchevskii has visited Bielefeld several times, supported by the Bielefeld Sonderforschungsbereich 343. He attended the conferences C1 and C3 under support by this project.

Results:

1. By means of the Tate algorithm the complete list of minimal Weierstrass equations for all non-dyadic local elliptic curves is found.

2. The complete description of the torsion part of the group of rational points (as an abstract Abelian group) for any non-dyadic local elliptic curve is obtained.
3. For the 2-torsion part of the Brauer group of any non-dyadic local elliptic curve a representation of all its elements by unramified quaternion division algebras is found.
4. For a non-dyadic local hyperelliptic curves the problem of the description of the m-torsion part of its Brauer group (as an abstract Abelian group) was reduced to the problem of the computation of m-torsion of the group of rational points of its Jacobian.
6. The abstract Abelian group structure of the 2-torsion part of the Brauer groups of non-dyadic local hyperelliptic curves is described.
7. The representation of the elements of the above 2-torsion part by means of unramified quaternion algebras for non-dyadic local hyperelliptic curves with good reduction was found.

Publications:

1. V. Yantchevskii, G. Margolin, Brauer groups of local elliptic and hyperelliptic curves and central division algebras over their function fields Preprint 95 - 42 Universität Bielefeld
2. V. Yanchevskii, G. Margolin. Brauer groups of local hyperelliptic curves with good reduction. Algebra and analysis v. 7.(1995)N6p 227-249 (in Russian).
3. [MRY-I] G. Margolin, U. Rehmann, V. Yanchevskii, Quaternion generation of the 2-torsion part of the Brauer group of a local quintic (the type of splitting (1,1,1,1)), preprint 1996 (27 pages, see also Participant 1)

[MRY-II] G. Margolin, U. Rehmann, V. Yanchevskii, Quaternion generation of the 2-torsion part of the Brauer group of a local quintic (the type of splitting (1,1,1,2)), preprint 1996 (25 pages, see also Participant 1).

Abstract: The splitting behavior of the Brauer group for the field extension given by the function field K of a quintic over some local field k is determined by giving the elements of the kernel of the natural map of Brauer groups $\mathrm{Br}(k) \to \mathrm{Br}(K)$.

[R-B-C] A.S. Rapinchuk, V.V. Benyash-Krivetz, V.I. Chernousov. Representation Varieties of the Fundamental Groups of Compact Orientable Surfaces. Israel Journal of Mathematics, **93** (1996) 29–71.

[B-C] V.V. Benyash-Krivetz, V.I. Chernousov. Representation Varieties of the Fundamental Groups of Compact Non-orientable Surfaces, in preparation, to be submitted to SFB 343.

Abstract of [R-B-C,B-C]: A description of varieties of n-dimensional representations and characters $R_n(\Gamma)$ for surface groups Γ and their representation varieties in characteristic zero is given and the structure of those varieties is described. The methods allow in particular to obtain new results on one-relator groups.

[C-M] V.I. Chernousov, A.S. Merkurjev. R-equivalence and Special Unitary Groups. Preprint 96–083, SFB 343, Bielefeld (1996).

Abstract: The group of R-equivalences is computed for special unitary groups in order to study rationality problems for such groups.

[C-P] V.I. Chernousov, V.P. Platonov. Provisional title: The Rationality Problem for Group Varieties over Arithmetic Fields, in preparation.

Abstract: It is proved that, over a p-adic or over a totally imaginary algebraic number field or a global field of positive characteristic, the variety of any simple algebraic group

not of type A_n is rational.
[C-R] V.I. Chernousov, A.A. Ryjkov. On Classification of Maximal Arithmetic Subgroups in Semisimple Simply Connected Groups, submitted to Mat. Sbornik.
Abstract: For simple simply connected groups over number fields, a criterion for maximality of arithmetic subgroups in terms of local parahoric subgroups is given.

Participant 5 – Alma Ata:

Participants are A. Dzhumadil'daev, K. Abdukhalikov, R. Kerimbaev. They contributed to the project by the below mentioned publications. Dzhumadil'daev has attended the conference C1 (supported by this project) and has visited Bielefeld in June/July 1996. For this last visit, his travel will be supported by this project, while his subsistence was supported by the Sonderforschungsbreich "Diskrete Strukturen in der Mathematik" at Bielefeld.

Results:

We have calculated the first and second cohomology groups of maximal graded nilpotent subalgebra $\mathcal{L}_1$ of Lie algebra L of Cartan Types W_n, S_n, H_n, K_{n+1}.
For $n > 2$, these groups are very large:

$$L = W_n,\ dim H^2(\mathcal{L}_1, \mathbf{C}) = n^2(n+1)(3n^3 + 3n^2 - 4n - 14)/24 \sim n^6/8,$$

$$L = S_n,\ dim H^2(\mathcal{L}_1, \mathbf{C}) = (n+1)n(n-1)(3n^3 + 6n^2 - 10n - 24)/24 \sim n^6/8,$$

$$L = H_n,\ dim H^2(\mathcal{L}_1, \mathbf{C}) = (2n-1)n(8n^4 + 28n^3 + 34n^2 + 5n - 21)/18 \sim 8n^6/9,$$

$$L = K_n,\ dim H^2(\mathcal{L}_1, \mathbf{C}) = (2n+1)(8n^5 + 20n^4 + 34n^3 + 7n^2 - 15n - 18)/18 \sim 8n^6/9.$$

But as modules over L_0 these groups have nice structures. The decomposition of $H^2(\mathcal{L}_1, \mathbf{C})$ into direct sum of irreducible submodules are quite observable. For example, for the Hamiltonian algebra $L = H_n,\ n > 1$, the group $H^2(\mathcal{L}_1, \mathbf{C})$, as a module over the symplectic subalgebra sp_n it can be decomposed into a direct sum of 5 irreducible components with multiplicity 1. This decomposition follows from the decomposition of sp_n-module $L_1 \otimes L_k$ into the direct sum of irreducible modules. The number of irreducible components of the last module is 10. It is the number of presentations of 3 as a sum of three nonnegative integers: $x + y + z = 3,\ x, y, z \in \mathbf{Z}_+$ and there is one-to-one correspondence between these partitions and irreducible components of the decomposition $L_1 \otimes L_k$.
The second homology group of $\mathcal{L}_1$ can be interpreted as a group of defining relations of the nilpotent Lie algebra $\mathcal{L}_1$.
All defining relations of $\mathcal{L}_1$ for $L = W_n, S_n, K_n, n > 2$, have degree 2. For $L = H_n, n > 1$ defining relations of $\mathcal{L}_1$ have degree 2 and 3. In case of $n = 1, 2$ higher degree ones are possible.
Some Corollaries.
Descriptions of the following three things are equivalent:

i)nonsplit extensions of Cartan Type Lie algebras (Virasoro Type Lie algebras);
ii)defining relations of maximal subalgebras of Cartan Type Lie algebras;
iii)invariant operations on $\wedge^k(\mathcal{L}_1) \to \mathbf{C}$
Any Cartan Type Lie algebras have *a finite number of irreducible tensor modules with nonzero second cohomologies.* For example, the Witt algebra $L = W_n = Der\mathbf{C}[[x_1, \ldots, n]]$ has exactly $5(n = 1), 8(n = 2, 3), 9(n > 3)$ tensor modules with nonsplit extensions. In particularly, nontrivial central extensions appear only in the case $n = 1$. The corresponding central extension is well known as a Virasoro algebra. We find new local deformations in addition to the Moyal bracket for the two-sided Hamiltonian Algebra. We also find new local deformations for Lie algebra without divergence in three variables.
These results were obtained by A. Dzhumadil'daev and his students R. Kerimbaev, S. Adbukasimova, A. Jolumbetova, Sh. Ibraev with assistance of K. Abduchalikov. Partially these results are published in [1-5].

Publications

1. A.S. Dzhumadil'daev. Virasoro Type Lie algebras and deformations, Zeitschrift für Physik, Ser. C, **72**(1996), No.3, 509-518.
2. A.S. Dzhumadil'daev, Symmetric (co)homologies of Lie algebras, Comptes Rendus Acad. Sci., Paris, Ser. mathem. (to appear).
3.A.S. Dzhumadil'daev, Noncommutative Lie algebras, Proc. of XXI Intern. Colloqium on Group Theoretical Methods in Physics", Goslar, 15-20 July, 1996.
4.A.S. Dzhumadil'daev, S. Abdukasimova, R.K. Kerimbaev. Defining relations of nilpotent subalgebras of Cartan Type Lie algebras. Presented to Comptes Rendus Acad. Sci., Ser. Mathem., Paris.
5.R.Kerimbaev, Second homology group of Hamiltonian Cartan Type Lie algebra Presented to Mathem.Sbornik.

Main seminar talks

1. Joint seminar of prof. B.Pareigis-J.Wess (Munich University), "Cohomologies of Lie algebras"
2. Seminar in Algebraic Geometry of prof. Yu.I.Manin (Max-Plank Institute,Bonn), "Virasoro Type Lie algebras: cohomologies and defining relations"
3. Conference "Algebraic groups" (U.Rehmann, Bielefeld), "Virasoro Type Lie algebras"
4. Algebraic Seminar of K. Roggenkamp (Stuttgart University), " Cohomologies and representations of modular Lie algebras"

Participant 6 – Tbilisi:
Participating researchers: J. Gubeladze, H. Inassaridze, M. Jibladze, T. Kandelaki, D. Pataria, T. Pirashvili. They contributed to the project as described below. T. Pirashvili visited the Sonderforschungsbereich Bielefeld for several months, his travel expenses were covered by this project. During the period covered by the project, M. Jibladze visited Max

Planck Institute for Mathematics, Bonn; Peripatetic Seminar on Sheaves and Logic, University of Edinburgh; Mathematical Departments of the Universities of Utrecht and Genoa; major part of his travel expenses has been covered by the project. Now J. Gubeladze is visiting the University of Osnabrück for one year (starting at August 1996) supported by Alexander v. Humboldt foundation, in order to do research in the area of toric varieties and related structures. His travel expenses from Tbilisi to Osnabrück were covered by this project.

Results:

i) J. Gubeladze.
By substantial changes and corrections in Demushkin's old paper the final positive answer to the isomorphism problem for monoid rings of submonoids of Z^r is obtained (i. e. two monoids are shown to be isomorphic if they have isomorphic monoid algebras with coefficients in arbitrary commutative rings). Basing on this result it is shown that if two discrete Hodge algebras (coefficients in any commutative ring) are isomorphic then their defining monomial ideal are same modulo a permutation of variables.

Publications:
[1] J. Gubeladze, The isomorphism problem for commutative monoid rings, J. Pure & Appl. Algebra, to appear.

ii) H. Inassaridze.
In [1], Quillen's algebraic K-theory of discrete rings is extended to the category of normed algebras over a commutative Banach ring with unit and its relationship with topological K-theory is established.
In [2] it is proved that Kasparov's bivariant K-functors form a right universal sequence of functors.

Publications:
[1]. H. Inassaridze, Algebraic K-theory of normed algebras, submitted to K-theory, 1996.
[2]. H. Inassaridze, Universal property of Kasparov bivariant K-theory, Georg. Math. J., to appear, 1996.

iii) M. Jibladze.
In [1] a cohomology theory is constructed giving in particular cases cohomology of small categories, Hochschild cohomology of algebras.
In [2] conditions on 2-categories are investigated that are equivalent to equipping their nerves with the structure of a cyclic set. This in particular yields description of a homotopy 2-type associated to a square group in the sense of Baues and Pirashvili.
In [3-4] a certain analog of the classical Hermite reciprocity for regular representations of finite cyclic groups is proved. In the course of giving a combinatorial explanation for it, relationship with bicolored necklaces and homogeneous generators of free Lie algebras is established.

Publications:

[1] M. Jibladze (with H.-J. Baues and A. Tonks), Cohomology of monoids in monoidal categories, In ‘Operads: Proceedings of Renaissance Conferences”, ed. J.-L. Loday, J. D. Stasheff, A. A. Voronov, Contemporary Mathematics vol. 202, Pub. AMS 1997.

[2]. M. Jibladze, Cyclic structures on nerves of 2-categories, in preparation.

[3] A. Elashvili and M. Jibladze, “Hermite reciprocity for the regular representations of cyclic groups”, submitted to Indagationes Mathematicae;

[4] A. Elashvili, M. Jibladze, D. Pataraia, “Combinatorics of necklaces and Hermite reciprocity”, submitted to the Journal of Algebraic Combinatorics”;

iv) T. Kandelaki.

It is constructed universal C^*-algebra generated by partial isometry.This algebra has essential Z_2-gradation. There is epimorphism from this algebra onto the Toeplitz algebra. Using gradation on this algebra we construct analogy of Cuntz algebras for each separable C^*-algebras, which gives homotopical interpretation Kasparov KK-groups. It is proved that this algebra is homotopic up to stabilization 2×2-matrices to the $M_2(C)$ in ungraded case (($M_2(C)$ with even gradation in graded case).Therefore those algebras are KK-isomorphic.

Publications:

[1] T. Kandelaki, On the universal C*-algebra generated by partial isometry, Georg. Math. J., to appear.

v)D. Pataraia. In this paper is introduced the notion of an internal category in a left exact cosimplicial category, generalizing ordinary internal categories. Then it is considered toposes, which admit the notion of ”a discrete internal category” by means of internal categories in a left exact cosimplicial categories. Then for such toposes is determined the notion of the fundamental group by means of the ”discrete” category corresponding to the terminal object. When the topos is the category of sheaves over a locally compact and locally simply connected space, then its fundamental group is the same as the fundamental group of the underlying space.

See also his joint work with M. Jibladze and A. Elashvili [4] in iii).

Publications: [1] D. Pataraia, Internal categories in the left exact cosimplicial category, Georg. Math. J., to appear.

vi) T. Pirashvili.

In [1], the ring structure of the Mac Lane cohomology of the ring of integers is calculated. The additive structure was already known by Bökstedt. However calculations in [8] do not use Bökstedt’s results, so it is somewhat an algebraic counterpart of his sophisticated topological methods.

In [2] it has been shown that the category of Leibniz representations of a classical semisimple Lie algebra has global dimension 2. In case of sl_2 the classification of all indecomposable Leibniz representations is given.

In [3], among other results is proved that, if G_* is a simplicial group of finite length, then H_nG_* also has finite length. Here the length of a simplicial group means the length of the corresponding Moore normalization and H_nG_* is a simplicial Abelian group given by $[k] \mapsto H_nG_k$. A similar fact is true if we replace G_* by a simplicial ring and we take the algebraic K-functors instead of group homology.

Publications: [1]. V. Franjou.; T. Pirashvili, On Mac Lane cohomology for the ring of integers, Topology, to appear.
[2]. Loday J.-L.; Pirashvili T., Leibniz representations of Lie algebras. J. algebra 181 (1996), 414-425.
[3]. T. Pirashvili, Simplicial degree of functors. submitted to Commentarii Mathematici Helvetici.

Applications of the results

The results can be applied to the study: of general (non-affine) toric varieties, to ($\mathbf{Z_2}$ graded) Kasparov KK-theory (including K-theory of commutative normed algebras), cohomology of algebraic theories (including Mac Lane cohomology), low dimensional homotopy classification problems, combinatoric of necklaces and Leibniz algebras.
Partial integration in the SFB-343 and AvH Foundation research projects, Cooperation with F. Waldhausen (Bielefeld), J. Herzog (Essen), W. Bruns, R. Vogt (Osnabrück), H. Kraft (Basel), J. L. Loday (Strasbourg), J. Cuntz (Heidelberg), H. J. Baues (Bonn).

Management and Financial Information

For Tbilisi group the INTAS funding was crucial in obtaining computer facilities. The funding was also very helpful in continuation of collaboration (travel costs) with colleagues in EU. The funding was an opportunity to obtain additional funding for already existing project.

Financial Information

Travel - 2244 ECU, Salary - 1246 ECU, Computers - 1496 ECU, Overhead - 554 ECU.
Total - 5540 ECU.

4.1.2 Planning

The time planning was essentially in accordance with which was indicated in the INTAS cooperation agreement.

4.1.3 Problems encountered and changes required to the co-operation agreement

No serious problems were encountered. The only changes to the cooperation agreement are that no participant so far wanted to use money for computer equipment, as it was more desirable to use the money in order to attend conferences or visit colleagues.

4.1.4 Use of equipment

Personal computers have been used in order to i) produce scientific manuscripts and ii) to maintain contacts with other partners by email and iii) to maintain support of scientific literature in so much this is distributed on the World Wide web. Inparticular this hold for the lab Tbilisi, where some of the INTAS grant was spent on computer equipment.

4.1.5 Analysis of the project

Given the framework of other projects (SFB, Euroconference programme, Oberwolfach Institute conferences, Alexander v. Humboldt foundation), the relationship between objectives, research activities and expected results as indicated in the INTAS co-operation agreement was realistic. The wealth of the results described in 4.1.1 of course is not only due to the INTAS funding.

However, participation of NIS scientists would have been very problematic, as e.g. the Euroconference programmes would not have allowed to include them, the SFB programme would have only allowed to invite them to Bielefeld, the Humboldt grants are usually dedicated to single persons, and the Oberwolfach conferences are just activities on some special place (and difficult to get, as there are always many applications for conferences). The INTAS funding for this project was initiative to perform a strongly desired integration of NIS scientists into a scientific project which already existed in Western Europe, but was only equipped with special funding which did not have enough flexibility for this integration process. Integration of NIS scientists meant an important strengthening of the project as a whole and an essential improvement of its scientific quality.

4.2 Financial Information

4.2.1 Cost breakdown for the period covered by this report

Labs:	travel, subsist.	salaries	comput. equipm.	overhead	total
1. Bielefeld (U. Rehmann)	800			200	1000
2. Besançon (E. Bayer)	1000				1000
3. Pomi St. Petersburg (A. Suslin)	3740	3180			6920
4. Minsk (V.I. Yantchevskii)	2216			554	2770
5. Alma-Ata (A. Dzhumadil'daev)	1246	624		374	2244
6. Tbilisi (T. Pirashvili)	2244	1246	1496	554	5540
Total					19474

All except the Alma-Ata lab have employed the total amount of money provided for them. Alma-Ata had originally suggested to use 900 ECU for computer equipment, but it turned out that it there was no useful computer for the purpose of this project available for this amount of money. Therefore 374 ECU (= 20 % of 1246 + 624) was paid as a contribution for overhead. It is also **suggested** to pay the difference of the proposed total and the money employed (526 = 2770 - 2244 ECU) as an additional salary for Kanat Abdukalikhov and Rashid Kerimbaev, who, by the present contract, are entitled to receive a salary (360 resp.

264 ECU). Since this salary was for just 6 months and since the whole project lasted for 18 months this seems to be justifiable.
At the time of the writing of this report, essentially 90 % of the total money has been transferred to the partners, with the exception of Minsk. On the occasion of the visit of V. Yantchevskii in December/January 1996 in Bielefeld, some remaining amount (ca. 900 ECU) will be paid. The remaining 10 % will be transferred after reception of the last payment by INTAS.

The funds were transferred to the NIS participants in cash on occasion of several visits of them at Bielefeld or on occasion of one of the above mentioned conferences C1, C2, C3, or by transactions to western bank accounts which some of them have. Funds for the western participants were transferred by ordinary bank transfer. In case of Alma Ata there was also a direct money transfer for one rate. No problems were encountered for all the money transfers.
Project finances other than the INTAS grant: Some visits of NIS participants at Bielefeld were supported by the Sonderforschungsbereich (SFB 343) "Diskrete Strukturen in der Mathematik", sponsored by the Deutsche Forschungsgemeinschaft (Subproject "Linear algebraic groups") and also by some Euro-Conference Project. For example, the funding for the conference C1 was done as follows: ca. 20000 DM was paid by SFB 343, ca. 15650 ECU was contributed by the Euro-Conference Project, and INTAS contributed as described above. The conference C2 was sponsored by the Oberwolfach Institute, and the conference C3 was sponsored, among others, by resources of the Belgium state, by the Euro-Conference Project mentioned above, and by INTAS. Also the German Alexander v. Humboldt foundation financed a one year research stay of A. Merkurjev (St. Petersburg) at Bielefeld (ending in August 1996) on the basis of a Humboldt prize for him (upon suggestion of the project coordinator of this INTAS project, U. Rehmann), and also a one years stay of A. Chernousov (Minsk) at Bielefeld as a Humboldt fellow, starting in June 1996.

5. Summary

Linear Algebraic Groups, Algebraic K-Theory, and related Structures in Algebra and Topology

Final Project Report on INTAS-93-2618, June 1, 1995 – November 30, 1996

Project Co-ordinator: Ulf Rehmann, University of Bielefeld

This project deals with the investigation of the rationality and splitting behavior of linear algebraic groups over arbitrary fields, the investigation of splitting properties of quadratic forms, the investigation of Lie algebras and their cohomology groups and their nonsplit and central extensions, in particular the computation of those groups, investigation of Azumaya algebras and Brauer groups of local hyperelliptic curves, investigation of algebraic K-theories for various objects including homogeneous spaces (of algebraic groups) and special normed algebras over normed commutative rings.

The main activities of the project were done by individual research in the above mentioned fields. Several interim results have been reported on the occasion of three international conferences mentioned below, also some individual visits of the NIS participants at various western universities. All fields mentioned in section 2. (Objectives) have been investigated intensively.

On the occasion of 3 international conferences, which were partially financed by this project, several participating individuals reported their results. These conferences were:

[C1] Euro-Conference on "Linear Algebraic Groups and Related Structures", organized by Ina Kersten (Bielefeld), Alexander Merkurjev (St. Petersburg) and Ulf Rehmann (Bielefeld), held at the Sonderforschungsbereich "Diskrete Strukturen in der Mathematik" at the University of Bielefeld (Germany) on July 3 - July 7, 1995.

[C2] Conference on "Algebraic K-Theory", organized by Dan Grayson (Urbana,Illinois) and Ulf Rehmann, held at the "Mathematisches Forschungsinstitut" Oberwolfach (Germany) on June 10 - 14, 1996.

[C3] Euro-Conference on "Linear Algebraic Groups and Related Structures", organized by Jean-Louis Colliot-Thélène (Paris), Eva Bayer-Fluckiger Besançon), Jean-Pierre Tignol (Louvain-la Neuve), Paul Van Praag, held at Louvain-la-Neuve (Belgium) on June 17 - 21, 1966.

Besides these conferences there are various research activities of the single labs partially in interaction with each other , which are described separately for each participant of the project below. The results of the project are or will be documented by 38 publications.