Quiver Grassmannians and Auslander varieties for wild algebras.

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Abstract. Let k be an algebraically closed field and Λ a finite-dimensional k-algebra. Given a Λ-module M, the set $\mathbb{G}_{\mathbf{e}}(M)$ of all submodules of M with dimension vector $\mathbf{e}$ is called a quiver Grassmannian. If D, Y are Λ-modules, then we consider $\operatorname{Hom}(D, Y)$ as a $\Gamma(D)$-module, where $\Gamma(D) = \operatorname{End}(D)^{\mathrm{op}}$, and the Auslander varieties for Λ are the quiver Grassmannians of the form $\mathbb{G}_{\mathbf{e}} \operatorname{Hom}(D, Y)$. Quiver Grassmannians, thus also Auslander varieties are projective varieties and it is known that every projective variety occurs in this way. There is a tendency to relate this fact to the wildness of quiver representations and the aim of this note is to clarify these thoughts: We show that for an algebra Λ which is (controlled) wild, any projective variety can be realized as an Auslander variety, but not necessarily as a quiver Grassmannian.

1. Introduction.

Let k be an algebraically closed field and Λ a finite-dimensional k-algebra. A *dimension vector* $\mathbf{d}$ for Λ is a function defined on the set of isomorphism classes of simple Λ-modules S with values d_S being non-negative integers. If M is a Λ-module, its dimension vector $\mathbf{dim}\, M$ attaches to the simple module S the Jordan-Hölder multiplicity $(\mathbf{dim}\, M)_S = [M : S]$.

Given a Λ-module M, the set $\mathbb{G}_{\mathbf{e}}(M)$ of all submodules of M with dimension vector $\mathbf{e}$ is called a quiver Grassmannian. Quiver Grassmannians are projective varieties and every projective variety occurs in this way (see the Appendix). If D, Y are Λ-modules, then we consider $\operatorname{Hom}(D, Y)$ as a $\Gamma(D)$-module, where $\Gamma(D) = \operatorname{End}(D)^{\mathrm{op}}$. The easiest way to define the Auslander varieties for Λ is to say that they are just the quiver Grassmannians $\mathbb{G}_{\mathbf{e}} \operatorname{Hom}(D, Y)$ (here, we rely on the Auslander bijections; the proper definition of the Auslander varieties would have to refer to right equivalence classes of right D-determined maps ending in Y, see [Ri]). The Auslander varieties are part of Auslander's approach of describing the global directedness of the category $\operatorname{mod} \Lambda$. Let as add that the quiver Grassmannians for Λ are special Auslander varieties, namely the Auslander varieties $\mathbb{G}_{\mathbf{e}} \operatorname{Hom}(D, Y)$ with $D = \Lambda$.

According to Drozd [D1], any finite dimensional k-algebra is either tame or wild (note that there are few tame algebras, most of the algebras are wild; for example, the path algebra of a connected quiver is tame only in case we deal with a Dynkin or an extended Dynkin quiver). It has been conjectured that wild algebras are actually controlled wild (the definition will be recalled in section 2). A proof of this conjecture has been announced by Drozd [D2] in 2007, but apparently it has not yet been published. We show that for a fixed (controlled) wild algebra Λ, any projective variety can be realized as an Auslander variety, but not necessarily as a quiver Grassmannian.

2. Controlled wild algebras.

We denote by $\operatorname{mod}\Lambda$ the category of all (finite-dimensional left) Λ-modules. Let rad be the radical of $\operatorname{mod}\Lambda$, this is the ideal generated by all non-invertible maps between indecomposable modules. If $\mathcal{C}$ is a collection of object of $\operatorname{mod}\Lambda$, we denote by $\operatorname{add}\mathcal{C}$ the closure under direct sums and direct summands. For every pair X, Y of Λ-modules, $\operatorname{Hom}(X, \mathcal{C}, Y)$ denotes the subgroup of $\operatorname{Hom}(X, Y)$ given by the maps $X \to Y$ which factor through a module in $\operatorname{add}\mathcal{C}$.

Here is now the definition. The algebra Λ is said to be *controlled wild* provided for any finite-dimensional k-algebra Γ, there is an exact embedding functor $F\colon \operatorname{mod}\Gamma \to \operatorname{mod}\Lambda$ and a full subcategory $\mathcal{C}$ of $\operatorname{mod}\Lambda$ (called the *control class*) such that for all Γ-modules X, Y, the subgroup $\operatorname{Hom}(FX, \mathcal{C}, FY)$ is contained in $\operatorname{rad}(FX, FY)$ and we have

$$\operatorname{Hom}(FX, FY) = F\operatorname{Hom}(X, Y) \oplus \operatorname{Hom}(FX, \mathcal{C}, FY).$$

In order to check that Λ is controlled wild, it is sufficient to exhibit such a functor F for just one suitable algebra Γ, for example for the 3-Kronecker algebra (this is the path algebra of the quiver with two vertices, say a and b, and three arrows $b \to a$).

We also mention that Λ is said to be *strictly wild* provided for any finite-dimensional k-algebra Γ, there is a full exact embedding functor $F\colon \operatorname{mod}\Gamma \to \operatorname{mod}\Lambda$ (thus, strictly wild algebras are controlled wild and we can take as control class $\mathcal{C}$ the zero subcategory). The 3-Kronecker algebra is a typical strictly wild algebra.

3. Auslander varieties

Proposition 1. *Let Λ be a finite-dimensional k-algebra which is controlled wild. Let V be any projective variety. Then there are Λ-modules D, Y and a dimension vector $\mathbf{e}$ for $\Gamma(D)$ such that $\mathbb{G}_{\mathbf{e}}\operatorname{Hom}(D, Y)$ is of the form V.*

The special case of a strictly wild algebras has been considered already in [Ri]. The proof of Proposition 1 will be given in this section. We start with the following Lemma.

Lemma 1. *Given finitely many modules $X_1, \dots, X_n$ and a set $\mathcal{C}$ of modules in $\operatorname{mod}\Lambda$, then there is a module C in $\operatorname{add}\mathcal{C}$ such that $\operatorname{Hom}(X_i, \mathcal{C}, X_j) = \operatorname{Hom}(X_i, C, X_j)$ for all i, j.*

Proof: Let $X = \bigoplus X_i$. It is sufficient to show that $\operatorname{Hom}(X, C, X) = \operatorname{Hom}(X, \mathcal{C}, X)$ for some module $C \in \operatorname{add}\mathcal{C}$. Since the subgroups $\operatorname{Hom}(X, C, X)$ with $C \in \operatorname{add}\mathcal{C}$ are subspaces of the finite-dimensional vector space $\operatorname{Hom}(X, X)$, there is $C \in \operatorname{add}\mathcal{C}$ such that $\operatorname{Hom}(X, C, X)$ is of maximal dimension. Let $C' \in \operatorname{add}\mathcal{C}$. Then also $C \oplus C'$ belongs to $\operatorname{add}\mathcal{C}$ and we have $\operatorname{Hom}(X, C, X) \subseteq \operatorname{Hom}(X, C \oplus C', X)$. The maximality of the dimension of $\operatorname{Hom}(X, C, X)$ implies that $\operatorname{Hom}(X, C, X) = \operatorname{Hom}(X, C \oplus C', X)$, and thus $\operatorname{Hom}(X, C', X) \subseteq \operatorname{Hom}(X, C, X)$. But $\operatorname{Hom}(X, \mathcal{C}, X) = \bigcup_{C'} \operatorname{Hom}(X, C', X)$.

Lemma 2. *Let R be a finite-dimensional algebra and e an idempotent in R. Let N be an R-module and* $\mathbf{c} = \mathbf{dim}\, ReN$. *If* $\mathbf{g}$ *is a dimension vector and U belongs to $\mathbb{G}_{\mathbf{g}+\mathbf{c}}N$, then*

$U \supseteq ReN$, and U/ReN is an element of $\mathbb{G}_{\mathbf{g}}(N/ReN)$. As a consequence, the varieties $\mathbb{G}_{\mathbf{g}+\mathbf{c}}N$ and $\mathbb{G}_{\mathbf{g}}(N/ReN)$ can be identified.

Proof. Let U be an element of $\mathbb{G}_{\mathbf{g}+\mathbf{c}}N$. We want to show that $U \supseteq ReN$. Given dimension vectors $\mathbf{d}, \mathbf{d}'$ for Λ, one writes $\mathbf{d}' \leq \mathbf{d}$ provided $\mathbf{d} - \mathbf{d}'$ has non-negative coefficients. Since $\mathbf{dim}\, U = \mathbf{g} + \mathbf{c}$, we have $\mathbf{dim}\, U \geq \mathbf{c}$. Let S be a simple R-module with $eS \neq 0$. Then

$$[U : S] = (\mathbf{dim}\, U)_S \geq \mathbf{c}_S = (\mathbf{dim}\, \Lambda eN)_S = [\Lambda eN : S],$$

and therefore $eN \subseteq U$, thus also $\Lambda eN \subseteq U$.

Proof of proposition 1. Let V be a projective variety. There is a finite-dimensional algebra Γ, a Γ-module M and a dimension vector $\mathbf{g}$ for Γ such that $\mathbb{G}_{\mathbf{g}}M$ is of the form V (see the Appendix). Since Λ is controlled wild, there is a controlled embedding F of $\operatorname{mod}\Gamma$ into $\operatorname{mod}\Lambda$, say with control class $\mathcal{C}$. Let $G = F({}_\Gamma\Gamma)$ and $Y = F(M)$. According to Lemma 1, there is $C \in \operatorname{add}\mathcal{C}$ such that $\operatorname{Hom}(G, C, G) = \operatorname{Hom}(G, \mathcal{C}, G)$ and $\operatorname{Hom}(G, C, Y) = \operatorname{Hom}(G, \mathcal{C}, Y)$. Let $D = G \oplus C$ and $R = \operatorname{End}(D)^{\mathrm{op}}$. Let e_G be the projection of D onto G with kernel C and $e = e_C$ the projection of D onto C with kernel G, both e_G, e_C considered as elements of R. Note that

$$\begin{aligned}\operatorname{Hom}(D, D) &= \operatorname{Hom}(G \oplus C, G \oplus C) = \operatorname{Hom}(G, G) \oplus \operatorname{Hom}(G, C) \oplus \operatorname{Hom}(C, G) \oplus \operatorname{Hom}(C, C) \\ &= F(\operatorname{Hom}(\Gamma, \Gamma)) \oplus \operatorname{Hom}(G \oplus C, \mathcal{C}, G \oplus C) \oplus \operatorname{Hom}(G, C) \oplus \operatorname{Hom}(C, G) \oplus \operatorname{Hom}(C, C),\end{aligned}$$

and

$$e \operatorname{Hom}(D, D)e = \operatorname{Hom}(G \oplus C, \mathcal{C}, G \oplus C) \oplus \operatorname{Hom}(G, C) \oplus \operatorname{Hom}(C, G) \oplus \operatorname{Hom}(C, C).$$

It follows that the map $\gamma \mapsto F(\gamma) \in e_G R e_G$ yields an isomorphism $\Gamma \to R/ReR$. Also, we are interested in the R-module $N = \operatorname{Hom}(G \oplus C, Y)$. Here, we have

$$\begin{aligned}N = \operatorname{Hom}(G \oplus C, Y) &= \operatorname{Hom}(G \oplus 0, Y) \oplus \operatorname{Hom}(0 \oplus C, Y) \\ &= F \operatorname{Hom}(\Gamma, M) \oplus \operatorname{Hom}(G \oplus 0, C, Y) \oplus \operatorname{Hom}(0 \oplus C, Y).\end{aligned}$$

If we multiply N with the element $e = e_C \in R$, we obtain

$$eN = \operatorname{Hom}(0 \oplus C, Y),$$

thus

$$ReN = R \operatorname{Hom}(0 \oplus C, Y) = \operatorname{Hom}(G \oplus 0, C, Y) \oplus \operatorname{Hom}(0 \oplus C, Y).$$

This shows that N/ReN is canonically isomorphic to $F \operatorname{Hom}(\Gamma, M)$ as an R-module. Of course, these modules are annihilated by e, thus they are R/ReR-modules and as we know $R/ReR = \Gamma$.

It remains to apply Lemma 2.

4. Quiver Grassmannians.

Proposition 2. *There are controlled wild algebras Λ such that not every projective variety can be realized as a quiver Grassmannian of a Λ-module.*

Proof. Let Λ be any local radical square zero k-algebra of dimension at least 4 (thus $\Lambda = k[T_1, \dots, T_n]/(T_1, \dots, T_n)^2$ with $n \geq 3$). It is well-known (and easy to see) that such an algebra is controlled wild. Let M be a Λ-module. The Grothendieck group $K_0(\Lambda)$ is free of rank one, thus the quiver Grassmannians are of the form $\mathbb{G}_i(M)$ with i a non-negative integer (the elements of $\mathbb{G}_i(M)$ are the submodules of M of dimension i). In order to determine the possible varieties $\mathbb{G}_i(M)$, we can assume that $i \leq \dim \operatorname{soc} M$. Namely, if $i > \dim \operatorname{soc} M$, then we consider the dual module M^* and the quiver Grassmannian $\mathbb{G}_{d-i}(M^*)$, where $d = \dim M = \dim M^*$. On the one hand, the varieties $G_i(M)$ and $G_{d-i}(M^*)$ are obviously isomorphic, on the other hand we have $d - i < \dim M - \dim \operatorname{soc} M \leq \dim M - \dim \operatorname{rad} M = \dim \operatorname{top} M = \dim \operatorname{soc}(M^*)$, here we have used that $\operatorname{rad} M \subseteq \operatorname{soc} M$.

Thus, let $i \leq s = \dim \operatorname{soc} M$. The submodules U of $\operatorname{soc} M$ of dimension i are just the subspaces of $\operatorname{soc} M$ of dimension i considered as a k-space, thus they form the usual Grassmannian $\mathbb{G}_i(\operatorname{soc} M) = \mathbb{G}_i(k^s)$, in particular, this is an irreducible (and rational) variety. Now let U be a submodule of M of dimension i which is not contained in $\operatorname{soc} M$. We claim that there is a projective line in $\mathbb{G}_i(M)$ which contains both U and a submodule U' of $\operatorname{soc} M$. Namely, let $b_1, \dots, b_t$ be a basis of $\operatorname{soc} U$, and extend it to a basis $b_1, \dots, b_i$ of U. Now $b_1, \dots, b_t$ are linearly independent elements of the socle of M. Since $i \leq \dim \operatorname{soc} M$, there is an i-dimensional subspace U' inside $\operatorname{soc} M$ which contains $\operatorname{soc} U' = U \cap \operatorname{soc} M$. We can extend the basis $b_1, \dots, b_t$ of $\operatorname{soc} U$ to a basis of U', say $b_1, \dots, b_t, b'_{t+1}, \dots, b'_i$, For $\lambda = (\lambda_0 : \lambda_1) \in \mathbb{P}^1$, define U_λ as the subspace of M with basis the elements $b_1, \dots, b_t$ as well as the elements $\lambda_0 b_j + \lambda_1 b'_j$ where $t < j \leq i$. Of course, U_λ is a submodule of M. On the one hand, we have $U_{(1:0)} = U$, on the other hand, $U_{(0:1)} = U'$ is an i-dimensional submodule of M which lies inside the socle of M.

Since the Grassmannian $\mathbb{G}_i(\operatorname{soc} M)$ is (rational and) connected and for any element $U \in \mathbb{G}_i(M)$ there is a $\mathbb{P}^1$-family of submodules which contains U and an element in $\mathbb{G}_i(\operatorname{soc} M)$, it follows that also $\mathbb{G}_i(M)$ is connected (even rationally connected, see [Ha]).

5. Open questions.

We have shown that any projective variety occurs as an Auslander variety for any (controlled) wild algebra. It seems that the Auslander varieties for the tame algebras are quite restrictive — is there a special property which all have? Such a result would provide a characterization of the tame-wild dichotomy in terms of Auslander varieties.

We have shown that dealing with local algebras with radical square zero, all quiver Grassmannians are rationally connected. Are there further properties which they share? On the other hand, the quiver Grassmannians for wild algebras should be of quite a general nature. Is there a class of varieties which can be realized as quiver Grassmannians for all wild algebras, but not for tame algebras?

Appendix. Every projective variety is a quiver Grassmannian.

We say that a k-module M is a *brick* provided $\mathrm{End}(M) = k$.

Proposition (Reineke). *Every projective variety is a quiver Grassmannian $\mathbb{G}_{\mathbf{e}}M$ for a brick M.*

Let us outline a proof, following Van den Bergh [L]. Let V be a projective variety. We can assume V is a closed subset of the projective space $\mathbb{P}^n$, defined by the vanishing of homogeneous polynomials $f_1, \dots, f_m$ of degree 2. Let Δ be the Beilinson quiver with 3 vertices, say a, b, c, with $n+1$ arrows $b \to a$ labeled $x_0, \dots, x_n$ as well as $n+1$ arrows $c \to b$, also labeled $x_0, \dots, x_n$. The path algebra of Δ with all the relations $x_i x_j = x_j x_i$ (whenever this makes sense) is called the Beilinson algebra. Let Λ be the factor algebra of this Beilinson algebra taking the elements $f_1, \dots, f_m$ as additional relations (obviously, these elements may be considered as linear combinations of paths of length 2 in the quiver Δ). Take $M = I_\Lambda(a)$, the indecomposable injective Λ-module corresponding to the vertex a, and take $\mathbf{e} = (1, 1, 1)$. Note that the elements of $\mathbb{G}_{\mathbf{e}}M$ are just all the serial Λ-modules, one from each isomorphism class (we call a module X *serial*, provided it has a unique composition series).

Here are some remarks on the history: The title of the appendix is also the title of a recent paper [Re] by Reineke, who answered in this way a question by Keller. The 2-page paper attracted a lot of interest, see for example blogs by Le Bruyn [L] (with the proof by Van den Bergh presented above) and by Baez [Bz]. Actually, the construction given in the proof of Van den Bergh is much older, it has been used before by several mathematicians dealing with related problems and may, of course, be traced back to Beilinson [Be].

The quiver Grassmannians play an important role in the representation theoretical approach to cluster algebras. Here one deals with the quiver Grassmannians $\mathbb{G}_{\mathbf{e}}M$, where M is a quiver representation without self-extensions. It has been asserted by Caldero and Reineke [CR] that the quiver Grassmannians $\mathbb{G}_{\mathbf{e}}M$, where M is a quiver representation without self-extensions, are very special: *if $\mathbb{G}_{\mathbf{e}}M$ is non-empty, then the Euler characteristic of $\mathbb{G}_{\mathbf{e}}M$ is positive*; a complete proof was given by Nakajima [N] and Qin [Q]. Note that an indecomposable quiver representation without self-extensions is a brick. If we consider the bricks M constructed by Van den Bergh as representations of the quiver Δ, then such a $k\Delta$-module will have self-extensions, but it seems to be remarkable to observe that M has no self-extensions when it is considered as a $k\Delta/\operatorname{Ann}(M)$-module, where $\operatorname{Ann}(M)$ is the annihilator of M in $k\Delta$ (in contrast, the examples constructed by Reineke are usually faithful quiver representations). Namely, we have $k\Delta/\operatorname{Ann}(M) = \Lambda$, and by construction, M is an injective Λ-module. We should recall that for any ring R, an R-module is said to be *quasi-injective* provided for any submodule U of M any map $U \to M$ can be extended to an endomorphism of M; for an artinian ring, a module M is quasi-injective if and only if M considered as an $R/\operatorname{Ann}(M)$-module is injective.

There is a tendency to relate the fact that every variety is a quiver Grassmannian to the tame-wild dichotomy as established by Drozd [D1]. For example, Baez [B] writes that one may suppose that this is *just another indication of the .wildness. of quiver representations once we leave the safe waters of Gabriel.s theorem.* The aim of this note was to clarify these thoughts.

Some mathematicians (see [L],[V]) refer in this context to “Murphy’s law”: *Anything that can go wrong, will go wrong* (as formulated in the Wikipedia [W]), or: *Anything that can happen, will happen.* But one should be aware that Murphy’s law may be a challenging assertion in daily life, but it is just a tautology when we consider mathematical questions. Indeed, in mathematics, if we know (that means: if we can prove) that something **does not** happen, then of course we have a proof that it **cannot** happen.

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