Some Auslander-Reiten quilts

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The lecture was dealing with the module categories of some special biserial algebras. Special biserial algebras were first studied by Gelfand and Ponomarev in 1968, they have provided the methods in order to classify all the indecomposable representations of such an algebra (the string modules and the band modules), and there is also known a precise recipe for obtaining all the irreducible maps (adding or deleting hooks and cohooks). The algebras which were considered in the lecture are the wind wheel algebras, they are obtained from the hereditary algebras of type $\widetilde{\mathbb{A}}_n$ by identifying suitable pairs of linearly oriented subquivers, the bars. The wind wheel algebras are minimal representation-infinite algebras.

The study of minimal representation-infinite k-algebras with k an algebraically closed field was one of the central themes of the representation theory around 1984 with contributions by Bautista, Gabriel, Roiter, Salmeron, Bongartz, Fischbacher and many others. Recent investigations of Bongartz [1] provide a new impetus for analyzing the module category of such an algebra and even seem to yield a basis for a classification of these algebras. Here is a short summary of this development. First of all, there are algebras with a non-distributive ideal lattice, such algebras have been studied already 1957 by Jans. Second, there are algebras with a good universal cover $\widetilde{\Lambda}$ and such that $\widetilde{\Lambda}$ has a convex subcategory which is a tame concealed algebra of type $\widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7$ or $\widetilde{\mathbb{E}}_8$; these were the algebras which have been discussed by Bautista, Gabriel, Roiter and Salmeron in 1984 (we say that the universal cover is *good* provided it is a Galois cover with free Galois group and is interval-finite). As Bongartz now has shown, the remaining minimal representation-infinite algebras also have a good cover $\widetilde{\Lambda}$, but all finite convex subcategories of $\widetilde{\Lambda}$ are representation-finite. These algebras can be shown to be special biserial and can be classified completely: After the separation of nodes, there are three different kinds: the hereditary algebras of type $\widetilde{\mathbb{A}}_n$, the wind wheel algebras, as well as the barbell algebras with non-serial bars. Whereas the barbell algebras are algebras with non-polynomial growth, the hereditary ones and the wind wheels are 1-domestic: this means that there is precisely one primitive 1-parameter family of indecomposable modules (and of course additional isolated indecomposables).

The aim of the lecture was to look at a wind wheel algebra W and to describe in detail first its Auslander-Reiten quiver, but then also the Auslander-Reiten quilt Γ of W (see [3]); the quilt is obtained from the set of Auslander-Reiten components which contain string modules by inserting suitable (infinite dimensional) indecomposable algebraically compact modules. These additional modules are constructed using $\mathbb{N}$-words and $\mathbb{Z}$-words, quite similar to the construction of the string modules using finite words, but for the infinite dimensional modules often some completion is necessary (see [2]).

We denote by rad the radical of the module category mod W, it is the ideal generated by the non-invertible homomorphisms between indecomposable W-modules.

Using transfinite induction, one defines powers $\operatorname{rad}^\lambda$ for any ordinal number. For example, for the first limit ordinal ω, one takes as $\operatorname{rad}^\omega$ the intersection of all finite powers rad^n — note that the Auslander-Reiten quiver of W is meant to display the factor category $\operatorname{mod} W/\operatorname{rad}^\omega$. In the same way, the factor category $\operatorname{mod} W/\operatorname{rad}^{\omega 2}$ (or at least part of it) is exhibited by the Auslander-Reiten quilt (here $\omega 2$ is the second limit ordinal).

It turns out that *the Auslander-Reiten quilt* Γ *of a wind wheel algebra is a connected orientable surface with boundary, its Euler characteristic is* $\chi(\Gamma) = -t$, *where* t *is the number of bars.*

The components of the Auslander-Reiten quiver of W which contain string modules are ramified components of type $\mathbb{A}_\infty^\infty$, the ramification data being given by a permutation π. Such a component is sewn together from partial translation quivers in the same way as one constructs Riemann surfaces in complex analysis. The permutations π which arise for the wind wheel algebras with t bars are precisely the commutators of two t-cycles.

As we have mentioned, the Auslander-Reiten quilt of any wind wheel algebra W is orientable (for example, for the wind wheel with only two simple modules we obtain a torus with one hole). On the other hand, it is easy to see that the category of W-modules contains as a full subcategory the module category of a representation-finite algebra L whose Auslander-Reiten quiver is homeomorphic to a Möbius strip. In order to understand the embedding $\operatorname{mod} L \to \operatorname{mod} W$, one may analyze in which way the irreducible maps of $\operatorname{mod} L$ are factorized inside $\operatorname{mod} W$ by looking at the quilt Γ. It turns out that a curious change of direction occurs when approaching some infinite dimensional W-modules which are not part of the quilt.

References

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[3] C.M. Ringel, *Infinite length modules. Some examples as introduction.* In: Infinite Length Modules (ed. Krause, Ringel), Birkhäuser Verlag. Basel (2000), p.1-73.

[4] C.M. Ringel, *The Minimal Representation-Infinite Algebras which are Special Biserial.* To appear in: Representations of Algebras and Related Topics, EMS Series of Congress Reports, European Math. Soc. Publ. House, Zürich, 2011 (Editors Andrzej Skowronski and Kunio Yamagata).