An introduction to the representation dimension of artin algebras.

Claus Michael Ringel (Oberwolfach Lecture, February 2008)

Let Λ be an artin algebra (this means that Λ is a module-finite k-algebra, where k is an artinian commutative ring). The modules to be considered will be left Λ-modules of finite length. Given a module M we denote by $\operatorname{add} M$ the class of modules which are direct summands of direct sums of copies of M.

The representation dimension of artin algebras was introduced by M.Auslander in his famous Queen Mary Notes, but remained a hidden treasure for a long time. Only very recently some basic questions concerning the representation dimension have been solved by Iyama and Rouquier, and now there is a steadily increasing interest in this dimension (in particular, see papers by Oppermann, and also Krause-Kussin, Avramov-Iyengar, and Bergh). This introduction will recall the basic setting and outline a general scheme in order to understand some of the artin algebras with representation dimension at most 3. But we should stress that the main focus at present lies on the artin algebras with representation dimension greater than 3.

1. Some basic results.

A module M is called a *generator* if any projective module belongs to $\operatorname{add} M$; it is called a *cogenerator* if any injective module belongs to $\operatorname{add} M$. It was Auslander who stressed the importance of the global dimension d of the endomorphism rings $\operatorname{End}(M)$, where M is both a generator and a cogenerator. Note that d is either 0 (this happens precisely when Λ is semisimple) or greater or equal to 2 (of course, it may be infinite). The representation dimension of an artin algebra Λ which is not semisimple is the smallest possible such value d; whereas the representation dimension of a semisimple artin algebra is defined to be 1.

The main tool for calculating the representation dimension is the following criterion due to Auslander (implicit in the Queen Mary Notes). Given modules M, X, denote by $\Omega_M(X)$ the kernel of a minimal right $(\operatorname{add} M)$-approximation $M' \to X$. By definition, the *M-dimension* $M\text{-}\dim X$ is the minimal value i such that $\Omega_M^i(X)$ belongs to $\operatorname{add} M$.

(A) **Theorem** (Auslander). *Let M be a Λ-module which is both a generator and a cogenerator and let $d \geq 2$. The global dimension of* $\operatorname{End}(M)$ *is less or equal to d if and only if $M\text{-}\dim X \leq d-2$ for all Λ-modules X.*

An immediate consequence is:

(B) **Theorem** (Auslander). *An artin algebra Λ is of finite representation type if and only if* $\operatorname{rep.dim.} \Lambda \leq 2$. This result was the starting observation and indicates that the representation dimension may be considered as a measure for the distance of being representation-finite.

There is the following characterization of the endomorphisms rings of modules which are both generators and cogenerators; its proof provides an important bicentralizer situation:

(C) **Theorem** (Morita-Tachikawa). *If M is a Λ-module which is a generator and cogenerator, then* $\operatorname{End}(M)$ *is an artin algebra with dominant dimension at least* 2 *and any artin algebra with dominant dimension at least* 2 *arises in this way.*

(D) **Theorem** (Iyama). *The representation dimension is always finite.* This asserts, in particular, that any artin algebra Λ can be written in the form $\Lambda = e\Lambda' e$, where Λ' is an artin algebra with finite global dimension; thus many homological questions concerning Λ-modules can be handled by dealing with modules for an algebra with finite global dimension.

(E) **Theorem** (Igusa-Todorov). *If* $\operatorname{rep.dim.}\Lambda \leq 3$, *then Λ has finite finitistic dimension.*

Until 2001, for all artin algebras Λ where the representation dimension was calculated, it turned out that $\operatorname{rep.dim.}\Lambda \leq 3$. Thus, there was a strong feeling that all artin algebras could have this property. If this would have been true, the finitistic dimension conjecture and therefore a lot of other homological conjectures would have been proven by (D).

(F) **Example** (Rouquier). *Let V be a finite-dimensional k-space, where k is a field, and $\Lambda(V)$ the corresponding exterior algebra. Then* $\operatorname{rep.dim.}\Lambda(V) = -1 + \dim V$.

2. Endomorphism rings of generator-cogenerators in case Λ is hereditary.

In case Λ is hereditary, one can determine the set of all possible values of the global dimension of endomorphism rings of Λ-modules which are generator-cogenerators. Let τ_Λ denote the Auslander-Reiten translation for the category $\operatorname{mod}\Lambda$.

Theorem (Dlab-Ringel). *Let Λ be a hereditary artin algebra and let $d \geq 3$ be in $\mathbb{N} \cup \{\infty\}$. There exists a Λ-module M which is both a generator and a cogenerator such that the global dimension of* $\operatorname{End}(M)$ *is equal to d if and only if there is a τ_Λ-orbit of cardinality at least d.*

3. Torsionless finite artin algebras.

We call an artin algebra Λ *torsionless-finite* provided there are only finitely many isomorphism classes of indecomposable modules which are torsionless (i.e. submodules of projective modules).

Theorem. *If Λ is torsionless-finite, then its representation dimension is at most* 3.

The proof follows again arguments by Auslander presented in the Queen Mary Notes. According to Auslander-Bridger a torsionless-finite artin algebra has also only finitely many isomorphism classes of indecomposable modules which are factor modules of injective modules. Let L be an additive generator for the subcategory of all torsionless modules, and F an additive generator for the subcategory of all factor modules of injective modules. Given any Λ-module X, let X' be the F-trace in X, thus the inclusion map $X' \to X$ is a right $(\operatorname{add} F)$-approximation of X. Let $p\colon X'' \to X$ be a right $(\operatorname{add} L)$-approximation of X. Then there is an exact sequence of the form $0 \to p^{-1}(X') \to X'' \oplus X' \to X \to 0$ which shows that $\Omega_{L\oplus F}(X)$ is a direct summand of $p^{-1}(X')$. Since $p^{-1}(X')$ is a submodule of X'', it follows that $\Omega_{L\oplus F}(X)$ is in $\operatorname{add} L$.

Many classes of artin algebras are known to be torsionless-finite: the hereditary algebras (Auslander), the algebras with $J^n = 0$ such that Λ/J^{n-1} is representation-finite, where J is the radical of Λ (Auslander), in particular: the algebras with $J^2 = 0$, but also the minimal representation-infinite algebras, then the artin algebras stably equivalent to hereditary algebras (Auslander-Reiten), the right glued algebras and the left glued algebras (Coelho, Platzeck; an artin algebra is right glued provided almost all indecomposable modules have projective dimension 1), as well as the special biserial (Schröer). Also, if Λ is a local algebra of quaternion type, then $\Lambda/\operatorname{soc}\Lambda$ is torsionless-finite, so that again its representation dimension is equal to 3 (Holm).

But it should be stressed that there are many classes of artin algebras with representation dimension 3 which are not necessarily torsionless-finite: for example the tilted algebras (Assem-Platzeck-Trepode), the trivial extensions of hereditary algebras (Coelho-Platzeck) as well as the canonical algebras (Oppermann).

Basic references

- M. Auslander: The representation dimension of artin algebras. Queen Mary College Mathematics Notes (1971)
- M. Auslander, M. Bridger: Stable module theory. Mem. Amer. Math. Soc. 94 (1969).
- K. Igusa, G. Todorov: On the finitistic global dimension conjecture for artin algebras. In: Representations of algebras and related topics. Fields Inst. Commun. 45 (2005), 201-204.
- O. Iyama: Finiteness of representation dimension. Proc. Amer. Math. Soc. 131 (2003), 1011-1014.
- R. Rouquier: Representation dimension of exterior algebras. Invent. Math. 168 (2006), 357-367.

- V. Dlab, C.M. Ringel: The global dimension of the endomorphism ring of a generator-cogenerator for a hereditary artin algebra. Preprint (2008).