Dieter Vossieck and the Development of the Representation Theory of Artin Algebras

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This is a pre-dinner-lecture, as part of the Bielefeld workshop *Discrete Categories in Representation Theory*, April 20 – 21, 2018. The workshop discusses the possible discreteness of additive categories: starting point is the celebrated paper *The algebras with discrete derived category* [9] by Dieter Vossieck. But there is another (more secrete) aim for this workshop: to celebrate the 60th birthday of Dieter Vossieck — he was born in September 1957. In this lecture, I want to give a short report on some of his contributions to representation theory, concentrating on his early Bielefeld years 1978 – 1986. Dieter Vossieck may be considered as one of my best PhD students, better: he is it, but unfortunately, he never handed in what was supposed to be his Bielefeld PhD thesis (see the comments below). Dieter provided a lot of important contributions to representation theory — actually there are only few publications, but his influence is much wider. The aim of my lecture is to draw the attention not only to his publications, but also to some of his ideas which he presented in lectures and in discussions and which are not available otherwise. He always was a perfectionist, so he refused to put forward any incomplete or unpolished result.

Dieter Vossieck studied at Bielefeld University, starting in 1978, he stayed here for eight years, then he went to Zürich, this was in 1986. He worked in Switzerland (in Zürich as well as in Basel) again for eight years. Then, from 1994 to 2008 he was in Mexico, at the UNAM in Mexico City and at the Universidad Michoacana des San Nicolás de Hidalgo in Morelia, interrupted for one year (2001 – 2002) by being professor at Beijing Normal University, in China. Since 2008 he is back in Bielefeld, partly as a private scholar, partly with some temporary university contracts.

1. The Happel-Vossieck list.

The first topic to be mentioned concerns the joint paper of Happel and Vossieck entitled *Minimal algebras of infinite representation type with preprojective component* [1]; it presents what is now the famous Happel-Vossieck list, namely the list of the frames of the tame concealed algebras.

Throughout the lecture, I will assume that we work over an algebraically closed field k. A *tame concealed* algebra is the endomorphism ring $A = \mathrm{End}({}_HT)$ of a preprojective tilting H-module T, where H is a (connected) tame hereditary algebra. It turns out that if A is tame concealed, then also the opposite algebra A^{op} is tame concealed.

Let us assume that H is a tame hereditary algebra and T a preprojective tilting H-module with endomorphism ring A.

If the quiver of H is a cycle with $p+q$ arrows, where p arrows point in one direction and $q \le p$ in the opposite direction, then H will be said to be of *type* $\widetilde{\mathbb{A}}_{pq}$. In this case, T has to be a slice module, thus A is hereditary again, and of the same type $\widetilde{\mathbb{A}}_{pq}$. If the quiver of H is not a cycle, then its underlying graph is one of the extended Dynkin diagrams $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$ and this is called the *type* of A. If H is of type $\widetilde{\mathbb{D}}_n$, then, up to change of orientation of arrows, there are at most four possibilities for the quiver of A. The most interesting algebras are those of type $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$. There are 56, 437, and 3 801 different isomorphism classes of tame concealed algebras of type $\widetilde{\mathbb{E}}_6$ $\widetilde{\mathbb{E}}_7$, and $\widetilde{\mathbb{E}}_8$, respectively. Now 3 801 different classes may seem to be difficult to overlook, but it is possible to reduce the numbers considerably by looking at *frames*. For $\widetilde{\mathbb{E}}_8$, there are 117 possible frames, 29 for $\widetilde{\mathbb{E}}_7$ and 5 for $\widetilde{\mathbb{E}}_6$, and it is easy to obtain the corresponding isomorphism classes up to duality by just replacing the arms by corresponding branches.

What is the relevance of this classification? The tame concealed algebras are important minimal representation-infinite algebras, where *minimal representation-infinite* means that the algebra A itself is representation-infinite, whereas for any non-zero idempotent e, the factor algebra $A/\langle e\rangle$ is representation-finite. What Happel and Vossieck show is that *an algebra A with a preprojective Auslander-Reiten component is minimal representation-infinite if and only if it is either tame concealed or else a wild generalized Kronecker algebra* (a generalized Kronecker algebra is the path algebra of a directed quiver with two vertices and say n arrows; it is wild provided $n \geq 3$).

There are numerous applications: First of all, the Happel-Vossieck list is used by Bongartz in order to characterize the representation-finite algebras (for this reason, the list is sometimes called the BHV-list). Secondly, the famous multiplicative basis paper by Bautista-Gabriel-Roiter-Salmeron strongly relies on the list (thus the paper reprints the list, but using a different ordering of the frames), as do the various proofs of the second Brauer-Thrall conjecture. But the list plays a role also in other settings, for example in the context of cluster categories.

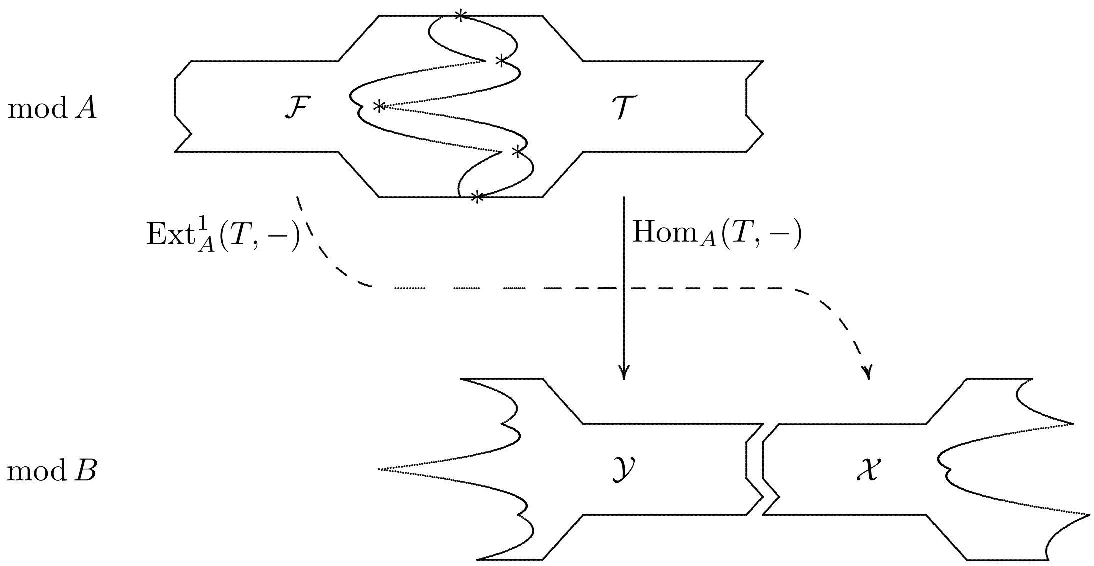
In 1984, Peter Gabriel gave a colloquium lecture at Bielefeld, devoted to the foundation of the representation theory of finite-dimensional algebras. At the end, he suggested that the Happel-Vossieck list has a similar relevance for representation theory as the list of the finite simple groups has for group theory. I should recall that at that time the classification of the finite simple groups was very popular — it was considered as one of the main achievements of the century; it was discussed even in newspapers. Gabriel's rating was felt as a real provocation at a mathematical faculty which was nicknamed outside of Bielefeld as a *Faculty of Group Theory* (since nearly all major mathematicians at Bielefeld were working on problems related to group theory, and at least one of them, Bernd Fischer, was strongly involved in the classification project).

The Happel-Vossieck classification was fully presented at a Luminy conference in September 1982 (and not *thereafter*, as formulated by Bongartz in *manuscripta mathematica* vol 46 (1984); actually there was a big discussion comparing the Bongartz and the Happel-Vossieck approach). It was published in 1983 and it is the center of Vossieck's Diplom thesis at Bielefeld. The thesis was handed in only in 1984, since Dieter was unhappy with a proof which relies on the use of a computer. Also, being a perfectionist, Dieter felt that the presentation of the frames should be based on some intrinsic partial ordering. He often came to my office showing me huge pieces of papers visualizing his attempts to outline such an ordering. Indeed, already a first glance at the various frames confirms the idea of a partial ordering of the tame concealed algebras of a fixed type, with the hereditary algebras as the maximal elements and the unique canonical algebra as the minimal element. Given a tame concealed algebra A, the Euler form q_A on the Grothendieck group of the module category is semidefinite with a positive radical generator $\mathbf{h}$ and the Happel-Vossieck list provides for each frame such a generator $\mathbf{h}$. The sum of the coefficients of $\mathbf{h}$ is an interesting invariant: it is maximal for A hereditary and minimal for the canonical algebra A. A similar invariant is what I want to call the *shortage* of A, it is the number of roots of q_A which are neither positive, nor negative: the shortage of A is zero iff A is hereditary, and is maximal iff A is the canonical algebra. Until now, no definite ordering of the tame concealed algebras has been described in detail; that is really a desideratum.

2. The foundation of tilting theory.

It seems that the first presentation of the Brenner-Butler tilting theory was given at a two day workshop in Bielefeld, in spring 1979, by Butler. His lecture was on the first day, and after this lecture, there was the immediate wish to see more details, thus a continuation was scheduled for the second day. Of course, special tilting functors were known at that time (first the Bernstein-Gelfand-Ponomarev reflection functors, then the APR-tilts, but also other constructions which were used for example by students of Gabriel). What was new and exciting was the axiomatic approach exhibited by Butler. It soon turned out that these axioms could be weakened considerably without changing the essential results (the Brenner-Butler tilting modules T had a projective direct summand which generates T).

What is called the Brenner-Butler theorem [2] is usually illustrated in the following picture. Here, we start with an A-module T which is *tilting*: it has projective dimension at most 1, no self-extension, and there is an exact sequence $0 \to {}_AA \to T' \to T'' \to 0$ with T', T'' in $\operatorname{add} T$. Let $B = \operatorname{End}(T)^{\mathrm{op}}$.



The class $\mathcal{T}$ is the class of A-modules generated by T and the class $\mathcal{Y}$ is the class of B-modules cogenerated by the dual $D(T)$ of T. What Brenner and Butler had observed, and this is the first step of tilting theory, is that the functor $\mathrm{Hom}_A(T,-)$ provides an equivalence between $\mathcal{T}$ and $\mathcal{Y}$ (the vertical arrow in the center of the picture).

But the tilting module T provides also a second equivalence, namely the equivalence (given by $\mathrm{Ext}^1_A(T,-)$ and shown by a bended dashed arrow) between the class $\mathcal{F}$ of all A-modules M with $\mathrm{Hom}(T,M) = 0$ and the class $\mathcal{X}$ of all B-modules N with $\mathrm{Hom}(N, D(T)) = 0$. After Butler's Bielefeld lecture, a lot of examples were studied in Bielefeld, and it took quite a while to realize that one always has the second equivalence. As I remember, the examples calculated by Dieter Vossieck were those which really were convincing and which paved the way to the final proof.

In this way, Dieter Vossieck was involved right from the beginning in the development of tilting theory. In the proceedings of ICRA III, Klaus Bongartz published an account of tilting theory which ended with the (slightly presumptuous) remark that *well-read mathematicians tend to understand the tilting theorem using spectral sequences.* Of course, this concerned the classical tilting modules of projective dimension at most 1. Thus, when the general notion of tilting modules with arbitrary finite projective dimension was introduced and studied by Miyashita, Happel, as well as Cline-Parshall-Scott, Dieter Vossieck (who always was well-read) took the initiative to analyze the corresponding tilting functors using spectral sequences. This was supposed to be his PhD-thesis at Bielefeld. In the summer 1986 he gave several lectures on his work. The manuscript was completely finished before he went to Zürich. Unfortunately, he hesitated to hand it in — apparently he wanted to show it first to Gabriel (but already in 1984, the relationship between Zürich and Bielefeld had started to be quite frosty). At the time, when Dieter was leaving Zürich in order to go to Mexico, I was invited to a lecture at Zürich. As usual, after such a lecture there was a common dinner and during the dinner Gabriel stressed that he did not understand why Dieter had not handed in the Bielefeld thesis (I have to say, he said this to my surprise, but may-be also Dieter's).

More than 15 years later, Brenner and Butler provided a contribution [11] for the *Handbook of Tilting Theory* with the title *A spectral analysis of classical tilting functors.* There, they write: *The spectral sequences in question seem first to have been written down, but nowhere published, by Dieter Vossieck in the mid-1980's and were re-discovered by the authors during the summer of 2002 whilest preparing the talk for the conference "Twenty Years of Tilting Theory" at Chiemsee in November 2002 on which this article is based. After that talk, Helmut Lenzing mentioned Vossieck's work, and kindly supplied a copy of his notes of a lecture in July 1986 by Vossieck at the University of Paderborn entitled "Tilting theory, derived categories and spectral sequences"* [... ;] *in the last two sections Vossieck briefly described the spectral sequences and filtration formulae which are stated and proved in the main part, Section 3, of this article.* In this way, a spectral sequence approach to tilting theory was finally made available (whereas the original manuscript of Dieter seems to be lost, as he told me — that is a pity).

3. Further topics.

As I mentioned at the beginning, this report is devoted mainly to the early Bielefeld years of Dieter Vossieck, but let me draw the attention at least to some of the topics he later was working on. Of course, he continued to look at questions in tilting theory: there are three very important papers written with Bernhard Keller, two in the Comptes Rendus [4,5], one in the Bulletin of the Belgian Mathematical Society [6], all three concern triangulated categories and aim to clarify and to complement the work of Dieter Happel, in particular his lecture notes on the use of triangulated categories in the representation theory of finite-dimensional algebras. Since tilting theory for module categories deals with special torsion pairs, one should, in the context of triangulated categories, investigate t-structures. And this is, what they do. Comparing the hearts of different t-structures, Keller and Vossieck obtained a common generalization of the Grothendieck-Roos duality for regular commutative rings and the tilting theory for finite-dimensional algebras. Also, tilting was generalized by Keller-Vossieck to silting (and silting theory became really popular in recent years).

Vossieck obtained his PhD at the University of Zürich, with a thesis [8] dealing with matrix problems (and not related at all to his Bielefeld PhD project). At that time, Nazarova and Roiter were very eager to be in contact with mathematicians from the West. They had developed many powerful, but technical reduction methods for matrix problems and hoped that for instance Gabriel would provide some fancy categorical or homological interpretation. The paper *Tame and wild subspace problems* [7] by Gabriel, Nazarova, Roiter, Sergejchuk and Vossieck has to be seen as part of this Kiev-Zürich cooperation.

For many applications, in particular for applications in Lie theory and in the theory of algebraic groups, quasi-hereditary algebras play a decisive role. Given a quasi-hereditary algebra A, say with the set Δ of standard modules, one is interested in the category $\mathcal{F}(\Delta)$ of all modules with a filtration with factors in Δ. Starting with a bimodule B, one may define the category of matrices $\operatorname{mat} B$ over B. In case B is what is called upper-triangular, $\operatorname{mat} B$ can be identified with the category $\mathcal{F}(\Delta)$ for some quasi-hereditary algebra A. Now the radical of a hereditary finite-dimensional algebra is upper-triangular, thus $\operatorname{mat} B = \mathcal{F}(\Delta)$ for some quasi-hereditary algebra A. A joint publication [10] with Hille provides in this case an explicit description of the quasi-hereditary algebra A.

Of course, there is the 2001 paper [9] on the algebras with discrete derived categories, which has attracted a lot of interest in the last years and which is the basis for the present workshop. Vossieck classified these algebra. Examples are the piecewise hereditary algebras of Dynkin type. Up to Morita equivalence, the remaining algebras with discrete derived categories are gentle algebras with a unique cycle in the quiver, and with an additional condition on the relations (the so-called clock condition). Soon after (in 2004) Bobiński, Geiß and Skowroński presented representatives for the derived equivalence classes of these algebras: besides the Dynkin algebras, there are the gentle algebras $\Lambda(r, n, m)$, with $1 \leq r \leq n$ and $0 \leq m$; they are given by an oriented cycle with n arrows and r zero relations of length 2, with an arm of length $m+1$ attached. They also described the Auslander-Reiten quiver of the derived category: there are $2r$ components of type $\mathbb{Z}\mathbb{A}_\infty$ and r components of type $\mathbb{Z}\mathbb{A}_\infty^\infty$. These algebras and categories have been studied further by several mathematicians and now are quite well understood.

In 2009, Dieter Vossieck organized at Bielefeld a reading course on ray categories (in particular about the freeness of the fundamental group, Roiter's vanishing theorem, and interval-finiteness) and this was followed by a sequence of lectures by him on multiplicative bases for subspace-finite vector space categories. Ray categories as introduced by Bautista, Gabriel, Roiter and Salmeron in their multiplicative basis paper have to be seen as the essential language for dealing with finite-dimensional algebras whose module categories are not too complicated or not too much entangled (whatever this means). The multiplicative basis paper and some consecutive papers provide a lot of fundamental structure theorems. Unfortunately, these papers are not so easy to follow, since they are very technical. They have scared away a lot of people (and the many recent books devoted to the representation theory of finite-dimensional algebras just avoid to touch these questions). Since no comprehensive presentation is yet available, any publication which provides help, provides improvement would be strongly appreciated. When Dieter delivered the lectures, he promised a written account, with all his fascinating examples which illuminate the results as well as the difficulties. But we are still waiting .... Such an account, even if it is incomplete and covers only specific parts, would be really valuable and definitely appreciated by the mathematical community.

4. Hammocks

There is just one joint publication of Dieter and myself. I like it very much, but it seems to be nearly forgotten — thus I want to use this occasion as an advertisement. The paper is called *Hammocks* [3] and has been published in the Proceedings of the London Mathematical Society in 1987. The concept of a hammock had been introduced before by Sheila Brenner in order to provide a combinatorial characterization of the function which attaches to any indecomposable module M the Jordan-Hölder multiplicity $[M\colon E]$ of some fixed simple module E.

A *hammock* is a finite translation quiver H with a unique source ω and with an additive function h with positive values (the *hammock function*) such that $h(x) = 1$ for any vertex x which is projective or injective. A typical example is the following: Let A be a representation-directed algebra and E a simple module. Then the full subquiver of the Auslander-Reiten quiver $\Gamma(A)$ of A given by the indecomposable modules M with $[M\colon E] \neq 0$ is a hammock with hammock function $h(M) = [M\colon E]$ and with ω being the projective cover of E (and $\Gamma(A)$ itself is just the union of these hammocks).

The main result of our hammock paper is as follows: *There is a bijection between the hammocks H and the subspace-finite posets X, where H corresponds to the poset X of all projective vertices $p \in H$ different from ω. Conversely, one attaches to a subspace-finite poset X the Auslander-Reiten quiver H of its subspace category.*

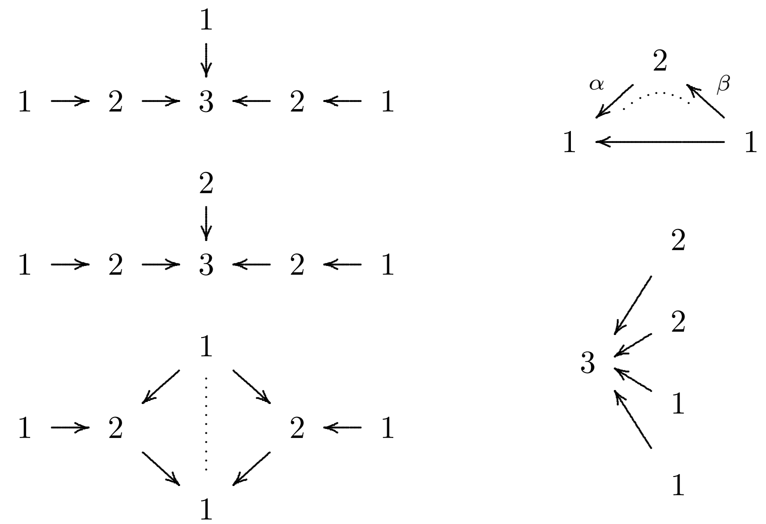
This shows that any hammock arising in whatever part of mathematics provides an interpretation of the corresponding setting in terms of subspace categories: a purely combinatorial invariant, the hammock function, points to a realization using subspaces of vector spaces.

Let us formulate an immediate consequence. We call an A-module M *locally indecomposable* provided for any primitive idempotent e of A, the vector space eM with its definable subspaces is an indecomposable S-space. (Recall that an S-space $(V; U_i)_{i\in I}$ is given by a vector space V and a set of subspaces U_i of V which are indexed by some set I. Such an S-space $(V; U_i)_{i\in I}$ is said to be *indecomposable* provided there is no proper decomposition $V = V' \oplus V''$ with $(U_i \cap V') + (U_i \cap V'') = U_i$, for all $i \in I$.)

Corollary. *Let A be a representation-directed algebra. Then any indecomposable A-module M is locally indecomposable.*

Let us consider some examples of indecomposable modules M and primitive idempotents e such that eM is 2-dimensional. To say that eM together with some subspaces U_j is an indecomposable S-space means in this case that (at least) 3 of these subspaces U_j are 1-dimensional and pairwise different.

Here are the examples: we draw quivers with relations (a commutative relation in the left column, a zero relation in the right one) and exhibit a dimension vector $\mathbf{d}$. The module to be considered is the (uniquely determined) indecomposable representation M with dimension vector $\mathbf{d}$.



On the left, we deal with representation-directed algebras, thus the Corollary can be applied: for any primitive idempotent e, the vector space eM with its definable subspaces is an indecomposable S-space.

In contrast, on the right, the 2-dimensional vector spaces (with their definable subspaces) are decomposable. In the upper case, the only non-zero proper subspace which is definable is the image of β (which is equal to the kernel of α). In the lower example, the 2-dimensional spaces again have only one non-zero proper subspace which is definable, namely (if we assume that we deal with subspaces and inclusion maps) the intersection of the two 2-dimensional subspaces.

In this way, one obtains an effective way to attach to a faithful indecomposable R-module M of a representation-finite basic algebra R its covering $\widetilde{R}$-module $\widetilde{M}$, where $\widetilde{R}$ is the universal cover of R. Namely, let $M = \bigoplus M_i$ be the Peirce decomposition of M (thus $M_i = e_i M$, where $e_i, \dots, e_n$ is a complete set of pairwise orthogonal, primitive idempotents of R). Consider M_i together with its definable subspaces and write it as the direct sum of indecomposable S-spaces M_{ij}. Then $\widetilde{M} = \bigoplus_{ij} M_{ij}$ is the Peirce decomposition of $\widetilde{M}$.

Above, we have exhibited an indecomposable representation M of a quiver with three vertices, three arrows and a zero relation (the upper case in the second column). Let us show the corresponding covering module $\widetilde{M}$:

Finale. In 1996, Bielefeld celebrated the 60th birthday of Bernd Fischer. One of the major speakers was John Horton Conway, well-known not only for his contributions to group theory, but also for his interest in numbers and games. He stayed at Bielefeld for nearly a week, always having a puzzle, the so-called Hanayama Cast Devil, in his pocket and asking anyone he was speaking to to separate the two identical (and quite innocent looking) metal pieces — a surprisingly difficult task. Actually, one may increase the difficulty by using three instead of the usual two pieces. It seems to me that this may be a suitable birthday present for Dieter. Happy Birthday!

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