# Invariant Subspaces of Nilpotent Linear Operators

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**The invariant subspaces of a nilpotent linear operator $T$.**

Let $k$ be a field.

Let $T$ be a linear operator on some vector space $V$.
**Example 1.** Take as $T$ the $(n \times n)$ Jordan block (with eigenvalue $0$)

$$J(n) = \begin{bmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{bmatrix},$$

it operates on $V = k^n$.
What are the invariant subspaces $U$ of $V$ ?
(invariant $=$ $T$-invariant, i.e. $T(U) \subseteq U$.)

The only possibilities for $U$ are the subspaces

$$0, \qquad k \times 0^{n-1}, \qquad k^2 \times 0^{n-2}, \quad \ldots \quad , \quad k^n,$$

they form a chain.

In general, $T$ will not be a single Jordan block, but a diagonal sum of Jordan blocks.

**Example 2.** The first interesting case is $T = J(3) \oplus J(1)$, this is the matrix

$$J(3) \oplus J(1) = \left[\begin{array}{cccc:c} 0 & 1 & & & \\ & 0 & 1 & & \\ & & 0 & & \\ \hdashline & & & & 0 \end{array}\right]$$

It operates on $V = k^4$. Let $e_1, \ldots, e_4$ be its canonical basis.

Let $U = \langle e_2 + e_4,\ e_3 \rangle$.
Obviously, $T(U) = \langle e_3 \rangle \subset U$, thus $U$ is an invariant subspace.

Claim: The triple $(V, T, U)$ is "indecomposable": If $V = V' \oplus V''$ with

$$T(V') \subseteq V', \quad T(V'') \subseteq V'', \quad \text{and} \quad U = (U \cap V') \oplus (U \cap V''),$$

then $V' = 0$ or $V'' = 0$.

## Nilpotent operators

A *nilpotent operator* is a pair $(V, T)$, where $V$ is a vector space and $T\colon V \to V$ is a linear map with $T^n =$ for some $n$.

Typical example: The pair $M(n) = (k^n, J(n))$, where $J(n)$ is the Jordan block $J(n)$.

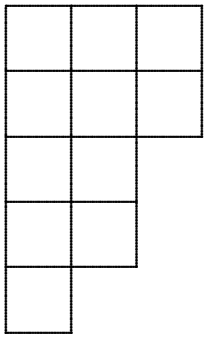
If $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t)$ is a decreasing sequence of natural numbers (a "partition"), let

$$M(\lambda) = M(\lambda_1) \oplus \cdots \oplus M(\lambda_t).$$

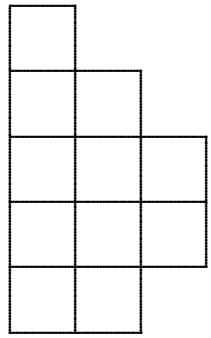
*Any nilpotent operator is isomorphic to* $M(\lambda)$ *for a* **unique** *partition* $\lambda$.

**Visualization:** Dealing with a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t)$, we draw the corresponding *Young diagram.*
Our convention: the parts correspond to the **columns**, the $i$-the column consists of $\lambda_i$ columns.

Example: $\lambda = (5, 4, 2)$

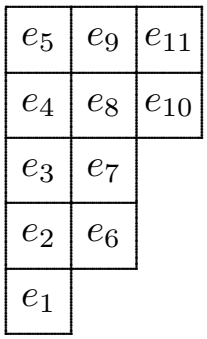


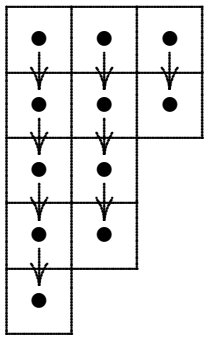
or, say



Consider the boxes as base vectors for $k^n$ with $T$ the shift downwards.

This yields
$J(5) \oplus J(4) \oplus J(2)$



Let $\mathcal{S}(n)$ be the class of *triples* $(V, T, U)$, where

$V$ is a finite-dimensional $k$-space $V$,
$T$ a linear operator $T\colon V \to V$ with $T^n = 0$,
$U$ a subspace of $V$ with $T(U) \subseteq U$, write $W = V/U$ (if needed)

An *isomorphism* between $(V, T, U)$ and $(V', T', U')$ is an invertible linear map $f\colon V \to V'$ with $f(U) = U'$ and $fT = T'f$.

**Aim. To classify the isomorphism classes of such triples.**

The *direct sum* of $X = (V, T, U)$ and $X' = (V', T', U')$ is the triple

$$X \oplus X' = (V \oplus V', T \oplus T', U \oplus U').$$

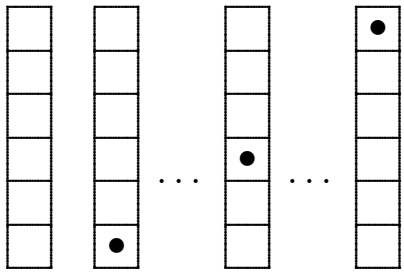
$(V, T, U)$ is said to be *indecomposable* provided it is not zero and not isomorphic to a direct sum $X \oplus X'$ with non-zero triples $X, X'$ (the *zero triple* is $(0, 0, 0)$).

Any triple is a direct sum of indecomposable triples and these direct summands are unique up to isomorphism (Krull-Remak-Schmidt property).

**Aim. To classify the isomorphism classes of indecomposable triples.**

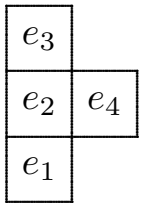
Consider again the previous two examples:

**Example 1 ("pickets"):**
The triples $(V, T, U)$ with $(V, T) = (k^n, J(n))$.
As we have mentioned,
the only invariant subspaces $U$
are the subspaces
$0, \quad k \times 0^{n-1}, \quad k^2 \times 0^{n-2}, \quad \ldots \quad , \quad k^n.$

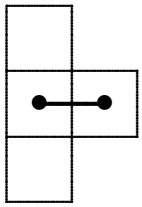


Easy to see: If $(V, T, U)$ is indecomposable and $\dim U = 1$, then $(V, T)$ is a picket.

**Example 2:**




with $U = \langle e_2 + e_4, e_2 \rangle$



Note: Both $U, W$ are of dimension 2 and indecomposable (with respect to $T$)
It follows: $(V, T, U)$ is an indecomposable triple.

The difficulty of classifying the indecomposable objects in $\mathcal{S}(n)$ increases with increasing $n$.

In $S(n)$ there are two special triples: $(k^n, J(n), 0)$ and $(k^n, J(n), k^n)$.

| $n$ | number of indecomposables | |
|---|---|---|
| 1 | 2 | $= 2 + \ \ 0$ |
| 2 | 5 | $= 2 + \ \ 3$ |
| 3 | 10 | $= 2 + \ \ 8$ |
| 4 | 20 | $= 2 + 18$ |
| 5 | 50 | $= 2 + 48$ |
| 6 | ? | |

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The **Online Encyclopedia of Integer Sequences** provides for 2, 5, 10, 20, 50 two entries:

A051109 Hyperinflation sequence of banknotes, next numbers: 100, 200

A124146 USA currency denominations in dollars, next numbers: 100, 500

For 0, 3, 8, 18, 48, . . . , there is no entry.

The difficulty of classifying the indecomposable objects in $\mathcal{S}(n)$ increases with increasing $n$.

In $S(n)$ there are two special triples: $(k^n, J(n), 0)$ and $(k^n, J(n), k^n)$.

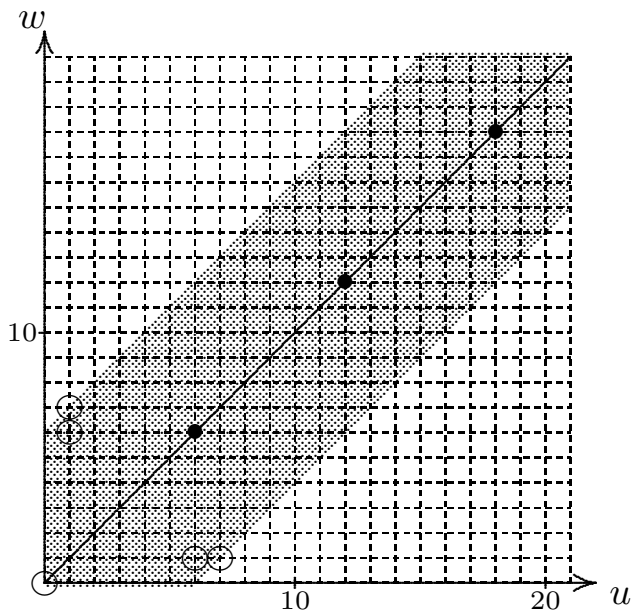
| $n$ | number of indecomposables | | |
|---|---|---|---|
| 1 | 2 | $= 2 + \ \ 0$ | $= 2 + 0$ |
| 2 | 5 | $= 2 + \ \ 3$ | $= 2 + \frac{3}{2} \cdot 2$ |
| 3 | 10 | $= 2 + \ \ 8$ | $= 2 + 2 \cdot 4$ |
| 4 | 20 | $= 2 + 18$ | $= 2 + 3 \cdot 6$ |
| 5 | 50 | $= 2 + 48$ | $= 2 + 6 \cdot 8$ |
| 6 | $\infty$ | | $= 2 + \frac{6}{0} \cdot 10$ |
| | | | $2 + \cdot \frac{6}{6-n} 2(n-1)$ |

We are going to study the case $n = 6$.

The *dimension pair* of a triple $(V, T, U)$ is the pair $(\dim U, \dim W)$ (where $W = V/U$).

Let $n = 6$.

**Theorem 1.** *A pair* $(u, w)$ *of natural numbers is the dimension pair of an indecomposable triple in* $\mathcal{S}(6)$ *if and only if* $(u, w)$ *satisfies* $|u - w| \leq 6$ *and is different from* $(0, 0)$, $(1, 6)$, $(1, 7)$, $(6, 1)$, $(7, 1)$.



We may reformulate part of Theorem 1 as follows:

*Assume* $(V,T,U)$ *is in* $\mathcal{S}(6)$ *and* $|\dim U - \frac{1}{2}\dim V| > 3$. *Then there are non-zero subspaces* $V', V'$ *with* $V = V' \oplus V''$ *such that*

$$T(V') \subset V', \quad T(V'') \subset V'' \quad \text{and} \quad U = (U \cap V') \oplus U \cap V'').$$

For $(V,T,U)$ indecomposable in $\mathcal{S}(6)$, the dimension of $U$ is roughly half of the dimension of $V$.

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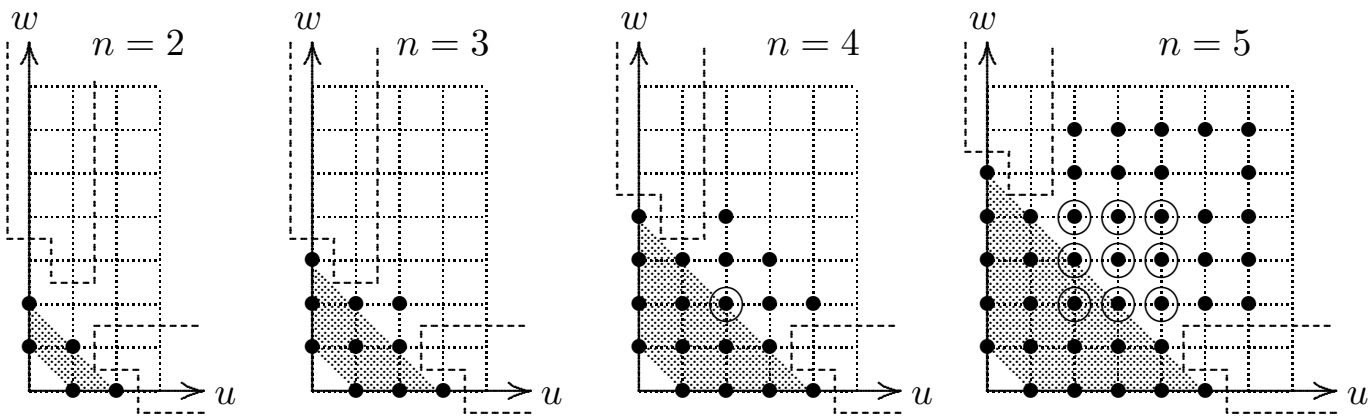
Why do we have to exclude the pairs $(1,6)$, $(1,7)$ $(6,1)$, $(7,1)$ ?

We have already noted that for $(V,T,U)$ indecomposable and $\dim U = 1$, then $(V,T)$ is indecomposable. This shows that $(1,6)$ and $(1,7)$ do not occur.

The pairs $(6,1)$ and $(7,1)$ are excluded, using duality: If $(V,T,U)$ is in $\mathcal{S}(n)$, then $(V^*, T^*, (V/U)^*)$ is also a triple in $\mathcal{S}(n)$.

For $n \leq 6$ and $(V,T,U) \in \mathcal{S}(n)$, we even have $|u-w| \leq n$,
and this bound is always optimal.

Here are the cases $n = 2,3,4,5$ (always, the picket region is shaded).




Encircled bullets: there are (precisely) two indecomposable triples.

Note: For $n \leq 5$, the pair $(n,n)$ does not occur as dimension pair!

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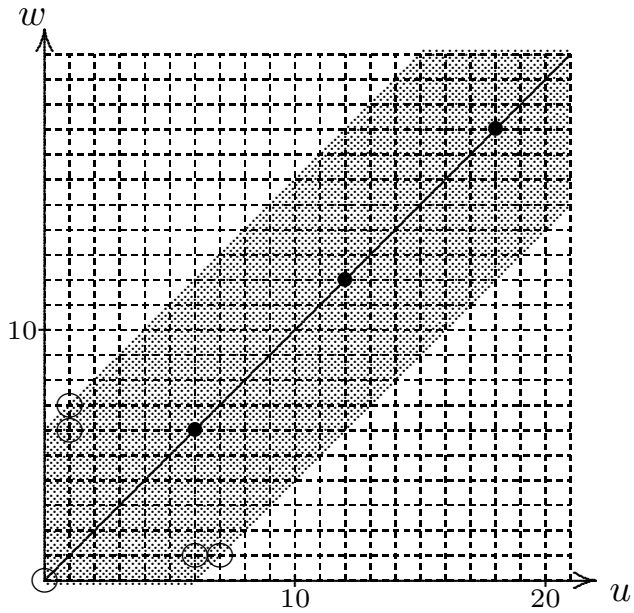
For $n \geq 7$, the numbers $|u-w|$ are not bounded,
but the possible dimension pairs are not yet known.

We return to $n = 6$.

**Theorem 2.** *If $(u, w)$ is not in $\mathbb{N}(6, 6)$, then the number of isomorphism classes of indecomposable objects in $\mathcal{S}(6)$ with dimension pair $(u, w)$ is finite (and independent of $k$).*




By contrast, the triples with dimension pair in $\mathbb{N}(6, 6)$ depend on the field $k$.

Let $(u, w) = t(6, 6)$ with $t \in \mathbb{N}_1$.

There are $t$ disjoint one-parameter families of indecomposable triples with dimension pair $(u, w)$.

$t = 1$

| | | |
|---|---|---|
| $e_6$ | | |
| $e_5$ | $e_{10}$ | |
| $e_4$ | $e_9$ | $e_{12}$ |
| $e_3$ | $e_8$ | $e_{11}$ |
| $e_2$ | $e_7$ | |
| $e_1$ | | |

generators of $U$: $e_8 + e_{11}$

$e_4 + (1 - c)e_9 - ce_{12}$

for any $c \in k$

$t = 2$

$V = M(6, 6, 4, 4, 2, 2)$

or

$V = M(6, 6, 5, 3, 3, 1)$

**Weakly homogeneous triples.** An indecomposable triple $(V,T,U)$ in $\mathcal{S}(n)$ is called *weakly homogeneous* provided $(V/U,T)$ is isomorphic to $(U,T)$ and $(V,T)$ is isomorphic to $(U,T)\oplus M(n)^t$ for some $t$. Then

$$\dim U + tn = \dim V = \dim U + \dim V/U = 2\dim U,$$

thus $\dim(V,T,U) = t(n,n)$.

For $n \le 5$, there are no weakly homogeneous triples. Return to $n=6$.

**Theorem 3.** *For any $t \in \mathbb{N}_1$, there are $t$ pairwise disjoint one-parameter families of weakly homogeneous triples in $\mathcal{S}(6)$ with dimension pair $t(6,6)$, each being indexed by $k \setminus \{0,1\}$.*
*If $k$ is algebraically closed, then there are only finitely many additional isomorphism classes indecomposable triples in $\mathcal{S}(6)$ with dimension pair $t(6,6)$* (and these triples are defined independently of $k$).

*If $(V,T,U)$ is weakly homogeneous, then $U = M(4,2)^r \oplus M(5,3,3,1)^s$ for some $r,s$* (and $t = r+2s$).

For $r>0, s>0$, there are **two** one-parameter families of weakly homogeneous triples with $U = M(4,2)^r \oplus M(5,3,3,1)^s$. Later we will see how to distinguish these two families.

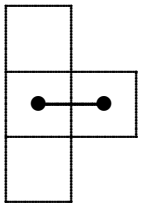
**Graded triples.** A grading of the triple $(V, T, U)$ is a direct decomposition $V = \bigoplus_{i \in \mathbb{Z}} V_i$ such that $T(V_i) \subseteq V_{i-1}$ and $U = \bigoplus (U \cap V_i)$.

**Theorem 4.** *For $n = 6$, any triple in $\mathcal{S}(6)$ can be graded.*
*For indecomposable triples, the grading is unique up to shift.*

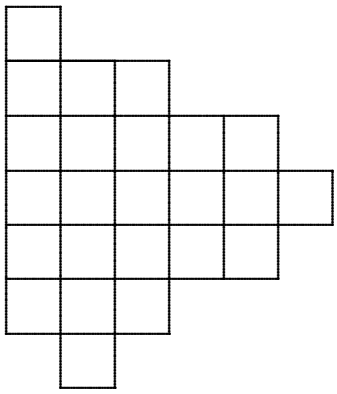
Interpretation. Write $(V, T) = M(\lambda)$ where $\lambda$ is a partition.
Visualize $M(\lambda)$ using the Young diagram of $\lambda$, present the parts as columns.
A grading of $(V, T, U)$ means to adjust the columns conveniently.

**Examples:**

The indecomposable triple with dim pair $(2, 2)$



The columns of a weakly homogeneous triple with $U = M(5, 3, 3, 1)$ are adjusted as follows:



The grading theorem is the essential result!
It provides a lot of new invariants: We can refine

$$u = \sum u_i, \quad w = \sum w_i,$$

where

$$u_i = \dim U \cap V_i, \quad w_i = \dim V_i/(U \cap V_i).$$

A graded triple is a system of vector spaces and linear maps as follows:

$$\begin{tikzcd}
\cdots & U_0 \arrow[d, "\iota"] & U_1 \arrow[l, "T"'] \arrow[d, "\iota"] & U_2 \arrow[l, "T"'] \arrow[d, "\iota"] & \cdots \arrow[l, "T"'] \\
\cdots & V_0 \arrow[l, "T"] & V_1 \arrow[l, "T"] & V_2 \arrow[l, "T"] & \cdots \arrow[l, "T"]
\end{tikzcd}$$

The squares are commutative (and $T^6 = 0$).

**A remark concerning the proof of all the results presented here:**

- One first deals with graded triples: Classification of the indecomposables, determination of the global structure of the category.
- The structure of the category of graded triples implies that any triple can be graded: Theorem 4.
- This then leads to the remaining assertions.

The classification of indecomposable triples follows a well-known procedure in the representation theory of finite dimensional algebras:

- "Covering".
- "Knitting".
- "Tilting".

However all these techniques had been available only for module categories itself — here we deal with a proper subcategory of a module category, thus the techniques had to be modified. (Note that any module category is abelian, the category $\mathcal{S}(n)$ is not abelian!)

Up to now, $\mathcal{S}(n)$ was considered only as a class of objects, not as a category. The appropriate notion of *maps* $(V, T, U) \to (V', T', U')$ is the following: take the linear maps $f\colon V \to V'$ with $fT = T'f$ and $f(U) \subseteq U'$.

**Categorical properties of $\mathcal{S}(6)$.** The category $\mathcal{S}(n)$ is additive, thus the endomorphisms of any triple form a ring.

Let $n = 6$. The endomorphism rings $\mathrm{End}(X)$ of an indecomposable triple is usually rather large, however the bulk of endomorphisms will be nilpotent with nilpotency index at most 8:

**Theorem 5.** *Let $X$ be an indecomposable triple in $\mathcal{S}(6)$. There is an ideal $I$ in $\mathrm{End}(X)$ with $I^8 = 0$ such that $\mathrm{End}(X)/I$ is a local uniserial ring.*

The ideal $I$ can be described as follows:
Given a Krull-Remak-Schmidt category such that the indecomposable objects have local endomorphism rings (such as $\mathcal{S}(n)$), its *radical* $\mathcal{R}$ is generated by all non-invertible maps between indecomposable objects.
The *infinite radical* is $\mathcal{I} = \bigcap_{i \in \mathbb{N}} \mathcal{R}^i$.

*In the category $\mathcal{S}(6)$, the infinite radical $\mathcal{I}$ is an idempotent ideal, that is, $\mathcal{I}^2 = \mathcal{I}$ holds. If $X$ is an indecomposable triple in $\mathcal{S}(6)$, then the ideal $\mathcal{I}(X, X)$ of $\mathrm{End}(X)$ has nilpotency index at most 8.*

The ideal $I$ to be used in Theorem 5 is $I = \mathcal{I}(X, X)$.

Recall: A graded triple is a representation of the following quiver:

$$\widetilde{Q}: \quad \begin{tikzcd}
\cdots & 0' \arrow[d, "\beta_0"] & 1' \arrow[l, "\alpha'_0"'] \arrow[d, "\beta_1"] & 2' \arrow[l, "\alpha'_1"'] \arrow[d, "\beta_2"] & \cdots \arrow[l, "\alpha'_2"'] \\
\cdots & 0 & 1 \arrow[l, "\alpha_0"] & 2 \arrow[l, "\alpha_1"] & \cdots \arrow[l, "\alpha_2"]
\end{tikzcd}$$

Usually, we will refrain from mentioning the indices of the arrows $\alpha_i, \alpha'_i, \beta_i$ and write $\alpha, \alpha', \beta$.

Let $\mathcal{S}(\widetilde{n})$ be the category of all representations of $\widetilde{Q}$ which satisfy

- the commutativity relations $\beta\alpha' = \alpha\beta$ (for all squares),
- the nilpotency relations $(\alpha')^n = \alpha^n = 0$
  (for all compositions of $n$ arrows $\alpha'$ or $\alpha$)
- and for which all the maps $\beta$ are realized by monomorphisms.

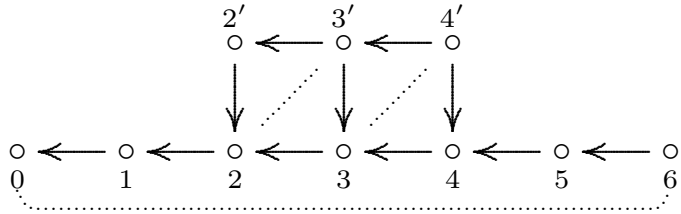
Note that this is just the category of triples in $\mathcal{S}(n)$ endowed with a grading.

Denote by $\sigma$ the shift of the grading (by 1). Then $\mathcal{S}(\widetilde{6})/\sigma$ are the triples in $\mathcal{S}(6)$ which are gradable.

**Aim. Classify the iso classes of indecomp representations of $\widetilde{Q}$.**

It is sufficient to look at a suitable finite subquiver of $\widetilde{Q}$. It turns out that the following one provides sufficient information (and can be handled):

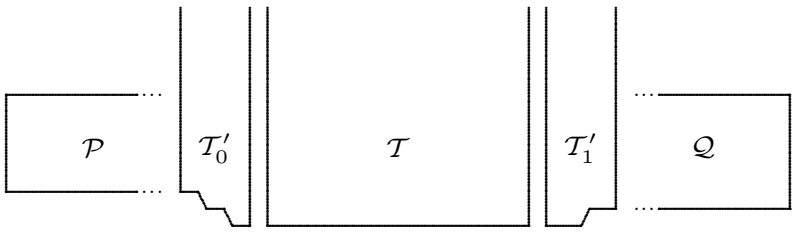
$Q_\Theta$:



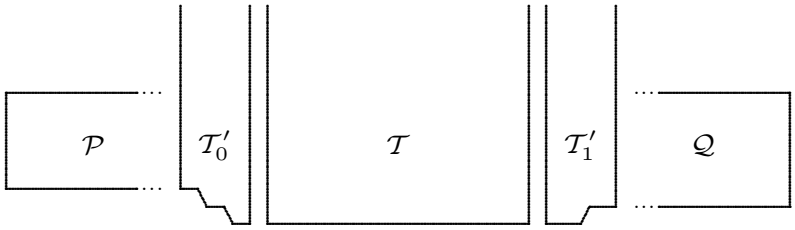

(with two commutativity relations and one zero relation).

The corresponding algebra $\Theta$ can be shown to be "tubular", and in this way we obtain a description of nearly all of the category $\mathcal{S}(\widetilde{6})$.

The shape of the category of $\Theta$-modules is as follows:

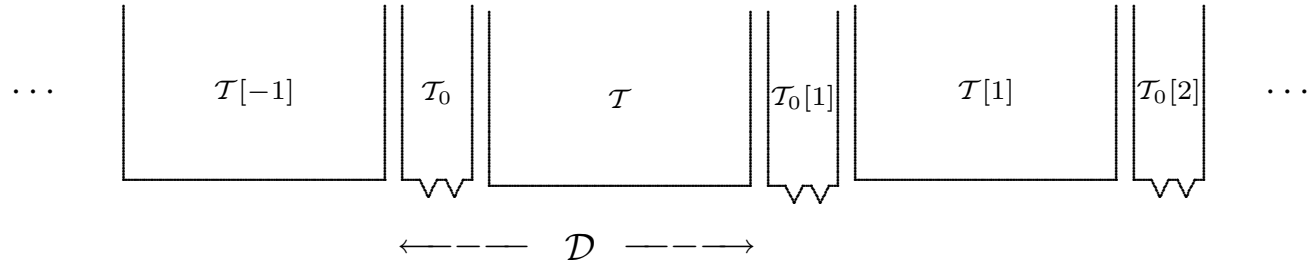


The shape of the category $\operatorname{mod}\Theta$:




The subcategories $\mathcal{T}_0'$ and $\mathcal{T}_1'$ will be partly identified in $\mathcal{S}(6)$.

The category $\mathcal{S}(\widetilde{6})$ can be described in the following way:




$\mathcal{D}$ is the fundamental domain for $\mathcal{S}(\widetilde{6})$ under the shift $\sigma$.

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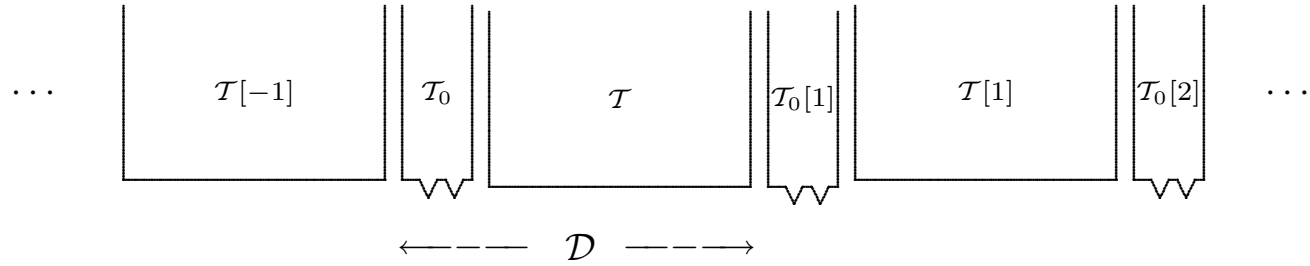


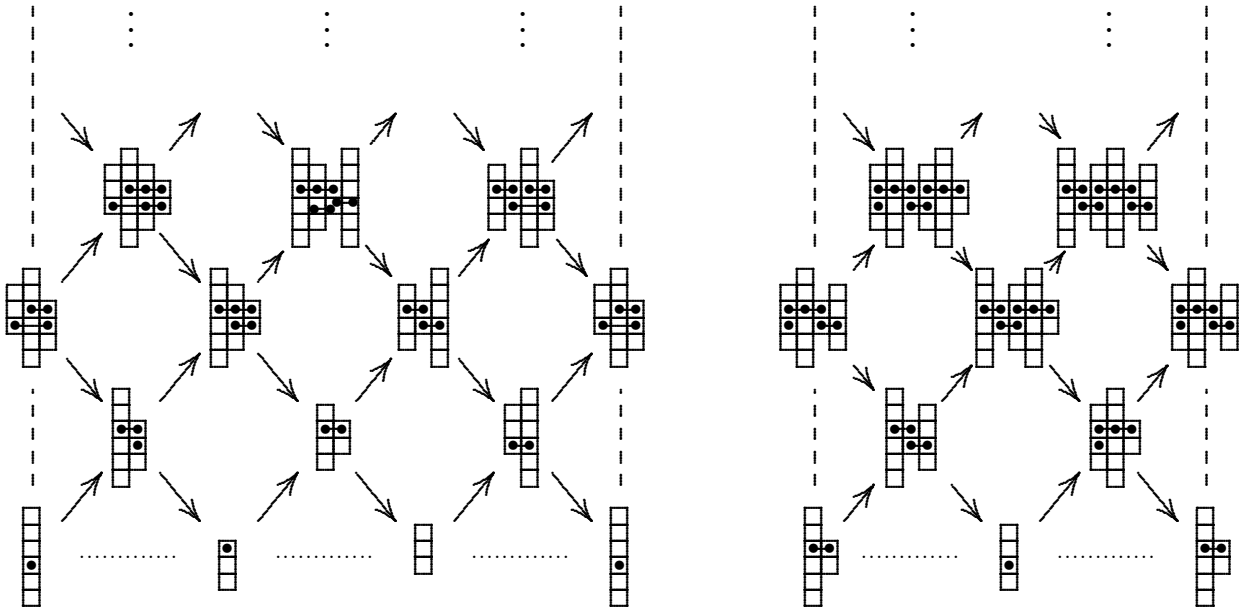

$\mathcal{D}$ is the fundamental domain for $\mathcal{S}(\widetilde{6})$ under the shift $\sigma$.

Since $\mathcal{S}(\widetilde{6})$ is "locally bounded", it follows that $\mathcal{S}(6) = \mathcal{S}(\widetilde{6})/\sigma$ (= the grading theorem).

Of essential importance is the central part $\mathcal{T}$,
there are **countably many stable tubular families** $\mathcal{T}_\gamma$ indexed by $\gamma \in \mathbb{Q}^+$,
each $\mathcal{T}_\gamma$ is **a** $\mathbb{P}_1(k)$**-family of tubes of type** $(6,3,2)$**.**

What is a tube? What is a tubular family?

**Two typical tubes in $\mathcal{S}(6)$:**



These are the exceptional tubes of rank 3 and 2 in $\mathcal{T}_0$

**The Auslander-Reiten quiver of categories such as $\mathcal{S}(n)$ or $\mathcal{S}(\widetilde{n})$.**

Let $\mathcal{C}$ a category such as $\mathcal{S}(n)$ or $\mathcal{S}(\widetilde{n})$.
The Auslander-Reiten quiver $\Gamma(\mathcal{C})$ describes
the factor category of $\mathcal{C}$ modulo its infinite radical, as follows:

- The **vertices** are the isomorphism classes $[X]$ of the indecomposable objects in $\mathcal{C}$.
- The number of **arrows** $[X] \to [X']$ is $\dim \operatorname{Hom}(X, X')/\operatorname{rad}(X, X')$
  Such arrows represent "irreducible maps". .
- In addition, there is a canonical bijection (the Auslander-Reiten translation) between the non-projective objects and the non-injective objects.

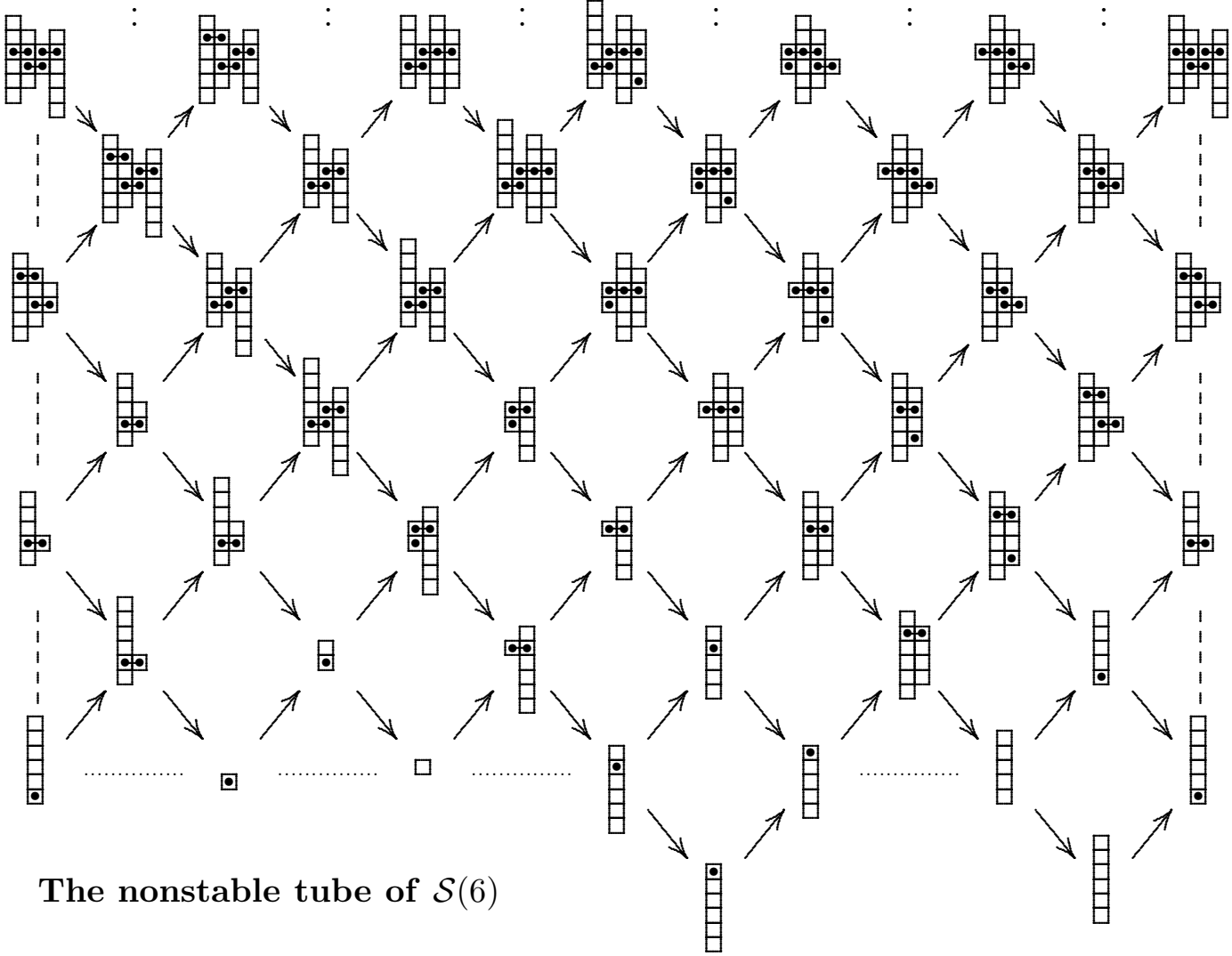
The translation is used to define relations (the "mesh relations") on $\Gamma(\mathcal{C})$.

In general, the Auslander-Reiten translation is not defined for all objects, some objects may be "projective" or "injective".

In the categories $\mathcal{S}(n)$, there are jut two triples of this kind, the triples

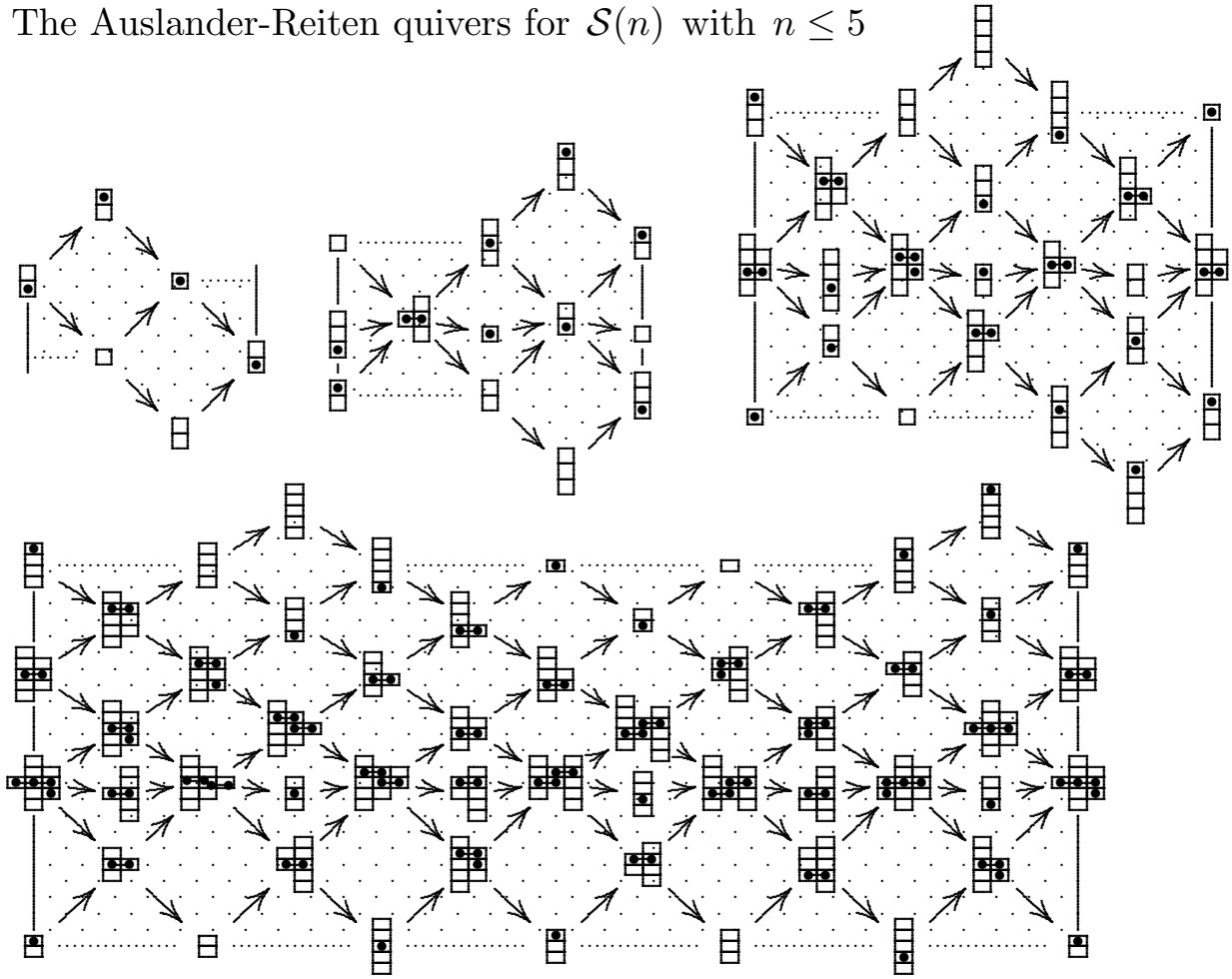
$$(k^n, J(n), 0), \quad \text{and} \quad (k^n, J(n), k^n),$$

both are "projective" and "injective" as well.



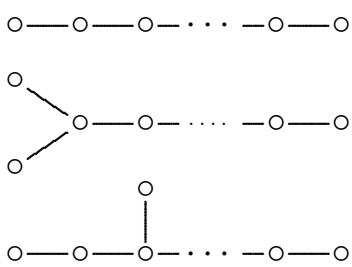
**The nonstable tube of $\mathcal{S}(6)$**

The Auslander-Reiten quivers for $\mathcal{S}(n)$ with $n \leq 5$



| $n$ | number of indecomposables | | | hidden Lie type |
|---|---|---|---|---|
| 1 | 2 | $= 2 + \ \ 0$ | $= 2 + 0$ | $\emptyset$ |
| 2 | 5 | $= 2 + \ \ 3$ | $= 2 + \frac{3}{2} \cdot 2$ | $A_2$ |
| 3 | 10 | $= 2 + \ \ 8$ | $= 2 + 2 \cdot 4$ | $D_4$ |
| 4 | 20 | $= 2 + 18$ | $= 2 + 3 \cdot 6$ | $E_6$ |
| 5 | 50 | $= 2 + 48$ | $= 2 + 6 \cdot 8$ | $E_8$ |
| 6 | $\infty$ | | | |

Here is the list of the simply laced Dynkin diagrams considered in Lie theory:



$A_n$

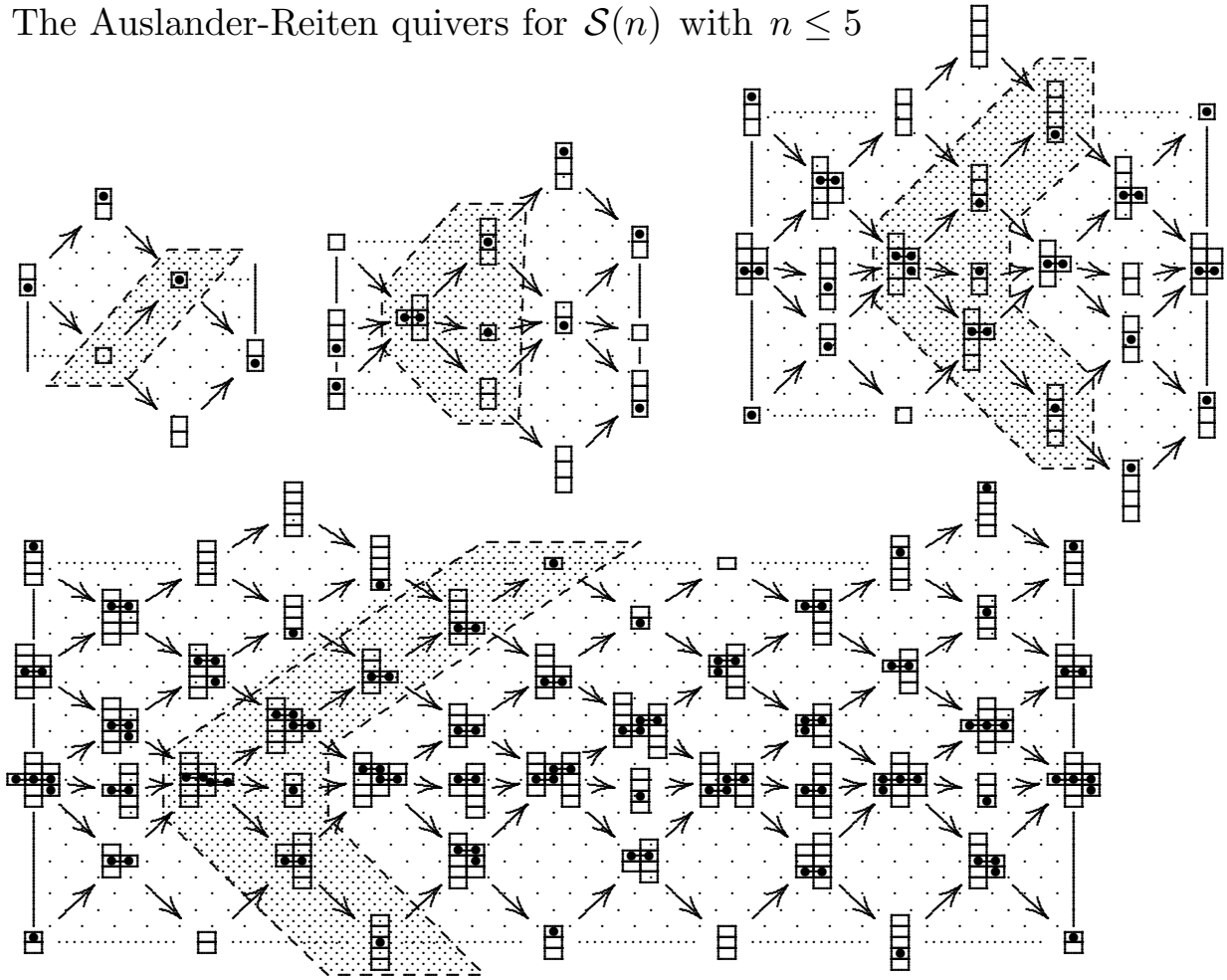


$D_n$



$E_m \quad (m = 6, 7, 8)$

The Auslander-Reiten quivers for $\mathcal{S}(n)$ with $n \leq 5$



**A more general setting.** For $\Lambda$ a ring, let $\mathcal{S}(\Lambda)$ denote the class of pairs $(M, U)$, with $M$ a finitely generated $\Lambda$-module and $U$ a submodule of $M$.

We consider the case where $\Lambda$ is a uniserial commutative ring of length $n$,
such as $\Lambda = k[T]/T^n$ (here, $k[T]$ is the polynomial ring in one variable $T$)
or $\Lambda = \mathbb{Z}/p^n$ (with $p$ a prime number).

Note that $\mathcal{S}(n) = \mathcal{S}(k[T]/T^n)$.

The problem to study $\mathcal{S}(\mathbb{Z}/p^n)$ was raised by Garrett Birkhoff in 1934:
To determine *relativ invariants of subgroups;*
*i.e. invariants under automorphisms* of the given group.

For any uniserial commutative ring $\Lambda$ of length $n$, the isomorphism classes of the $\Lambda$-modules correspond bijectively to the partitions with parts of size at most $n$, thus we can use the same box diagrams as in the special case of nilpotent operators.

**Theorem 6.** *For $n \leq 5$, the Auslander-Reiten quiver of $\mathcal{S}(\Lambda)$ depends only on $n$, not on $\Lambda$.*

| $n$ | number of indecomposables | | |
|---|---|---|---|
| 1 | 2 | $= 2 + \ \ 0$ | |
| 2 | 5 | $= 2 + \ \ 3$ | |
| 3 | 10 | $= 2 + \ \ 8$ | 1984 Hunter-Richman-Walker |
| 4 | 20 | $= 2 + 18$ | 1999 Richman-Walker |
| 5 | 50 | $= 2 + 48$ | 1999 Richman-Walker |
| 6 | $\infty$ | | 1934 (Birkhoff) |

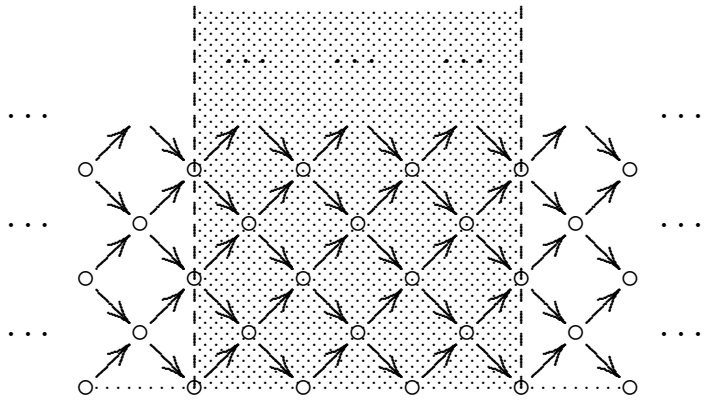
Birkhoff has shown that the indecomposable objects $(M, U)$ in $\mathcal{S}(\mathbb{Z}/p^6)$ with $M$ of partition type $(6, 4, 2)$ depend on the prime $p$ (and the number of isomorphism classes tends to infinity, if $p \to \infty$)

But a complete classification of the indecomposable objects in $\mathcal{S}(\mathbb{Z}/p^6)$ is still unknown (in contrast to $\mathcal{S}(k[T]/T^6) = \mathcal{S}(6)$).

We return to consider $\mathcal{S}(6)$. Here we deal mostly with **stable tubes.**

**Stable tubes of rank $r$.**

here: $r = 3$



Vertices: the pairs $(x, y) \in \mathbb{Z}^2$
with $y \geq 0$ and $x \equiv y \bmod 2$

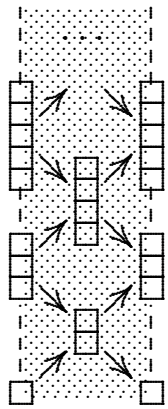
Arrows: $(x, y) \to (x + 1, y + 1)$
and $(x, y) \to (x + 1, y - 1)$

Translation $\tau(x, y) = (x-2, y)$

Identify $(x, y)$ with $(x + 2r, y)$

A *homogeneous tube* is a tube of rank 1

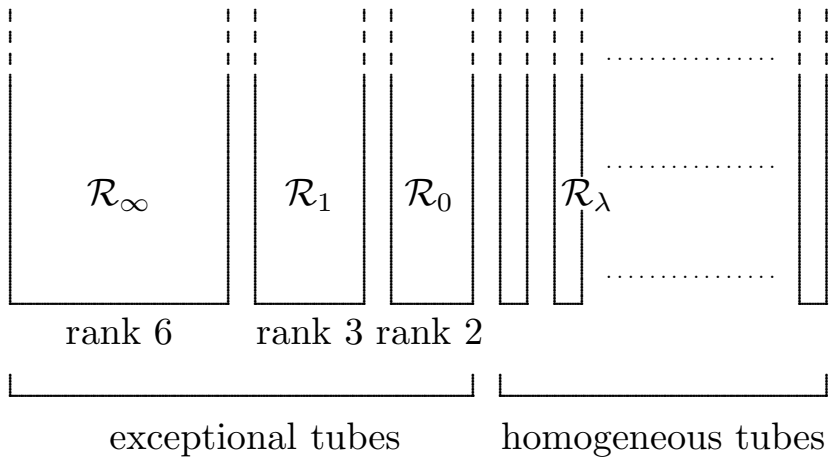
The Auslander-Reiten quiver
of the category of nilpotent operators
is a typical homogeneous tube!



Recall what we have mentioned for the case $n = 6$ :

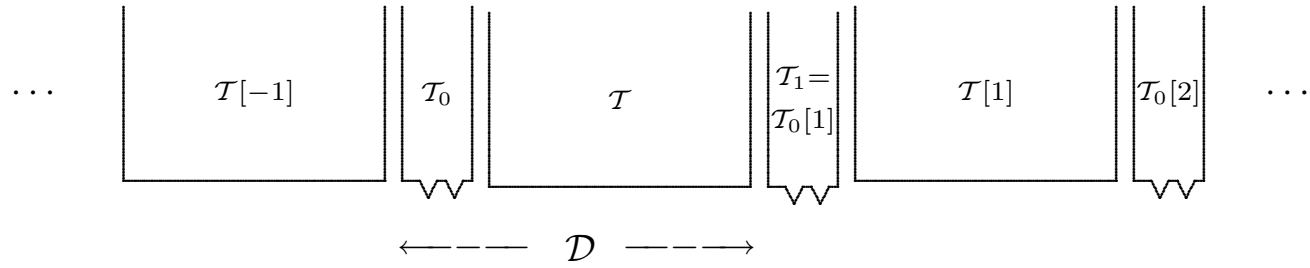
There are countable many stable tubular families in $\mathcal{T}$ , all of type $(6, 3, 2)$.

**A tubular family of type $(6, 3, 2)$ has the following form:**



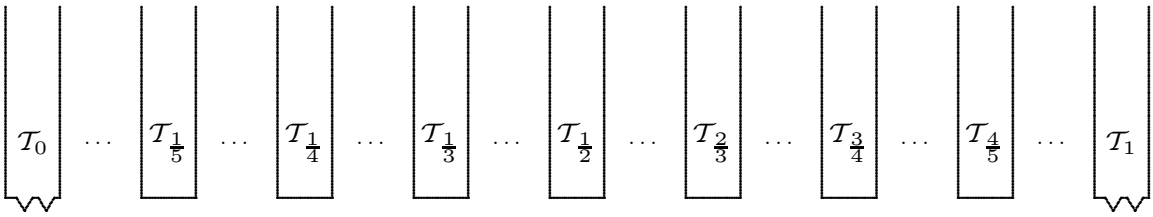

The index set for the tubular families we are interested in, will always be $\mathbb{P}_1(k)$.

Here is the category $\mathcal{S}(\widetilde{6})$, and $\mathcal{D}$ is a fundamental domain for the shift $\sigma$.



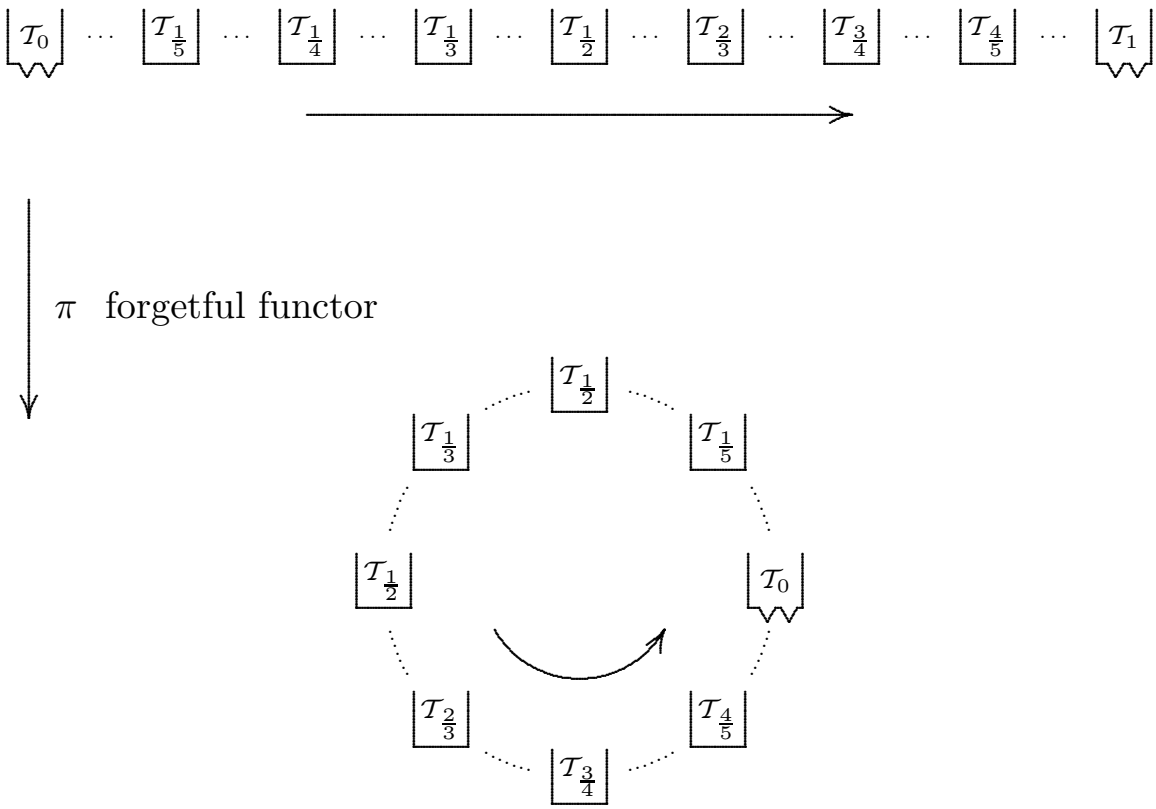

Any object in $\mathcal{S}(\widetilde{6})$ has a "slope" $\gamma \in \mathbb{Q}$, the shift $\sigma$ increases the slope by $1$. The objects with slope $\gamma$ form the subcategory $\mathcal{T}_\gamma$.

The part containing the objects with slope in $\mathbb{Q}^+ \cap [0,1]$ looks as follows:




The graded triples with a fixed slope form a tubular family of type $(6,3,2)$.

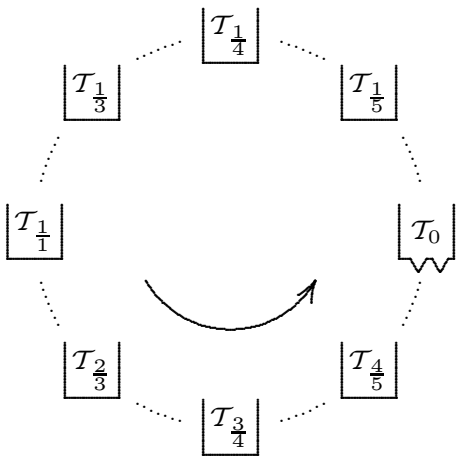
## Forgetting the grading



**To repeat: The classification of the indecomposable triples in $\mathcal{S}(6)$** (in case $k$ is algebraically closed):

There are two projective-injective triples, with dimension pair $(0,6)$ and $(6,0)$.

For the remaining triples, one needs three invariants .




First invariant: The slope,
a rational number $0 \leq \gamma < 1$

Second invariant: The spectral parameter $c$,
an element of $\mathbb{P}_1(k) = k \cup \{\infty\}$

Third invariant: A vertex $x$ in a tube.
If $c \notin \{0,1,\infty\}$, then $x \in \mathbb{N}$
If $c \in \{0,1,\infty\}$, then $x = (i,m)$, $m \in \mathbb{N}$

$1 \leq i \leq 2$ for $c = 0$
$1 \leq i \leq 3$ for $c = 1$
$1 \leq i \leq 6$ for $c = \infty$

**The endomorphism theorem (Theorem 4).**

Denote by $\pi\colon \mathcal{S}(\widetilde{6}) \to \mathcal{S}(6)$ the forgetful functor (= forgetting the grading).

Recall that $\sigma$ denotes the shift of the grading by $1$.

The main formula is the following: Let $X, Y$ be graded triples. Then

$$\mathrm{Hom}_{\mathcal{S}(6)}(\pi(X), \pi(Y)) = \bigoplus\nolimits_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathcal{S}(\widetilde{6})}(X, \sigma^i(Y)).$$

Now, in $\mathcal{S}(\widetilde{6})$ the following holds true: If $X$ has slope $\gamma$ and $X'$ has slope $\gamma'$, then $\mathrm{Hom}(X, X') = 0$ unless $\gamma \leq \gamma'$

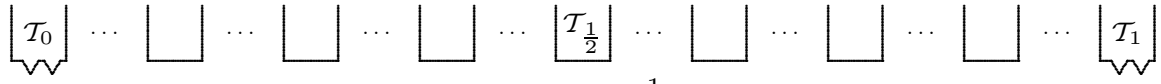
Therefore,

$$\begin{aligned}\mathrm{End}_{\mathcal{S}(6)}(\pi(X)) &= \bigoplus\nolimits_{i\geq 0} \mathrm{Hom}_{\mathcal{S}(\widetilde{6})}(X, \sigma^i(X)) \\ &= \mathrm{End}_{\mathcal{S}(\widetilde{6})}(X) \oplus \bigoplus\nolimits_{i\geq 1} \mathrm{Hom}_{\mathcal{S}(\widetilde{6})}(X, \sigma^i(X))\end{aligned}$$

If $X$ is an indecomposable graded triple, then $\mathrm{End}_{\mathcal{S}(\widetilde{6})}(X)$ is a local uniserial ring, and $\mathrm{Hom}_{\mathcal{S}(\widetilde{6})}(X, \sigma^i(X)) = 0$ for $i \geq 8$.

Recall: Almost all indecomposable triples $(V, T, U)$ with fixed dimension pair are **weakly homogeneous** (i.e. $U \simeq V/U,\ V \simeq U \oplus M(6)^t$ ), and then $U = M(4,2)^r \oplus M(5,3,3,1)^s$ for some pair $r, s$.

Let $X = (V, T, U)$ be weakly homogeneous with slope $\gamma$.




For $\gamma = 0$
$U = M(4,2)^r$

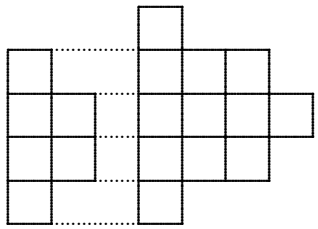
For $\gamma = \frac{1}{2}$
$U = M(5,3,3,1)^s$

For $\gamma = 1$
$U = M(4,2)^r$

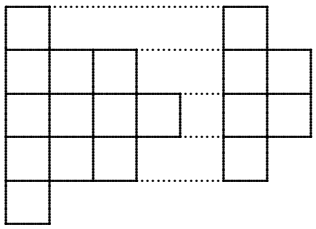
For $0 < \gamma < \frac{1}{2}$
there is an exact sequence
$0 \to X' \to X \to X'' \to 0$
slope $0$ slope $\frac{1}{2}$

For $\frac{1}{2} < \gamma < 1$
there is an exact sequence
$0 \to X' \to X \to X'' \to 0$
slope $\frac{1}{2}$ slope $1$

Adjustment
of the columns



even odd



odd even

Recall: $(V,T,U)$ indecomposable in $\mathcal{S}(6)$. Then $|\dim U - \frac{1}{2}\dim V| \leq 3$.
This means: The dimension of $U$ is **roughly** half of the dimension of $V$.

If $(V,T,U)$ is weakly homogeneous, then we even have: $\dim U = \frac{1}{2}\dim V$.

The structure theorem for weakly homogeneous triples asserts:
$V = M(4,2)^r \oplus M(5,3,3,1)^s \oplus M(6)^{r+2s}$. This implies:

$$\begin{aligned}
\dim \operatorname{Ker} T &= \tfrac{1}{4}\dim V \\
\dim \operatorname{Ker} T^3 &= \tfrac{2}{3}\dim V \\
\dim \operatorname{Ker} T^5 &= \tfrac{11}{12}\dim V
\end{aligned}$$

$$\begin{aligned}
\tfrac{11}{24}\dim V \leq \dim \operatorname{Ker} T^2 \leq \tfrac{1}{2}\dim V \\
\tfrac{19}{24}\dim V \leq \dim \operatorname{Ker} T^4 \leq \tfrac{5}{6}\dim V
\end{aligned}$$

$$\dim \operatorname{Ker} T^4 / \operatorname{Ker} T^2 = \tfrac{1}{3}\dim V$$

Again, for indecomposable triples which are not weakly homogeneous, these (in)equalities are **roughly** true: they hold up to small differences ...

**References:**

Ringel-Schmidmeier: Invariant subspaces of nilpotent linear operators. I. J. Reine Ang. Math. (to appear).

Ringel-Schmidmeier: Invariant subspaces of nilpotent linear operators. II. (in preparation).

## Appendix

*Jetzt eine provokative Frage: ist in der Mathematik eigentlich nicht schon alles erforscht? Diesen Eindruck kann man ja in der Schule und vielleicht auch im Grundstudium bekommen.*

Ziegler: Ja, diesen Eindruck kann man bekommen, und ich finde das sehr schade. Er hat damit zu tun, dass der Ausschnitt, den man von der Mathematik zu sehen bekommt, sehr klein ist. In der Linearen Algebra von endlichdimensionalen Vektorräumen ist in der Tat alles erforscht. Dasselbe gilt in der Analysis von ein oder zwei Veränderlichen, da gibt es nichts Neues zu entdecken.

When I challenged this statement, Ziegler modified the assertion as follows:

> Nicht einmal in der Linearen Algebra von endlichdimensionalen Vektorräumen ist alles erforscht.

But even this seems to me misleading.

**The teaching of linear algebra:**

The wealth of open problems is usually not discussed.

Often not even the present knowledge is mentioned.

Let me start with a typical result included in every Linear Algebra course:

*If $U_1, U_2$ are finite-dimensional subspaces of a vector space $V$, then*

$$\dim(U_1 \cap U_2) = \dim U_1 + \dim U_2 - \dim(U_1 + U_2).$$

Also the following ingredient for the proof is rightly considered as essential:

*If $U_1, U_2$ are finite-dimensional subspaces of $V$, then there is a basis $\mathcal{B}$ of $V$ which is compatible with the subspaces $U_i$* (i.e. $\mathcal{B} \cap U_i$ is a basis of $U_i$).

Question: If $n$ subspaces $U_i$ of $V$ are given, is there still a basis $\mathcal{B}$ of $V$ which is compatible with the subspaces $U_i$ ?

(1) Already for $n = 3$ the answer is **no.**
(2) But the answer is **yes** if the subspaces form two chains.

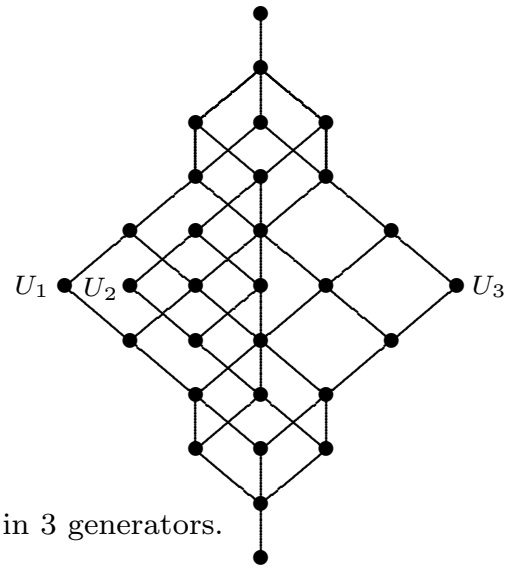
**The $n$-subspace problem.**

Let us assume that $n$ subspaces $U_i$ of $V$ are given. Forming inductively sums and intersections, how many subspaces can we obtain?

Dedekind (1900) has shown:
For $n = 3$, we get at most 30 subspaces.
(*Über die von 3 Elementen erzeugte Dualgruppe)*

This can be realized with $V = k^{10}$

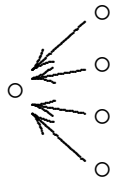
$$U_1 = 0\,k\,0\,0\,k\,0\,0\,k\,k\,k$$
$$U_2 = 0\,0\,k\,0\,0\,k\,k\,0\,k\,k$$
$$U_3 = 0\,0\,0\,k\,x\,x\,k\,k\,0\,k$$




The free modular lattice in 3 generators.

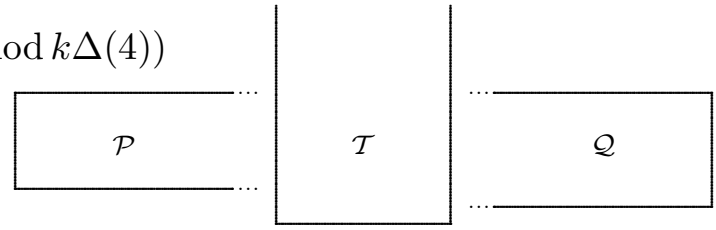
For the cases $n \geq 4$, see Gelfand-Ponomarev (1969, ... ):
the cubicles, the perfect elements, ...

**The Auslander-Reiten quiver of the $4$-subspace problem.**



4-subspace quiver
$\Delta(4)$

$\Gamma(\operatorname{mod} k\Delta(4))$




a preprojective component

a preinjective component

a stable tubular family of type $(2,2,2)$

We get again a "tame" category!

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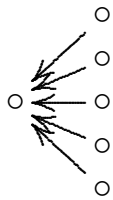


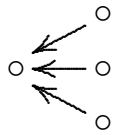
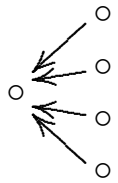
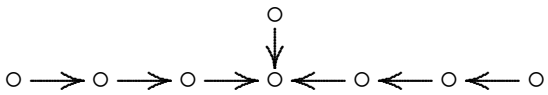
5-subspace quiver
$\Delta(5)$

Here, we get a "wild" category!

**The representation types.**

| | finite | tame | wild |
|---|---|---|---|
| $n$ endomorphisms | $0$ | $1$ | $\geq 2$ |
| $n$-Kronecker quiver $\circ \overset{\longrightarrow}{\underset{\longrightarrow}{\vdots}} \circ$ | $\leq 1$ | $2$ | $\geq 3$ |
| $n$-subspace problem | $\leq 3$ | $4$ | $\geq 5$ |
| $\mathcal{S}(n)$ | $\leq 5$ | $6$ | $\geq 7$ |

Problems arising in linear algebra tend to have a combinatorial nature! Unfortunately, this message is seldom revealed in linear algebra courses.

| **The relationship to Lie theory:** | **Lie type** |
|---|---|
| 3-subspace quiver  | $D_4$ |
| 4-subspace quiver  | $\widetilde{D}_4$ |
| The algebra $\Theta$ used above is tilting equivalent to  | $\widetilde{E}_7$ |

For describing the indecomposable objects, one uses the root systems, the qudratic forms known from Lie theory (= the combinatorial part of Lie theory).

But the relationship is even stronger: The semisimple complex Lie algebras themselves can be reconstructed using the representation theory of quivers.