The representation theory of Dynkin quivers and the Freudenthal-Tits magic square.

Theorem. *If Q is a quiver of Dynkin type*

$$\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8,$$

then Q is representation-finite. Any connected quiver which is not of Dynkin type has a subquiver $\widetilde{Q}$ of Euclidean type

$$\widetilde{\mathbb{A}}_n, \widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8,$$

and thus is representation-infinite.

If Q is of Dynkin type, then *there exists a maximal indecomposable representation M.*

If $\widetilde{Q}$ is of Euclidean type, *there are countably many 1-parameter families of indecomposable reps.*

(Ref: Gabriel, or better: Yoshii-Bäckström-Gabriel-Kleiner. And Donovan-Freislich, Nazarova for the structure of $\operatorname{rep}\widetilde{Q}$.)

The types $\mathbb{A}_n, \mathbb{D}_n$, and $\widetilde{\mathbb{A}}_n, \widetilde{\mathbb{D}}_n$ are easy to handle.

Our aim: To discuss the exceptional cases

$$\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8 \quad \text{and} \quad \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8.$$

To be more precise, this is **our aim:**

For $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$: To construct the max. indec. rep M.

For $\widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8$: To construct a representation in each τ-orbit of simple regular representations
(in particular, to construct a primitive one-parameter family of indecomposable representations).

The structure of rep $\widetilde{Q}$ is well-known:

There are the preinjective and the preinjective representations, they are obtained from the projective or, respectively, the injective representations using the Auslander-Reiten translation τ.

In addition, there are the regular representations. This is an abelian exact subcategory $\mathcal{R}$ which is serial. The (relative) simple objects in $\mathcal{R}$ are called the simple regular representations: these are the representations which one needs to know.

It is sufficient to to construct at least one element in each τ-orbit of simple regular representations.

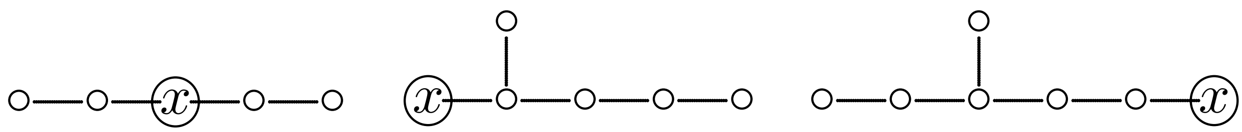
(Note that all the τ-orbit of simple regular representations are periodic, and the period is 1 for all but at most three τ-orbits.)

(1) Special triples.

Let Q be a connected quiver and x a fixed vertex. A *special triple for* (Q, x) is a set of representations $A(1), A(2), A(3)$ of Q which are Hom-orthogonal, indecomposable, and $A(i)_x \neq 0$ for $1 \leq i \leq 3$.

Theorem. *There are precisely three cases* (Q, x) *with a unique special triple, namely*

$\mathbb{A}_5$ $\mathbb{D}_6$ $\mathbb{E}_7$



Example: The special triple for $\mathbb{A}_5$ with linear orientation (left), with the center the unique sink (right):

$$11111,\ 01110,\ 00100 \qquad 11100,\ 01110,\ 00111.$$

For $\mathbb{E}_7$ with subspace orientation:

$$\begin{smallmatrix} & & 1 & & & \\ 1 & 1 & 2 & 2 & 1 & 1 \end{smallmatrix}, \quad \begin{smallmatrix} & & 2 & & & \\ 1 & 2 & 3 & 2 & 2 & 1 \end{smallmatrix}, \quad \begin{smallmatrix} & & 0 & & & \\ 0 & 1 & 1 & 1 & 1 & 1 \end{smallmatrix}.$$

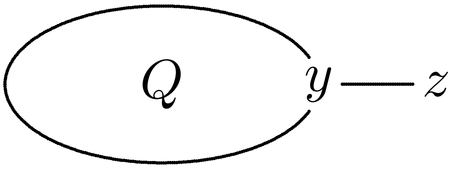
Remark: The important word is **unique**. Usually (for example if Q is of infinite representation type), there will be **many** special triples. For very few pairs (Q, x), there is **no** special triple.

Two steps of the proof.

(a) The hammock for x has to be thin. Otherwise, there is a first indecomposable module M with $\dim M_x = 2$. Its direct predecessors are a special triple, its direct successors are a special triple. Contradiction.

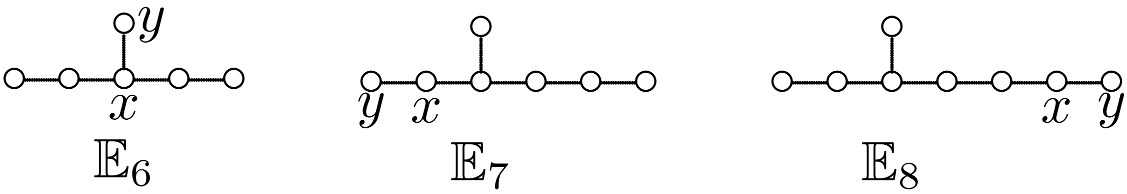
(b) Thus x is a vertex of $\mathbb{A}_n$, a branch vertex of $\mathbb{D}_n$, a branch vertex of $\mathbb{E}_6$ of rank 3, or the branch vertex of $\mathbb{E}_7$ of rank 4.

Recall that the one-point extension of a quiver Q at a vertex y is of the form

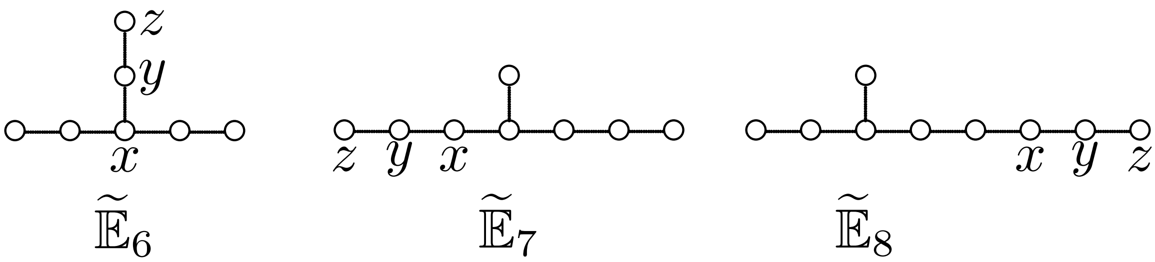


the vertex z is called the extension vertex.

The three cases of our Theorem yield as one-point extensions of Δ at x quivers of the following form:



A further one-point extension now at y yields:



Altogether, there is the following extension scheme:

$$\begin{array}{ccccc} (\Delta, x) & \mapsto & Q & \mapsto & \widetilde{Q} \\ & & & & \\ \mathbb{A}_5 & \mapsto & \mathbb{E}_6 & \mapsto & \widetilde{\mathbb{E}}_6 \\ \mathbb{D}_6 & \mapsto & \mathbb{E}_7 & \mapsto & \widetilde{\mathbb{E}}_7 \\ \mathbb{E}_7 & \mapsto & \mathbb{E}_8 & \mapsto & \widetilde{\mathbb{E}}_8 \end{array}$$

Thick subcategories. Given a quiver Q and pairwise orthogonal representations $A(1), \dots, A(m)$ with endomorphism ring k.
Let $\mathcal{E}(A(1), \dots, A(m))$ be the extension closure and $\Gamma(A(1), \dots, A(m))$ the Ext-quiver of $A(1), \dots, A(m)$.
It has vertices $1,\dots,m$;
the number of arrows $i \to j$ is $\dim \operatorname{Ext}^1(A(i), A(j)))$.

Then:

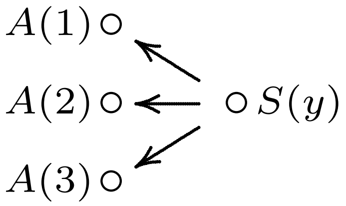
$$\mathcal{E}(A(1), \dots, A(n)) \ \simeq\ \operatorname{rep} \Gamma(A(1), \dots, A(n))$$

Example 1. $A(1), A(2), A(3)$ a special triple. Then

$$\circ \qquad \circ \qquad \circ \qquad\qquad \Gamma = \Gamma(A(1), A(2), A(3)).$$

$\operatorname{rep} \Gamma$ is a semisimple category with three simples.

Example 2. The Q-reps $A(1), A(2), A(3), S(y)$:



$$\Gamma = \Gamma(A(1), A(2), A(3), S(y))$$

The quiver Γ is of type $\mathbb{D}_4$.

Γ has a unique maximal indecomposable representation, its dim vector is $\begin{matrix} 1 & \\ 1 & 2 \\ 1 & \end{matrix}$.

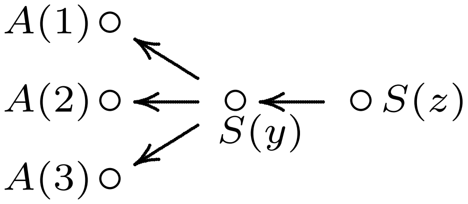
This corresponds in $\mathcal{E}(A(1), A(2), A(3), S(y))$ to an indecomposable module M with

$$M|\Delta = A(1) \oplus A(2) \oplus A(3) \quad \text{and} \quad \dim M_y = 2.$$

In this way, we obtain as M the maximal Q-module.

Let us stress a special feature of $\mathbb{D}_4$: It is the only Dynkin diagram with an automorphism of order 3.

Example 3. The $\widetilde{Q}$-reps $A(1), A(2), A(3), S(y), S(z)$:



$\Gamma = \Gamma(A(1), A(2), A(3), S(y), S(y))$

This is an extended Dynkin quiver of type $\widetilde{D}_4$. It has a one-parameter family with dimension vector $\begin{smallmatrix} 1 & & \\ 1 & 2 & 1 \\ 1 & & \end{smallmatrix}$.

This corresponds in $\mathcal{E}(A(1), A(2), A(3), S(y)), S(z)$ to a 1-parameter family of indecomposable modules $M(\lambda)$ with

$$M(\lambda)|\Delta = A(1) \oplus A(2) \oplus A(3)$$
$$\text{and} \quad \dim M(\lambda)_y = 2, \ \dim M(\lambda)_z = 1.$$

Similarly for all the multiples of $\begin{smallmatrix} 1 & & \\ 1 & 2 & 1 \\ 1 & & \end{smallmatrix}$.

In this way, we obtain the 1-parameter families of $\widetilde{Q}$-modules.

And the representations $\overline{A(i)}$ with

$$\overline{A(i)}|\Delta = A(i) \quad \text{and} \quad \dim \overline{A(i)}_y = 1, \ \overline{A(i)}_z = 0$$

are simple regular reps in the three regular τ-orbits.

Since $\overline{A(i)}_z = 0$, the representations $\overline{A(i)}$ are representations of Q. Let us add some remarks about the relationship between M and the representations $A(i)$ and $\overline{A(i)}$.

First of all, there is an exact sequence of the form

$$0 \to A(1) \oplus A(2) \oplus A(3) \to M \to S(y)^2 \to 0.$$

It follows, that $\overline{A(i)}$ is a factor module of M (for example, $\overline{A(1)} \simeq M/(A(2) \oplus A(3))$).

In this way, we obtain an exact sequence

$$0 \to M \to \overline{A(1)} \oplus \overline{A(2)} \oplus \overline{A(2)} \to S(y) \to 0.$$

The projection map $M \to \overline{A(i)}$ is the composition of the irreducible maps in a sectional path. Note that M is a wing module (as defined in SLNM 1099), and the representations $\overline{A(1)}, \overline{A(2)}, \overline{A(3)}$ belong to the boundary of the Auslander-Reiten quiver of kQ.

Deleting x, we deal with $\mathbb{A}_2 \sqcup \mathbb{A}_2,\ \mathbb{A}_5,\ \mathbb{D}_6.$

One may construct the special triple $A(1)$,$A(2)$,$A(3)$ starting with the representations $A(i)|\Delta\backslash\{x\}$, see [R].

Starting with $\Delta\backslash\{x\}$ the one-point extension scheme of our Dynkin diagrams looks as follows:

$$\begin{array}{ccccc} \Delta \setminus \{x\} & & \Delta & & Q \\ \\ \mathbb{A}_2 \sqcup \mathbb{A}_2 & \mapsto & \mathbb{A}_5 & \mapsto & \mathbb{E}_6 \\ \mathbb{A}_5 & \mapsto & \mathbb{D}_6 & \mapsto & \mathbb{E}_7 \\ \mathbb{E}_6 & \mapsto & \mathbb{E}_7 & \mapsto & \mathbb{E}_8 \end{array}$$

We discuss this arrangement of Dynkin types later (this is part of the Freudenthal-Tits magic square).

(2) The numbers $t = 2, 4, 8$.

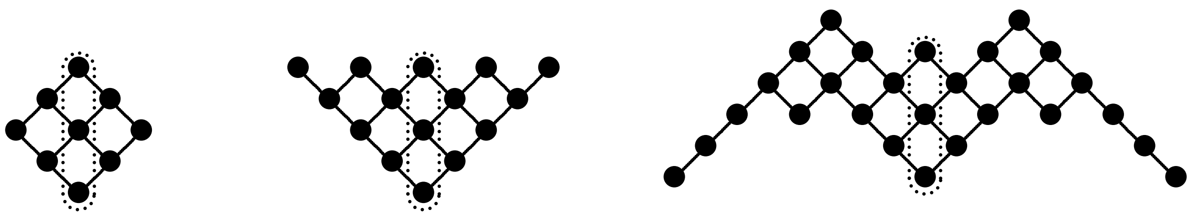
(2.1) For $\Delta = \mathbb{A}_5, \mathbb{D}_6, \mathbb{E}_7$, we have

$$P(x) = \tau^t I(x) \quad \text{with} \quad t = \begin{cases} 2 \\ 4 \\ 8 \end{cases}.$$

(2.2) The number of indecomposable Δ-modules N with $N_x \neq 0$ is

$$3(t+1) = \begin{cases} 9 \\ 15 \\ 27 \end{cases}.$$

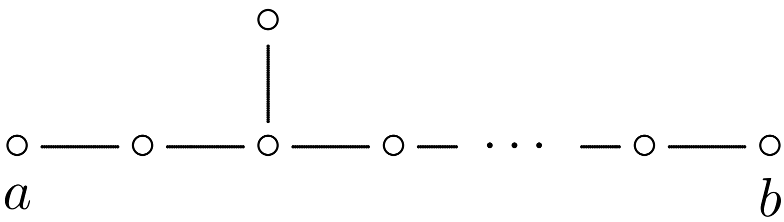
These Δ-modules can be arranged as follows (non-zero maps go from left to right):



$(\mathbb{A}_5, x)$ $(\mathbb{D}_6, x)$ $(\mathbb{E}_7, x)$

Always, the special triple is seen in the middle.

(2.3) Consider $Q = \mathbb{E}_6,\ \mathbb{E}_7,\ \mathbb{E}_8$



$$|\{M \mid \text{indec.}, \dim M_a = 1, \dim M_b = 0\}| = 4t,$$
$$|\{M \mid \text{indec.}, \dim M_a = 1, \dim M_b = 1\}| = 4t.$$

A famous theorem of Hopf [11], Bott and Milnor [4] and Kervaire [12] asserts:

There is a t-dimensional real division algebra with $t \geq 2$ *iff*

$$t = \ 2,\ 4,\ 8.$$

Hurwitz (1898/1923): *the only real normed division algebras different from* $\mathbb{R}$ *are*

$$\mathbb{C},\ \mathbb{H},\ \mathbb{O}$$

The quaternions $\mathbb{H}$ were found by Hamilton in 1843. In the same year, Graves discovered the octonions, two years later also Cayley. The octonions $\mathbb{O}$ form an 8-dimensional real vector spaces which is a division algebra, but the multiplication is not associative.

We have exhibited already several $2-4-8$ assertions in the representations theory of $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$.

Question. Is there a relationship between the module categories of the exceptional quivers

$$\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$$

and the division algebras

$$\mathbb{C}, \mathbb{H}, \mathbb{O}.$$

(3) The Freudenthal-Tits magic square.

Recall: The one-point extension scheme looks as follows:

$$\begin{array}{ccccc} \mathbb{A}_2^2 & \mapsto & \mathbb{A}_5 & \mapsto & \mathbb{E}_6 \\ \mathbb{A}_5 & \mapsto & \mathbb{D}_6 & \mapsto & \mathbb{E}_7 \\ \mathbb{E}_6 & \mapsto & \mathbb{E}_7 & \mapsto & \mathbb{E}_8 \end{array}$$

Note:

— It is a symmetric matrix arrangement.
— It is the right lower corner of the Freudenthal-Tits magic square

Here is the Freudenthal-Tits magic square: it concerns certain Dynkin types:

$$\begin{array}{cccc} \mathbb{A}_1 & \mathbb{A}_2 & \mathbb{C}_3 & \mathbb{F}_4 \\ \mathbb{A}_2 & \mathbb{A}_2^2 & \mathbb{A}_5 & \mathbb{E}_6 \\ \mathbb{C}_3 & \mathbb{A}_5 & \mathbb{D}_6 & \mathbb{E}_7 \\ \mathbb{F}_4 & \mathbb{E}_6 & \mathbb{E}_7 & \mathbb{E}_8 \end{array}$$

The magic square is naturally divided into its right lower (3×3)-corner and the upper left boundary: in the right lower corner, there are only simply laced diagrams, and the rank increase is always 1.

This is the part we are interested in.

The entries should be indexed by the pairs (A, B) where A, B are the division rings $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. Better: index the columns using $J(B) = J_3(B)$, the (Jordan) algebra of Hermitian (3×3)-matrices over B:

	$J(\mathbb{R})$	$J(\mathbb{C})$	$J(\mathbb{H})$	$J(\mathbb{O})$
$\mathbb{R}$	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{C}_3$	$\mathbb{F}_4$
$\mathbb{C}$	$\mathbb{A}_2$	$\mathbb{A}_2^2$	$\mathbb{A}_5$	$\mathbb{E}_6$
$\mathbb{H}$	$\mathbb{C}_3$	$\mathbb{A}_5$	$\mathbb{D}_6$	$\mathbb{E}_7$
$\mathbb{O}$	$\mathbb{F}_4$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$

(Hermitian means: the transpose is the conjugate matrix.)

Due to Freudenthal and Tits, later also Vinberg and others, there is a well-defined construction

$$L(A, B) = \mathrm{Der}(A) \;\oplus\; A_0 \otimes J(B)_0 \;\oplus\; \mathrm{Der}(J(B))$$

which yields the Lie algebra in question.
(Here, A_0 are the purely imaginary elements in A, and $J(B)_0$ the trace zero matrices).

Slogan. *The exceptional Lie algebras only exist since the octonions exist.*

The exceptional Lie groups $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$ may be considered as the isometry groups of something like the projective planes over $\mathbb{C}, \mathbb{H}, \mathbb{O}$ (see e.g. J. Baez).

If B is of dimension $t = 1, 2, 4, 8$, then $J(B)$ is of dimension

$$3(t+1) = 6,\ 9,\ 15,\ 27.$$

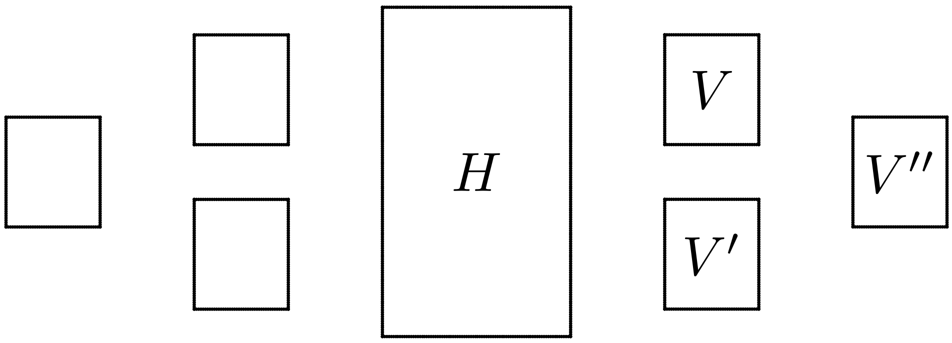
(the diagonal entries are real, this gives the dimension 3, the upper triangular entries are arbitrary and determine uniquely the lower ones, this gives $3t$).

Recall that the numbers $9,\ 15,\ 27$ have appeared in (2.2).

Rubenthaler (TAMS 2008) has shown how to recover the octonions inside the Lie algebra $\mathbf{g} = \mathbf{g}(\mathbb{E}_6)$.

Let $\mathbf{h}$ be a Cartan subalgebra and $\Phi \subset \mathbb{R}^6$ the root system. Let $\Phi(a, b)$ be the set of roots of the form $a{-}*{-}\overset{*}{\overset{|}{*}}{-}*{-}b$ and $\mathbf{g}(a, b) = \bigoplus_{\alpha \in \Phi(a,b)} \mathbf{g}_\alpha$.

Then $\mathbf{g}$ decomposes as follows:



where $H = \mathbf{h} + \mathbf{g}(0, 0)$, and

$$V = \mathbf{g}(1, 0),\ V' = \mathbf{g}(0, 1),\ V'' = \mathbf{g}(1, 1).$$

The Lie multiplication is a bilinear pairing

$$[-, -] \colon V \otimes V' \to V''.$$

Note that this bilinear pairing is non-degenerate!

Instead of looking at this bilinear form, we also look a the corresponding "triality":

$$V \otimes V' \otimes (V'')^* \to k.$$

Rubenthaler shows that there are natural identifications $V \simeq V'$ and $V \simeq V''$ (again using just Lie products, or Weyl group elements) such that the bilinear pairing

$$[-,-]\colon V \otimes V' \to V''.$$

yields a bilinear pairing

$$-\bullet-\colon V \otimes V \to V$$

which is precisely the octonion multiplication.

Actually, Rubenthaler exhibits explicit root vectors a, b, c in $\mathbf{g}$ with the following property:

If one defines for $v_1, v_2 \in V$

$$v_1 \bullet v_2 = [a, [v_1, [b, [c, v_2]]]],$$

then

$$-\bullet-\colon V \otimes V \to V$$

is the octonion multiplication.

Question: Is it possible to use the Hall algebra construction for $\mathbb{E}_6$, in order to obtain a quantized version of the octonions?

A baby model for the sequence $t = 1, 2, 4, 8, \infty$.

Let t be the number of indecomposable representations M of the m-subspace quiver Q with odd dimensional total space M_0:

m	0	1	2	3	4
Q	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{D}_4$	$\widetilde{\mathbb{D}}_4$
t	1	2	4	8	∞

Detailed information:

m	0	1	2	3	4
Q	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{A}_3$	$\mathbb{D}_4$	$\widetilde{\mathbb{D}}_4$
$\dim M_0 = 1$	1	2	4	8	16
$\dim M_0 = 2$	0	0	0	1	∞
$\dim M_0 = 3$	0	0	0	0	16
$\dim M_0 = 3$	0	0	0	0	∞

(for $\widetilde{\mathbb{D}}_4$, there are 16 indecomposables with $\dim M_0 = i$ if i is odd and infinitely many if i is even).

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