6. Covering theory.

Covering theory provides an important method to construct indecomposable representations of a directed quiver with cycles using for example the universal cover of the quiver.

6.1. Locally finite-dimensional quivers.

A quiver Q is said to be *locally finite* provided any vertex x is head or target of only finitely many arrows. A representation M of a quiver Q is said to be *locally finite-dimensional* provided all the vector spaces M_x are finite-dimensional. Let us denote by $\operatorname{Mod} kQ$ the category of locally finite-dimensional representations of Q (and by $\operatorname{MOD} kQ$ the category of all the representations of Q).

If a module M is a (not necessarily finite) direct sum of modules with local endomorphism rings, say $M = \bigoplus_{i\in I} M_i$, we say that any indecomposable module occurs with *finite multiplicity*, provided for any module N, the number of indices $i \in I$ such that M_i is isomorphic to N is finite. Recall that the theorem of Krull-Remak-Schmidt-Azuyama asserts that the number of indices $i \in I$ with M_i isomorphic to N does not depend on the decomposition.

Theorem. *Any indecomposable representation in* $\operatorname{Mod} kQ$ *has local endomorphism ring, any representation in* $\operatorname{Mod} kQ$ *is the direct sum of indecomposable representations, each one occurring with finite multiplicity.*

Proof. First, let M be an indecomposable locally finite-dimensional representation of Q. We show that for any endomorphism $f = (f_x)_{x\in Q_0}$ of M either all the maps f_x are nilpotent (in this case f is said to be *locally nilpotent*) or else that all non-zero maps f_x are automorphisms. Recall that given a finite-dimensional vector space V and an endomorphism ϕ of V, there is a (unique!) direct decomposition $V = V' \oplus V''$ of vector spaces such that $\phi(V') \subseteq V', \phi(V'') \subseteq V''$ so that the restriction $\phi' = \phi|V'$ is bijective, whereas the restriction $\phi'' = \phi|V''$ is nilpotent.

Looking at the vector space endomorphism f_x of M_x, we obtain in this way a direct decomposition $M_x = M'_x \oplus M''_x$ such that $f_x(M'_x) \subseteq M'_x$ and $f_x(M''_x) \subseteq M''_x$, with $f'_x = f_x|M'_x$ bijective, and $f''_x = f_x|M''_x$ nilpotent. Let $\alpha\colon x \to y$ be an arrow of Q, thus there is given $M_\alpha\colon M_x \to M_y$. With respect to the direct decompositions $M_x = M'_x \oplus M''_x$ we can write f_x in matrix form $\left[\begin{smallmatrix} (f'_x)^t & 0 \\ 0 & (f''_x)^t \end{smallmatrix}\right]$. Similarly, we use the direct decompositions $M_x = M'_x \oplus M''_x$ and $M_y = M'_y \oplus M''_y$ in order to write M_α in matrix form $M_\alpha = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$. Since f is an endomorphism of M, there is the commutativity condition $M_\alpha f_x = f_y M_\alpha$, thus also $M_\alpha f_x^t = f_y^t M_\alpha$ for all $t \geq 0$. In terms of matrices, this means that

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} (f'_x)^t & 0 \\ 0 & (f''_x)^t \end{bmatrix} = \begin{bmatrix} (f'_y)^t & 0 \\ 0 & (f''_y)^t \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

in particular, we have

$$B(f''_x)^t = (f'_y)^t B, \quad \text{and} \quad C(f'_x)^t = (f''_y)^t C,$$

for all $t \geq 0$. Since f''_y is nilpotent, the maps $B(f''_y)^t$ and $(f''_y)^t C$ are zero for t large. Since f'_x is invertible, we conclude that $B = 0,\ C = 0$.

This shows that we have vector space decompositions $M_x = M'_x \oplus M''_x$ such that for any arrow $\alpha\colon x \to y$, the map M_α maps M'_x into M'_y and M''_x into M''_y, thus $M = M' \oplus M''$ is a direct decomposition of kQ-modules. By assumption M is indecomposable, thus either $M = M'$ or $M = M''$. In the first case, all non-zero maps f_x are automorphisms, in the second case, f is locally nilpotent.

It follows as usual that the set of locally nilpotent endomorphisms of M is an ideal in the endomorphism ring $\operatorname{End}(M)$ of M and that this ideal is the unique maximal ideal of $\operatorname{End}(M)$, thus $\operatorname{End}(M)$ is a local ring.

Now let us consider arbitrary locally finite-dimensional representations M of Q, where Q is a locally finite quiver. Without loss of generality, we may assume that Q is connected, thus clearly Q_0 is a countable set. Since the assertion of the theorem is well-known for finite quivers, we assume that Q is infinite, thus we can label the vertices as an infinite sequence $x(1), x(2), \ldots$ and we put $\mathcal{X}(t) = \{x(1), x(2), \ldots, x(t)\}$.

If $\mathcal{X}$ is a set of vertices of Q, we say that $M \in \operatorname{Mod} kQ$ is *$\mathcal{X}$-indecomposable* provided for any direct decomposition $M = M' \oplus M''$, we have $M'_x = 0$ for all $x \in \mathcal{X}$ or $M''_x = 0$ for all $x \in \mathcal{X}$. Note that we do not require that $M_x \neq 0$, not even that $M \neq 0$, thus a direct summand of an $\mathcal{X}$-indecomposable module is $\mathcal{X}$-indecomposable.

A finite direct decomposition $M = \bigoplus_{i=1}^m M(i)$ is called an *$\mathcal{X}$-decomposition* provided all the $M(i)$ are $\mathcal{X}$-indecomposable. If $\mathcal{X}$ is a finite set, then any $M \in \operatorname{Mod} kQ$ has an $\mathcal{X}$-decomposition (but note that the direct summands $M(i)$ in an $\mathcal{X}$-decompositions are usually not unique, not even up to isomorphism).

Now we start with a module $M \in \operatorname{Mod} kQ$ and want to decompose it. This is done inductively, looking at the vertices $x(1), x(2), \ldots$. The direct summands $M(v)$ of M obtained in step t will be indexed by a set $I(t)$ of sequences $v = (v_1, \ldots, v_t)$ of natural numbers. In step 1, take an $\mathcal{X}(1)$-decomposition

$$M = \bigoplus_{v_1=1}^{m(1)} M(v_1) = \bigoplus_{I(1)} M(v)$$

where $I(1)$ is the set of numbers $1 \leq v_1 \leq m(1)$. Assume we have constructed already an $\mathcal{X}(t)$-decomposition $M = \bigoplus_v M(v_1, \ldots, v_t)$, then we take for any module $M(v_1, \ldots, v_t)$ an $\{x(t+1)\}$-decomposition

$$M(v_1, \ldots, v_t) = \bigoplus_{v_{t+1}=1}^{m(v_1,\ldots,v_t)} M(v_1, \ldots, v_t, v_{t+1}).$$

Of course, since $M(v_1, \ldots, v_t)$ is $\mathcal{X}(t)$-indecomposable, all the modules $M(v_1, \ldots, v_t, v_{t+1})$ are $\mathcal{X}(t+1)$-indecomposable. Thus, we obtain in this way an $\mathcal{X}(t+1)$-decomposition

$$M = \bigoplus_{v \in I(t+1)} M(v).$$

with $I(t+1)$ the set of sequences $(v_1, \dots, v_{t+1})$ such that $(v_1, \dots, v_t)$ belongs to $I(t)$ and $1 \le v_{t+1} \le m(v_1, \dots, v_t)$.

Let I be the set of infinite sequences $v = (v_1, v_2, \dots)$ such that $(v_1, \dots, v_t)$ belongs to $I(t)$, for all t. For $v \in I$, we define the module $M(v)$ as $M(v) = \bigcap_t M(v_1, \dots, v_t)$. Note that the restriction of $M(v)$ to the full subquiver with vertices in $\mathcal{X}(t)$ is equal to the restriction of $M(v_1, \dots, v_t)$ to this subquiver. This shows that $M(v)$ is $\mathcal{X}(t)$-indecomposable, for all t and that $M = \bigoplus_I M(v)$. Since $M(v)$ is $\mathcal{X}(t)$-indecomposable for all t, we see that $M(v)$ is either zero or else indecomposable. If we denote by I' the set of indices v such that $M(v) \neq 0$, then $M = \bigoplus_{I'} M(v)$ is a direct decomposition with indecomposable direct summands.

It remains to stress that multiplicities have to be finite: If M, N are locally finite-dimensional kQ-modules and $N_x \neq 0$ for some vertex x, then in any direct decomposition of M, the number of direct summands which are isomorphic to N is bounded by $\dim M_x$.

Remark. In the proof given above, we had to single out at the end the indices $v \in I$ with $M(v) = 0$, thus replacing the index set I by I'. One may wonder whether one can avoid this. Given a module M in $\operatorname{Mod} kQ$, we have used $\{x\}$-decompositions $M = \bigoplus M^{(i)}$. Of course, in case $M_x = 0$, we may require to use as decomposition just the trivial one $M = M$, and for $M_x \neq 0$, we may require that $M_x^{(i)} \neq 0$ for all i. In general, this will reduce the size of I, but still some of the summands $M(v)$ with $v \in I$ may be zero.

As an example, consider the quiver of type $\mathbb{A}_\infty$ with arrows from right to left, and the following representation M:

$$k^2 \xleftarrow{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} k^2 \xleftarrow{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}} k^2 \leftarrow \cdots$$

In the first decomposition $M = M(1) \oplus M(2)$, we may assume that one of the direct summands, say $M(1)$, is simple projective. In the second step, we can decompose $M(2) = M(2,1) \oplus M(2,2)$ with $M(2,1)$ of length 2, then, in the next step, $M(2,2) = M(2,2,1) \oplus M(2,2,2)$ with $M(2,2,1)$ of length 2, and so on. In this way, we obtain as index set I' the set of sequences of the form $(2, \dots, 2, 1, 1, \dots)$ starting with $s \ge 0$ entries equal to 2, all others equal to 1, with $M(1,1,\dots)$ the simple projective module, all other modules $M(2,\dots,2,1,1,\dots)$ of length 2. But the set I contains in addition the constant sequence $(2,2,\dots)$ with $M(2,2,\dots) = 0$.

6.2. Automorphism of a quiver Q which operate freely on Q_0.

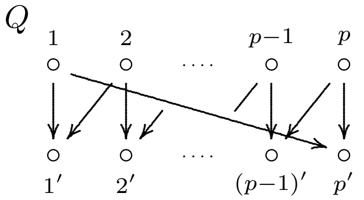
Let Q be a quiver and g an automorphism of Q. We say that g *operates freely* on Q_0 provided given a vertex x of Q and a natural number s with $g^s x = x$, we have $g^s = 1$. Of course, then G acts also freely on the arrow set Q_1 (namely, if $\alpha \colon x \to y$ is an arrow with $g(\alpha) = \alpha$, then $g(x) = x$, thus $g = 1$).

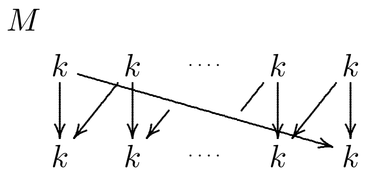
If M is a representation of Q, there is a representation M^g defined as follows: $(M^g)_x = M_{gx}$ for any vertex x, and $(M^g)_\alpha = M_{g\alpha}$ for any arrow α.

Lemma. *Assume that g is an automorphism of Q which acts freely on Q_0 and has infinite order. If $M \neq 0$ is a finite-dimensional representation of Q, then M^g is not isomorphic to M.*

Proof. Let x be a vertex of Q. Since g acts freely on Q_0 and has infinite order, the elements $x, gx, g^2x, \dots$ are pairwise different. Let $M \neq 0$ be a finite-dimensional representation of Q. Let x be a vertex of Q with $M_x \neq 0$. If M^g is isomorphic to M, then $\dim(M^g)_y = \dim M_y$ for all vertices y. Thus $\dim M_{gx} = \dim(M^g)_x = \dim M_x \neq 0$. By induction, it follows that $M_{g^t x} \neq 0$ for all $t \geq 0$. But then M has infinite support (since the set $x, gx, g^2x, \dots$ is infinite). This contradicts the fact that M is finite-dimensional.

Example. Let us present a typical example of a quiver Q with an automorphism g of finite order $p \geq 2$ acting freely on Q_0 and an indecomposable representation M of Q such that $M^g = M$





Here, M is defined by $M_x = k$ for all vertices x and $M_\alpha = 1_k$ for all arrows α. The automorphism g of Q is defined by $g(i) = i{+}1$ and $g(i') = (i{+}1)'$ (modulo p). Note that $q_Q(\mathbf{dim}\, M) = 0$, thus $\mathbf{dim}\, M$ is not a real root. This has to be the case, as the following proposition shows.

Proposition. *Let Q be a locally finite quiver and $g \neq 1$ an automorphism of Q which acts freely on Q_0. If $M \neq 0$ is a finite-dimensional indecomposable representation of Q with $q(\mathbf{dim}\, M) = 1$, then M^g is not isomorphic to M.*

Proof. If q has infinite order, then use the Lemma. Thus, we assume that g has finite order $p \geq 2$. Let Q_0' be a set of representatives of the various g-orbits in Q_0, and Q_1' a set of representatives of the various g-orbits in Q_1. Let $\mathbf{d} = \mathbf{dim}\, M$. Since M is isomorphic to M^g, we have $\mathbf{d}_{g(x)} = \mathbf{d}_x$ for any vertex x. If α is an arrow, then $t(g(\alpha)) = g(t(\alpha))$ and $t(h(\alpha)) = g(h(\alpha))$, thus also $\mathbf{d}_{t(g(\alpha))} = \mathbf{d}_{t(\alpha)}$ and $\mathbf{d}_{h(g(\alpha))} = \mathbf{d}_{h(\alpha)}$. It follows that

$$q(\mathbf{d}) = \sum_{x \in Q_0} \mathbf{d}_x^2 - \sum_{\alpha \in Q_1} \mathbf{d}_{t(\alpha)} \mathbf{d}_{h(\alpha)} = p\Big(\sum_{x \in Q_0'} \mathbf{d}_x^2 - \sum_{\alpha \in Q_1'} \mathbf{d}_{t(\alpha)} \mathbf{d}_{h(\alpha)} \Big)$$

is divisible by p. Since $p \geq 2$, this contradicts the assumption that $q(\mathbf{dim}\, M) = 1$.

6.3. Groups operating freely on a quiver.

Assume now that there is given an automorphism group G of a quiver Q which operates freely on Q_0. We denote by Q/G the orbit quiver: if x is a vertex of Q, let $\pi(x)$ be the G-orbit of x, if $\alpha\colon x \to y$ is an arrow, let $\pi(\alpha)\colon \pi(x) \to \pi(y)$ be the G-orbit of α, thus $\pi\colon Q \to Q/G$ is a morphism of quivers.

Our interest concerns the functor $\pi_\lambda\colon \operatorname{mod} kQ \to \operatorname{mod} k(Q/G)$ which is defined as follows: If N is a finite-dimensional representation of Q, then

$$(\pi_\lambda N)_z = \bigoplus_{x \in \pi^{-1}(z)} N_x, \quad (\pi_\lambda N)_\gamma = \bigoplus_{\alpha \in \pi^{-1}(\gamma)} N_\alpha,$$

for all vertices z and all arrows γ of Q; this functor π_λ is usually called the *push-down functor.*

> Any kQ-module N may be considered as a G-graded $k(Q/G)$-module so that $\pi_\lambda N$ is the corresponding $k(Q/G)$-module obtained from M by forgetting the grading. Thus, such a push-down functor is just a **forgetful functor**. Parallel to the development of covering theory by Gabriel, Bongartz and Riedtmann, the (equivalent) theory of dealing with group-graded algebras and the corresponding graded modules was developed by Gordan and Green.

Theorem. *Let Q be a locally finite quiver and G a group of automorphisms of Q which acts freely on Q_0. Let N be a finite-dimensional indecomposable representation of Q such that the representations N^g for $g \in G$ are pairwise non-isomorphic. Then $\pi_\lambda N$ is an indecomposable representation of Q/G. If N' is a finite-dimensional indecomposable representation of Q such that the representations $\pi_\lambda N$ and $\pi_\lambda N'$ are isomorphic, then there is $g \in G$ such that N' is isomorphic to N^g.*

Proof. We need a further functor, namely $\pi_i \colon \operatorname{mod} kQ \to \operatorname{Mod} kQ$ which is defined as follows: If M is a finite-dimensional $k(Q/G)$-module, then

$$(\pi_. M)_x = M_{\pi(x)}, \quad (\pi_. M)_\alpha = M_{\pi(\alpha)},$$

for all vertices x and all arrows α of Q; this functor is called the *pull-up functor* (note that by definition $\pi_. M$ is locally finite-dimensional, thus in $\operatorname{Mod} kQ$). Of course, for any element $g \in G$, we have $(\pi_. M)^g = \pi_. M$.

Given a finite-dimensional kQ-module N, let us consider $\pi_. \pi_\lambda N$. For x a vertex of Q, we have

$$(\pi_. \pi_\lambda N)_x = (\pi_\lambda N)_{\pi(x)} = \bigoplus_{y \in \pi^{-1}\pi(x)} N_y.$$

The set $\pi^{-1}\pi(x)$ is by definition the G-orbit of x, thus

$$(\pi_. \pi_\lambda N)_x = \bigoplus\nolimits_{g\in G} N_{g(x)} = \bigoplus\nolimits_{g\in G} N^g_x = \Big(\bigoplus\nolimits_{g\in G} N^g\Big)_x.$$

It follows that

$$\pi_. \pi_\lambda N = \bigoplus\nolimits_{g\in G} N^g.$$

Since by assumption the modules N^g are pairwise non-isomorphic (and indecomposable), we have obtained in this way a direct decomposition of $\pi_. \pi_\lambda N$ using pairwise non-isomorphic modules with local endomorphism rings.

Let us show now that π_λ is indecomposable. Thus, assume there is given a direct decomposition $\pi_\lambda = M \oplus M'$ of $k(Q/G)$-modules. Then

$$\bigoplus\nolimits_{g\in G} N^g = \pi_. \pi_\lambda N = \pi_. M \oplus \pi_. M'.$$

According to 6.1 and the Krull-Remak-Schmidt-Azumaya theorem, there is a subset $H \subseteq G$ such that $\pi_. M$ is isomorphic to $\bigoplus_{h\in H} N^h$. If $M \neq 0$, then H is not empty. Since

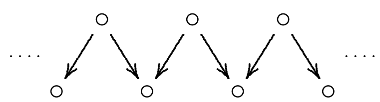
$(\pi_. M)^g = \pi_. M$ for all $g \in G$, it follows that $\pi_. M$ is also isomorphic to $\bigoplus_{h\in H} N^{hg}$, for all $g \in G$, thus there is a direct decomposition of $\pi_. M$ into indecomposable modules such that one of the summands is isomorphic to N. Assume now that both M, M' are non-zero. Then there is a direct decomposition of $\pi_. M \oplus \pi_. M' = \pi_. \pi_\lambda N$ into indecomposable modules such that two of the summands are isomorphic to N. This contradicts the theorem of Krull-Remak-Schmidt-Azuyama.

Finally, assume that N, N' are finite-dimensional indecomposable representation of Q such that the representations $\pi_\lambda N$ and $\pi_\lambda N'$ are isomorphic. Then $\pi_\lambda N$ is isomorphic to $\bigoplus_{g\in G} N^g$ as well as to $\bigoplus_{g\in G} (N')^g$. Again using the assumption that the modules N^g are pairwise non-isomorphic (and indecomposable) and that N' is indecomposable, the theorem of Krull-Remak-Schmidt-Azumaya implies that N' is isomorphic to some N^g. This completes the proof.

Corollary. *Let Q be a locally finite quiver and G a torsionfree group of automorphisms of Q which acts freely on Q_0. Then π_λ provides an injective map from the set of G-orbits of isomorphism classes of indecomposable kQ-modules to the set of isomorphism classes in indecomposable $k(Q/G)$-modules.*

Proof. This is a direct consequence of the theorem using the lemma in 6.2.

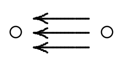
Remark. The map given by π_λ is injective, but usually not surjective. A typical example is provided the quiver Q of type $\mathbb{A}_\infty$ with bipartite orientation



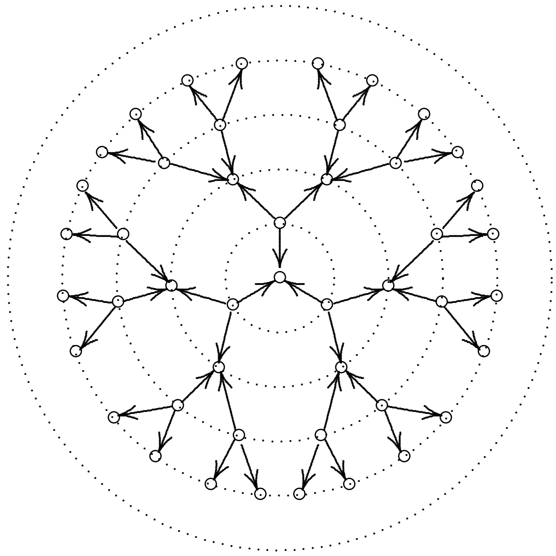
and the shift automorphism g so that there are precisely two g-orbits, the sources and the sinks. Then Q/G is the Kronecker quiver. Obviously, the indecomposable representation M of Q/G with $M_x = k$ for both vertices and $M_\alpha = 1_k$ for both arrows is not isomorphic to a representation of the form $\pi_\lambda N$.

6.4. Representations of the 3-Kronecker quiver.

In order to study the 3-Kronecker quiver



we may start with its universal cover Q, this is the 3-regular tree with bipartite orientation (mentioned already in the second lecture):



The free group G with two generators operates freely on Q such that Q/G is just the 3-Kronecker quiver. Since G is torsionfree, we can use the last corollary in order to see that any finite-dimensional indecomposable representation N of Q yields a finite-dimensional indecomposable representation $\pi_\lambda N$ of Q/G and that given finite-dimensional indecomposable representations N, N' of Q, the representations $\pi_\lambda N$ and $\pi_\lambda N'$ are isomorphic if and only if N and N' belong to the same G-orbit.

One obtains in this way many different representations of Q/G, for example all the indecomposable preprojective or preinjective representations and this reveals some of the internal structure of these representations.

In the case of a finite directed quiver Q'', Coxeter functors $C^-_{Q''}$ and $C^+_{Q''}$ have been introduced above. In order to use such functors also for our quiver Q, we proceed as follows: let N be a finite-dimensional representation of Q, let Q'_0 be a finite subset of Q_0 containing all vertices x with $N_x \neq 0$ and let Q'' be the full subquiver of Q whose vertices y are those elements of Q_0 with a path of length at most 2 in $\overline{Q}$ starting in Q'_0 and ending in y. Then Q'' is a finite directed quiver, thus the Coxeter functors $C^-_{Q''}$ and $C^+_{Q''}$ do exist and it is not difficult to see that the modules $C^-_{Q''}(N)$ and $C^+_{Q''}(N)$ do not depend on the choice of Q'_0, thus we just write $C^-(N)$ and $C^+(N)$, respectively.

Also it is easy to see that for any finite-dimensional representation N of kQ, we have $C^-(\pi_\lambda N) = \pi_\lambda C^- N$ and $C^+(\pi_\lambda N) = \pi_\lambda C^+ N$. At the end of this section, we will exhibit the dimension vectors first of a simple projective kQ-module S, then those of C^-S and $(C^-)^2S$. Under the push-down functor π_λ, the $k(Q/G)$-modules $\pi_\lambda S$, $\pi_\lambda C^-S$ and $\pi_\lambda(C^-)^2S$ are the indecomposable preprojective $k(Q/G)$-modules with dimension vectors $(1,0)$, $(8,3)$, $(55,21)$, respectively.

As an example, let us draw the attention to the kQ-module $(C^-)^2S$. The vector spaces of dimensions 55 and 21 used in the $k(Q/G)$-module $\pi_\lambda(C^-)^2S$ are decomposed in $(C^-)^2S$ into quite small subspaces: the 21-dimensional space into three subspaces of dimension 3 and 12 subspaces of dimension 1, the 55-dimensional space into one subspace of dimension 7, six subspaces of dimension 4 and 24 subspaces of dimension 1.

Recall that the numbers 21, 55, and, more generally, the dimensions of the vector spaces used in any preprojective $k(Q/G)$-modules are even index Fibonacci numbers. Thus, the covering theory provides partition formulas for the even index (and also for the odd index) Fibonacci numbers, see [Fahr-Ringel].

S

$C^- S$

$(C^-)^2 S$