## Introduction.

A **root system** is a finite set of vectors in a Euclidean vector space satisfying some strong symmetry conditions. Root systems are used as convenient index sets when dealing with semi-simple complex Lie algebras or algebraic groups, but play an important role also in other parts of mathematics. The root systems have been classified by Killing and Cartan at the end of the 19th century, the different types of irreducible root systems are labeled by the Dynkin diagrams $\mathbb{A}_n, \mathbb{B}_n, ..., \mathbb{G}_2$. As we mentioned, the definition of the root systems refers to symmetry properties, but it turns out that there are further hidden symmetries which are not at all apparent at first sight. They have been discovered only quite recently and extend the use of root systems considerably.

Always, $\Lambda$ will be a hereditary artin algebra. If $\Lambda$ is of finite representation type, it is well-known that the indecomposable $\Lambda$-modules correspond bijectively to the positive roots of a root system. The positive roots form in a natural way a poset, these posets are called the **root posets.** In the setting of $\Lambda$-modules, the ordering is given by looking at subfactors. Root posets play a decisive role in many parts of mathematics: of course in Lie theory, in geometry (hyperplane arrangements) and group theory (reflection groups), but also say in singularity theory, in topology, and even in free probability theory (non-crossing partitions). The aim of the lectures will be to report on combinatorial properties of the root posets which have been found in recent years by various mathematicians, in view of these applications. Of course, whenever possible, we will focus the attention to the relevance of these properties in the representation theory of hereditary artin algebras. Several of the results which we will discuss have been generalized to the Kac-Moody root systems, but often we will restrict to the (finite) root systems.

**Outline of the lectures.** A root system $\Phi$ is a finite subset of a Euclidean space $V$. If $x$ is a root, we denote by $A_x$ the hyperplane orthogonal to $x$, and by $\rho_x$ the reflection at $A_x$. In this way, we attach to $\Phi$ (or $\Phi_+$) a finite set $\mathcal{R}(\Phi)$ of hyperplanes in $V$, such sets are called hyperplane arrangements. The reflections $\rho_x$ generate the corresponding Weyl group $W$. Using the reflections, one defines on $W$ a partial ordering, the so-called absolute ordering $\leq_a$. Given a Coxeter element $c$ in $W$, the set $\mathrm{NC}(W, c)$ of all element $w \in W$ with $w \leq_a c$ is called the lattice of generalized non-crossing partitions. In the case of a root system of type $\mathbb{A}$, one just obtains the usual lattice of non-crossing partitions, as introduced by Kreweras and now used for example in free probability theory.

As we have mentioned, we always will consider a hereditary artin algebra $\Lambda$. Let $\operatorname{mod}\Lambda$ be the category of all (left) $\Lambda$-modules of finite length. Recall that a full subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ is said to be thick provided it is closed under kernels, cokernels, and extensions, thus it is an abelian exact subcategory, and we say that a thick subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ is *exceptional* provided its quiver is directed. We denote by $A(\operatorname{mod}\Lambda)$ the poset of exceptional subcategories of $\operatorname{mod}\Lambda$, this will be the main object of interest. The central result to be shown asserts that

$$\boxed{A(\operatorname{mod}\Lambda) \simeq \mathrm{NC}(W(\Lambda), c(\Lambda)),}$$

where $W(\Lambda)$ and $c(\Lambda)$ are the Weyl group and the Coxeter element, respectively, corresponding to $\Lambda$.

Lecture 1 is devoted to numbers which arise from counting problems dealing with a representation-finite hereditary artin algebra $\Lambda$. The numbers we are interested in will depend just on the Dynkin type of $\Lambda$ (and not on the orientation). Thus, here we deal with what we call Dynkin functions: A Dynkin function $f$ attaches to any Dynkin diagram $\Delta$ an integer, or more generally a real number, sometimes even a set or a sequence of real numbers (for example the sequence of exponents); thus we get four sequences of numbers, namely $f(\mathbb{A}_n)$, $f(\mathbb{B}_n)$, $f(\mathbb{C}_n)$, $f(\mathbb{D}_n)$ as well as five additional single values $f(\mathbb{E}_6)$, $f(\mathbb{E}_7)$, $f(\mathbb{E}_8)$, $f(\mathbb{F}_4)$, $f(\mathbb{G}_2)$. Typical Dynkin functions are the number of indecomposable modules, the number of tilting modules, the number of complete exceptional sequences. We will analyze some of these Dynkin functions, looking for example for the prime factorization of their values.

As we will see, there is a unified, but quite mysterious way to deal with some of these Dynkin functions, namely to invoke the exponents of $\Delta$. As Shapiro and Kostant (1959) have shown, the root poset can be used in order to determine the exponents: Namely, if $r_t$ is the number of roots of height $t$, then $(r_1, r_2, \dots)$ is an integer partition and the dual partition is the partition of the exponents.

Lecture 2 will provide a deeper understanding of the role of the exponents. Usually, they seem to fall from heaven: either by looking at the invariant theory of the action of the Weyl group on the ambient space of the root system (Chevalley 1955), of by dealing with the eigenvalues of a Coxeter element (Coxeter, 1951). But there is a third possibility to obtain the exponents, it fits to the interpretation in terms of the height partition of the roots: to built the exponents by looking at subsets of the hyperplane arrangement $\mathcal{R}(\Phi)$, namely going up step by step in a chain of poset ideals of $\Phi_+$. We report on old investigations of V. I. Arnold (1979) and K. Saito (1979/1981) and on a recent paper by Abe-Barakat-Cuntz-Hoge-Terao on the freeness of hyperplane arrangements.

Lecture 3 concerns the classical tilting theory, the study of (finitely generated) tilting modules for a hereditary artin algebra. As we will see in this lecture, already the basic setting of tilting theory should be refined, replacing the usually considered torsion pair by a torsion triple or even a torsion quadruple, thus putting tilting theory into the realm of the stability theory of King. The study of tilted algebras turns out to be just the study of sincere exceptional subcategories. We also will study perpendicular pairs of exceptional subcategories. Altogether we obtain a wealth of bijections (the Ingalls-Thomas bijections) between sets of modules and subcategories. These bijections explain why we obtain the same Dynkin functions when dealing with quite different counting problems.

Lecture 4 is devoted to the study of the poset $A(\operatorname{mod}\Lambda)$ of all exceptional subcategories of $\operatorname{mod}\Lambda$, or, equivalently, of all exceptional antichains in $\operatorname{mod}\Lambda$. Using the results of lecture 3, it will be shown that this poset is self-dual (it has a self-duality with square being essentially the Coxeter transformation). Also, any interval in $A(\operatorname{mod}\Lambda)$ is again of the form $A(\operatorname{mod}\Lambda')$ for some hereditary artin algebra $\Lambda'$, and the maximal chains in $A(\operatorname{mod}\Lambda)$ correspond bijectively to the complete exceptional sequences of $\Lambda$-modules. On the other hand, we will see that the posets $\mathrm{NC}(W(\Lambda), c(\Lambda))$ and $A(\operatorname{mod}\Lambda)$ can be identified.

In this way the theory of generalized non-crossing partitions has to be seen as part of the representation theory of hereditary artin algebras. I should stress that the essential parts of the lectures 3 and 4 are due to Ingalls and Thomas (2009), as well as Igusa and Schiffler (2010). We end the lecture by returning to the special case of the Dynkin types $\mathbb{A}_n$ and review some classical problems which are related to the maximal chains in $A(\operatorname{mod}\Lambda)$: namely, to count labeled trees (Sylvester, 1857, Borchardt, 1860, Cayley, 1889), as well as parking functions (Pyke, 1959, Konheim-Weiss, 1966, Stanley, 1997).

Altogether, the lectures concern the **Catalan combinatoric** and the corresponding Narayana numbers. One may also say that these lectures are about the **cluster complex.** Actually, I will mention the cluster complex only in passing by, but one should be aware that the cluster combinatorics in the Dynkin case is really the combinatoric of the representation-finite hereditary artin algebras. Of course, we deal with the **categorification of combinatorial data,** this is the essence of the lectures.

**Too late?** This report comes late, very late, maybe too late. It concerns objects which have been in the mainstream of representation theory 40 years ago, now they seem to be standard and well understood. The first lecture will focus the attention to a lot of numbers; such numbers had been calculated in the early days of representation theory, but as it seems, never systematically, and only few records are available (by Gabriel-de la Peña, Bretscher-Läser-Riedtmann, and U. Seidel, a student of Happel). As I. Assem wrote to me: there should be many student theses at various universities devoted to such calculations, but one did not dare to publish them. The mathematicians working in the representation theory of algebras felt that there would not be an independent interest in these numbers, the only exception may have been Gabriel: he pointed out that here the Catalan numbers play a role — but as far as I know never in lectures to a mathematical audience, just in a text written for amateurs and enthusiasts. To repeat: a survey similar to the first lecture may (and should) have been given in the seventies or early eighties.

But the numbers presented here were discussed in mathematics: outside of representation theory — a development which was ignored by all of us. For example, F. Chapoton presenting already in 2002 (thus more than 10 years ago) the numbers of clusters, positive clusters and exceptional sequences, and there is a corresponding survey by S. Fomin and Reading written 2005). Actually, some of the numerology can be traced much further back, namely to considerations by Brieskorn and Deligne in the seventies.

Of course, there is an advantage of a late presentation: we are able to present a rather complete picture. But be aware: There are still many questions open. In particular, one misses an interpretation of the numerical data in terms of the exponents (see the lecture 1). Also, given a hereditary artin algebra of Dynkin type $\Delta$, it is not clear how to relate the antichains in the category $\operatorname{mod}\Lambda$ and the antichains in the poset $\Phi_+(\Delta)$, thus to relate non-crossing and non-nesting partitions in a satisfactory way.

**The approach.** I will try to be as elementary as possible. I will prefer to consider individual modules in contrast to subcategories (thus, instead of dealing with thick subcategories, I usually will work with antichains: a thick subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ is an abelian exact subcategory closed under extensions, the corresponding antichain is given by the simple objects of $\mathcal{C}$, and $\mathcal{C}$ is obtained back from the antichain as its extension closure).

Given an artin algebra $\Lambda$, I will prefer to work with its module category $\operatorname{mod}\Lambda$ and will not touch the corresponding derived category $D^b(\operatorname{mod}\Lambda)$. I know that triangulated categories are now well-known and well-appreciated, but they will not be needed in an essential way.

The lectures were addressed to mathematicians working in the representation theory of finite-dimensional algebras, and they deal with a topic all participants were familiar with, namely the representation theory of hereditary artin algebras: first we consider just representation-finite ones, say corresponding to quivers of finite type, or, more generally, to species of finite type, later than hereditary artin algebras in general. The literature usually restricts to quivers, and avoids species. As I mentioned, I want to be as elementary as possible, but nevertheless we will take into account species. The reason is the following: There is the division between the series $\mathbb{A}_n, \mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n$, and the exceptional cases. Let me look at the series: always, the case $\mathbb{A}_n$ is considered as the basic case, the three remaining cases $\mathbb{B}_n, \mathbb{C}_n, \mathbb{D}_n$ are deviations of $\mathbb{A}_n$ (note that for these three cases a large part is of type $\mathbb{A}$, there is a difference only at one of the ends). For many problems, for example the counting problems as they will be discussed in the first lecture, the cases $\mathbb{B}_n$ and $\mathbb{C}_n$ yield the same answer, and the formulas which one obtains are really neat, condensed and surprisingly easy to remember, whereas the formulas for $\mathbb{D}_n$ usually look much more complicated, and often they may be considered as variations of the $\mathbb{B}_n$ formulas. Thus, in order to understand the formulas for $\mathbb{D}_n$ properly, it seems to be advisable to look first at the numbers for $\mathbb{A}$ and $\mathbb{B}$ and only afterwards to the case $\mathbb{D}$. This is the reason that we definitely want to include the case $\mathbb{B}$ (and $\mathbb{C}$), thus to work not only with path algebras of directed quivers, but with hereditary artin algebras in general.

**References.** This is a survey dealing with contributions by a large number of mathematicians, I will try to indicate the main sources, but to name all contributors seems to be a nearly hopeless task. The material to be covered is vast and I am not at all an expert in several of the subjects, thus sometimes I have to be vague, and provide just some indications. I am grateful to many mathematicians for introducing me to various topics, see the acknowledgments at the end of the introduction.

We will deal with a large set of counting problems, and it will turn out that several of these problems yield the same answer. This is of course of great interest and asks for some explanation: to provide natural bijections between the objects in questions. However, this also tends to be a source for priority fights: just think of say 100 equivalent counting problems (see N1). Now any problem can be solved individually (so there may be 100 proofs), or else one can show the equivalence to a similar problem where the answer is known (there are $\binom{100}{2} = 4950$ equivalence proofs). But the situation may be even more complex: in case we deal with a Dynkin function, one may have to consider it case by case, or one can find a unified proof; one may need to rely on computer calculations or find a conceptual proof. And the answer may be given by a magic formula, say in terms of the exponents, and a final proof should explain this!

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Of course, all my considerations concerning representations of hereditary artin algebras rely on the old collaboration with V. Dlab, those on tilted algebras on the collaboration with D. Happel. I am grateful to H. Krause and L. Hille for stressing the relevance of thick subcategories, and of King's stability theory, respectively. I have learned from G. Röhrle the basic induction principle for hyperplane arrangements.

But as the main driving force I have to mention F. Götze, the chairman of the Bielefeld CRC 701. He advised me already several years ago to study non-crossing partitions. He organized joint study groups of the Bielefeld research groups in probability theory and in representation theory in order to raise the mutual interest — for a long time, this seemed to be a hopeless endeavor. One of the topics he always stressed were the parking functions, but I realized only now, when writing up these notes, the direct bijection between the parking functions and complete exceptional sequences for the directed quiver of type $\mathbb{A}_n$ (see the end of lecture 4): I had been working on exceptional sequences without being aware of such a relationship (but he seemed to know). Thus, I have to thank the Bielefeld CRC 701 who has supported me in this way (see also N2). I should add that the presentation has gained from the Bielefeld workshop on *Non-crossing Partitions in Representation Theory* organized in June 2014 by B. Baumann, A. Hubery, H. Krause, Chr. Stump, and the Bielefeld CRC has to be praised for providing the financial support.

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**Notes.**

**N1.** This is not an exaggeration: there is the famous list by R. Stanley on problems which yield the Catalan numbers. He exhibits 66 different problems in Enumerative Combinatorics, vol. 2, and many additional ones in his Catalan Addendum, see http://www-math.mit.edu/∼rstan/ec/catadd.pdf.

**N2.** The author was project leader at the CRC 701 until June 2013, thus he wants to thank the DFG for the corresponding financial support. This was ended due to the age discrimination of the DFG and the University of Bielefeld: there was strong pressure not even to apply for further funding.