Degeneration of modules and the construction of Prüfer modules

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Let $\Lambda$ be an artin algebra (this means that $\Lambda$ is a module-finite $k$-algebra, where $k$ is an artinian commutative ring). Bautista-Pérez [BP] and Smalø [S] have recently shown the following: Let $W, W'$ be $\Lambda$-modules of finite length with isomorphic tops and isomorphic first syzygy modules. If $W$ and $W'$ have no self-extensions, then $W$ and $W'$ are isomorphic. This is well-known in case $k$ is an algebraically closed field, but it is of interest to know such a result also for example for $\Lambda$ being a finite ring. Actually, for $k$ an algebraically closed field, the usual algebraic geometry arguments allow a stronger conclusion: If $W$ has no self-extension, then $W'$ is a degeneration of $W$ (in the following sense: $W'$ belongs to the closure of the orbit of $W$ in the corresponding module variety). The first aim of the lecture was to show a corresponding result for general $\Lambda$, using the notion of a degeneration as introduced by Riedtmann-Zwara [Z1]: the module $W'$ is said to be a degeneration of $W$ provided there is an exact sequence of finite length modules of the form: $0 \to X \to W \to W' \to 0$ (in case $k$ is algebraically closed, the notions coincide, as Zwara [Z2] has shown).

**Proposition 1.** Let $U_0, U_1$ be finite length modules, and $w, w': U_0 \to U_1$ monomorphisms. Denote by $W, W'$ the cokernels of $w, w'$, respectively. If $W$ has no self-extensions, then $W'$ is a degeneration of $W$.

Let us describe in which way one obtains a corresponding Riedtmann-Zwara sequence. Actually, let us consider a slightly more general setting for the following tower construction: Start with a pair of maps $w_0, v_0: U_0 \to U_1$ between finite length modules, such that $w_0$ is a proper monomorphism with cokernel $W$. Forming inductively pushouts, we obtain a sequence of maps $w_i, v_i: U_i \to U_{i+1}$ with $i \geq 0$, such that all the maps $w_i$ are monomorphisms with cokernel $W$ (and such that $w_{i+1}v_i = v_{i+1}w_i$ for all $i$). We form the direct limit $U_\infty$ of all the modules $U_i$ with respect to the monomorphisms $w_i$ (and we may assume that these maps $w_i$ are inclusion maps), and consider also the module $U_\infty/U_0$.

If we assume that $W$ has no self-extensions, then $U_\infty/U_0$ is an (infinite) direct sum of copies of $W$, and this implies that one of the inclusion maps $w_i$ is a split monomorphism: thus $U_{i+1}$ is isomorphic to $U_i \oplus W$. Now, if $v_0$ is also a monomorphism, say with cokernel $W'$, then the inductive construction of the module $U_{i+1}$ yields an exact sequence $0 \to U_i \to U_{i+1} \to W' \to 0$. As we have seen, we can replace $U_{i+1}$ by $U_i \oplus W$, thus we deal with a Riedtmann-Zwara sequence. This completes the proof of proposition 1.

Let us return to the general setting of dealing with a pair of maps $w_0, v_0: U_0 \to U_1$ between finite length modules, such that $w_0$ is a proper monomorphism with cokernel $W$. The maps $v_i: U_i \to U_{i+1}$ yield a map $v_\infty: U_\infty \to U_\infty$ which maps $U_0$ into $U_1$ and which induces an isomorphism $\overline{\nu}: U_\infty/U_0 \to U_\infty/U_1$. If we compose the canonical projection $U_\infty/U_0 \to U_\infty/U_1$ with the inverse of $\overline{\nu}$, we obtain a locally nilpotent surjective endomorphism of $U_\infty/U_0$ with kernel $W$. Let us call a module $M$ a Prüfer module with basis $W$, provided there exists a locally nilpotent surjective endomorphism of $M$ with kernel $W$ of finite length; thus $U_\infty/U_0$ is a Prüfer module with basis $W$. 
A module $M$ is said to be of finite type provided it is a direct sum of copies of a finite number of indecomposable modules of finite length (thus if and only if $M$ is both endo-finite and pure-projective). Note that for the tower construction exhibited above, the module $U_\infty$ is of finite type if and only if the Prüfer module $U_\infty/U_0$ is of finite type. We are interested in Prüfer modules which are not of finite type, since there is the following result:

**Proposition 2.** Let $M$ be a Prüfer module which is not of finite type, and let $I$ be an infinite set. Then the product module $M^I$ has an indecomposable direct summand $G$ which is of infinite length and endo-finite.

Recall that a module $N$ is said to be endo-finite provided it is of finite length when considered as a module over the opposite of its endomorphism ring. Indecomposable infinite length modules which are endo-finite have been called generic modules by Crawley-Boevey [CB]. He has shown that the existence of a generic module implies that there are infinitely many isomorphism classes of indecomposable finite-length modules of some fixed endo-length $d$ (and actually the proof shows that there are infinitely many natural numbers $d$ such that there are infinitely many isomorphism classes of indecomposable finite-length modules of endo-length $d$).

Proposition 2 is based on previous investigations of Krause [K], see also [R1]: Let $M$ be a Prüfer module, then there is a surjective locally nilpotent endomorphism $f$ with kernel of finite length; denote by $W[n]$ the kernel of $f^n$. Then $M^I$ contains the union $U = \bigoplus_n W[n]^I$. This submodule is a direct sum of copies of $M$, and it is a direct summand of $M^I$, say $M^I = U \oplus U'$. The module $U'$ is endo-finite, thus a direct sum of copies of finitely many indecomposable endo-finite modules. In case the latter modules all are of finite length, then one can show that $M$ is of finite type. This then completes the proof of proposition 2.

We want to use the tower construction in order to obtain a wealth of Prüfer modules. For this, one needs submodules $U_0 \subset U_1$ with additional homomorphisms (or even embeddings) $U_0 \to U_1$, and of special interest seems to be the take-off part of the category of all $\Lambda$-modules of finite length (as introduced in [R2]).

**References**


