Ubiquity and Universality of Quiver Grassmannians

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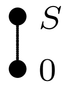
k algebraically closed field.

Algebras are k-algebras, modules are finite-dimensional.

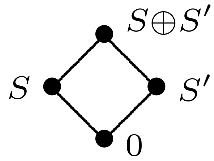
Submodule lattices. M a module, $\mathcal{S}M$ its submodule lattice.

Examples: $\mathcal{S}M$

S simple module



$S \oplus S'$, S, S' simple, $S \not\simeq S'$



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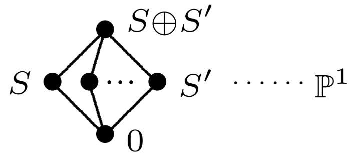
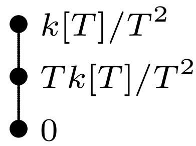
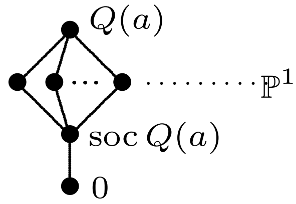


$k[T]/T^2$

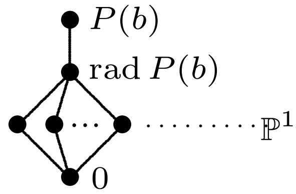


Kronecker algebra  $\mathcal{S}M$

$M = Q(a)$ (injective)

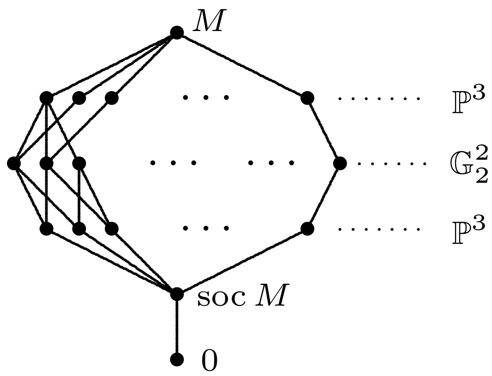


$M = P(b)$ (projective)



4-Kronecker algebra 

$M = Q(a)$ (injective)



Λ finite-dimensional k-algebra,
W.l.o.g.: Λ is basic

basic means: the simple Λ-modules have dimension 1, or equivalently, Λ is a factor algebra of the path algebra of a finite quiver.

M module, $\mathcal{S}M$ the lattice of submodules.

Let e be a natural number and $\mathbb{G}_e(M)$ the set of submodules of M of dimension (= length) equal to e.

$\mathbb{G}_e(M)$ is a subset of the Grassmannian variety $\mathbb{G}_e({}_kM)$ (the set of all subspaces of dimension e of the vector space ${}_kM$)

$\mathbb{G}_e({}_kM)$ is a projective variety and the subset $\mathbb{G}_e(M) \subseteq \mathbb{G}_e({}_kM)$ is closed, thus it is also a projective variety and we have

$$\boxed{\mathcal{S}M = \bigsqcup_e \mathbb{G}_e(M)}$$

The projective varieties $\mathbb{G}_e(M)$ are usually not irreducible, even not connected.

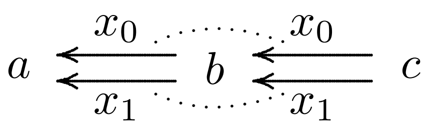
There is a disjoint decomposition

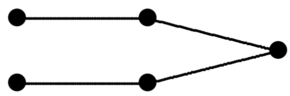
$$\boxed{\mathbb{G}_e(M) = \bigsqcup_{\sum e_i = e} \mathbb{G}_{\mathbf{e}}(M)}$$

here, $\mathbb{G}_{\mathbf{e}}(M)$ is the set of submodules of M with dim vector $\mathbf{e} = (e_i)_i$, it is called a *quiver Grassmannian.*

Again: The quiver Grassmannians $\mathbb{G}_{\mathbf{e}}(M)$ are usually not irreducible, even not connected.

Example 1.

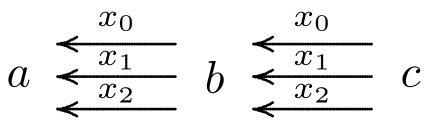




$P(c)$

$\mathbb{G}_{(1,1,0)}(P(c)) = \{*\} \sqcup \{*\}$.

Example 2.



with relations:
$x_i x_j = x_j x_i$ for $i \neq j$
and $x_1 x_2 = 0$

$\mathbb{G}_{(1,1,1)}(Q(a)) = V_{\mathbb{P}^2}(x_1 x_2)$.

To repeat:

$\mathbb{G}_{\mathbf{e}}(M)$ is the set of submodules of M with dimension vector $\mathbf{e}$, it is a projective algebraic variety,

and we have the disjoint decomposition:

$$\boxed{\mathcal{S}M = \bigsqcup_{\mathbf{e}} \mathbb{G}_{\mathbf{e}}(M)}$$

Some References.

Quiver Grassmannians have been considered by

Schofield (1992), Crawley-Boevey (1996)

Relevance for cluster algebras.

Zelevinsky

Caldero-Chapoton (2006)

Caldero-Keller (2006)

The Euler characteristic of quiver Grassmannians of exceptional modules are used to define cluster variables in cluster algebras of acyclic type.

Ubiquity

The Auslander varieties

What are submodules? A fancy (but useful) description is given by the categorical definition:
the submodules of Y are the right equivalence classes of injective maps with target Y.

Here is the definition of right equivalence:
Given two maps $f\colon X \to Y$ and $f'\colon X \to Y$, define
$f \preceq f' \iff$ there exists $h\colon X \to X'$ such that $f = f'h$.

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle h}\searrow & & \nearrow{\scriptstyle f'} \\ & X' & \end{array}$$

f, f' are *right equivalent,* provided $f \preceq f' \preceq f$.

We denote the right equivalence class of f by $[f\rangle$.

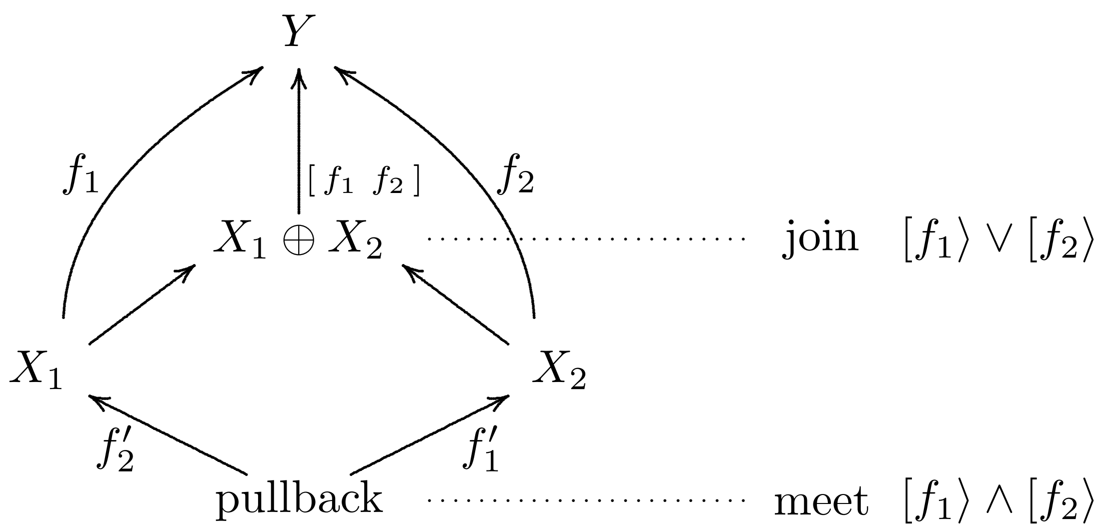
The set $[\to Y\rangle$ of right equivalence classes of maps ending in Y is a poset with respect to the relation $\preceq$.

Λ fin-dim k-algebra, $\operatorname{mod}\Lambda$ the fin-dim Λ-modules.

$f\colon X \to Y$ is *right minimal* provided
$X = X' \oplus X'',\ f(X'') = 0 \implies X'' = 0$

Any right equivalence class contains a right minimal map f.

$[\to Y\rangle$ *is a lattice:* Given $f_1\colon X_1 \to Y,\ f_2\colon X_2 \to Y.$



Remark: Even if f_1, f_2 are right minimal,
the maps $[f_1 f_2]$ and $f_2' f_1$ usually are not right minimal.

Exercise. Show that the lattice $[\to Y\rangle$ is modular.

This means: $[f_1\rangle \vee [f_2\rangle \wedge [f_3\rangle$ is independent of bracketing, provided $f_1 \preceq f_3$.

The global posets and the local groups.

The poset $[\to Y\rangle$ describes the set of right minimal maps with target Y, but only the corresponding right equivalence classes. What information is lost in this way?

How do right equivalence classes look like? They are groupoids (categories with all maps being invertible)!

Let $f\colon X \to Y$ be right minimal. Define its *right automorphism group* as

$$\text{r-Aut}(f) = \{h \in \operatorname{End}(X) \mid fh = f\} \ \subseteq\ \operatorname{Aut}(X).$$

Then $\text{r-Aut}(f) = 1 + u\operatorname{Hom}(X, K)$, where $K = \operatorname{Ker}(f)$ und $u\colon K \to X$ is the inclusion map.

The classification problem for right minimal maps is split into the global problem of describing the lattices $[\to Y\rangle$ and the local problem of dealing with the groups $\text{r-Aut}(f)$.

Define for any pair of modules C, Y

$$\eta_{CY} \colon [\to Y\rangle \longrightarrow \mathcal{S}\operatorname{Hom}(C, Y)$$

$$\begin{aligned}\eta_{CY}(f \colon X \to Y) &= \operatorname{Im}\operatorname{Hom}(C, f) \\ &= f\operatorname{Hom}(X, Y) \\ &= \{g \in \operatorname{Hom}(C, Y) \mid g \text{ factors through } f\}.\end{aligned}$$

Note: If $f \preceq f'$, then $\eta_{CY}(f) \subseteq \eta_{CY}(f')$.
$\big($namely $f\operatorname{Hom}(X,Y) = f'h\operatorname{Hom}(X,Y) \subseteq f'\operatorname{Hom}(X',Y)\big)$.

In particular, η_{CY} is well-defined on the equivalence classes.

Right C-determination. $f\colon X \to Y$ morphism, C a module.

f is *right C-determined* provided any $f'\colon X' \to Y$ such that $f'\phi$ factors through f for all $\phi\colon C \to X'$, factors through f.

If $\operatorname{add} C = \operatorname{add} C'$, then f right C-determined iff f right C'-determined.

f right C-determined $\implies$ f right $(C \oplus C')$-determined.

Here is the main definition: Let ${}^{C}[\to Y\rangle$ be the set of right equivalence classes of right C-determined maps ending in Y.

The subset ${}^{C}[\to Y\rangle$ of $[\to Y\rangle$ is closed under meets.

Warning. Usually it is not closed under joins.

Auslander's completeness theorem. *The set* $[\to Y\rangle$ *is the filtered union of meet-semilattices:*

$$\boxed{[\to Y\rangle = \bigcup\nolimits_C {}^C[\to Y\rangle.}$$

(This just means: every morphism is right C-determined by some module C.)

The Auslander bijections. *The map* η_{CY} *defined by* $\eta_{CY}(f) = \operatorname{Im}\operatorname{Hom}(C, f)$ *is a poset isomorphism*

$$\boxed{\eta_{CY}\colon\ {}^C[\to Y\rangle \longrightarrow \mathcal{S}\operatorname{Hom}(C, Y).}$$

(It is easy to see that the restriction of the map η_{CY} to ${}^C[\to Y\rangle$ is injective and preserves meets. As a consequence, it preserves and reflects the ordering. The essential assertion is the surjectivity.)

References:

M. Auslander: *Functors and morphisms determined by objects,* and *Applications of morphisms determined by objects.* In: Lecture Notes in Pure Appl. Math. 37. Marcel Dekker (1978), 1-244, 245-327 (see also Chapter XI of the Auslander-Reiten-Smalø book).

Ringel: *The Auslander bijections: How morphisms are determined by modules.* arXiv:1301.1251

Transfer from $\mathcal{S}\,\mathrm{Hom}(C,Y)$ to ${}^{C}[\to Y\rangle$.

$$\boxed{\eta_{CY}\colon\ {}^{C}[\to Y\rangle \longrightarrow \mathcal{S}\,\mathrm{Hom}(C,Y).}$$

$\mathcal{S}\,\mathrm{Hom}(C,Y)$ is a modular lattice of finite height,
thus ${}^{C}[\to Y\rangle$ is a modular lattice of finite height.

$\mathcal{S}\,\mathrm{Hom}(C,Y)$ has as distinguished element the zero-submodule.
thus ${}^{C}[\to Y\rangle$ has a unique minimal element $[f\rangle$.

There is a Krull-Remak-Schmidt Theorem for ${}^{C}[\to Y\rangle$.

Transfer II: The Jordan-Hölder Theorem for ${}^C[\to Y\rangle$.

The composition series of $\operatorname{Hom}(C,Y)$ correspond to maximal C-factorizations of right minimal right C-determined maps ending in Y.

Composition factors of $\operatorname{Hom}(C,Y)$ correspond to "C-neighbors" in ${}^C[\to Y\rangle$, these are pairs of of maps (fh,f) ending in Y.

Composition factors of $\operatorname{Hom}(C,Y)$ are simple $\Gamma(C)$-modules, thus of the form $\operatorname{top}\operatorname{Hom}(C,C_0)$, where C_0 is an indecomposable direct summand of C (the $\Gamma(C)$-modules $\operatorname{Hom}(C,C_0)$ are the indecomposable projective $\Gamma(C)$-modules).

The Auslander bijection attaches to each $[f\rangle \in {}^C[\to Y\rangle$ a dimension vector (this is a dimension vector for $\Gamma(C)$-modules).

Let ${}^C[\to Y\rangle_{\mathbf{e}}$ be the set of elements in ${}^C[\to Y\rangle$ with dimension vector $\mathbf{e}$. Under the Auslander bijection, this set corresponds bijectively to $\mathbb{G}_{\mathbf{e}}(\operatorname{Hom}(C,Y))$, we call it an *Auslander variety.*

$\mathcal{S}\operatorname{Hom}(C,Y)$ is the disjoint union of finitely many quiver Grassmannian — correspondingly, ${}^C[\to Y\rangle$ is the disjoint union of finitely many Auslander varieties ${}^C[\to Y\rangle_{\mathbf{e}}$.

The special case $C = \Lambda$. *The Auslander bijection $\eta_{\Lambda Y}$ is the obvious identification of both*

$$ {}^{\Lambda}[\to Y\rangle \qquad \text{and} \qquad \mathcal{S}\operatorname{Hom}(\Lambda, Y) $$

with $\mathcal{S}Y$, namely:

$$
\begin{array}{ccccc}
{}^{\Lambda}[\to Y\rangle & \xrightarrow{\ \eta_{\Lambda Y}\ } & \mathcal{S}\operatorname{Hom}(\Lambda, Y) & \qquad & \operatorname{Hom}(\Lambda, Y) \\
 & {\scriptstyle \operatorname{Im}} \searrow \quad \swarrow {\scriptstyle \mathcal{S}\epsilon} & & & \downarrow {\scriptstyle \epsilon} \\
 & \mathcal{S}Y & & & Y
\end{array}
$$

On the left: right determined by Λ means just monomorphism, right equivalence of monomorphisms means just same image, thus right equivalence classes of monos are just submodules.

On the right: ϵ is the evaluation at 1, it yields an isomorphism of Λ-modules.

Corollary. *Any quiver Grassmannian arises as an Auslander variety:*

$$ \boxed{\ {}^{\Lambda}[\to Y\rangle_{\mathbf{e}} \leftrightarrow \mathbb{G}_{\mathbf{e}}({}_{\Lambda}Y)\ } $$

As we later will mention, any projective variety arises as a quiver Grassmannian, thus any projective variety arises as an Auslander variety.

Wildness. Let Δ be a finite connected quiver, neither Dynkin, nor extended Dynkin.

Theorem. *Given any finite-dimensional algebra Λ and any Λ-module M, there are $k\Delta$-modules C, Y such that ${}^{C}[\to Y\rangle$ is just $\mathcal{S}M$, and therefore ${}^{C}[\to Y\rangle_{\mathbf{e}} \leftrightarrow \mathbb{G}_{\mathbf{e}}(M)$ for any $\mathbf{e}$.*

Proof: We use that Δ is strictly wild.
Thus, given Λ, there is a full and exact embedding functor $F\colon \operatorname{mod}\Lambda \to \operatorname{mod} k\Delta$.

Let $C = F({}_{\Lambda}\Lambda)$ and $Y = F(M)$.

Then $\Lambda = \operatorname{End}({}_{\Lambda}\Lambda)^{\mathrm{op}} \simeq \operatorname{End}(C)^{\mathrm{op}} = \Gamma(C)$ and
$M = \operatorname{Hom}({}_{\Lambda}\Lambda, M) \simeq \operatorname{Hom}(C, Y)$.

(Warning: $\mathcal{S}M$ and $\mathcal{S}(F(M))$ are usually very different; $\mathcal{S}M$ canonically embeds into $\mathcal{S}(F(M))$, but usually this is a proper inclusion.)

More generally, *if Λ' is any wild algebra, then for any finite-dimensional algebra Λ, any Λ-module M, and any dimension vector $\mathbf{e}$, there are Λ'-modules C, Y and a dimension vector $\mathbf{e}'$ such that ${}^{C}[\to Y\rangle_{\mathbf{e}'}$ is just $\mathbb{G}_{\mathbf{e}}(M)$.*

Triangulated categories.

Let $\mathcal{C}$ be a Hom-finite triangulated k-category with split idempotents and with a Serre functor.

Recall (Reiten-Van den Bergh): A *Serre functor* is a self-equivalence of $\mathcal{C}$ with a natural isomorphism

$$D\,\mathrm{Hom}(X,-) \simeq \mathrm{Hom}(-,SX)$$

for all $X \in \mathcal{C}$; here, $D = \mathrm{Hom}_k(-,k)$.

$[\rightarrow Y\rangle$ the poset of right equivalence classes of maps to Y,

${}^{C}[\rightarrow Y\rangle$ the subset of maps which are right C-determined.

Also here: $\mathrm{Hom}(C,Y)$ is a $\Gamma(C)$-module, where $\Gamma(C) = \mathrm{End}(C)^{\mathrm{op}}$.

Krause's completeness theorem.

$$\boxed{[\to Y\rangle = \bigcup_C {}^C[\to Y\rangle,}$$

(the morphism f is right determined by $S^{-1}\,\mathrm{cone}(f)$).

The Krause Bijections. *There is a poset isomorphism*

$$\boxed{\eta_{CY}\colon\ {}^C[\to Y\rangle \longrightarrow \mathcal{S}\,\mathrm{Hom}(C,Y)}$$

defined by $\eta_{CY}(f) = \mathrm{Im}\,\mathrm{Hom}(C,f)$

Reference:

Krause. In: The Abel Symposium 2011(to appear). arXiv:1110.56250

Universality

Theorem. *Any projective variety is the quiver Grassmannian $\mathbb{G}_{\mathbf{e}}(M)$ of a Schurian representation M of an acyclic quiver Q, with $\mathbf{e}$ a thin dimension vector.*

(A representation M is called *Schurian* provided $\mathrm{End}(M) = k$.)

References:
Markus Reineke: arXiv:1204.5730

blog (2.5.2012): the bourbaki code (Lieven Le Brujn): Quiver Grassmannians can be anything.
with many contributions. In particular
Michel Van den Bergh, using the Beilinson quiver.

blog (4.5.2012): The n-Category Cafe (John Baez): Quivering with Exitement: Every projective variety is the Grassmannian of a quiver.

Lutz Hille: *Tilting line bundles and moduli of thin sincere representations of quivers.* An. St. Univ. Ovidius Constantza 4 **(1996)**, 76-82. See Example on the last page.

Birge Huisgen-Zimmermann: *The geometry of uniserial representations...*, J. Pure Applied Algebra 127 **(1998)**, 39-72, *Varieties of uniserial representations... IV*, TAMS 353 (2001), 2091-2113, (with Derksen and Weyman) *Top stable degenerations of finite dimensional representations II.* to appear).

Note that dealing with cluster algebras only quiver Grassmannians $\mathbb{G}_{\mathbf{e}}M$ with M an exceptional module are of interest.

We recall that a representation M of a quiver is called *exceptional* provided

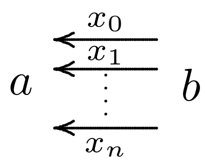
$$\mathrm{Ext}^1(M, M) = 0$$

(any indecomposable exceptional representations of a quiver are Schurian).

The quiver Grassmanniangs $\mathbb{G}_{\mathbf{e}}(M)$ with M exceptional are very special:

Theorem (Caldero-Reineke). *If M is exceptional, and $\mathbb{G}_{\mathbf{e}}(M)$ is non-empty, then its Euler characteristic is positive.*

Example: $K(n+1)$
(a generalized Kronecker quiver)



Consider the submodules U of $Q(a)$ of length 2, thus the submodules with dimension vector $(1,1)$: we consider $\mathbb{G}_{(1,1)}(Q(a))$.

The set $\mathbb{G}_{(1,1)}(Q(a))$ *is a* $\mathbb{P}^n$*, the coordinization is given by the maps* $x_0, \ldots, x_n$.

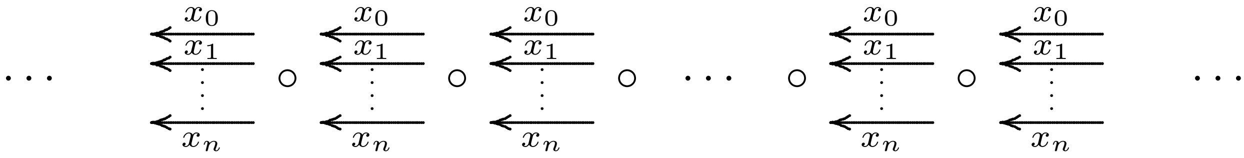
Namely, let U be a submodule of $Q(a)$ of length 2. Given $0 \neq u \in U_1$, the elements $x_0(u), \ldots, x_n(u)$ form a non-zero element of k^{n+1} and different non-zero elements $u, u' \in U_1$ yield scalar multiples of each other.

Thus a length 2 submodule U of $Q(a)$ yields an element of $\mathbb{P}^n$ and

$$\boxed{\mathbb{G}_{(1,1)}(Q(a)) = \mathbb{P}^n.}$$

Also note: Any module with simple socle, in particular any indecomposable module of length 2 can be embedded into $Q(a)$. In this way, $\mathbb{G}_{(1,1)}(Q(a))$ may be considered as the set of isomorphism classes of the indecomposable length 2 modules.

$B(n)$ the Beilinson quiver
vertex set $\mathbb{Z}$ and $n+1$ arrows from a to $a-1$ for $a \in \mathbb{Z}$, always labeled $x_0, x_1, \ldots, x_n$



Relations: the set K of the equations $x_i x_j = x_j x_i$
(whenever this makes sense)

$B(n, d)$ the full subquiver with vertices $0, 1, 2, \ldots, d$.
$\Lambda = kB(n, d)/\langle K \rangle$ (called *Beilinson algebra*)

Given $0 \neq \lambda = (\lambda_0, \ldots, \lambda_n) \in k^{n+1}$,
let $U(\lambda) = U_{n,d}(\lambda)$ be the following representation of $B(n, d)$:
$U(\lambda)_a = k$ for $0 \leq a \leq d$ and
$U(\lambda)_{x_i} = \lambda_i$ for all arrows x_i

$$B(n,d) \qquad 0 \;\; \begin{matrix} \xleftarrow{x_0} \\ \xleftarrow{x_1} \\ \vdots \\ \xleftarrow[x_n]{} \end{matrix} \;\; 1 \;\; \begin{matrix} \xleftarrow{x_0} \\ \xleftarrow{x_1} \\ \vdots \\ \xleftarrow[x_n]{} \end{matrix} \;\; 2 \quad \cdots \quad d{-}1 \;\; \begin{matrix} \xleftarrow{x_0} \\ \xleftarrow{x_1} \\ \vdots \\ \xleftarrow[x_n]{} \end{matrix} \;\; d$$

$U(\lambda)_a = k$ for $0 \leq a \leq d$ and
$U(\lambda)_{x_i} = \lambda_i$ for all arrows x_i

The representations $U(\lambda)$ are sincere and serial
(sincere = all simple modules appear as composition factors)
(serial = there is a unique composition series)

Lemma. *For $d \geq 1$, any sincere serial representation of $B(n,d)$ is isomorphic to some $U(\lambda)$. And $U(\lambda)$, $U(\lambda')$ are isomorphic if and only if $\lambda' = c\lambda$ for some $c \in k^*$.*

This means: The quiver Grassmannian $\mathbb{G}_{(1,1,\ldots,1)}({}_{B(n,d)}Q(0))$ is the set of the sincere serial $B(n,d)$-modules (one from each iso-class), and

$$\boxed{\mathbb{G}_{(1,1,\ldots,1)}({}_{B(n,d)}Q(0)) = \mathbb{P}^n}$$

.

Let V be a projective variety: a closed subset of some $\mathbb{P}^n$, thus defined by the vanishing of a finite set J of homogeneous polynomials in the variables $x_0, x_1, \ldots, x_n$.

We assume that these polynomials have degree at most d.

Consider the algebra Λ with quiver $B(n,d)$ and the relations K and J.

Lemma. *The sincere serial Λ-modules are the modules $U(\lambda)$ with $\lambda \in V$.*

This means: The quiver Grassmannian $\mathbb{G}_{(1,1,\ldots,1)}({}_\Lambda Q(0))$ is the set of the sincere serial Λ-modules (one from each iso-class), and

$$\boxed{\mathbb{G}_{(1,1,\ldots,1)}({}_\Lambda Q(0)) = V}$$

.

Recall that a module M is *quasi-injective,* provided any map $f\colon U \to M$ defined on a submodule U of M can be extended to a map $M \to M$, or, equivalently (for Λ left artinian), provided M considered as $\Lambda/\operatorname{Ann}(M)$-module is injective ($\operatorname{Ann}(M)$ is the annihilator of M in Λ).

Thus: If M is quasi-injective module, then

$$\operatorname{Ext}^1_{\Lambda/\operatorname{Ann}(M)}(M, M) = 0.$$

Dealing with the algebra Λ and its module $Q(0)$ we have seen:

Theorem. *Any projective variety is the quiver Grassmannian $\mathbb{G}_{\mathbf{e}}(M)$ of a Schurian quasi-injective representation M of an acyclic quiver Q.*

What we encounter is the strict difference between the conditions

$$\operatorname{Ext}^1_{\Lambda/\operatorname{Ann}(M)}(M, M) = 0.$$

and

$$\operatorname{Ext}^1_{\Lambda}(M, M) = 0.$$

The second assumption puts severe constraints on the Grassmannians $\mathbb{G}_{\mathbf{e}}(M)$, the first one not at all!