08.12.2023

Brick chain filtrations

Claus Michael Ringel

We deal with the category of finitely generated modules over an artin algebra A. Recall that an object in an abelian category is said to be a *brick* provided its endomorphism ring is a division ring. Simple modules are, of course, bricks, but in case A is connected and not local, there do exist bricks which are not simple. The aim of this survey is to focus the attention to filtrations of modules where all factors are bricks, with bricks being ordered in some definite way.

In general, a module category will have many oriented cycles. Recently, Demonet has proposed to look at so-called brick chains in order to deal with a very interesting directedness feature of a module category. These are the orderings of bricks which we will use.

The following survey relies on investigations by a quite large group of mathematicians. We have singled out some important observations and have reordered them in order to obtain a completely self-contained (and elementary) treatment of the relevance of bricks in a module category. (Most of the papers we rely on are devoted to what is called τ-tilting theory, but for the results we are interested in, there is no need to deal with τ-tilting, or even with the Auslander-Reiten translation τ).

Outline. This is a report on a very important development in the last 10 years which focuses the attention to the use of bricks in order to describe the structure of arbitrary modules over artin algebras. It relies on the work of a quite large number of mathematicians, see section 12 (but there are further related papers). We have singled out some important observations and have reordered them in order to obtain a completely self-contained (and elementary) treatment of the relevance of bricks in a module category.

The first three sections describe the main results presented in this survey. There is Theorem 1.2 (with its strengthening 3.2) and the corresponding finiteness theorem 3.3; this concerns the existence of brick chain filtrations. Theorem 2.2 asserts that looking at finitely generated torsion classes, one only has to deal with torsion classes generated by semibricks. Theorem 2.5 deals with the lower neighbors of a finitely generated torsion class. The proofs of these results are given in Sections 4 to 9. Section 11 extends the view to torsion classes which are not necessarily finitely generated (but is irrelevant for the brick chain filtrations).

1. All modules have brick chain filtrations.

1.1. We deal with an artin algebra A; the modules to be considered are the left A-modules of finite length.

Given a class $\mathcal{X}$ of modules, we denote by $\mathcal{E}(\mathcal{X})$ the class of modules which have a filtration with all factors in $\mathcal{X}$. If $M_1, \ldots, M_m$ are modules, let $\mathcal{E}(M_1, \ldots, M_m) = \mathcal{E}(\{M_1, \ldots, M_m\})$ (such a convention will be used throughout the paper in similar situations).

We recall that a *brick* is a module whose endomorphism ring is a division ring. A finite sequence $(B_1, \ldots, B_m)$ is a *brick chain,* if all B_i are bricks and $\operatorname{Hom}(B_i, B_j) = 0$ for $i < j$. A filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ will be called a *brick chain filtration,* provided there is a brick chain $(B_1, \ldots, B_m)$ (its *type*) such that M_i/M_{i-1} belongs to $\mathcal{E}(B_i)$, for $1 \leq i \leq m$.

1.2. Theorem. *Any module has brick chain filtrations.*

The result will be strengthened in 3.2.

1.3. Some examples.

(1) If $S_1, \ldots, S_n$ are the simple A-modules, then $(S_1, \ldots, S_n)$ is obviously a brick chain. If $\operatorname{Ext}^1(S_i, S_j) = 0$ for all $i > j$, then any sincere A-module M has a brick chain filtration $(M_i)_i$ of type $(S_1, \ldots, S_n)$ (here, M_i is the maximal submodule of M whose composition factors are of the form $S_1, \ldots, S_i$).

In particular, recall that A is said to be *directed,* provided the simple modules $S_1, \ldots, S_n$ can be ordered in such a way that $\operatorname{Ext}^1(S_i, S_j) = 0$ for all $i \geq j$. For such a directed algebra A, all sincere A-modules M have a brick chain filtration of type $(S_1, \ldots, S_n)$ with the additional property that the factors of the filtration are semisimple.

(2) If A is a cyclic Nakayama algebra with simple modules $S_1, \ldots, S_n$ such that $\operatorname{Ext}^1(S_i, S_{i-1}) \neq 0$ for all $1 \leq i \leq n$ (where we write $S_0 = S_n$), then any indecomposable module M, has a brick chain filtration with at most two factors: Let us assume that the top of M is S_n. If the length of M is at most n, then M itself is a brick. Now assume that the length of M is $an + r$ with $a \geq 1$ and $0 \leq r < n$. Let H be the module of length n with top S_n. If $r = 0$, then M has a brick chain filtration of type (H). If $r \neq 0$, then M has a brick chain filtration of type (B, H), where B is the factor module of H of length r.

(3) In contrast to many results in representation theory, here it is not helpful to consider first indecomposable modules. Namely, arbitrary brick chain filtrations of modules M and M' do not determine a brick chain filtration of $M \oplus M'$.

1.4. Duality. Let us denote by D the usual duality functor. Given a brick chain filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ of type $(B_1, \ldots, B_m)$, then clearly D yields a corresponding brick chain filtration for $\mathrm{D}\, M$, namely

$$0 = \mathrm{D}\, M/\mathrm{D}\, M_m \subset \mathrm{D}\, M/\mathrm{D}\, M_{m-1} \subset \cdots \subset \mathrm{D}\, M/\mathrm{D}\, M_1 \subset \mathrm{D}\, M/\mathrm{D}\, M_0 = \mathrm{D}\, M,$$

and its type is $(\mathrm{D}\, B_m, \ldots, \mathrm{D}\, B_1)$.

1.5. The proof of Theorem 1.2 and its strengthening 3.2 will be given in Section 7, and will be based on the use of torsion classes. Definition and properties of torsion classes will be recalled in the next section. Our construction of a brick chain filtration of a module M will yield quite special filtrations, namely what we call "torsional" ones. Let us note already here: if a filtration $(M_i)_i$ of a module M is torsional, then the top of any module M_i is generated by the top of M. Thus, even in the case of a directed algebra, the brick chain filtrations which we will construct are usually different from the obvious filtrations mentioned in example (1) above.

2. Finitely generated torsion classes.

2.1. A class $\mathcal{T}$ of modules is said to be a *torsion class* provided $\mathcal{T}$ is closed under factor modules and under extensions. The set of all torsion classes is a complete lacctive; the meet of a set of torsion classes is just the set-theoretical intersection. Given a class $\mathcal{X}$ of modules, we denote by $T(\mathcal{X})$ the smallest torsion class which contains $\mathcal{X}$ (thus the set-theoretical intersection of all torsion classes containing $\mathcal{X}$). According to the Noether theorems, $T(\mathcal{X})$ is just the class of modules which have a filtration whose factors are factor modules of objects in $\mathcal{X}$.

A torsion class $\mathcal{T}$ is said to be *finitely generated* provided there is a module M with $\mathcal{T} = T(M)$. Of course, any torsion class $\mathcal{T}$ is the set-theoretical union of the finitely generated torsion classes contained in $\mathcal{T}$.

Bricks B_i are said to be Hom-orthogonal provided $\operatorname{Hom}(B_i, B_j) = 0$ for all $i \neq j$. A finitely generated module X is said to be a *semibrick* iff it is a direct sum of bricks such that non-isomorphic bricks are Hom-orthogonal. A module M is said to be *basic* provided it has no direct summand of the form $N \oplus N$ with N indecomposable. Thus, a finitely generated module X is the direct sum $\bigoplus_i B_i$ of finitely many pairwise orthogonal modules iff X is a basic semibrick.

2.2. Theorem. *For any artin algebra A, the map $X \mapsto T(X)$ provides a bijection between the isomorphism classes of basic semibricks and the finitely generated torsion classes.*

In particular, any finitely generated torsion class is generated by a finite set of bricks. Of course, this implies that *any torsion class if generated by a set of bricks.*

The proof of Theorem 2.2 is given in 5.6 (the surjectivity of the map) and 8,8 (the injectivity of the map). Actually, in Section 5, we construct explicitly an inverse of the map, thus let us add:

Addendum. *Any module M has a factor module X which is a semibrick and such that $T(M) = T(X)$,* see Proposition 5.6. Namely, we introduce for any module M its "iterated endotop" $\operatorname{et}^\infty M$, see 5.4, and we show that $X = M/\operatorname{et}^\infty M$ is a semibrick with $T(M) = T(X)$. The indecomposable direct summands of X will be called the *top bricks* of M, see 5.7.

2.3. The algebra A is said to be *brick finite* provided there is only a finite number of isomorphism classes of bricks, and *torsion class fiite* provided there is only a finite number of torsion classes.

Corollary. *An algebra is brick finite iff it is torsion class finite, and in this case any torsion class is finitely generated.*

Proof. Clearly, an algebra which is brick finite has also only finitely many isomorphism classes of basic semi-bricks. Assume that A is brick finite, let $\mathcal{T}$ be any torsion class. We start to construct an inclusion chain of torsion classes $\mathcal{T} = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_t$ where $\mathcal{T}_i = \mathcal{T}(B_0, \dots, B_t)$ with bricks $B_0, B_1, \dots, B_t$. If $\mathcal{T}_t$ is a proper subset of $\mathcal{T}$, there is a brick $B_{t+1} \in \mathcal{T} \setminus \mathcal{T}_t$, thus we let $\mathcal{T}_{t+1} = \mathcal{T}(B_0, \dots, B_{t+1})$. Since A is brick finite, the procedure stops, thus $\mathcal{T}$ is generated by a finite number of bricks.

The bijection of Theorem 2.2 asserts that the number of isomorphism classes of semi-bricks is equal to the number of finitely generated torsion classes. □

2.4. Remark. The bijection provided by Theorem 2.2 is of great interest, since it allows to consider the set of isomorphism classes of basic semibricks as a partially ordered set, using the natural partial ordering of the set of torsion classes due to set-theoretical inclusion.

This poset structure of the class of semibricks (thus also of the class of bricks) provides the foundation for the notion of a brick chain as used in Theorem 1.2.

2.5. A pair of torsion classes $\mathcal{T}' \subset \mathcal{T}''$ will be said to be *neighbors* provided there is no torsion class $\mathcal{T}$ with $\mathcal{T}' \subset \mathcal{T} \subset \mathcal{T}''$; here, $\mathcal{T}'$ is called a *lower neighbor* of $\mathcal{T}''$ and $\mathcal{T}''$ is called an *upper neighbor* of $\mathcal{T}'$.

Given a module N, let ${}^{\perp}N$ be the class of all modules M with $\operatorname{Hom}(M, N) = 0$. It is easy to see that ${}^{\perp}N$ is closed under extensions and under factor modules, thus it is a torsion class.

2.6. Theorem. *Let M be a module. The map $B \mapsto T(M) \cap {}^{\perp}B$ provides a bijection between the isomorphism classes of the top bricks B of M and the lower neighbors of $T(M)$. Any torsion class properly contained in $T(M)$ is contained in at least one of the torsion classes $T(M) \cap {}^{\perp}B$.*

The module B is the only brick C in $T(M)$ with $T(M) \cap {}^{\perp}B = T(M) \cap {}^{\perp}C$.

Theorem 2.6 asserts that the module B is the only brick C in $T(M)$ with $T(M) \cap {}^{\perp}B = T(M) \cap {}^{\perp}C$.The brick B lies, of course, in $T(M) \setminus {}^{\perp}B$, but it is not necessarily the only brick in $T(M) \setminus {}^{\perp}B$. Example: Take the A_2-quiver $1 \leftarrow 2$ and consider $M = 1 \oplus 2$; let $B = 2$. Then $T(M)$ is the class of all modules, ${}^{\perp}B$ are the modules with top in $\operatorname{add} 1$, thus the two bricks 2 and $\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$ both belong to $T(M) \setminus {}^{\perp}B$.

The proof of 2.6 will be given in section 8.

2.7. A torsion class $\mathcal{T}$ is said to be *completely join irreducible* provided the join $\mathcal{T}_*$ of the torsion classes properly contained in $\mathcal{T}$ is still properly contained in $\mathcal{T}$ (and thus $\mathcal{T}_*$ a lower neighbor of $\mathcal{T}$).

Corollary. *The map $B \mapsto \mathcal{T}(B)$ provides a bijection between the isomorphism classes of the bricks and the completely join irreducible torsion classes.*

Proof. Theorem 2.2 sends a brick to the torsion class $\mathcal{T}(B)$; according to 2.6, $\mathcal{T}(B)$ has a unique lower neighbor, namely $\mathcal{T}_* = \mathcal{T}(B) \cap {}^{\perp}B$ and any torsion class properly contained in $\mathcal{T}$ is contained in $\mathcal{T}_*$. This shows that $\mathcal{T}(B)$ is completely join irreducible.

Conversely, assume that $\mathcal{T}$ is a completly join irreducible torsion class. Clearly, $\mathcal{T}$ is finitely generated: Let M be any module in $\mathcal{T} \setminus \mathcal{T}_*$, where $?T_*$ is the join of the torsion classes properly contained in $\mathcal{T}$, then $\mathcal{T} = \mathcal{T}(M)$. Let $B_1, \ldots, B_t$ be the top bricks of M, thus $\mathcal{T} = \mathcal{T}(M) = \mathcal{T}(B_1, \ldots, B_t)$. According to 2.6, $\mathcal{T}$ has t lower neighbors. Since $\mathcal{T}$ is completely join irreducible, we have $t = 1$, thus $\mathcal{T}$ is generated by a brick. $\square$

2.8. Remark. Let $\mathcal{T}(M)$ be a finitely generated torsion class and B a top brick of M. The torsion class $T(M) \cap {}^\perp B$ is not necessarily finitely generated. As a typical example, consider the Kronecker algebra A and let $M = B$ be a simple regular Kronecker module. Then $T(M) \cap {}^\perp B$ is the torsion class of all preinjective Kronecker modules (and this torsion class is not finitely generated).

2.9. What about upper neighbors? According to Theorem 2.5, any finitely generated torsion class has only finitely many lower neighbors. But a finitely generated torsion class may have infinitely many upper neighbors! For example, let A be the Kronecker algebra and M any non-zero regular module. If R is simple regular and not a factor module of M, then $T(M, R)$ is an upper neighbor of $T(M)$, and non-isomorphic simple regular modules $R.R'$ lead to different upper neighbors $T(M, R)$ and $T(M, R')$. It follows that the number of upper neighbors of $T(M)$ is $\max(|k|, \aleph_0)$.

Whereas a lower neighbor of a finitely generated torsion class does not have to be finitely generated, any upper neighbor of a finitely generated torsion class is (trivially) finitely generated: Namely, if $\mathcal{N}$ is an upper neighbor of the torsion class $T(M)$, then $\mathcal{N} = T(M, N)$ for any module N in $\mathcal{N} \setminus T(M)$. Of course, there are even bricks B in $\mathcal{N} \setminus T(M)$, namely suitable top bricks of N, where N belongs to $\mathcal{N} \setminus T(M)$.

2.10. Remark. Throughout the paper, we usual draw the attention to finitely generated torsion classes. For the benefit of the reader, let us mention here some typical (and quite diverse) examples of torsion classes which are **not** finitely generated. We take as A the Kronecker algebra, or, more generally, an n-Kronecker algebra with $n \geq 2$.

(a) The class $\mathcal{I}$ of preinjective modules is a torsion class. which is not finitely generated. The torsion class $\mathcal{I}$ is generated by any infinite set of indecomposable preinjective modules. Here, we deal with the union of an increasing sequence of finitely generated torsion classes. Note that $\mathcal{I}$ has no lower neighbor.

(b) The modules without a non-zero preprojective direct summand is the torsion class $T(\mathcal{R})$, where $\mathcal{R}$ denotes the class of all regular modules. It is easy to see that $T(\mathcal{R})$ is not finitely generated and that it has no upper neighbor.

Let us discuss the case $n = 2$ in more detail. Let $\mathcal{X}$ be a non-empty set of pairwise non-isomorphic simple regular Kronecker modules (thus $\mathcal{X}$ is a set of pairwise Hom-orthogonal bricks). The torsion classes $\mathcal{T}(\mathcal{X})$ are the torsion classes $\mathcal{T}$ with $\mathcal{I} \subset \mathcal{T} \subseteq T(\mathcal{R})$, thus the torsion classes $\mathcal{T}$ with $\mathcal{I} \subseteq \mathcal{T} \subseteq T(\mathcal{R})$ correspond bijective to the subsets of the set of isomorphism classes of simple regular modules. Note that any torsion class $\mathcal{T}(\mathcal{X})$ has infinitely many neighbors: maybe only finitely many lower neighbors (this happens iff $\mathcal{X}$ is finite, thus iff $\mathcal{T}(\mathcal{X})$ is finitely generated) or only finitely many upper neighbors (this happens iff $\mathcal{X}$ contains representatives from almost all isomorphism c?asses of simple regular modules), but altogether the number of neighbors is $\max(|k|, \aleph_0)$.

For the Kronecker algebra, all torsion classes $\mathcal{T}$ but one are of the form $T(\mathcal{B})$, where $\mathcal{B}$ is a (may-be infinite) set of pairwise orthogonal bricks; the only exception is $\mathcal{T} = \mathcal{I}$.

3. Torsional brick chain filtrations.

We have mentioned above that Theorem 1.2 can be strengthened. We need some definitions.

3.1. A submodule U of a module M is said to be *torsional* provided U belongs to $T(M)$. If M has a brick chain filtration $(M_i)_i$ of type $(B_1, \dots, B_m)$, and all the submodules M_i are torsional submodules of M, then also all the bricks B_i belong to $T(M)$.

Looking at filtrations $(M_i)_i$ of a module M, one may request that all the submodules M_i are torsional submodules of M. There is the following stronger property: A filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m$ will be said to be *torsional* provided M_{i-1} is a torsional submodule of M_i, for all $1 \le i \le m$. If $(M_i)_i$ is a torsional filtration of M, then M_{i-1} belongs to $T(M_i)$, for all $1 \le i \le t$, thus we have the inclusion chain $0 = T(M_0) \subseteq T(M_1) \subseteq \cdots \subseteq T(M_m) = T(M)$, and therefore all the submodules M_i are torsional submodules of M (but the converse is not true: let M be a serial module with composition factors $1, 2, 2, 1, 2$ going upwards, with an endomorphism with image of length 2; and take the filtration $(M_i)_{0 \le i \le 3}$ with M_i of length $0, 2, 3, 5$ for $i = 0, 1, 2, 3$; then, all the submodules M_i are torsional submodules of M, but M_1 is not a torsional submodule of M_2).

3.2. Theorem. *Any module has torsional brick chain filtrations.*

As we will see in 9.2, the torsional brick chain filtrations of a module M can be constructed easily by induction: Let B be a top brick of M and M' minimal with M/M' in $\mathcal{E}(B)$. Since M' is a proper submodule of M, by induction there is a torsional brick chain filtration of M', say $0 = M_0 \subset M_1 \subset \cdots \subset M_{m-1} = M'$. Let $M_m = M$. Then $(M_i)_{0 \le i \le m}$ is a torsional brick chain filtration of M. As a consequence of this procedure, we have:

3.3. Theorem. *Any module has only finitely many torsional brick chain filtrations.*

Note that this means that any module M determines a finite set of bricks which can be used as building blocks in order to reconstruct the module M, namely the bricks which occur in the types of the finitely many torsional brick chain filtrations of M.

If $(M_i)_i$ is a torsional brick chain filtration of type $(B_1, \dots, B_m)$, then by definition all the bricks B_i belong to $T(M)$. Now, the brick B_m is (obviously) even a factor module of M, but the remaining bricks B_i do not have to be factor modules of M. Here is a typical example: Let M be serial with composition factors going up: $1, 2, 2, 1, 2$, with torsional brick chain filtration $0 \subset M_1 \subset M$, where $M_1 = B_1$ is the submodule of length three: here, M_1 is not generated by M.

4. Some preliminaries.

4.1. Lemma. *Let M' be a non-zero module in $T(M)$. Then* $\operatorname{Hom}(M, M') \neq 0$.

Proof: M' has a filtration $0 = M'_0 \subset M'_1 \subset \cdots \subset M_m = M$, where all the factors M_i/M_{i-1} are non-zero factor modules of M. Since M'_1 it is a factor module of M, we get a non-zero homomorphism $M \to M'_1 \to M'$. □

4.2. Examples of non-isomorphic bricks B', B with $B' \in T(B)$. According to Lemma 4.1, $\operatorname{Hom}(B, B') \neq 0$. (On the other hand, we will see in 6.4 that $\operatorname{Hom}(B', B) = 0$.)

Example 1: $B = \begin{smallmatrix}2\\1\end{smallmatrix}$ and $B' = 2$. Here, we have an epimorphism $B \to B'$. (Or, if we want to have the same support: Let $B = \begin{smallmatrix}2\\1\\1\end{smallmatrix}$, and $B' = \begin{smallmatrix}2\\1\end{smallmatrix}$.)

Example 2: $B = \begin{smallmatrix}2\\1\end{smallmatrix}$, and $B' = \begin{smallmatrix}2\\2\\1\end{smallmatrix}$. Here, we have a monomorphism $B \to B'$ and B, B' have the same support.

Example 3: $B = \begin{smallmatrix}2\\1\end{smallmatrix}$, and $B' = \begin{smallmatrix}3\\2\end{smallmatrix}$. Here, we have a non-zero map $B \to B'$ which is neither epi nor mono.

4.3. Lemma. *A non-zero module M is a brick iff it has no non-zero proper torsional submodule.*

Proof. Let M be a module. If M is not a brick, there is an endomorphism f of M such that $f(M)$ is non-zero and a proper submodule. Since $f(M)$ belongs to $T(M)$, we see that $f(U)$ is a torsional submodule of M.

Conversely, assume that U is a non-zero proper submodule which is torsional. Since U belongs to $T(M)$, there is a non-zero submodule U' of U which is a factor module of M. We get a non-zero and not invertible endomorphism $M \to U' \subseteq U \subset M$, thus M is not a brick. □

5. The endotop and the iterated endotop of a module.

We are going to show the surjectivity assertion of Theorem 2. We need the notion of the endotop $\operatorname{et} M$ of a module M.

5.1. Endotop. Denote by $E = \operatorname{End}(M)$ the endomorphism ring of M (operating on the left of M), and $\operatorname{rad} E$ its radical. Then $(\operatorname{rad} E)M$ is a submodule of M and we define $\operatorname{et} M = M/(\operatorname{rad} E)M$, and call it the *endotop* of M; by definition, the endotop of M is a factor module of M.

5.2. Examples: Let A be the local algebra $k\langle x, y\rangle$ with $\operatorname{rad}^3 = 0$. *If M is indecomposable,* $\operatorname{et} M$ *may be decomposable:* Let M is the 3-dimensional indecomposable module with simple socle and top of length 2, then $\operatorname{et} M$ is the direct sum of two copies of the simple module, Also, $\operatorname{et} M$ *may not be a semibrick:* Let M be uniserial of length 3 with $M/\operatorname{rad}^2 M$ not isomorphic to $\operatorname{rad} M$. Then $\operatorname{et} M = M/\operatorname{rad}^2 M$ is a serial module of length 2, thus not a brick. This leads us below to consider not only et, but the iterations et^i, see 5.4.

If A is the Kronecker algebra, and M a regular Kronecker module, then $\operatorname{et} M$ is just the regular top of M.

5.3. Proposition. *Let M be a module. Then M belongs to $T(\operatorname{et} M)$, therefore $T(M) = T(\operatorname{et} M)$. The kernel of the canonical map $M \to \operatorname{et} M$ is torsional.*

Proof. Let $f_1, \dots, f_t$ be a basis of $E = \operatorname{rad} \operatorname{End} M$. Let $(\operatorname{rad} \operatorname{End} M)^m = 0$. The image of $g = (f_i) \colon \bigoplus_i M \to M$ is $(\operatorname{rad} E)M = \operatorname{rad}_E M = M_1$ and $\operatorname{et} M = M/M_1$. Let $M_{j+1} = g(M_j)$ for all $j \geq 0$ with $M_0 = M$. Then $M_m = 0$. By induction, all modules M_j/M_{j+1} are generated by $\operatorname{et} M$. This shows that $T(M) \subseteq T(\operatorname{et} M)$. On the other hand, we also have $T(\operatorname{et} M) \subseteq T(M)$, since $\operatorname{et} M$ is a factor module of M. Thus M and $\operatorname{et} M$ generate the same torsion-class.

The kernel M' of the canonical map $M \to \operatorname{et} M$ is by definition the image of the map g, thus generated by M. Therefore M' belongs to $T(M)$. □

5.4. We iterate the construction et and get epimorphisms

$$M \to \operatorname{et} M \to (\operatorname{et})^2 M \to \cdots .$$

Since M is of finite length, the sequence stabilizes eventually; in this way we get the *iterated entotop* $\operatorname{et}^\infty M = \operatorname{et}^a M$ for $a \gg 0$.

Example. Let A be a suitable artin algebra with two simple modules 1 and 2. For $n \geq 0$, let $M[n]$ be a serial module of length $n+2$, with composition factors going up: $(1, \dots, 1, 2, 1)$ (thus starting with n factors of the form 1). Then, for $0 \leq i \leq n$, we have $\operatorname{et}^i M[n] = M[n-i]$. For $0 \leq i < n$, the module $M[i]$ is not a brick, but $\operatorname{et}^n M[n] = M[0]$ is a brick (with composition factors $(2,1)$).

5.5. Proposition. *Let M be a module. The iterated endotop $X = \operatorname{et}^\infty M$ is a semibrick and $T(M) = T(X)$; the kernel of the canonical map $M \to \operatorname{et}^\infty M$ is a torsional submodule of M.*

Proof. It is obvious that the iterated endotop of a module is always a semibrick, since the sequence $M \to \operatorname{et} M \to (\operatorname{et})^2 M \to \cdots$ stabilizes precisely when $\operatorname{End}(\operatorname{et}^a M)$ is semisimple. Proposition 5.3 yields that the torsion classes $T(\operatorname{et}^i M)$ are equal, for all $i \geq 0$.

The kernel K of the canonical map $M \to \operatorname{et}^\infty M$ has a filtration whose factors are the kernels K_i of the canonical maps $\operatorname{et}^i M \to \operatorname{et}^{i+1} M$, for all $i \geq 0$. According to 5.3, all modules K_i belong to $T(M)$, thus K belongs to $T(M)$. □

5.6. Corollary. *A torsion class $\mathcal{T}$ is finitely generated iff there is a semibrick X with $\mathcal{T} = T(X)$.* □

Corollary 5.6 shows that the map $X \mapsto T(X)$ from the class of semibricks X to the set of finitely generated torsion classes is surjective. This is part of the assertion of Theorem 2.2.

5.7. Since the iterated endotop of a module M is a semibrick, the indecomposable direct summands of the iterated endotop are bricks and will be called the *top bricks* of M.

6. Extensions of modules in ${}^{\perp}B$ by modules in $\mathcal{E}(B)$, where B is a brick.

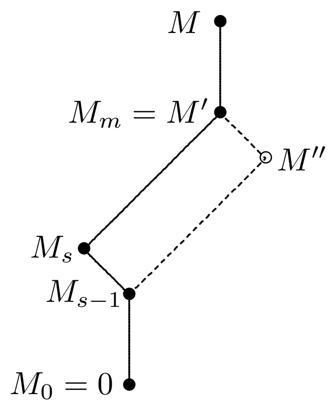
6.1. Proposition. *Let B be a brick and Y a module in ${}^{\perp}B$. Let $X = B \oplus Y$. If M is in $T(X)$, then M has a submodule M' in $T(Y)$ such that M/M' belongs to $\mathcal{E}(B)$.*

Of course, if $\operatorname{Hom}(M,B) \neq 0$, then M' is a proper submodule of M. And conversely, if M' is a proper submodule of M, then $\operatorname{Hom}(M,B) \neq 0$.

6.2. Proof. Let M' be a submodule of M which belongs to $T(X)$ with $M/M' \in \mathcal{E}(B)$, and minimal with these two properties. We claim that M' belongs to ${}^{\perp}B$.

Thus, assume for the contrary that there is a non-zero map $f\colon M' \to B$. Since M' belongs to $T(B,Y)$, there is a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M'$ such that all factors $F_i = M_i/M_{i-1}$ are factor modules of B or of Y. Let s be minimal such that $f|M_s$ is non-zero. Thus, f vanishes on M_{s-1} and induces a map $\overline{f}\colon M'/M_{s-1}$ with non-zero restriction to $F_s = M_s/M_{s-1}$. Let us denote by $u\colon F_s \to M'/M_{s-1}$ the inclusion map. Thus, the composition $\overline{f}\cdot u\colon F_s \to B$ is a non-zero map.

Now F_s is a factor module of some B or of Y. Since $\operatorname{Hom}(Y,B) = 0$, F_s cannot be a factor module of Y, thus F_s is a factor module of B. Also, since B is a brick, there is no non-zero map from a proper factor module of B to B, thus we see that $F_s = B$ and that the composition $\overline{f}\cdot u\colon B = M_s/M_{s-1} \subseteq M'/M_{s-1} \to B$ is an isomorphism. This shows that u is a split monomorphism. It follows that there is a submodule M'' of M' with $M_{s-1} \subseteq M''$, such that $M_s \cap M'' = M_{s-1}$ and $M_s + M'' = M'$.



It follows that $M'/M'' \simeq M_s/M_{s-1} = B$, and that $M''/M_{s-1} \simeq M'/M_s$. Since M/M' and M'/M'' belong to $\mathcal{E}(B)$, also M/M'' belongs to $\mathcal{E}(B)$. On the other hand, $M''/M_{s-1} \simeq M'/M_s$ has a filtration by factors isomorphic to F_i with $s+1 \le i \le t$ and M_{s-1} has the filtration with factors F_i where $1 \le i \le s-1$. Since all the factors F_i belong to $T(X)$, also M'' belongs to $T(X)$. Altogether we see that M'' is a submodule of M which belongs to $T(X)$ and such that $M/M' \in \mathcal{E}(B)$, Since M'' is a proper submodule of M', this contradicts the minimality of M'. It follows that M' belongs to ${}^{\perp}B$. □

6.3. Corollary. *Let B be a brick and X a semibrick such that B is a direct summand of X. Let M be in $T(X)$, Then any non-zero map $M \to B$ is surjective,*

Proof. We can assume that X is basic. Let $X = B \oplus Y$, then Y belongs to ${}^{\perp}B$. Given M in $T(X)$, we can apply Proposition 6.1. Let $f\colon M \to B$ be a non-zero map. According to 6.1, there is a submodule M' of M which belongs to ${}^{\perp}B$ such that M/M' belongs to $\mathcal{E}(B)$. Since f vanishes on M', we get an induced map $\overline{f}\colon M/M' \to B$, and $\overline{f}$ is non-zero. However, any non-zero map in $\mathcal{E}(B)$ with target B is an epimorphism. Since $\overline{f}$ is surjective, also f is surjective. □

6.4. Corollary. *Let B, B' be non-isomorphic bricks, and assume that B' is in $T(B)$, Then* $\operatorname{Hom}(B', B) = 0$.

Proof. Assume there is a non-zero map $f\colon B' \to B$. According to 6.3, the map f is surjective. Since B' belongs to $T(B)$, we know from 4.1 that there is a non-zero map $g\colon B \to B'$. Since f is surjective, the composition $gf\colon B' \to B \to B'$ is non-zero. Since B' is a brick, this means that gf is an isomorphism. Thus f is a (split) monomorphism. Altogether we see that f is bijective, thus B and B' are isomorphic. □

Remark. If B is a brick and Y a module in ${}^{\perp}B$, then $T(Y)$ is usually properly contained in $T(B, Y) \cap {}^{\perp}B$. For example, consider the quiver with vertices $1, 2$, one arrow $1 \leftarrow 2$ and a loop at 2. Let $Y = 0$ and M a serial module with composition factors going up $1, 2, 2$. Let B be the submodule of M of length two. Then M belongs to $T(B) \cap {}^{\perp}B$.

7. The existence of torsional brick chain filtrations.

7.1. Proposition. *Let B be a top brick of the module M. Then M has a proper submodule M' which belongs to $T(M) \cap {}^{\perp}B$, such that M/M' belongs to $\mathcal{E}(B)$.*

Proof. Let $X = \operatorname{et}^{\infty} M$. Then B is a direct summand of X. Then $T(M) = T(X)$ by Proposition 5.5, thus M belongs to $T(X)$. Now use Proposition 6.1. □

7.2. Proof of Theorem 3.2. According to 7.1, M has a proper submodule M' in $T(M) \cap {}^{\perp}B$ such that M/M' belongs to $\mathcal{E}(B)$ for some brick B.

By induction, M' has a brick chain filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_{m-1} = M'$ of type $(B_1, \dots, B_{m-1})$ such that any M_{i-1} is in $T(M_i)$ for all $1 \le i \le m-1$. Note that we have $T(M_0) \subseteq T(M_1) \subseteq \cdots T(M_{m-1}) = T(M')$.

Let $M_m = M$ and $B_m = B$. Now, for $1 \le i \le m-1$, the module M_i maps onto B_i. But $M_i \in T(M') \subseteq {}^{\perp}B$. As a consequence, $\operatorname{Hom}(B_i, B) = 0$. This shows that $(B_1, \dots, B_m)$ is a brick chain. Of course, the filtration M_i is of type $(B_1, \dots, B_m)$. Also, M_{i-1} is in $T(M_i)$ for all $1 \le i \le m-1$, by induction, and for $i = m$ by 7.1. □

7.4. Some examples of torsional brick chain filtrations.

(1) If M is a brick, the filtration presented by Theorem 3.2 is the trivial one $0 \subset M$. Namely, assume M is a brick and $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ is a torsional brick chain filtration. Then M_1 belongs to $T(M)$, thus $\operatorname{Hom}(M, M_1) \neq 0$, (see Lemma 3.1). A non-zero map $M \to M_1$ gives rise a non-zero composition $M \to M_1 \subseteq M,$, thus the inclusion $M_1 \subseteq M$ is a split epimorphism and therefore the identity map.

(2) Not every brick chain filtration $(M_i)_i$ of a module M is torsional. For example, let A be the quiver $1 \leftarrow 2$ and M the sincere indecomposable module. There is the brick chain filtration $0 \subset M_1 \subset M_2 = M$, where M_1 is the socle of M. Of course, M_1 is not contained in $T(M)$.

(3) Here is a module with two torsional brick chain filtrations. We start with the quiver with vertices $1, 2, 3$, two arrows $1 \leftleftarrows 2$, one arrow $2 \leftarrow 3$ and one zero relation, and form a node $1 = 3$. The injective module $I(1)$ is of length four, with socle 1, second layer $2 \oplus 2$, and third layer 1. There is the filtration with the following factors going up: first the socle 1, then an indecomposable module of length two, finally a copy of 2. Another filtration has only two factors going up: first the injective envelope of 1 in the category of modules of Loewy length at most two, then the simple module 1.

(4) For a Nakayama algebra, any indecomposable module M has only one torsional brick chain filtration $(M_i)_i$, and this filtration has length at most 2. Namely, let S be the top of M. Then all bricks in $T(M)$ have top S. Assume that M has precisely m composition factors of the form S, and U is the unique submodule of M with top S which is a brick. Then either M is in $\mathcal{E}(U)$, then $0 \subset M$ is the only torsional brick chain filtration of M. Else $0 \subset U \subseteq M$ is the only torsional brick chain filtration of M.

(5) Duality. We have mention in Section 1 that using the duality functor D, we obtain from a brick chain filtration $(M_i)_i$ of M a corresponding brick chain filtration for $\mathrm{D}\,M$. But we should stress: If the filtration $(M_i)_i$ is torsional, the dual filtration does not have to be torsional. As a typical example, let A be a connected Nakayama algebra with two simple modules and an indecomposable module M of length three, let U be its socle. Then M has the brick chain filtration $(0 \subset U \subset M)$. This filtration is torsional, whereas the dual filtration is not torsional.

There are brick chain filtrations $(M_i)_i$ of modules such that neither the filtration $(M_i)_i$ nor the dual filtration $(\mathrm{D}\,M/\,\mathrm{D}\,M_i)$ is torsional. Here is an example: Let A be a connected Nakayama algebra with three simple modules and M indecomposable of length four. Let U be the submodule of M of length two. Then $(0 \subset U \subset M)$ is a brick chain filtration, however neither this filtration nor its dual is torsional.

8. The lower neighbors of a finitely generated torsion class.

We are going to prove Theorem 2.5, thus we determine the lower neighbors of the torsion class $T(M)$. Let $B_1, \dots, B_t$ be the top bricks of M and let $X = \bigoplus_i B_i$ (thus X is a brick and $T(M) = T(X)$, see Section 5). Let $\mathcal{T}_i = T(X) \cap {}^{\perp}B_i$ for $1 \le i \le m$.

8.1. Since B_i belongs to $T(X)$, but not to ${}^{\perp}B_i$, we see that $\mathcal{T}_i$ *is a proper subclass of* $T(X)$. Let $j \neq i$. Since $\mathrm{Hom}(B_j, B_i) = 0$, we see that B_j belongs to ${}^{\perp}B_i$, thus to $\mathcal{T}_i$ and therefore $\mathcal{T}_j \neq \mathcal{T}_i$. It follows that *the torsion classes $\mathcal{T}_i$ are pairwise different.*

8.2. *For any module N in $T(X) \setminus {}^{\perp}B_i$, there is an epimorphism $N \to B_i$, thus B_i belongs to $T(N)$.*

For the proof, we can assume that $i = 1$. Since N is not in ${}^{\perp}B_1$, there is a non-zero map $f\colon N \to B_1$. Corollary 6.3 asserts that f is surjective. Thus B_1 is in $T(N)$. □

8.3. Let us show that $\mathcal{T}_i$ *is properly contained in $T(X)$ and that the torsion classes $\mathcal{T}_i$ are pairwise different.* By definition, $\mathcal{T}_i$ is contained in $T(X)$. Since B_i belongs to $T(X)$

and not to ${}^{\perp}B_i$, we see that $\mathcal{T}_i$ is a proper subclass of $T(X)$. Next, let us assume that $i \neq j$. Now B_i does not belong to ${}^{\perp}B_i$. But B_i is orthogonal to B_j, thus B_i does belong to ${}^{\perp}B_i$. This shows that $\mathcal{T}_i \neq \mathcal{T}_j$. □

8.4. *Any torsion class $\mathcal{N}$ which is properly contained in $T(X)$ is contained in some $\mathcal{T}_i$.* Proof. Assume, for the contrary, that $\mathcal{N}$ is properly contained in $T(X)$, but not contained in any $\mathcal{T}_i$. Then $\mathcal{N}$ is not contained in any of the torsion classes ${}^{\perp}B_i$. Thus, for any i, there is a module M_i which is contained in $\mathcal{N}$, thus in $T(X)$, and not in ${}^{\perp}B_i$. Since M_i is not contained in ${}^{\perp}B_i$, there is a non-zero map $f_i\colon M_i \to B_i$ According to 6.3, f_i is surjective. This shows that B_i is contained in $T(M_i) \subseteq \mathcal{N}$. Therefore, $T(X) = T(B_1, \dots, B_m) \subseteq \mathcal{N}$. The reverse inclusion is given by assumption. Altogether, we see that $\mathcal{N} = T(X)$, a contradiction. □

8.5. *The torsion class $\mathcal{T}_i$ is a lower neighbor of $T(X)$ and any lower neighbor of $T(X)$ is obtained in this way.* Proof. Let $\mathcal{N}$ be a torsion class with $\mathcal{T}_i \subseteq \mathcal{N} \subset T(X)$. Since $\mathcal{N}$ is properly contained in $T(X)$, we can use 8.4 in order to see that $\mathcal{N} \subseteq \mathcal{T}_j$ for some j. Therefore $\mathcal{T}_i \subseteq \mathcal{N} \subseteq \mathcal{T}_j$. According to 8.3, we have $i = j$ and $\mathcal{T}_i = \mathcal{N}$. This shows that $\mathcal{T}_i$ is a lower neighbor of $T(X)$.

Conversely, if $\mathcal{N}$ is a lower neighbor of $T(X)$, then we use again 8.4. We see that $\mathcal{N}$ is contained in $\mathcal{T}_i$ for some i. It follows that $\mathcal{N} = \mathcal{T}_i$. Thus, the classes $\mathcal{T}_i$ are the only lower neighbors of $T(X)$. □

8.6. We claim that B_i *is the only brick C in $T(X)$ with $\mathcal{T}_i = T(X) \cap {}^{\perp}C$.* Thus, let C be a brick in $T(X)$ with $\mathcal{T}_i = T(X) \cap {}^{\perp}C$. Since $\mathcal{T}_i \subseteq {}^{\perp}C$, we see that C is not in ${}^{\perp}B_i$. Since C is in $T(X)$, but not in ${}^{\perp}B_i$, Corollary 6.3 provides an epimorphism $f\colon C \to B_i$. Also, B_i is not in ${}^{\perp}C$, that means $\operatorname{Hom}(B_i, C) \neq 0$. The composition with the epimorphism $f\colon C \to B_i$ yields a non-zero map $C \to B_i \to C$. Since C is a brick, the composition has to be an isomorphism, thus the map $f\colon C \to B_i$ is a split monomorphism. Since B_i is indecomposable, it follows that B_i and C are isomorphic. □

8.7. Summary. The assertions 8.3 and 8.5 show that the map $B_i \mapsto \mathcal{T}_i$ provide a bijection between the bricks B_i and the lower neighbors of $T(X)$. Note that for any module M with $T(M) = T(X)$, the bricks B_i are just the top bricks of M. 8.4 shows that any torsion class $\mathcal{N}$ which is properly contained in $T(X)$ is contained in some $\mathcal{T}_i$. For the final assertion of Theorem 3.5, see 8.6. This concludes the proof of Theorem 2.6. □

8.8. Proof of the injectivity assertion in Theorem 2.2. *The map $[X] \mapsto T(X)$ from the set of isomorphism classes of basic semibricks to the set of torsion classes is injective.* Thus, let X, X' be basic semibricks with $T(X) = T(X')$. If B' is a brick which is a direct summand of X, then according to Theorem 3.5, the torsion class $T(X) \cap {}^{\perp}(B')$ is a lower neighbor of $T(X) = T(X')$, thus this torsion class is of the form $T(X) \cap {}^{\perp}B$ for some indecomposable direct summand B of X, and since B' is a brick in $T(X)$ with $T(X) \cap {}^{\perp}B = T(X) \cap {}^{\perp}(B')$, the bricks B and B' are isomorphic. Thus, any indecomposable direct summand of X' occurs as a direct summand of X, and similarly, any any indecomposable direct summand of X occurs as a direct summand of X'. This shows that X and X' are isomorphic. □

8.9. Given a pair $\mathcal{T}' \subset \mathcal{T}$ of neighbors with $\mathcal{T}'$ finitely generated, also $\mathcal{T}$ is finitely generated (but the converse is not true). Theorem 2.5 allows to attach to any pair $\mathcal{T}' \subset \mathcal{T}$ of neighbors, where $\mathcal{T}$ is finitely generated, a brick C, namely the brick C with $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}C$. This is the **brick labeling** procedure for pairs of neighbors. To repeat: For the brick labeling of $\mathcal{T}' \subset \mathcal{T}$ we use the uniquely determined brick B in $\mathcal{T}$ with $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B$.

9. The torsional brick chain filtrations.

9.1. Proposition. *Let $(M_i)_i$ be a brick chain filtration of M of type $(B_1, \ldots, B_m)$, and assume that all the submodules M_i are torsional submodules of M. Then B_m is a top brick of M.*

Proof. Since M_i belongs to $T(M)$, also its factor module B_i belongs to $T(M)$. This shows that $T(B_1, \ldots, B_m) \subseteq T(M)$. On the other hand, M has a filtration with factors B_i, therefore $T(M) \subseteq T(B_1, \ldots, B_m)$. This shows that

$$T(M) = T(B_1, \ldots, B_m) = T(Y \oplus B),$$

where $Y = \bigoplus_{i=1}^{m-1} B_i$ and $B = B_m$. We note: Since $(B_1, \ldots, B_m)$ is a brick chain, we have $\mathrm{Hom}(Y, B) = 0$.

Now, $T(Y \oplus B) = T(\mathrm{et}^{\infty}(Y \oplus B))$, We calculate inductively $\mathrm{et}^{a}(Y \oplus B)$ for all $a \geq 0$. We claim that $\mathrm{et}^{a}(Y \oplus B) = Y_a \oplus B$, where Y_a is a factor module of Y with $\mathrm{Hom}(Y_a, B) = 0$. For $a = 0$, we put $Y_a = Y$. Assume we have $\mathrm{et}^{a}(Y \oplus B) = Y_a \oplus B$, where Y_a is a factor module of Y with $\mathrm{Hom}(Y_a, B) = 0$. Since $\mathrm{Hom}(Y_a, B) = 0$, the radical maps in the endomorphism ring of $Y_a \oplus B$ map into Y_a. If U_a is the sum of these images, then $\mathrm{et}^{a}(Y \oplus B) = Y_{a+1} \oplus B$ with $Y_{a+1} = Y_a/U_a$. Also, we have $\mathrm{Hom}(Y_{a+1}, B)$, since any non-zero homomorphism $Y_{a+1} \to B$ would yield a non-zero homomorphism $Y_a \to Y_{a+1} \to B$. Since we deal with modules of finite length, there is some a such that $U_a = 0$, and therefore $\mathrm{et}^{\infty}(Y \oplus B) = Y_a \oplus B$. This shows that B is a direct summand of $\mathrm{et}^{\infty}(Y \oplus B) = Y_a \oplus B$.

Since B is a direct summand of $\mathrm{et}^{\infty}(Y \oplus B)$, it is a direct summand of $\mathrm{et}^{\infty} M$, see But this means that $B = B_m$ is a top brick of M. □

9.2. Corollary. *Let M be a non-zero module with top bricks $T_1, \ldots T_t$. For $1 \leq i \leq t$, let $M^{(i)}$ be the maximal submodule of M which belongs to $T(M) \cap {}^{\perp}T_i$. The torsional brick chain filtrations of M are the filtrations obtained from a torsional brick chain filtration of $M^{(i)}$ by adding the inclusion $M^{(i)} \subset M$.* □

9.3. Proof of Theorem 3.3. For any module M, let $\phi(M) \in \mathbb{N} \cup \{\infty\}$ be the number of torsional brick chain filtrations of M. Of course, we have $\phi(0) = 1$. For $M \neq 0$, let $M^{(i)}$ be the maximal submodule of M which belongs to $T(M) \cap {}^{\perp}T_i$. Then, according to 9.2,

$$\phi(M) = \sum_i \phi(M^{(i)}).$$

This shows that $\phi(M)$ is finite, for all modules M, as asserted in Theorem 3.3. □

10. More about brick chain filtrations.

10.1. A module M will be said to be *homogeneous* provided M belongs to $\mathcal{E}(B)$ for some brick B. Note that if M is non-zero and belongs to $\mathcal{E}(B)$, where B is a brick, then B is uniquely determined, as the following Lemma shows.

Lemma. *If B is a brick and M is a non-zero module in $\mathcal{E}(B)$, then B is the image of an endomorphism of M and B is the only brick which occurs as image of an endomorphism of M.*

Proof. First, we show that B occurs as the image of an endomorphism of M. Since M belongs to $\mathcal{E}(B)$, there is a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ of M such that all factors are isomorphic to B. A corresponding map $M \to M/M_{m-1} \simeq B \simeq M_1 \subseteq M$ is an endomorphism of M wich image isomorphic to B.

Conversely, let f be an endomorphism of M whose image is a brick. Since $\mathcal{E}(B)$ is an exact abelian subcategory, the image M' of f belongs to $\mathcal{E}(B)$. Now M' is a non-zero module in $\mathcal{E}(B)$. As we have seen in the first part of the proof, M' has an endomorphism with image $f(M')$ being isomorphic to B. But we assume that M' is a brick, thus the image of an endomorphism of M' is either zero or M' itself. This shows that $M' = f(M')$, thus M' is isomorphic to B. □

Examples. If A is a local algebra, then all modules are, obviously, homogeneous. But it is interesting to observe that also for the Kronecker algebra, all indecomposable modules are homogeneous.

10.2. A filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M$ will be said to be *directed* provided $\operatorname{Hom}(M_i/M_{i-1}, M_j/M_{j-1}) = 0$ for all $1 \le i < j \le m$.

Proposition. *Let M be a module with a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$. Then $(M_i)_i$ is a brick chain filtration iff $(M_i)_i$ is a directed filtration and all the factors are homogeneous.*

Proof. First, assume that $(M_i)_i$ is a brick chain filtration, say of type $(B_1, \ldots, B_m)$. Since M_i/M_{i-1} belongs to $\mathcal{E}(B_i)$, all the factors of the filtration are homogeneous. Also, for $i < j$, we have $\operatorname{Hom}(B_i, B_j) = 0$. Therefore $\operatorname{Hom}(M_i/M_{i-1}, M_j/M_{j-1}) = 0$.

Conversely, assume that $(M_i)_i$ is a directed filtration (with proper inclusions) and all factors are homogeneous. Since $F_i = M_i/M_{i-1}$ is a homogeneous module, there is a brick B_i with $F_i \in \mathcal{E}(B_i)$. Since F_i is non-zero, B_i occurs both as a submodule and as a factor module of F_i. Thus, any non-zero homomorphism $f\colon B_i \to B_j$ yields a non-zero homomorphism $F_i \to F_j$. Since the given filtration is directed, we see that $\operatorname{Hom}(B_i, B_j) = 0$ for $i < j$. Thus, $(B_1, \ldots, B_m)$ is a brick chain. □

10.3. The composition factors which occur in the top of a module M give rise to interesting brick chain filtrations of M:

Proposition. *Let M be a module. If S is a simple module which occurs in the top of M, then M has a brick chain filtration of type $(B_1, \ldots, B_m)$ with $B_m = S$.*

Proof. Let M' be the minimal submodule of M such that M/M' has only S as composition factor, thus M/M' belongs to $\mathcal{E}(S)$ and S does not occur in the top of M'.

Now take a torsional brick chain filtration $(M_i)_{1\le i\le m-1}$ of M', say of type $(B_1,\dots,B_{m-1})$ and let $M_m = M$. Since we deal with a torsional filtration of M', the modules M_i, thus also the bricks B_i are in $\mathcal{T}(M')$, thus the top of B_i is generated by M'. As a consequence, $\operatorname{Hom}(B_i, S) = 0$. This shows that $(B_1,\dots,B_m)$ with $B_m = S$ is a brick chain, and that the filtration $(M_i)_{1\le i\le m}$ is a brick chain filtration of type $(B_1,\dots,B_m)$. □

10.4. A module M has usually several brick chain filtrations, and the length of these filtrations seem to be unrelated. As a typical example, let A be the path algebra of the directed quiver of type $\mathbb{A}_n$ and M the indecomposable sincere A-module. It is easy to see that M has brick chain filtrations of length m, for any $1 \le m \le n$.

By definition, a module M is homogeneous iff $(0 \subseteq M)$ is a brick chain filtration. A homogeneous module which is not a brick has only one brick chain filtration, namely $(0 \subseteq M)$. But bricks usually have several brick chain filtrations:

Proposition. *A brick which is not simple has at least two brick chain filtrations.*

Proof. Let M be a brick. Then $(0 \subset M)$ is a brick chain filtration of length 1.

Let S be a simple module which occurs in the top of M. According to 10.3, there is a brick chain filtration $(M_i)_{1\le i\le m}$ with M_m/M_{m-1} in $\mathcal{E}(S)$. We claim that $m \ge 2$. Namely, if $m = 1$, then M itself belongs to $\mathcal{E}(S)$. But since M is a brick, we must have $M = S$, thus M is simple. □

10.5. Question. *Are there modules with infinitely many brick chain filtrations?*

11. Brick labeling in general.

We consider now also torsion classes which are not necessarily finitely generated.

11.1. Proposition. *Let $\mathcal{T}' \subset \mathcal{T}$ be a pair of torsion classes.*

(a) *If M belongs to $\mathcal{T} \setminus \mathcal{T}'$ and is of minimal length, then M is a brick and $\mathcal{T}' \subseteq {}^{\perp}M$.*

(b) *If B is a brick and $\mathcal{T}' \subseteq {}^{\perp}B$, let $\mathcal{N} = T(\mathcal{T}', B)$ and $\mathcal{N}' = \mathcal{N} \cap {}^{\perp}B$, then*

$$\mathcal{T}' \subseteq \mathcal{N}' \subset \mathcal{N} \subseteq \mathcal{T}$$

and the torsion classes $\mathcal{N}' \subset \mathcal{N}$ are neighbors. Any module $M \in \mathcal{N}$ has a submodule $M' \in \mathcal{N}'$ such that $M/M' \in \mathcal{E}(B)$. In particular, B is the unique module in $\mathcal{N} \setminus \mathcal{N}'$ of smallest length.

Proof. Let M be in $\mathcal{T} \setminus \mathcal{T}'$, then according to 5.5, also $X = \operatorname{et}^{\infty} M$ is in $\mathcal{T} \setminus \mathcal{T}'$ (since $T(M) = T(X)$). There is an indecomposable direct X' of X which belongs to $\mathcal{T} \setminus \mathcal{T}'$. We have epimorphisms $M \to X \to X'$. Thus, if we assume that M is of minimal length in $\mathcal{T} \setminus \mathcal{T}'$, then $M = X'$ is a brick. The minimality condition also implies that $\mathcal{T}' \subseteq {}^{\perp}M$. Namely, given a module M' in $\mathcal{T}'$ and a homomorphism $f\colon M' \to M$, then $f(M)$ belongs to $\mathcal{T}'$, thus $M/f(M)$ does not belong to $\mathcal{T}'$. By the minimality of M we must have $f(M) = 0$.

Next, assume that B is a brick and that $\mathcal{T} \subseteq {}^{\perp}B$, let $\mathcal{N} = T(\mathcal{T}, B)$. It is trivial that $\mathcal{T} \subseteq \mathcal{N}' \subseteq \mathcal{N} \subseteq \mathcal{T}$, and $\mathcal{N}' \neq \mathcal{N}$, since B belongs to $\mathcal{N}$ and not to ${}^{\perp}B$.

Let M be any module in $\mathcal{N} \setminus \mathcal{N}'$. We claim that there is a submodule M' of M which belongs to $\mathcal{N}'$ such that $M/M' \in \mathcal{E}(B)$. Since M is in $\mathcal{N} = T(\mathcal{T}', B)$, there is $Y \in \mathcal{T}'$ such that M belongs to $T(Y, B)$. According to 6.1, there is $M' \in T(Y, B) \cap {}^{\perp}B$ such that $M/M' \in \mathcal{E}(B)$. But $T(Y, B) \cap {}^{\perp}B \subseteq T(T', B) \cap {}^{\perp}B = \mathcal{N}'$.

It follows that $\mathcal{N}' \subset \mathcal{N}$ are neighbors: namely, if $M \in \mathcal{N} \setminus \mathcal{N}'$, then its submodule M' is a proper submodule, thus M/M' maps onto B, thus $T(\mathcal{N}', M)$ contains $\mathcal{T}'$ as well as B, and therefore is equal to $\mathcal{N}$. □

11.2. Remark. If we label the pair $\mathcal{T}' \subset \mathcal{T}$ of neighbors by the brick B with $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B$, then we have on the one hand: B belongs to $\mathcal{T}$ and not to $\mathcal{T}'$. On the other hand, for every module M in $\mathcal{T}'$, in particular for the bricks in $\mathcal{T}'$, we have $\operatorname{Hom}(M, B) = 0$.

Thus we obtain in this way the Hom-condition which is used in the definition of a brick-chain: If $\mathcal{T}_1 \subset \mathcal{T}_2 \subseteq \mathcal{T}_3 \subset \mathcal{T}_4$ is a chain of torsion classes with $\mathcal{T}_1 \subset \mathcal{T}_2$ as well as $\mathcal{T}_3 \subset \mathcal{T}_4$ being neighbors, and B is the label for $\mathcal{T}_1 \subset \mathcal{T}_2$, whereas B' is the label for $\mathcal{T}_3 \subset \mathcal{T}_4$, then $\operatorname{Hom}(B, B') = 0$.

12. Final remarks: History and references.

12.1. The terminology "semibrick" seems to be due to Assai [A]. I used to call the indecomposable direct summands of a basic semibrick an "antichain" of bricks, but this is in conflict with Demonet's important notion of a brick chain (and to say that "an antichain of bricks is a brick chain", sounds rather odd).

12.2. The results presented here are usually considered as part of the so-called τ-tilting theory (what-ever this means). There seems to be a strange reluctance to deal with bricks. For example, many authors prefer to speak about τ-tilting finiteness instead of brick finiteness (these properties are equivalent, see [DIJ]): here, τ-tilting finiteness means that there are only finitely many τ-tilting modules (whatever this means). Whereas brick finiteness is very easy to grasp, τ-tilting finiteness is much less intuitive!

12.3. Torsion pairs. Torsion pairs $(\mathcal{T}, \mathcal{F})$ were introduced by Dickson [Di] as a generalization of the use of torsion and p-torsion subgroups in abelian group theory, thus generalizing a feature of the category of $\mathbb{Z}$-modules to R-modules, were R is an arbitrary ring. In this paper, torsion classes play a decisive role, but we never mention the corresponding torsionfree class. Of course, since the dual of a torsion class is a torsionfree class, any result about torsion classes provides a corresponding result about torsionfree classes. In this way, the paper yields many assertions about torsionfree classes. But we should mention an intriguing feature of our topic: if we dualize the bijection between chains of torsion classes and brick chains, we obtain a corresponding bijection between chains of torsionfree classes and again brick chains, since the dual of a brick chain is a brick chain.

Throughout the paper, we have used the notation ${}^{\perp}\mathcal{N}$ for the torsion class of modules M with $\operatorname{Hom}(M, N) = 0$ for all $N \in \mathcal{N}$, where $\mathcal{N}$ is an arbitrary class of modules. Correspondingly, given a class $\mathcal{M}$ of modules, one writes $\mathcal{M}^{\perp}$ for the (torsionfree) class of all modules N with $\operatorname{Hom}(M, N) = 0$ for all $M \in \mathcal{M}$. In this way, one obtains all torsion pairs as $({}^{\perp}\mathcal{N}, ({}^{\perp}\mathcal{N})^{\perp})$, or also as $({}^{\perp}(\mathcal{M}^{\perp}), \mathcal{M}^{\perp})$, starting with arbitrary module classes $\mathcal{N}, \mathcal{M}$. Of course, ${}^{\perp}(\mathcal{M}^{\perp})$ is nothing else than the torsion class $T(\mathcal{M})$ generated by $\mathcal{M}$ (and

${}^{\perp}\mathcal{N})^{\perp}$ is the torsionfree class generated by $\mathcal{N}$). The torsion pairs $(\mathcal{T},\mathcal{F})$ were introduced to focus the attention, for any module M, to the largest submodule U of M which belongs to $\mathcal{T}$, its $\mathcal{T}$-torsion submodule (then M/U is the largest factor module of M which belongs to $\mathcal{F}$) In this light, Proposition 6.1 deals with torsion pair $({}^{\perp}B,({}^{\perp}B)^{\perp})$, namely with the ${}^{\perp}B$-torsion submodule M' of M, and asserts that the (torsion-free) factor module M/M' belongs to $\mathcal{E}(B)$.

Note that the main results 1.2 and 3.2 are shown by an iterative use of 6.1: In this way, we deal with a chain of torsion classes in order to obtain a filtration $(M_i)_i$ of the given module M with factors M_i/M_{i-1} in module classes of the form $\mathcal{E}(B_i)$.

12.4. In contrast to the classical example, torsion classes in general are not hereditary (where *hereditary* means that the torsion class $\mathcal{T}$ is closed under submodules). Of course, the torsion classes $T(M)$ considered in our paper are usually not hereditary, and we take care of this feature when we focus the attention to what we call "torsional" submodules.

The brick-chain theorems 1.2 and 3.2 should be seen in the light of the original example of abelian group theory: any finitely generated module M has a filtration $(M_i)_{0\le i\le m}$ where the factors M_i/M_{i-1} with $0 \le i < m$ are in $\mathcal{E}(\mathbb{Z}/p_i\mathbb{Z})$, for pairwise different prime numbers p_i, whereas M_m/M_{m-1} is in $\mathcal{E}(\mathbb{Z})$. In abelian group theory, this filtration always splits. In our case, we cannot expect that the filtrations provided in 1.2 and 3.2 split, just look at indecomposable modules M which are not bricks.

12.5. The relevance of torsion classes when dealing with finite length categories was seen already by Auslander and Smalø [AS].

12.6. Wide subcategories and torsion classes. Given an abelian category, the exact abelian subcategories which are closed under extensions are now usually called *wide* subcategories. The rather obvious relationship between semibricks and wide subcategories was mentioned in [R1] under the name "simplification". The search for semibricks (or wide subcategories) which generate a given torsion class was initiated by Ingalls and Thomas [IT]. Theorem 2.2 generalizes some of their considerations. Actually, the injectivity of the map in 2.2 has been shown by Marks and Stovičeek in [MS].

The relevance of the endotop of a module is well-known and was stressed by Asai when looking at τ-rigid modules (our proof of 5.5 follows closely Asai [A]).

12.7. Brick labeling, The brick labeling as presented in sections 8 and 11 is due to Barnard, Carroll and Zhu [BCZ]. Actually, neighbor pairs $\mathcal{T}'\subset\mathcal{T}''$ of torsion classes have attracted a lot of interest and several different denominations are used in the literature: that $\mathcal{T}''$ covers $\mathcal{T}'$, that there is an arrow $\mathcal{T}''\to\mathcal{T}'$ in the Hasse quiver of the lattice of torsion classes, or one speaks about minimal inclusions of torsion classes. The brick B used as label is called a *minimal extending module* for $\mathcal{T}'$ in [BCZ].

The bijection 2.7 between bricks and completely join irreducible torsion classes has been exhibited in Theorem 1.0.5 in [BCZ].

12.8. Brick chains. As we have seen, given a chain of torsion classes, the brick labeling of the neighbor torsion classes yields a brick chain. This observation was used by Demonet [De] to consider not only the finite brick chains considered in the paper, but to deal with arbitrarily large totally ordered sets of bricks with the corresponding Hom-condition, called again brick chains. Of special interest are those brick chains which cannot be further refined, since they correspond bijectively to the chains of torsion classes which cannot be refined. This bijection is essential for an understanding of the lattice of all torsion classes.

The relevance of this bijection is the following: Since the set of all torsion classes is, in a natural way, a partially ordered set, using the set-theoretical inclusion, the bijection transfers this ordering to bricks and semibricks.

12.9. Upper neighbors. We were able to discuss the lower neighbors of a finitely generated torsion class, but (unfortunately) not its upper neighbors. As we have seen, the lower neighbors of a finitely generated torsion class are easily determined. To determine the upper neighbors of a torsion class $\mathcal{T}$ is much more subtle and is related to the question whether $\mathcal{T}$ is functorially finite or not. This is the place where τ-tilting theory (and mutations) come into play! If $\mathcal{T}$ is functorially finite, $\mathcal{T}$ has only finitely many upper neighbors and there is also no problem to determine them. It would be nice if the following assertion were true: *If a finitely generated torsion class is not functorially finite, then it has infinitely many upper neighbors.* Actually, given a finitely generated torsion class T, it may be important to look not only at its upper neighbors, but to look at the upper neighbors of any lower neighbor of T. Altogether, one should be aware that here we are in the realm of the second Brauer-Thrall conjecture.

12.10. Special brick chain filtrations have been used already a long time ago. In particular, we have shown in [R2] that for a hereditary k-algebra, where k is an algebraically closed field, any exceptional module is a tree module, The basis of the proof is Schofield induction, dealing with brick chain filtrations of length 2.

12.11. In this report, we have assumed to be in the context of artin algebras. Actually, nearly all the results presented here are valid more generally in arbitrary length categories, thus for example for finitely generated modules over left artinian rings.

Acknowledgement. The author is grateful to S. Asai and W. Crawley-Boevey for some useful remarks concernig the presentation.

13. References.

[AIR] T. Adachi, O. Iyama, I. Reiten. τ-tilting theory. Compos. Math., 150(3), 415–452 (2014).

[A] S. Asai. Semibricks. Int. Math. Res. Not., (16), 4993–5054 doi.org/10.1093/imrn/rny150 (2020).

[AS] M. Auslander, S. O. Smalø. Addendum to: Almost split sequences in subcategories. J. Algebra, 71, 592–594 (1981).

[BCZ] E. Barnard, A. Carroll, S. Zhu. Minimal inclusions of torsion classes. Algebr. Comb., 2(5), 879–901 (2019).

[De] L. Demonet. Maximal chains of torsion classes. Appendix to: B. Keller. A survey on maximal green sequences, In: Representation theory and beyond, Contemp. Math. 758, Amer. Math. Soc., Providence, RI, 267?-286 (2020).

[DIJ] L. Demonet, O. Iyama, G. Jasso. τ–tilting finite algebras, bricks, and g-vectors. Int. Math. Res. Not. IMRN, (3), 852–892 (2019).

[Di] S. E. Dickson. A torsion theory for Abelian categories. Trans. Amer. Math. Soc., 121, 223–235 (1966).

[IT] C. Ingalls, H. Thomas. Noncrossing partitions and representations of quivers. Comp. Math. 145, 1533?156 (2009).

[MS] F. Marks, J. Stovìček. Torsion classes, wide subcategories and localisations. Bulletin of the London Mathematical Society. Bull. Lond. Math. Soc. 49, 405–416, doi:10.1112/blms.12033 (2017).

[R1] C. M. Ringel. Representations of K-species and bimodules. J. Algebra 41, 269–302 (1976).

[R2] C. M. Ringel. Exceptional modules are tree modules. Lin. Alg. Appl. 275-276, 471–493 (1998).

C. M. Ringel
Fakultät für Mathematik, Universität Bielefeld
POBox 100131, D-33501 Bielefeld, Germany

ringel@math.uni-bielefeld.de