Distinguished bases of exceptional modules.

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An indecomposable representation M of a quiver $Q = (Q_0, Q_1)$ is said to be exceptional provided $\operatorname{Ext}^1(M, M) = 0$. And it is called a tree module provided one can choose a set $\mathcal{B}$ of bases of the vector spaces M_x ($x \in Q_0$) such that the coefficient quiver $\Gamma(M, \mathcal{B})$ is a tree quiver; we call $\mathcal{B}$ a tree basis of M. It is known that exceptional modules are tree modules. A tree module usually has many tree bases and the corresponding coefficient quivers may look quite differently. The aim of this note is to introduce a class of indecomposable modules which have a distinguished tree basis, the "radiation modules" (generalizing an inductive construction considered already by Kinser). For a Dynkin quiver, nearly all indecomposable representations turn out to be radiation modules, the only exception is the maximal indecomposable module in case $\mathbb{E}_8$. Also, the exceptional representations of the generalized Kronecker quivers are given (via the universal cover) by radiation modules. Consequently, with the help of Schofield induction one can display all the exceptional modules of an arbitrary quiver in a nice way.

Let $Q = (Q_0, Q_1)$ be a locally finite quiver. We will consider finite-dimensional representations of Q (thus kQ-modules, where kQ is the path algebra of Q). An indecomposable representation M of Q is said to be *exceptional* provided $\operatorname{Ext}^1(M, M) = 0$. And it is called a *tree module* provided one can choose a set $\mathcal{B}$ of bases $\mathcal{B}_x$ of the vector spaces M_x ($x \in Q_0$) such that the coefficient quiver $\Gamma(M, \mathcal{B})$ is a tree quiver; in this case, we call $\mathcal{B}$ a *tree basis* of M. Let me recall the definition of the coefficient quiver $\Gamma(M, \mathcal{B})$ as introduced in [R3], it is a quiver whose vertices and arrows are labeled by elements of Q_0 and Q_1, respectively. Its vertex set is the disjoint union of the sets $\mathcal{B}_x$, the elements of $\mathcal{B}_x$ being labeled by x. The arrows of $\Gamma(M, \mathcal{B})$ are obtained as follows: For an arrow $\alpha\colon x \to y$ in Q_1 and $b \in \mathcal{B}_x$, write $M_\alpha(b) = \sum_{b' \in \mathcal{B}_y} c_{b'b} b'$ with coefficients $c_{b'b} \in k$; there is an arrow $b \to b'$ in $\Gamma(M, \mathcal{B})$ with label α provided $c_{b'b} \neq 0$.

It is known [R3] that exceptional modules are tree modules. But even if we know that a module M is a tree module, it often seems to be difficult to find directly a tree basis. The aim of this note is to provide for the exceptional modules an algorithm for obtaining a tree basis. It turns out that we should start by looking at indecomposable representations M with a thin vertex (a vertex x is said to be *thin* for M provided the vector space M_x is one-dimensional). The first modules which we will consider are what we call the radiation modules, see sections 2 and 3. They are inductively defined, in any step one constructs an indecomposable module with a thin vertex. This generalizes a construction introduced by Kinser [K] for rooted tree quivers (using the name "reduced representations").

As we will see in section 4, nearly all indecomposable representations of a Dynkin quiver are radiation modules, the only exception is the maximal indecomposable module in case $\mathbb{E}_8$. The radiation modules which are our main concern are the preprojective and preinjective representations of a tree without leaves with bipartite orientation (a leaf of a tree is a vertex with a single neighbor). These modules are studied in sections 5. Typical examples of trees without leaves are the n-regular trees with $n \geq 2$, note that they occur as the universal cover of a generalized Kronecker quiver. Thus, using covering theory, one obtains a distinguished tree basis for any exceptional representation of a generalized Kronecker quiver. With the help of Schofield induction we then get a nice tree basis for any exceptional module, see the outline in the last section 6.

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1. Some exceptional representations M with a thin vertex.

If M is a representation of Q and $\dim M_x = 1$ for some vertex x, we say that x is a *thin* vertex for M.

Let Q be a quiver with underlying graph $\overline{Q}$ being a tree. Let x be a vertex of Q and let Q^x be obtained from Q by deleting the vertex x and all the arrows involving x.

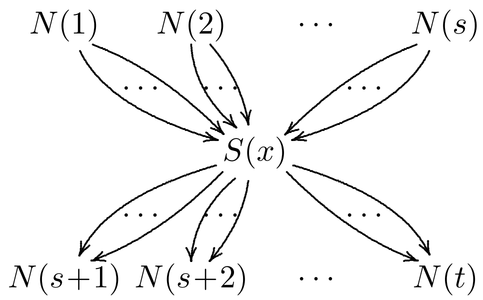
We say that a family of indecomposable modules $N(1), \dots, N(t)$ is an *exceptional family,* provided the modules are pairwise non-isomorphic and $\operatorname{Ext}^1(N(i), N(j)) = 0$ for all i, j. The family $N(1), \dots, N(t)$ is an *orthogonal family* provided $\operatorname{Hom}(N(i), N(j)) = 0$ for all $i \neq j$.

Proposition 1. (a) *Let $N(1), \dots, N(t)$ be an orthogonal family of modules with endomorphism ring k, such that $N(i)_x = 0$ for all i, and assume that for any index i, there is a neighbor $y(i)$ of x with $n(i) = \dim N(i)_{y(i)} > 0$. Then there is a module M with endomorphism ring k, unique up to isomorphism, with thin vertex x such that the restriction of M to Q^x is $\bigoplus_{i=1}^t N(i)^{n(i)}$. If the family $N(1), \dots, N(t)$ is in addition exceptional, then also M is exceptional.*

(b) *Conversely, let M be an exceptional representation of Q, and let x be a vertex of Q with $M_x \neq 0$. Then the restriction M' of M to Q^x is of the form $M' = \bigoplus_{i=1}^t N(i)^{n(i)}$, where $N(1), \dots, N(t)$ is an exceptional family of indecomposable modules and $n(i) > 0$ for all i. For any index i, there is a unique neighbor $y(i)$ of x with $\dim N(i)_{y(i)} \neq 0$. If $\dim M_x = 1$, then $n(i) \leq \dim N(i)_{y(i)}$ for all $1 \leq i \leq t$. If $\dim M_x = 1$ and $\dim N(i)_{y(i)} = 1$ for $i \in I$, where I is a subset of $\{1, 2, \dots, t\}$, then the modules $N(i)$ with $i \in I$ form an orthogonal family.*

Proof: (a) Let $N(1), \dots, N(t)$ be an orthogonal family of modules with endomorphismring k such that $N(i)_x = 0$, and assume that for any index i, there is a neighbor $y(i)$ of x with $n(i) = \dim N(i)_{y(i)} > 0$. We can assume that there are arrows $y(i) \to x$ for $1 \leq i \leq s$ and arrows $x \to y(i)$ for $s+1 \leq i \leq t$. Since $N(i)_x = 0$, we see that the modules $S(x), N(1), \dots, N(t)$ are orthogonal bricks, thus we can use the process of simplification (see [R1]): it asserts that the subcategory $\mathcal{A}$ of representations of Q which have a filtration with factors in $\{S(x), N(1), \dots, N(t)\}$ is an exact abelian subcategory of $\operatorname{mod} kQ$ which is closed under extensions, and which has the simple objects $S(x), N(1), \dots, N(t)$. Since the simple objects of $\mathcal{A}$ have endomorphism ring k, the category $\mathcal{A}$ is equivalent to the category of locally nilpotent representation of a quiver $Q(\mathcal{A})$ with $t+1$ vertices which are labeled $S(x), N(1), \dots, N(t)$. For every pair N, N' of vertices $Q(\mathcal{A})$, the number of arrows from N to N' is equal to $\dim \operatorname{Ext}^1(N, N')$. For $1 \le i \le s$, we have $\dim \operatorname{Ext}^1(N(i), S(x)) = \dim N(i)_{y(i)} = n(i)$ and $\dim \operatorname{Ext}^1(S(s), N(i)) = 0$. For $s+1 \le i \le t$, we have $\dim \operatorname{Ext}^1(S(x), N(i)) = \dim N(i)_{y(i)} = n(i)$ and $\dim \operatorname{Ext}^1(N(i), S(s)) = 0$, thus the quiver $Q(\mathcal{A})$ is obtained from the following quiver Q' by adding arrows $N(i) \to N(j)$ for $1 \le i, j \le t$, according to the extension groups $\operatorname{Ext}^1(N(i), N(j))$ in question.

Q'



Let me repeat that the number of arrows of Q' between $N(i)$ and $S(x)$ is $n(i)$, for $1 \le i \le t$.

However, we are only interested in the subcategory $\mathcal{A}'$ of $\mathcal{A}$ consisting of the representations of Q whose restriction to Q^x is a direct sum of modules of the form $\bigoplus_i N(i)^{m(i)}$ with $m(i) \in \mathbb{N}_0$. This subcategory $\mathcal{A}'$ is an exact abelian subcategory of $\mathcal{A}$ with the same simple objects and $\mathcal{A}'$ is equivalent to the category of locally nilpotent representation of the quiver Q' itself.

Now the quiver Q' has the following real root:

$$\begin{array}{cccc} n(1) & n(2) & \cdots & n(s) \\ & & 1 & \\ n(s+1) & n(s+2) & \cdots & n(t) \end{array}$$

thus in $\mathcal{A}'$ there is precisely one indecomposable object M with one factor $S(x)$ and $n(i)$ factors $N(i)$, for $1 \le i \le t$ and the endomorphism ring of M is k.

In case the family $N(1), \dots, N(t)$ is in addition exceptional, then $\mathcal{A}' = \mathcal{A}$ and therefore $Q(\mathcal{A}) = Q'$. In this case, the object M is an exceptional module. Namely, the module M is always exceptional when considered as an object in $\mathcal{A}'$; now $\mathcal{A}' = \mathcal{A}$, and $\mathcal{A}$ is closed under extensions in $\operatorname{mod} kQ$.

(b) We use the following lemma:

Lemma 1. *Let X' be a submodule of X and $X'' = X/X'$. We assume that* $\operatorname{Ext}^1(X, X) = 0$ *and* $\operatorname{Hom}(X', X'') = 0$. *Then* $\operatorname{Ext}^1(X', X') = 0$ *and* $\operatorname{Ext}^1(X'', X'') = 0$.

Proof: The inclusion map $X' \to X$ yields an epimorphism $\operatorname{Ext}^1(X,X) \to \operatorname{Ext}^1(X',X)$, thus $\operatorname{Ext}^1(X',X) = 0$. Applying $\operatorname{Hom}(X',-)$ to the exact sequence $0 \to X' \to X \to X'' \to 0$ yields an exact sequence

$$\operatorname{Hom}(X',X'') \to \operatorname{Ext}^1(X',X') \to \operatorname{Ext}^1(X',X).$$

Since the end terms are zero, also the middle term is zero. This shows that $\operatorname{Ext}^1(X',X') = 0$. By duality, also $\operatorname{Ext}^1(X'',X'') = 0$.

Assume now that M is an exceptional representation of Q. Let $y_1,\dots,y_n$ be the neighbors of x, say with arrow $y_i \to x$ for $1 \le i \le r$ and $x \to y_i$ for $r+1 \le i \le n$. The quiver Q^x is the disjoint union of two parts Q^+ and Q^-, where Q^- is the union of the connected components of Q^x which contain the vertices $y_1,\dots,y_r$, and Q^+ is the union of the connected components of Q^x which contain the vertices $y_{r+1},\dots,y_n$. Let M^+ be the restriction of M to Q^+ and M^- the restriction of M to Q^-. Note that M^+ is a submodule of M, whereas M^- is a factor module of M, say $M^- = M/X$. Thus, M has the following chain of submodules $M^+ \subseteq X \subseteq M$ and X/M^+ is a direct sum of copies of $S(x)$. Now we apply Lemma 1 to the submodule X of M, this is possible, since X and M/X has disjoint support, thus $\operatorname{Hom}(X, M/X) = 0$. We see that $\operatorname{Ext}^1(X,X) = 0$ and $\operatorname{Ext}^1(M^-,M^-) = 0$. Next, we apply Lemma 1 to the submodule M^+ of X, again using that we deal with modules M^+ and X/M^+ with disjoint support. We conclude that $\operatorname{Ext}^1(M^+,M^+) = 0$. Since there is no arrow between vertices of Q^- and Q^+, thus $\operatorname{Ext}^1(M^-,M^+) = 0 = \operatorname{Ext}^1(M^+,M^-)$. Note that $M' = M^+ \oplus M^-$. Altogether we have shown that $\operatorname{Ext}^1(M',M') = 0$, thus the modules $N(1),\dots,N(t)$ form an exceptional family.

Now consider a module $N(i)$. Since the support of M is connected, and M is indecomposable, at least one of the vertices y_j must belong to the support of $N(i)$. Since $\overline{Q}$ is a tree, there is just one such vertex. This shows that there is a unique neighbor $y(i)$ of x such that $N(i)_{y(i)} \neq 0$, namely $y(i) = y_j$.

From now on, let us assume that $\dim M_x = 1$. Let us show that $n(i) \le \dim N(i)_{y(i)}$ for all $1 \le i \le t$. We assume that for $1 \le i \le s$, the arrow between x and $y(i)$ ends in x, whereas for $s+1 \le i \le t$, it starts in x. First, consider some i with $s+1 \le i \le t$, say $i = r+1$. Then $\dim \operatorname{Ext}^1(S(x), N(r+1)) = \dim N(r+1)_{y(r+1)}$. There is the exact sequence

$$0 \to N(r+1)^{n(r+1)} \oplus X' \to X \to S(x) \to 0 \quad \text{with} \quad X' = \bigoplus_{i=r+2}^{t} N(i)^{n(i)}.$$

If $n(i) > \dim \operatorname{Ext}^1(S(x), N(r+1))$, then X splits off a copy of $N(r+1)$ (see the Ext-Lemma in [R4]). But this is impossible, since X is indecomposable. The dual argument works for $1 \le i \le r$. This shows that $n(i) \le \dim N(i)_{y(i)}$ for all indices $1 \le i \le t$.

In particular, we see that $\dim N(i)_{y(i)} = 1$ implies $n(i) = 1$. Now assume in addition that there are two different indices i,j with $\dim N(i)_x = 1 = \dim N(j)_x$, say $i = 1, j = 2$. We want to show that $\operatorname{Hom}(N(1), N(2) = 0$. Thus, assume that $\operatorname{Hom}(N(1), N(2)) \neq 0$. Then the support of $N(1)$ and $N(2)$ is contained in the same connected component of Q^x, thus $y(1) = y(2)$. Let $y = y(1) = y(2)$.

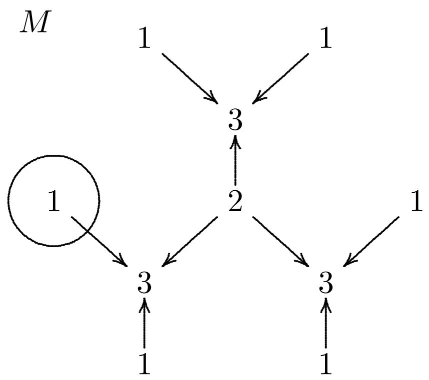
Let us assume that the arrow α between x and y starts in x. Since $\dim M_x = 1$, we can assume that $M_x = k$ and consider the element $M_\alpha(1) \in M_y$. Note that

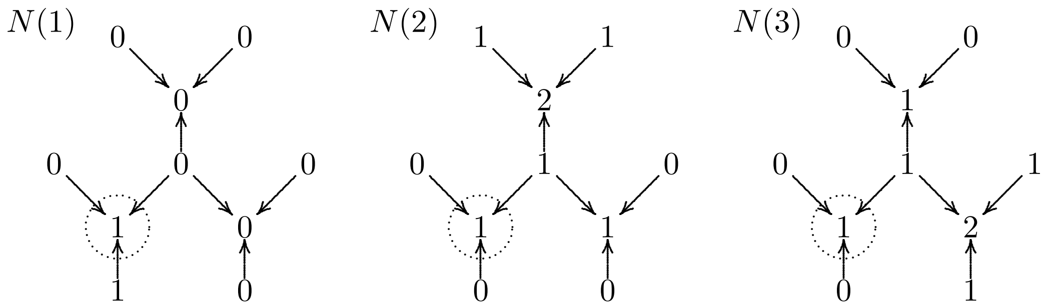
$$M_y = M'_y = \bigoplus_{i=1}^{t} N(i)_y^{n(i)} = N(1)_y \oplus N(2)_y \oplus N'_y \quad \text{with} \quad N' = \bigoplus_{i=2}^{t} N(i)_y^{n(i)},$$

thus we can write $M_\alpha(1) = (a_1, a_2, a_3)$ with $a_1 \in N(1)_y,\ a_2 \in N(2)_y,\ a_3 \in N'_y$. Note that both a_1, a_2 are non-zero, since otherwise M would split off $N(1)$ or $N(2)$, respectively. Let $\phi\colon N(1) \to N(2)$ be a non-zero homomorphism. Since $\operatorname{Ext}^1(N(2), N(1)) = 0$, we know that ϕ has be a monomorphism or an epimorphism. It follows from $\dim N(1)_y = 1 = \dim N(2)_y$, that ϕ_y is bijective. Replacing if necessary ϕ by a scalar multiple, we can assume that $\phi_y(a_1) = a_2$. Let $N'' \subseteq N(1) \oplus N(2)$ be the graph of ϕ, thus $N''_z = \{(b, \phi(b)) \mid b \in N(1)_z\}$ for all vertices z of Q. The module N'' is a submodule of $N(1) \oplus N(2)$ which has $0 \oplus N(2)$ as a direct complement inside $N(1) \oplus N(2)$. Since we see that M_α maps M_x into $N''_y \oplus N'_y$, it follows that M splits off $N(2)$, namely $M = Y \oplus N(2)$, where $Y_x = M_x$ and the restriction of Y to Q^x is equal to $N'' \oplus N'$. This contradicts the indecomposability of M.

We use the dual considerations in case the arrow between x and y starts in y and terminates in x. This concludes the proof.

Example 1. Consider the following exceptional representation M, encircled is a vertex x with $\dim M_x = 1$, note that x has a unique neighbor y. The lower line exhibits the dimension vectors of the modules $N(i)$, here the vertex y is encircled by a dotted circle.



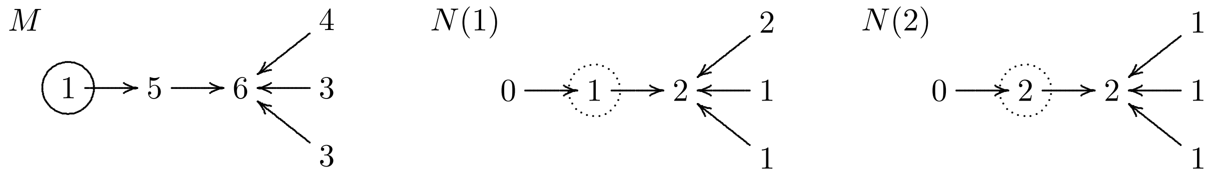


(A related representation, namely the representation $P(c, 3)$ for the 3-regular tree with bipartite orientation and c a source, will be discussed in detail towards the end of section

5. The representation M considered above is the restriction of $P(c,3)$ to the ball with center c and radius 2.)

We mainly will be interested in exceptional modules M with a thin vertex x such that the restriction M' to Q^x decomposes as the direct sum of indecomposable modules $N(i)$ with $\dim N(i)_{y(i)} = 1$, so that also $n(i) = 1$. But let us exhibit here an example with $\dim N(i)_{y(i)} > 1$ for some index i (further examples will be provided in section 3, namely part 2 of example 3, as well as example 4 — whereas the indecomposable direct summands of M in example 2 are orthogonal, those in section 3 are not).

Example 2. An exceptional module M with a thin vertex x such that the restriction M' of M to Q^x is of the form $N(1)\oplus N(2)^2$ for an exceptional orthogonal family $N(1), N(2)$.



2. Radiation modules.

Let us assume that we deal with a quiver Q which is a tree. We consider pairs (M,x) where M is an indecomposable module and x is a vertex with $\dim M_x = 1$ (called the *origin* of M or better of the pair (M,x)). We define the class of *radiation modules* as well as the corresponding *radiation quivers* inductively as follows.

First of all, for any vertex x, the pair $(S(x),x)$ is a radiation module, its radiation quiver $R(S(x),x))$ is the quiver with a single vertex (and no arrow), the vertex being labeled x. Second, if M is an indecomposable module of length at least 2 and x is a vertex with $\dim M_x = 1$, then (M,x) is a radiation module provided the restriction of M to the quiver obtained from Q by deleting the vertex x is the direct sum $\bigoplus N(i)$ of an orthogonal family of indecomposable modules $N(i)$ such that for any index i there is a neighbor $y(i)$ of x with $(N(i),y(i))$ being a radiation module. The radiation quiver $R(M,x)$ of (M,x) is obtained from the disjoint union of the radiation quivers $R(N(i),y(i))$ by adding a vertex with label x and for every index i we connect x and $y(i) \in R(N(i),y(i))_0$ by an arrow which points in the same direction as the arrow in Q between x and $y(i)$, and we use this arrow of Q also as the label of the arrow in $R(M,x)$. Note that the labels of the vertices and the arrows of $R(M,x)$ provide a quiver homomorphism $R(M,x) \to Q$.

Let (M,x) be a radiation module, and b a non-zero element of M_x (note that M_x is one-dimensional). For any radiation module (M,x) we define a basis $\mathcal{B}(M,x)$ as follows, we call it a *radiation basis* containing b. Assume that the restriction of M to the quiver obtained by deleting the vertex x is $\bigoplus N(i)$ with indecomposable representations $N(i)$. For every index i, there is a vertex $y(i)$ which is a neighbor of x such that $(N(i),y(i))$ is a radiation module. Since $x, y(i)$ are neighbors, there is an arrow say α which connects them. If $\alpha\colon x \to y(i)$, let $\mathcal{B}(i)$ be a radiation basis of $(N(i),y(i))$ containing the element $M_\alpha(b)$. If $\alpha\colon y(i) \to x$, let $\mathcal{B}(i)$ be a radiation basis of $(N(i),y(i))$ containing the element

$M_\alpha^{-1}(b)$. Finally, let $\mathcal{B}(M,x)$ be the set containing the element b as well as all the elements in the (disjoint!) sets $\mathcal{B}(i)$.

Proposition 2. *If (M,x) is a radiation module, then the endomorphism ring of M is k. A radiation basis $\mathcal{B}(M,x)$ of M is a tree basis and the corresponding coefficient quiver is $R(M,x)$.*

Proof. The first assertion is shown by induction. If $M = S(x)$, then the endomorphism ring of M is k. If M is of length at least 2, then the restriction of M to the quiver Q^x is the direct sum of radiation modules $(N(i), y(i))$, and by induction the endomorphism ring of $N(i)$ is k. Now the family of modules $N(i)$ is an orthogonal family of modules with endomorphism ring k and for any index i there is the neighbor $y(i)$ of x with $n(i) = \dim N(i)_{y(i)} = 1$. Thus M is the module constructed in Proposition 1(a) using the vertex x and the modules $N(i)$. According to this proposition, the endomorphism ring of M is k.

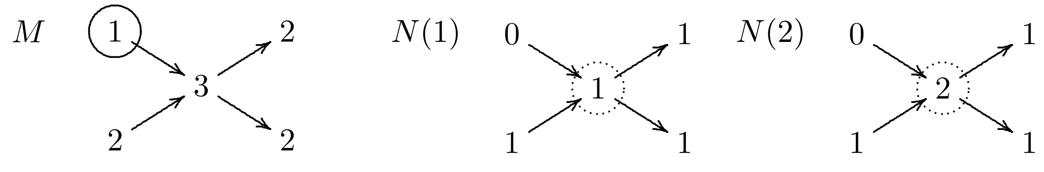
Also the remaining assertions are straight-forward: Namely, assume that there is given a finite set of trees $T(i)$ and for every i a vertex $y(i)$ in $T(i)$. If we take the disjoint union of the graphs $T(i)$, an additional vertex x and connect x with the vertices $y(i)$ by a single edge, then we obtain again a tree. In order to see that $R(M,x)$ is a coefficient quiver of M, one observes by induction that $\mathcal{B}(M,x)$ is a basis of M, the corresponding coefficient quiver is just $R(M,x)$. r is just $R(M,x)$.

Remark. In the special situation of Q being a tree quiver with precisely one root, radiation modules have been considered before by Kinser [K], he called them reduced representations. His detailed investigation of these modules is of great interest. Note that they play a prominent role in his study of the representation ring of such a quiver. See also the paper [KM] by Katter and Mahrt for a further study of this class of radiation modules.

3. Exceptional radiation modules.

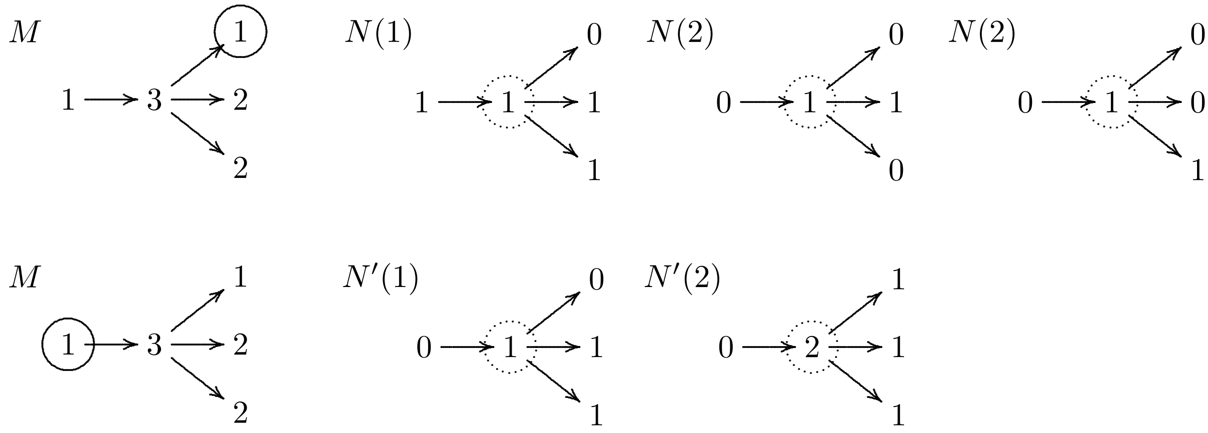
We are mainly interested in modules which are both exceptional as well as radiation modules, thus in exceptional modules with a thin vertex. We should stress that by construction radiation modules have a thin vertex, but may not be exceptional, and conversely, there are exceptional modules with a thin vertex which are not radiation modules. Here are corresponding examples:

Example 3. An exceptional module M with a thin vertex, such that (M,x) is not a radiation module for any thin vertex x. The module M which we exhibit is an indecomposable preinjective module with a unique thin vertex. On the right, we show the indecomposable direct summands $N(1), N(2)$ of M', here, the neighbor $y = y(i)$ of x is encircled using a dotted circle. Note that the modules $N(1), N(2)$ form an exceptional family. Since $\operatorname{Hom}(N(1), N(2)) \neq 0$, the family is not orthogonal.

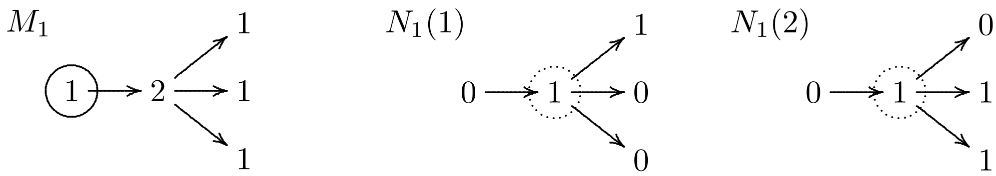


Also, one may look at this example in the light of Proposition 1(b): If we write $M' = N(1)^{n(1)} \oplus N(2)^{n(2)}$, then $n(1) = n(2) = 1$. Now $\dim N(2)_{y(i)} = 2$, thus $n(2) < \dim N(2)_{y(i)}$.

Example 4. This is an example of an exceptional module M with vertices x, x' such that (M, x) is a radiation module, whereas (M, x') is not. The module M which we present here is indecomposable and preinjective, the vertices x, x' are encircled. We write M^x for the restriction of M to Q^x, and $M^{x'}$ for its restriction to $Q^{x'}$. On the right, we show the indecomposable direct summands $N(i)$ and $N'(i)$ of M^x and $M^{x'}$ respectively. Any of the vertices x, x' has a unique neighbor y, this vertex is encircled using a dotted circle. Note that the modules $N(1), N(2), N(3)$ form an orthogonal exceptional family, whereas $N'(1), N'(2)$ is an exceptional family which is not orthogonal (namely, we have $\dim \operatorname{Hom}(N'(1), N'(2)) = 1$).



Example 5. Radiation modules are usually not exceptional and, in contrast to exceptional modules, not determined by the dimension vector. Here we exhibit a radiation module M_1 (the origin x is encircled) and the decomposition of the restriction $M_1' = N_1(1) \oplus N_1(2)$ to Q^x. This module M_1 is an indecomposable regular module and its dimension vector shows that it has self-extensions. Note that there are corresponding radiation modules M_2 and M_3 with the same dimension vector obtained by permuting the arms on the right.



We assume now that (M, x) is a radiation module such that M is exceptional. We want to discuss properties of the corresponding radiation basis of M. Let us start with a general (and well-known) property of exceptional modules:

Lemma 2. *Let M be a representation of the quiver Q with $\operatorname{Ext}^1(M, M) = 0$. Then for every arrow α in Q, the linear map M_α is injective or surjective.*

Proof: Let $\alpha\colon x \to y$ be an arrow such that M_α is neither injective nor surjective. Write $M_x = M'_x \oplus M''_x$ with M'_x the kernel of M_α and $M_y = M'_y \oplus M''_y$ with M'_y the image of M_α. By assumption M'_x and M''_y are non-zero, thus there is a non-zero linear map $N\colon M_x \to M_y$ such that $N(M'_x) \subseteq M''_y$ and $N(M''_y) = 0$.

In order to construct a non-trivial self-extension of M, start with the direct sum $M \oplus M$ and replace the map $(M \oplus M)_\alpha = \begin{bmatrix} M_\alpha & 0 \\ 0 & M_\alpha \end{bmatrix}$ by the map $\begin{bmatrix} M_\alpha & N \\ 0 & M_\alpha \end{bmatrix}$. The properties of N show that the rank of this map is larger then the rank of $(M \oplus M)_\alpha$, thus the new representation cannot be isomorphic to $M \oplus M$, and obviously, it is an extension of M by itself.

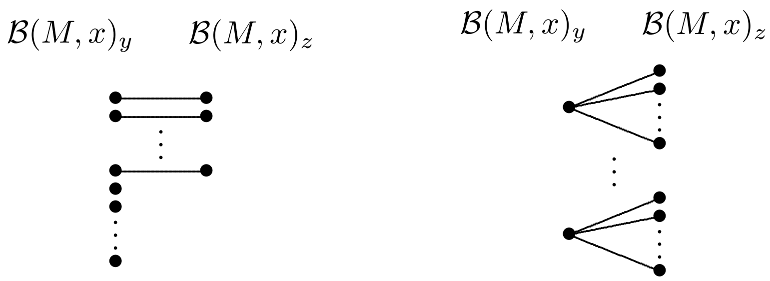
Proposition 3. *Let (M, x) be an exceptional radiation module with radiation basis $\mathcal{B}(M, x)$. Consider an edge $\{y, z\}$ in $\overline{Q}$ such that $d(x, y) + 1 = d(y, z)$. Then any element of $\mathcal{B}(M, x)_z$ is connected to precisely one element of $\mathcal{B}(M, x)_y$ by an arrow.*

If $\dim M_y \leq \dim M_z$, then any element of $\mathcal{B}(M, x)_y$ is connected to at least one element of $\mathcal{B}(M, x)_z$. If $\dim M_y \geq \dim M_z$, then any element of $\mathcal{B}(M, x)_y$ is connected to at most one element of $\mathcal{B}(M, x)_z$.

Note that here we consider a path

$$x \;\text{——}\quad \cdots \quad \text{——}\; y \overset{\alpha}{\text{——}} z$$

in the tree $\overline{Q}$. It follows that the edges between $\mathcal{B}(M, x)_y$ and $\mathcal{B}(M, x)_z$ are of one of the following forms:



Corollary 1. *Let (M, x) be an exceptional radiation module. Consider an edge $\{y, z\}$ in $\overline{Q}$ such that $d(x, y) + 1 = d(y, z)$. Then the number of arrows between $\mathcal{B}(M, x)_y$ and $\mathcal{B}(M, x)_z$ is precisely* $\dim M_z = |\mathcal{B}(M, x)_z|$.

Corollary 2. *Let (M, x) be an exceptional radiation module. Let α be an arrow which connects the vertices y and z. If $\dim M_y = \dim M_z$, then with respect to a suitable ordering of the bases $\mathcal{B}(M, x)_y$ and $\mathcal{B}(M, x)_z$, the map M_α is given by the identity matrix.*

Proof of Proposition 3. Write $\mathcal{B} = \mathcal{B}(M, x)$. First, we claim that any element of $\mathcal{B}_z$ is connected to precisely one element of $\mathcal{B}_y$ by an arrow. This is an immediate consequence of the way the radiation quiver is constructed. Namely, any vertex b of $R(M, x)$ occurs at a certain step as an origin. In the next step, this vertex b is connected by an arrow to the new origin, say b'. In later steps, no further arrows are attached which involve b. Thus, let $d(x, z) = i$ and let D be the maximal distance of vertices to x. We start with step 0, namely with the disjoint union of radiation quivers consisting of a single vertex, one for each vertex with distance D to x. The basis $\mathcal{B}_z$ is constructed in step $D-i$, all the elements of $\mathcal{B}_z$ are origins of radiation quivers. In the step $D-i+1$, the basis $\mathcal{B}_y$ is constructed: here, any vertex $b \in \mathcal{B}_z$ is connected by an edge to precisely one vertex $b' \in \mathcal{B}_y$.

For the second assertion, we have to take into account the orientation of the arrow α which connects y and z, as well as whether M_α is injective or surjective.

(1) *If α starts at y and M_α is injective, then any vertex in $\mathcal{B}_y$ is connected by an edge to at least one vertex in $\mathcal{B}_z$.* Proof: If $b \in \mathcal{B}_y$ is not connected by an edge to a vertex in $\mathcal{B}_z$, then b belongs to the kernel of M_α.

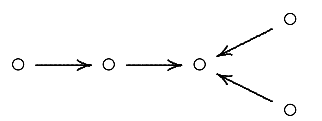
(2) *If α starts at y and M_α is surjective, then any vertex in $\mathcal{B}_y$ is connected by an edge to at most one vertex in $\mathcal{B}_z$.* Proof: If there are vertices $b' \neq b''$ in $\mathcal{B}_z$ which are connected by edges to $b \in \mathcal{B}_y$, then neither b' not b'' belong to the image of M_α.

(3) *If α starts at z and M_α is injective, then any vertex in $\mathcal{B}_y$ is connected by an edge to at most one vertex in $\mathcal{B}_z$.* Proof: If there are vertices $b' \neq b''$ in $\mathcal{B}_z$ which are connected by edges to $b \in \mathcal{B}_y$, then $b' - b''$ belongs to the kernel of M_α.

(4) *If α starts at z and M_α is surjective, then any vertex in $\mathcal{B}_y$ is connected by an edge to at least one vertex in $\mathcal{B}_z$.* Proof: If $b \in \mathcal{B}_y$ is not connected by an edge to a vertex in $\mathcal{B}_z$, then b is not in the image of M_α.

We should stress that an exceptional radiation module may have additional tree bases which are not radiation bases.

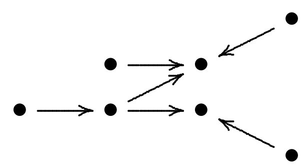
Example 6. We deal with the quiver Q of type $\mathbb{D}_5$ with subspace orientation.



Q

Let M be the maximal indecomposable module. There are three vertices x with $\dim M_x = 1$, namely the leaves of the quiver Q. It is easy to see directly that (M, x) is a radiation module for any leaf x of Q, this is a special case of Proposition 4 shown in the next section.

Here is a coefficient quiver for M, we denote it by Γ:



Γ

This cannot be a radiation quiver! Let Q^x be obtained by removing any leaf x from Q, the restriction M' of M to Q^x is the direct sum of two indecomposable modules, thus if $R(M, x)$ is a radiation quiver of M, then there are two arrows starting in x. But in Γ, only one arrow starts at any leaf of Q. This shows that Γ cannot occur as a radiation quiver.

We also may invoke Corollary 2 above. Observe that there is an arrow $\alpha \colon y \to z$ such that $\dim M_y = \dim M_z = 2$. The matrix presentation given by Γ involves three non-zero coefficients for M_α, whereas Corollary 2 assert that the matrix presentation of M_α with respect to a radiation basis has only two non-zero coefficients.

4. Dynkin quivers.

Proposition 4. *Let Q be a Dynkin quiver and M an indecomposable module with a thin vertex x. Then (M, x) is a radiation module.*

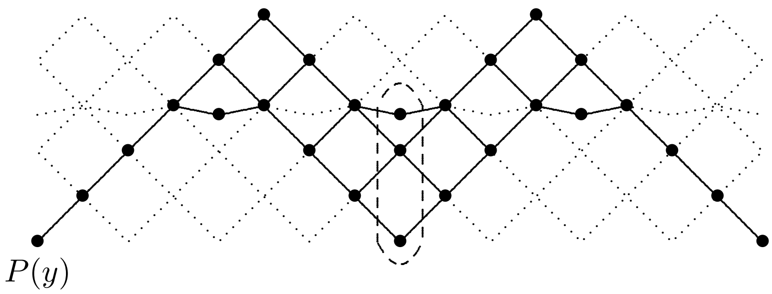
This is an immediate consequence of the following lemma.

Lemma 3. *Let Q be a Dynkin quiver and M an indecomposable module with thin vertex x. Let M' be the restriction of M to the quiver obtained by deleting the vertex x. Then M' decomposes as the direct sum of orthogonal indecomposable modules $N(i)$. For every index i, there is a unique neighbor $y(i)$ of x with* $\dim N(i)_{y(i)} \neq 0$ *and we have* $\dim N(i)_y = 1$.

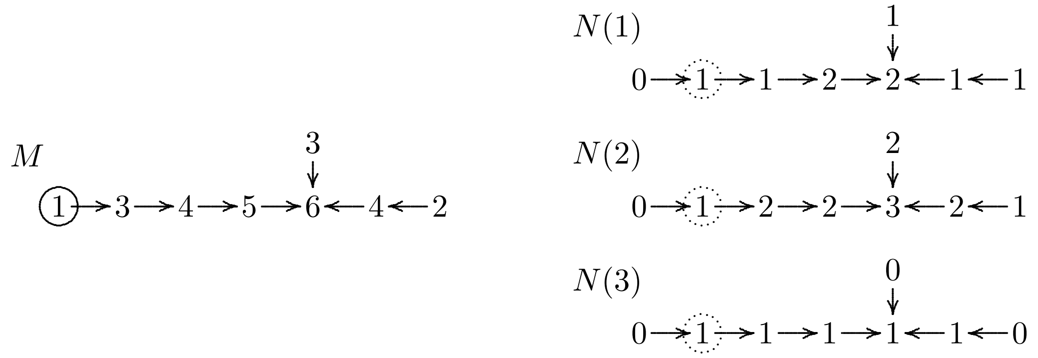
Proof of lemma 3. For a Dynkin quiver, all indecomposable representations are exceptional, thus we can apply proposition 1(b) and decompose $M' = \bigoplus N(i)^{n(i)}$ where $N(1), \dots, N(t)$ is an exceptional family. For every i, there is a unique neighbor y of x with $\dim N(i)_y \neq 0$. If $\dim N(i)_{y(i)} \geq 2$ for some i, then the process of simplification (see [R1]) asserts that the full subcategory of all modules with a filtration with factors $S(x)$ and $N(i)$ is representation infinite, impossible. This shows that the modules $N(i)$ form an orthogonal exceptional family.

Note that for Δ a Dynkin quiver, the only exceptional modules without a thin vertex are the maximal indecomposable modules for the quivers of type $\mathbb{E}_8$.

Example 7. Consider the quiver of type $\mathbb{E}_8$ with subspace orientation and let M be the maximal indecomposable representation with a thin vertex, say the vertex x. Then x has a unique neighbor y and $\dim M_y = 3$. It follows that the restriction M' of M to Q^x is the direct sum of 3 indecomposable modules $N(1), N(2), N(3)$. The easiest way to determine these modules $N(i)$ explicitly is as follows: Consider the hammock H_y for the path algebra kQ^x, this is the support of the functor $\mathrm{Hom}(P(y), -)$ and has the following shape

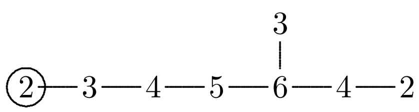


(it is a poset with partial ordering going from left to right). There is only one triple of pairwise incomparable elements, it is surrounded by a dashed line. The corresponding modules $N(1), N(2), N(3)$ are the direct summands of M' which we are looking for:



We have mentioned already that an exceptional representation M of a Dynkin quiver without a thin vertex is the maximal indecomposable representation of a quiver Q of type $\mathbb{E}_8$. Let us show that the methods presented here provide also for these modules a quite nice tree basis.

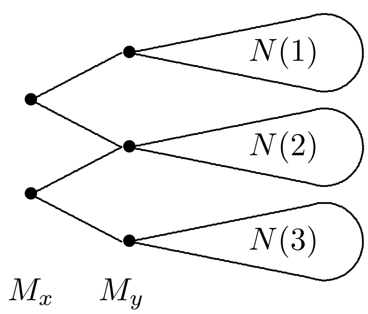
Note that $\mathbf{dim}\, M$ looks as follows



We denote by x the encircled vertex and by y its neighbor. We consider again the quiver Q^x obtained from Q by deleting the vertex x (as well as the arrow involving x) and by M' the restriction of M to Q^x. According to Proposition 1(b), we can write $M' = \bigoplus_{i=1}^t N(i)^{n(i)}$, where $N(1), \dots, N(t)$ is an exceptional family of indecomposable modules, all $n(i) \geq 1$, and all $\dim N(i)_y \geq 1$.

Now Q^x is a quiver of type $\mathbb{E}_7$, thus any indecomposable representation N of Q^x satisfies $\dim N_y \leq 1$. This shows that $\dim N(i)_y = 1$ for all i, and therefore $t = 3$. It follows that $N(1), N(2), N(3)$ is an orthogonal family, again we refer to Proposition 1(b).

As we have seen in Example 7, the hammock H_y for such a quiver of type $\mathbb{E}_7$ has a unique triple of incomparable elements: the corresponding representations of Q^x are our modules $N(1), N(2), N(3)$. Proposition 4 asserts that the pairs $(N(i), y)$ are radiation modules. We obtain a tree basis of M by using the radiation bases of the pairs $(N(i), y)$ and connecting the origins (they form a basis of M_y) with basis elements of M_x as follows:



Of course, the dimension vectors of the modules $N(i)$ (in particular the actual radiation bases of the $N(i)$) depend on the orientation of Q.

Remark. In the special cases of dealing with a Dynkin quiver with a unique sink, Proposition 4 has been obtained independently in the Diplom thesis of V. Katter, see [KM].

5. Bipartite trees without leaves

Let us now draw the attention to infinite quivers, but we will assume that the quivers are locally finite without infinite paths. For a general discussion of the Auslander-Reiten components of infinite quiver we may refer to [BLP].

The quiver Q is said to be *bipartite* provided that every vertex is a sink or a source. Note that any graph which is a tree can be endowed with precisely two orientations so that we obtain a bipartite tree quiver. A vertex x of a tree quiver is called a *leaf* provided there is at most one arrow α which starts or ends in x. Note that a tree quiver without leaves has to be infinite. As typical examples one should take the n-regular tree with bipartite orientation, where $n \geq 2$. These are the quivers which we later will use when dealing with the generalized Kronecker quivers. We should stress that the case $n = 3$ was already discussed in [FR].

If Q is a quiver, let Q^* be the opposite quiver: it has the same vertices, but every arrow $\alpha\colon x \to y$ is replaced by an arrow $\alpha^*\colon y \to x$.

Let Q be a bipartite tree quiver. Given two vertices x, y, let $d(x, y)$ be the distance between x and y. We denote by $N(x)$ the set of neighbors of x (thus the set of vertices y with $d(x, y) = 1$).

If $x \in Q_0, t \in \mathbb{N}_0$, let $B(x, t)$ be the set of vertices y with $d(x, y) \leq t$, we may call it the *ball* with center x and radius t. If necessary, we will consider $B(x, t)$ as a full subquiver of Q. We will be interested in the balls $B(x, t)$ where x is a sink and t is even, as well as in those where x is a source and t is odd. Note that for these pairs x, t, the boundary $\{y \mid d(x, y) = t\}$ of $B(x, y)$ consists of sinks (if x is a sink, then y is also a sink iff $d(x, y)$ is even. If x is a source, then y is a sink iff $d(x, y)$ is odd).

We denote by ρ^- the composition of all BGP-reflection functors at all the sources of Q, this is a functor $\rho^-\colon \operatorname{mod} kQ \to \operatorname{mod} kQ^*$. Note that $\rho^-\rho^-$ is just the Auslander-Reiten translation τ^-, see Gabriel [G] (here we use that $\overline{Q}$ has no cyclic paths of odd length).

Given a bipartite tree quiver Q without leaves, let us define indecomposable representations $P(x, t) = P_Q(x, t)$ for certain $x \in Q_0, t \in \mathbb{N}_0$ as follows: If x is a sink, let $P(x, 0) = S(x) = P(x)$, the one-dimensional representation with support $\{x\}$. If x is a source, let $P(x, 1) = P(x)$, the indecomposable projective module with top $S(x)$; it is the unique thin module with support $B(x, 1)$. For $t \geq 2$, define $P(x, t) = \tau^- P(x, t-2)$. Thus, by definition

$$\begin{aligned} P(x, 2t) &= \tau^{-t} P(x, 0), \\ P(x, 2t+1) &= \tau^{-t} P(x, 1). \end{aligned}$$

Looking at the same time at the quivers Q and Q^*, we see that for all $t \geq 1$

$$P_Q(x, t) = \rho^- P_{Q*}(x, t-1).$$

We call the modules of the form $P(x, t)$ the *preprojective* modules.

Lemma 4.

(a) *The support of* $P(x,t)$ *is* $B(x,t)$.

(b) *If* $t \geq 1$ *and* $d(x,y) \in \{t-1, t\}$, *then* $\dim P(x,t)_y = 1$.

(c) *Let* X *be an indecomposable representation of* Q. *Then* $\operatorname{Hom}(X, P(x,t)) \neq 0$ *if and only if* X *is preprojective and the support of* X *is contained in the support of* $P(x,t)$.

(d) *Let* $P = P(x,t)$ *and* z *a vertex with* $P_z = 0$. *Then* $d(x,z) = t+1$ *if and only if* $\operatorname{Ext}^1(S(z), P) \neq 0$ *and then* $\dim \operatorname{Ext}^1(S(z), P) = \dim P_y = 1$.

Proof: (b) We use induction, the case $t = 0$ being trivial. Let $t \geq 1$. We know that $P_Q(x,t) = \rho^- P_{Q*}(x, t-1)$. Let y be a vertex of Q with $d(x,y) = t$. There is a unique neighbor y' of y with $d(x,y') = t-1$ and $\dim P_{Q*}(x,t-1)_{y'} = 1$ (this is clear for $t = 1$, whereas for $t > 1$ this is the induction hypothesis). The definition of ρ^- shows that $P(x,t)_y = P_{Q*}(x,t-1)_{y'}$, thus $\dim P(x,t)_y = 1$. If $t \geq 1$ and y' is any vertex of Q with $d(x,y') = t-1$, then ρ^- does not change the vector space at the position y', thus $\dim P(x,t)_{y'} = \dim P_{Q*}(x,t-1)_{y'} = 1$.

(a) The induction procedure mentioned in (b) shows that the support of $P(x,t)$ is contained in $B(x,t)$, and as we have see, it contains all the vertices y with $d(x,y) = t$. But the support of an indecomposable module has to be connected, thus it has to be all of $B(x,t)$.

(c) Let X be an indecomposable representation of Q with $\operatorname{Hom}(X, P(x,t)) \neq 0$. We write $P(x,t) = \tau^{-s}P$ for some indecomposable projective module (namely $P = \tau^s P(x,t)$ and $s = \lfloor \frac{t}{2} \rfloor$). If $\tau^s X \neq 0$, then $\operatorname{Hom}(\tau^s X, P) \neq 0$ and therefore $\tau^s X$ is projective. Altogether, we see that $X = \tau^{-i} P'$ for some indecomposable projective module $P' = P(y)$ and $0 \leq i \leq s$. Since $\operatorname{Hom}(P', \tau^{-s+i} P) = \operatorname{Hom}(P', \tau^i P(x,t)) \neq 0$, it follows that y belongs to the support of $\tau^i P(x,t)$, thus to the support $B(x,t)$ of $P(x,t)$.

Conversely, consider the quiver $B(x,t)$. Since this is a tree quiver, the category of representations of $B(x,t)$ has a preprojective component. Let $\mathcal{C}(x,t)$ be the full subcategory of the indecomposable representations of $B(x,t)$ which are predecessors of $P(x,t)$, we may consider this as a subcategory of the category of representations of Q. As we have seen, any predecessor X of $P(x,t)$ in the category $\operatorname{mod} kQ$ belongs to $\mathcal{C}(x,t)$.

(d) Let $P = P(x,t)$ and z a vertex with $P_z = 0$ and $\operatorname{Ext}^1(S(z), P) \neq 0$. Since $P_z = 0$, we must have $d(x,z) > t$. Since $\operatorname{Ext}^1(S(z), P) \neq 0$, we see that $d(x,z) = t+1$. Then $\dim \operatorname{Ext}^1(S(z), P) = \dim P_y = 1$, according to (b).

Here is a description of the Auslander-Reiten sequences starting in $P(x,t)$

$$0 \to P(x,t) \to \bigoplus\nolimits_{y \in N(x)} P(y, t+1) \to P(x, t+2) \to 0.$$

We recall the inductive definition of the *reachable* objects of a length category: First of all, the simple projective objects are reachable. Second, if M is indecomposable, but not simple projective, then M is reachable provided that there exists a minimal right almost split map $M' \to M$ such that all the indecomposable direct summands of M' are reachable.

Proposition 5. *The preprojective modules are reachable and they form a component of the Auslander-Reiten quiver.*

It is of interest to stress the following property of this preprojective component $\mathcal{P}$: for any indecomposable object X in $\mathcal{P}$, there are sectional paths starting in a simple projective module S and ending in X, namely, there is a sectional path from $S = P(y, 0)$ to $X = P(x, t)$, provided $d(x, y) = t$.

Proposition 6. *For $t \geq 1$, the pairs $(P(x, t), y)$ with $d(x, y) \in \{t, t-1\}$ are radiation modules.*

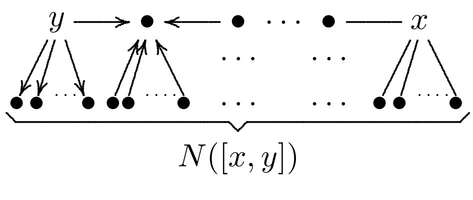
Proof. According to Lemma 4(b), we know that the vertices y with $d(x, y) \in \{t, t-1\}$ are thin vertices for $P(x, t)$. For the proof of the proposition, we use induction with respect to t, the case of $t = 1$ being obvious.

Thus, consider for $t \geq 2$ the pairs $(P(x, t), y)$ with $d(x, y) \in \{t, t-1\}$. The essential case to deal with is $d(x, y) = t - 1$. Namely, if y is a vertex with $d(x, y) = t$, then there is a unique neighbor y' of y inside the support of $P(x, y)$, and we have both $\dim P(x, t)_y = 1$ and $\dim P(x, t)_{y'} = 1$. Note that $d(x, y') = t-1$. If we know that $(P(x, t), y')$ is a radiation module, then clearly also $(P(x, t), y)$ is a radiation module.

Let us assume now that $d(x, y) = t - 1$. Let U be the submodule of $P(x, t)$ with $P(x, t)/U = S(y)$, this is just the restriction of $P(x, t)$ to Q^y. We write $U = \bigoplus_{i \in I} N(i)^{n(i)}$, with pairwise non-isomorphic indecomposable modules $N(i)$ and integers $n(i) \geq 1$. Since U is a submodule of $P(x, t)$, we know that these submodules $N(i)$ are again preprojective modules, see Lemma 4(c). Also, any such module $N(i)$ satisfies $N(i)_y = 0$. Of course, we also have $\mathrm{Ext}^1(S(y), N(i)) \neq 0$, and the extensions in $\mathrm{Ext}^1(S(y), N(i))$ are furnished by an arrow $y \to y(i)$ of Q, namely by the unique arrow which connects y with the support of $N(i)$. According to Lemma 4(d), we know that $\dim N(i)_{y(i)} = 1$ for all $i \in I$. Thus, we can apply Proposition 1(b) in order to see that the modules $N(i)$ form an orthogonal exceptional family. It remains to look at the modules $N(i)$. First of all, if $y(i)$ is a neighbor of y with $d(x, y(i)) = t$, then $N(i) = S(y(i))$ and the pair $(N(i), y(i))$ is a trivial radiation module. For the remaining modules $N(i)$ we have $d(x, y(i)) = t - 2$. Since $N(i)$ is preprojective, it is of the form $N(i) = P(x', t')$ for some vertex x' and $t' \leq t$. Since the support of $N(i)$ is properly contained in the support of $P(x, t)$, we even have $t' < t$. The vertex $y(i)$ has to belong to the boundary of the support of $N(i)$. By induction we know that $(N(i), y(i))$ is a radiation module. Thus we see that $(P(x, t), y)$ is a radiation module.

We have seen in the proof of Proposition 6 that for a source y and $d(x, y) = t-1$, the kernel of the canonical epimorphism $P(x, t) \to S(y)$ is a direct sum of orthogonal bricks. Let us describe this kernel in more details.

Given a pair x, y of vertices of Q, we denote by $[x, y]$ the set of vertices lying on the path between x and y. If S is a set of vertices of Q, we denote by $N(S)$ the set of neighbors of S: these are the vertices z which do not belong to S but such that there is a vertex $z' \in S$ with $d(z, z') = 1$. We will be interested in the sets $N([x, y])$, where x, y are vertices with y a source:



(This picture reminds on centipedes.)

We consider vertices $y \neq z$, where y is a source, and look at the modules $S(y)$ and $P(z, d(y, z) - 1)$.

Lemma 5. *Let y be a source and $z \neq y$ some other vertex. Then $S(y), P(z, d(y, z)-1)$ is an orthogonal pair with*

$$\dim \operatorname{Ext}^1(P(z, d(y, z) - 1), S(y)) = 0$$
$$\dim \operatorname{Ext}^1(S(y), P(z, d(y, z) - 1)) = 1.$$

Proof. According to Lemma 4(a), we have $P(z, d(y, z) - 1)_y = 0$, thus the pair $S(y), P(z, d(y, z) - 1)$ is orthogonal. We have $\operatorname{Ext}^1(P(z, d(y, z) - 1), S(y)) = 0$, since $S(y)$ is injective. According to Lemma 4(d), we have $\dim \operatorname{Ext}^1(S(y), P(z, d(y, z) - 1)) = 1$.

If $d(x, y) = t - 1$ and we look at $z \in N([x, y])$ (thus z is one of the legs of the centipedes), then we have

$$d(y, z) - 1 = t - d(x, z).$$

(Namely, let z' be the neighbor of z which belongs to $[x, y]$, then $d(y, z) - 1 = d(y, z')$ and $t - d(x, z) = 1 + d(x, y) - d(x, z') - 1 = d(x, y) - d(x, z') = d(y, z')$.) This explains that we will have to consider modules $P(z, t')$ with $t' = t - d(x, z)$.

Proposition 7. *Let y be a source and x any vertex. Let $t = d(x, y) + 1$. Then there is the following exact sequence:*

$$0 \to \bigoplus_{z \in N([x,y])} P(z, t - d(x, z)) \to P(x, t) \to S(y) \to 0$$

The modules $S(y)$ and $P(z, t - d(x, z))$ with $z \in N([x, y])$ are pairwise orthogonal bricks and the map $P(x, t) \to S(y)$ is the projective cover of $S(y)$ in the full subcategory

$$\mathcal{F}\Big(S(y); P(z, t - d(x, z)), z \in N([x, y])\Big).$$

Proof. We use induction on t. In the case $t = 1$, we have $x = y$ and there is the exact sequence

$$0 \to \bigoplus_{z \in N(x)} P(z, 0) \to P(x, 1) \to S(x) \to 0.$$

Now assume the assertion is true for some $t \geq 1$. Consider a pair of vertices x, z_0 with $d(x, z_0) = t$. There is a unique vertex y with $d(x, y) = t-1$ and $d(y, z_0) = 1$. By induction, there is the exact sequence

$$(*) \qquad 0 \to \bigoplus\nolimits_{z \in N([x,y])} P(z, t - d(x, z)) \to P(x, t) \to S(y) \to 0.$$

Choose some neighbor z_0 of y with $d(x, z_0) = t$. There is (up to isomorphism) a unique module M with top $S(y)$ and socle $S(z_0)$ and we may rearrange the factors in $(*)$ in order to obtain an exact sequence of the form

$$0 \to \bigoplus_{z \in N([x,y]), z \neq z_0} P(z, t - d(x, z)) \to P(x, t) \to M \to 0.$$

Now apply the functor ρ^-. We obtain the sequence

$$(**) \qquad 0 \to \bigoplus_{z \in N([x,y]), z \neq z_0} P(z, t + 1 - d(x, z)) \to P(x, t + 1) \to \rho^- M \to 0.$$

The module $\rho^- M$ has top $S(z_0)$ and its socle is the direct sum of the modules $S(z)$ where $z \neq y$ is a neighbor of z_0. Of course, for these vertices z, we have $S(z) = P(z, 0)$ and $d(x, z) = t + 1$. Since these modules $P(z, 0)$ are projective, we obtain from $(**)$ an exact sequence of the form

$$0 \to U \to P(x, t + 1) \to S(z_0) \to 0$$

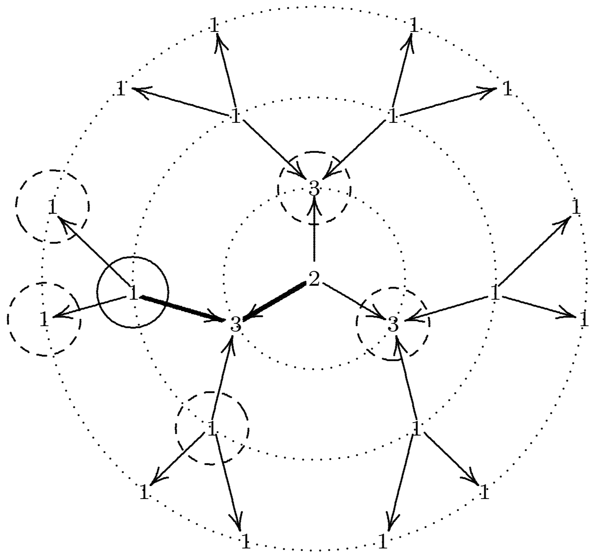
where U is the direct sum of the modules $P(z, t + 1 - d(x, z))$ with $z \in N([x, y]), z \neq z_0$ as well as those of the form $P(z, 0) = P(z, t + 1 - d(x, z))$ with $z \neq y$ a neighbor of z_0. It remains to observe that the elements in $N([x, z_0])$ are precisely the vertices $z \in N([x, y]), z \neq z_0$ and the neighbors $z \neq y$ of z_0. Thus we see that we obtain for the pair x, z_0 the required sequence.

Now let us show that the modules $S(y)$ and $P(z, t - d(x, z))$ and $z \in N([x, y])$ are orthogonal and that the only non-trivial extension groups between these modules are the groups $\mathrm{Ext}^1(S(y), P(z, t - d(x, z))$ and these are one-dimensional. We can assume that $t \geq 2$ and we denote by y' the unique vertex with $d(y, y') = 1$ and $d(y', x) = t - 2$. If $d(x, z) = t$, then $P(z, t - d(x, z)) = P(z, 0) = S(z)$ is simple projective, not isomorphic to $S(y)$ and $\dim \mathrm{Ext}^1(S(y), S(z)) = 1$. Also, for $z \neq z'$, the modules $S(z), S(z')$ are orthogonal. Thus, assume that $d(x, z) < t$. Let $z' \in [x, y]$ with $d(z, z') = 1$. Write $a = d(y, z')$ and $b = d(z', x)$, thus $a + b = t - 1$ and $d(z, x) = b + 1$, thus $t - d(x, z) = a$. Since $d(x, z) < t$, we see that $z' \neq y$, thus $d(z, y') = a$ and therefore $P(z, a)_y = 0$ and $\dim P(z, a)_a = 1$. According to Lemma 4(d), we have $\dim \mathrm{Ext}^1(S(y), P(z, a)) = 1$. It follows from $P(z, a)_y = 0$ that $P(z, a)_{z'} = 0$ for $d(x, z') = t$, thus $P(z, a)$ and $P(z', 0)$ are orthogonal, and also that $\mathrm{Ext}^1(P(z, a), P(z', 0)) = 0$. Assume now that the we deal with two vertices $z_1 \neq z_2$ in $N([x, y])$ with $d(x, z_2) \leq d(x, z_1) < t$, let $a_1 = t - d(x, z_1)$, and $a_2 = t - d(x, z_2)$, thus $a_1 \leq a_2$. In order to see that there are no homomorphisms or extensions between $P(z_1, a_1)$ and $P(z_2, a_2)$, we can apply ρ^{a_1}, thus we have to consider $P(z_1, 0)$ and $P(z_2, a_2 - a_1)$. Note that $d(z_1, z_2) = a_2 - a_1 + 2$, thus z_1 is not in the support of $P(z_2, a_2 - a_1)$ and not even a neighbor of this support. This completes the proof.

Corollary. *Let y be a source and x any vertex. The family of modules $P(z, d(y, z) - 1)$ with $z \in N([x, y])$ is an exceptional orthogonal family of bricks.*

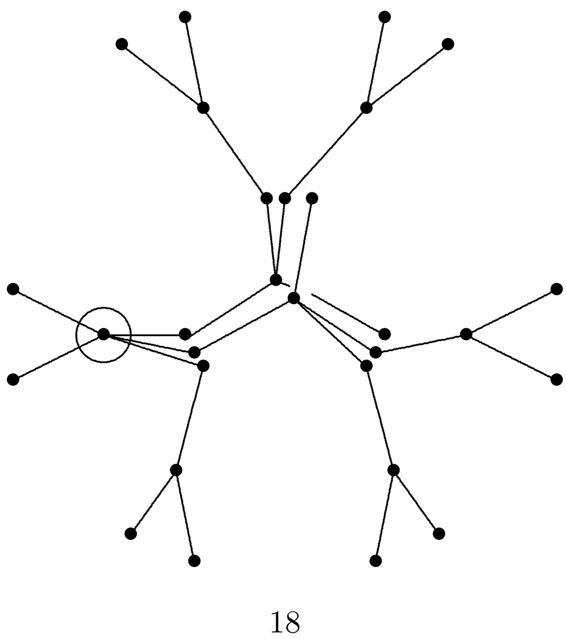
As an example, let us consider the 3-regular tree with bipartite orientation as studied already in [FR]. Let us display the dimension vector $\mathbf{dim}\, P(x, 3)$ for a source x. As vertex

y we take a source with $d(x,y)=2$, thus $(P(x,3),y)$ is a radiation module.

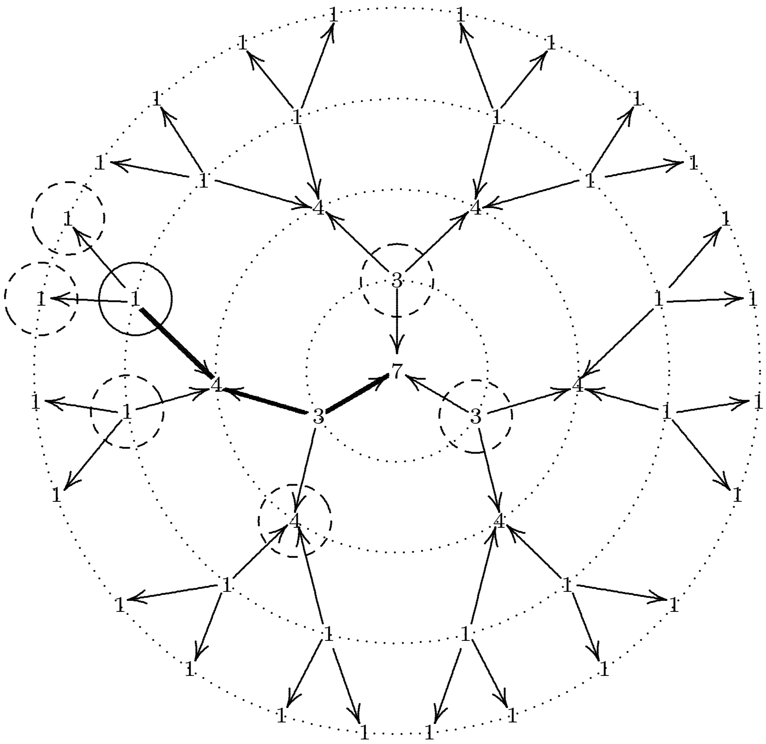


Here, x is the vertex at the center. The vertex y is encircled using a solid circle, the vertices $z \in N([x,y])$ are encircled using dashed circles.

Here is the radiation quiver $R(P(x,3),y)$, the vertex y has been encircled.



Finally, let us show the dimension vector $\mathbf{dim}\, P(x, 4)$ with x a sink.



Again, x is the vertex at the center. We have chosen a source y such that $d(x, y) = 3$, the vertex y is encircled using a solid circle, whereas the dashed circles mark the vertices $z \in N([x, y])$. Altogether, we see that there is an exact sequence of the form

$$0 \to U \to P(x, 4) \to S(y) \to 0$$

with

$$U = P(z_1, 0) \oplus P(z_2, 0) \oplus P(z_3, 1) \oplus P(z_4, 2) \oplus P(z_5, 3) \oplus P(z_6, 3)$$

6. The generalized Kronecker quivers and Schofield induction.

The generalized Kronecker quivers are the quivers $K(n)$ with 2 vertices, a sink and a source, and n arrows (going from the source to the sink); the case $n = 2$ is the ordinary Kronecker quiver, its representations have been studied by Weierstraß and Kronecker. The universal cover $Q(n)$ of the quiver $K(n)$ is the n-regular tree with bipartite orientation. Using the push-down functor, any indecomposable representation of the quiver $Q(n)$ yields an indecomposable representation of $K(n)$. In particular, the representation $P_{Q(n)}(x, t)$ of $Q(n)$ defined in the last section provides an indecomposable representation $P_{K(n)}(t)$ and one obtains in this way just the preprojective $K(n)$-modules (the special case $n = 3$ has been discussed in detail in [FR], all the considerations presented there can easily be adapted to the general case). Obviously, under the push-down functor a tree basis is sent to a tree basis. Since all the modules $P(x, t)$ are radiation modules we obtain in this way distinguished tree bases for the preprojective $K(n)$-modules. The dual considerations yield distinguished tree bases for the preinjective $K(n)$-modules.

Let us consider now an arbitrary quiver Q and M an exceptional representation of Q. It has been shown in [R3] (see also [R5]) that M is a tree module. Here we want to outline in which way we can use the previous assertions in order to exhibit a nice tree basis of M. In order to exhibit a tree basis for an exceptional module M, one uses Schofield induction, thus one considers exact sequences of the form

$$0 \to Y^y \to M \to X^x \to 0$$

with indecomposable middle term, where (X, Y) is an orthogonal exceptional pair with $\dim \mathrm{Ext}^1(X, Y) = e > 0$ and (x, y) is the dimension vector of a sincere preprojective or preinjective representation E of the e-Kronecker quiver $K(e)$. Note that the triple $(X, Y; E)$ uniquely determines M and one obtains in this way inductively all the exceptional representations of Q. In order to construct a tree basis of M we need to know tree bases of X, Y and E. For the procedure to obtain a tree basis of M from the tree bases of X, Y, E we refer to [R3] (see also [R5]).

A required tree basis for E has been exhibited already in [R3], but the present note provides an intrinsic way for obtaining such a tree basis. Let us repeat: Instead of working with the e-Kronecker quiver itself, we consider its universal cover, the e-regular tree $Q(e)$ with bipartite orientation. Let E be obtained from the representation $\widetilde{E}$ of $Q(e)$ by the push-down functor. Now $Q(e)$ is a bipartite tree quiver without leaves and $\widetilde{E}$ is a preprojective or preinjective representation of Q. Thus, there is a vertex x such that the pair $(\widetilde{E}, x)$ is a radiation quiver. But this means that $\widetilde{E}$ has a distinguished tree basis. Under the push-down functor, we obtain a distinguished tree basis for E.

References

[BLP] R. Bautista, S. Liu, and C. Paquette, Representation theory of strongly locally finite quivers, Proc. London Math. Soc. (to appear).

[FR] Ph. Fahr, C. M. Ringel: Categorification of the Fibonacci numbers using representations of quivers. Journal of Integer Sequences. Vol. 15 (2012), Article 12.2.1

[G] P. Gabriel: Auslander-Reiten sequences and representation-finite algebras. In Representation theory I. Springer LNM 831, Berlin (1980) 1-71.

[K] R. Kinser: Rank functions on rooted tree quivers. Duke Math. J. 152 (2010), 27-92.

[KM] V. Katter, N. Mahrt: Reduced representations of rooted trees. (To appear).

[R1] C. M. Ringel: Representations of K–species and bimodules. J. Algebra 41 (1976), 269-302.

[R2] C. M. Ringel: Tame algebras and integral quadratic forms. Springer LNM 1099 (1984).

[R3] C. M. Ringel: Exceptional modules are tree modules. Lin. Alg. Appl. 275-276 (1998) 471-493.

[R4] C. M. Ringel: The Gabriel-Roiter measure. Bull. Sci. math. 129 (2005), 726-748.

[R5] C. M. Ringel: Indecomposable representations of the Kronecker quivers. Proc. Amer. Math. Soc. 141 (2013), 115-121.

[RV] Ringel, C. M., Vossieck, D.: Hammocks. Proc. London Math. Soc. (3) 54 (1987), 216-246.

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