Algebra at the Turn of the Centuries

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ABSTRACT. At the turn of the centuries it seems to be appropriate to pause and to try to envision future possibilities. We want to discuss the prospects of algebra. To look into the future requires an understanding of the past, the longstanding aims, but also the difficulties which have been encountered. We are going to review part of the history of algebra in order to outline its present state. It is important to notice the missed opportunities and to analyze the reasons. To recognize the present possibilities requires to be aware of the tools which now are available and which may not yet have been used in an optimal way. We urge the reader to focus attention to the need for algebraic considerations in all parts of mathematics but also outside of mathematics. Of course, a view back should also strengthen the interest in classical open problems which now may be feasible to attack.

1. The development of algebra. What is algebra? The word is derived from the title of an Arabic text-book labeled AL-JABR W AL-MUQĀBALA by AL-KHWĀRIZMĪ, the meaning of the title is *to move back and forth* and it refers to the usual methods of solving linear equations, the book was concerned with linear and some quadratic equations: AL-JABR is the addition of equal terms to both sides of an equation in order to eliminate negative terms, AL-MUQĀBALA is the subtraction of equal terms from both sides of an equation [vW]. Note that also the name of the author AL-KHWĀRIZMĪ is present in todays terminology, namely in the word *algorithm.* The Chinese name for algebra, DÀI SHÙ reveals another aspect, the use of variables which stand for numbers or related entities. The sets of entities to be handled by algebraic operations was successively extended during the centuries, by CARDANO and GAUSS, by HAMILTON, GRASSMANN and CAYLEY: to larger number systems such as the complex numbers, the quaternions, but also to operators and so on.

The 20-th century has seen the development of what has been called *Modern Algebra,* again this is the title of a book, this times by VAN DER WAERDEN; its emphasis lies on the study of algebraic structures such as groups, rings, fields, in contrast to individual equations. Starting from the fourth edition (1955), the book was renamed *Algebra* pretending that the so called modern algebra is the universal approach to algebraic questions. The two aspects of algebra which we have mentioned can also be seen in the modern approach: on the one hand, there is the axiomatic method: the axioms of an algebraic structure fix the rules for calculations. On the other hand, one often needs presentations of algebraic objects by generators and relations, say as factor structures of free structures, and the free structures are usually generated by "variables": typical examples are the polynomial rings $k[T_1,\dots,T_n]$ in the theory of commutative k-algebras, or the free algebras $k\langle X_1,\dots,X_n\rangle$ when dealing with non-commutative k-algebras.

This is a the written version of a lecture presented at the National Conference on Algebra VII, held in Beijing Normal University, October 1999.

It is essential to consider the relationship between algebra and geometry. A first intuitive approach would try to stress the differences of algebraic and geometrical thinking, but at least since Descartes the algebraization of geometry has turned out to be an essential tool in geometry and an important source of algebraic objects. The possibility of using coordinates in geometry (thus algebraic tools) allows a description of geometric objects such as curves or surfaces in algebraic terms. But even the coordinate free approach to geometry, the study of geometric objects with respect to their internal symmetries relies on the use of algebraic structures: of groups, of Lie algebras and so on. In the 19-th century, a main task of algebra was to determine all or at least some of the invariants say of curves with respect to suitable symmetry operations; algebra was considered as "invariant theory". As it has turned out, there really is no difference between algebraic geometry and commutative algebra, there is the mutually possibility to translate back and forth. This is one of the great achievements of mathematics! And the main challenge for the future seems to be to establish a corresponding identification between algebra in general and a "non-commutative geometry" which still has to be developed. There are already several proposals for translation schemes, but all are partial and wait for further investigations. If we look at the commutative case, the basic setting is to deal with the polynomial ring $k[T_1, \dots, T_n]$ and its factor rings, or, equivalently, with the ideals of the ring $k[T_1, \dots, T_n]$, thus commutative algebra sometimes has been called "ideal theory". In dealing with the non-commutative analogy, the free algebra $k\langle X_1, \dots, X_n\rangle$, it seems to be advisable to consider not only ideals or left ideals, but, more generally, arbitrary modules; thus one needs a general "module theory" or "representation theory"[1].

As one of the main tools of algebra we have to mention the combinatorial considerations. Such considerations have been used in many different ways, let us mention just a few. There is the combinatorics of words (of sequences of letters) which is necessary for example for describing monomials. Second, finite configurations have played an important role in algebra, the main reference book for the advances of algebra in the 19-th century, Weber's algebra [W] is a splendid source: look at the study of cubic surfaces via the configuration of the 27 lines and its intersection pattern (every line meets precisely 10 others), or the study of quartic curves via the configuration of the 28 double tangents. Also, recall the use of Young diagrams and Young tableaux in order to handle questions concerning symmetric groups and general linear groups and their linear operations on vector spaces. The most prominent example is the classification of the finite dimensional semisimple complex Lie algebras via their root systems, thus via the Dynkin diagrams. The four series A_n, B_n, C_n, D_n correspond to the classical geometries, and there are the additional cases E_6, E_7, E_8, F_4 and G_2. It should be noted that the combinatorial configurations mentioned above are related to Dynkin diagrams: the 27 lines to the case E_6, the 28 double tangents to E_7. Of course, the Young diagrams and Young

[1] Given a ring R, one may distinguish between R-modules and representations of R, but usually we will not do so: a representation of R is given by a ring homomorphism $\varphi\colon R \to \mathrm{End}_{\mathbb{Z}}(M)$ from R into the endomorphism ring of an abelian group M, an R-module by a suitable map $R \times M \to M$; given a representation $\varphi\colon R \to \mathrm{End}_{\mathbb{Z}}(M)$, we obtain via $(r,m) \mapsto \varphi(r)(m)$ a corresponding R-module structure on M, and all are obtained in this way. It is just a matter of taste which point of view is stressed.

tableaux refer to the cases A_n.

Classical algebra had reached a peak at the end of the 19-th century. The vivid development of algebra during the centuries was followed by a sudden recline, the so called "death of invariant theory". It was based on decisive difficulties which were encountered and which could not be overcome at that time. Three difficulties have to be mentioned: First, the difficulties of calculations. Invariants (and these are just algebraic expressions) which had been determined turned out to have too many summands and too large coefficients. For example, SALMON's book [Sa] needs eight pages in order to tabulate the coefficients of one single invariant. Second, the difficulties of visualization. Dealing say with curves, one would like to see their points, but this is difficult to achieve. Several books were devoted to provide recipes for drawing at least part of such curves, see for example BRILL [Br], but this still was not very satisfactory. Third, theoretical difficulties. The translation of geometry into algebra was accomplished by dealing with the ring of regular functions on a geometrical object, and this ring is commutative and thus one can use the usual rules of calculations. However, it soon turned out to be quite decisive to work not only with regular functions but also with other data, for example with differential operators, and this could mean: to work with non-commutative structures. Clearly, for any calculation it is necessary to know the rules which are allowed, but such systems of axioms had not yet been developed.

What is the situation 100 years later? At the end of the 20-th century all the difficulties mentioned have been overcome. New tools are available. First of all: Computer algebra. Formal calculations can be done easily, the results can be stored easily, and all the information needed can be retrieved easily. The algorithmic approach to algebra is an essential one, from the dawn of algebra, and the computer algebra packages allow to overcome all the difficulties of actual computations. Second: Computer graphic. The shape of curves, of surfaces can be plotted very well. Let us recall that *Algebra is Geometry is Algebra,* thus any visual observation should also lead to further algebraic knowledge. Third, the "modern algebra" was conceived just in order to overcome the theoretical difficulties, to provide the general framework in order to deal with all kinds of algebraic structures, in particular also with non-commutative ones. The corresponding systems of axioms are well-understood and they allow to describe algebraic structures by generators and relations, a by now standard technique[2]. Thus, to repeat: all the difficulties have been overcome, all the necessary tools do exist, they just have to be used.

The need to study non-commutative structures comes from several quite different areas. As we have mentioned already, there are the algebras of differential operators, for example the Weyl algebras, which have to be handled. Second, in operator theory, it clearly is of interest to handle arbitrary projections in a given Hilbert space, but they will not commute. Of course, the representation theory of Lie algebras is the same as the representation theory of their enveloping algebras, and these enveloping algebras are non-commutative. Actually, in some sense these enveloping algebras still belong to the realm of commutative algebra – the reason being that an important operation, the comultiplication **is** (co-)commutative (thus the dual space is a commutative algebra). The really non-commutative versions are

[2] Actual there are two different proposals how to resolve the theoretical difficulties, one is entitled "modern algebra", the other one was conceived by mathematical logicians: a unification of these two approaches is still needed.

the corresponding quantum groups, here both multiplication and comultiplication are non-commutative. By now it is common to use the attribute *quantum* in order to express the fact that one replaces some commutative structure by a more general non-commutative one, that one invokes commutators which beforehand were supposed to vanish. There is quantum information theory, quantum coding theory, quantum probability theory and so on, these quantum theories seem to be quite fashionable!

One basic example of a non-commutative algebra should be mentioned here, the so called Kronecker algebra

$$K := \begin{bmatrix} k & 0 \\ k^2 & k \end{bmatrix}$$

here k is a field. It is a four-dimensional k-algebra and it occurs in many different surroundings. Its representations are just pairs of matrices of the same size, say $(m \times n)$-matrices, thus it may be used to classify matrix pencils, pairs of symmetric bilinear forms (in characteristic different from 2), and also the vector bundles over the projective line $\mathbb{P}_1$. The problem of classifying such pairs of matrices was considered by Weierstraß, but he found only part of the solution. The problem then was solved by Kronecker in 1890. It is curious to see that an essential part of the classification is that of finding what is called the Jordan normal form (in case k is algebraically closed, or the rational normal form in the general case) of a square matrix, but one has to take into account an additional possible eigenvalue ∞. To phrase it differently, the category of K-modules contains as a full exact subcategory the category of all $k[T]$-modules, where $k[T]$ is the polynomial ring in one variable T, and actually there are several such embeddings which overlap in the same way as affine lines overlap in $\mathbb{P}_1$. One may consider the (non-commutative algebra) K as a sort of compactification of the (commutative) algebra $k[T]$. It is a typical (but quite trivial) example for explaining the geometrical meaning of results from the representation theory of non-commutative rings. The Kronecker algebra K really may be considered as one of the main starting example for non-commutative geometry.

In our further discussion, we are going to present some other typical, but again very easy, examples of behavior in the general setting of non-commutative algebra. We will restrict to rings and modules, usually to (associative) algebras which are defined over some field k and to their representations. Similar considerations seem to apply to groups, to semigroups, to non-associative algebras and so on. Note that our restriction is not too severe, since after all many questions concerning these parallel theories can be recovered in terms of algebras and their representations using group algebras, semigroup algebras or enveloping algebras. We had planned to discuss also more advanced topics such as derived categories, A_∞-algebras, formal groups, or conformal algebras, but we refrain from doing so: it seems that the central considerations can be explained well using classical (and elementary) algebraic structures.

2. An example. Let us start with a problem posed by Krull in 1932: Does the Krull-Remak-Schmidt property (KRS) hold for artinian modules? The answer is No, and this was established only in 1995 in a joint paper of Facchini, Herbera, Levy and Vamos. We recall the following. Let R be any ring (associative, with 1) and M any R-module M. Such a module M is said to be indecomposable provided $M \neq 0$ and for any direct decomposition $M = M' \oplus M''$ either $M' = 0$ or $M'' = 0$. Of course, if M is of finite length[3], then there are indecomposable submodules $N_1, \dots, N_t$ such that $M = \bigoplus_{i=1}^t N_i$. The theorem of Krull-Remak-Schmidt asserts that such a decomposition is essentially unique: given a second set of indecomposable submodules $N_1', \dots, N_s'$ such that $N = \bigoplus_{i=1}^s N_i'$, then $s = t$ and there is a permutation π such that $N_i \simeq N'_{\pi(i)}$ for all i. Note that a module M is of finite length if and only if M is both noetherian and artinian. To be noetherian means that the ascending chain condition is satisfied for submodules of M, thus that for any sequence $M_1 \subseteq M_2 \subseteq \cdots \subseteq M$ of submodules there exists an index i with $M_i = M_{i+1}$. Similarly, to be artinian means that the descending chain condition is satisfied for submodules of M: given any sequence $M_1 \supseteq M_2 \supseteq \cdots$ of submodules M there exists an index i with $M_i = M_{i+1}$. If M satisfies at least one of the properties of being noetherian or artinian, then M can be written as a finite direct sum of indecomposables, and one may ask whether such a decomposition is essential unique, as in the case of a finite length module. Now, one knows that this is not the case for noetherian modules, in general. Typical examples are obtained by taking the direct sum of two suitable ideals M_i in a noetherian domain R which is not a principal ideal ring. Of course, if we consider the ring $R = \mathbb{Z}$ of all integers, thus R-modules are just abelian groups, then KRS holds for noetherian modules, but it is known since 1945 that it does not hold for torsionfree abelian groups of finite rank.

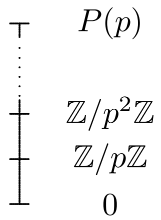
Now let us consider artinian modules. The usual examples of artinian modules which are not of finite length are the so called Prüfer groups: For any prime number p, there are embeddings

$$\mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathbb{Z}/p^2\mathbb{Z} \hookrightarrow \mathbb{Z}/p^3\mathbb{Z} \hookrightarrow \cdots$$

and we may form the union (or better, the direct limit)

$$P(p) := \varinjlim \mathbb{Z}/p^i\mathbb{Z}.$$

The group which we obtain in this way is such a Prüfer group, its only subgroups are the obvious ones, thus the submodule lattice has the form



in particular, $P(p)$ is artinian. It is easy to show that

$$\mathbb{Q}/\mathbb{Z} = \bigoplus_p P(p),$$

where the direct sum extends over all prime numbers p, the copy $P(p)$ being obtained from the subgroup of $\mathbb{Q}$ of all rational numbers with denominator a power of p by

[3] This means that there exist submodules $0 = M_0 \subset M_1 \subset \cdots \subset M_n = M$ such that if N is a submodule of M with $M_{i-1} \subset N \subseteq M_i$ for some i, then $N = M_i$.

factoring out $\mathbb{Z}$. Of course, this may be used as an alternative definition of $P(p)$. If one denotes by $\mathbb{Z}_{(p)}$ the localisation of $\mathbb{Z}$ at the prime ideal generated by p, thus $\mathbb{Z}_{(p)}$ is the set of all rational numbers with denominator prime to p, then also $\mathbb{Q}/\mathbb{Z}_{(p)} = P(p)$.

We denote[4] by $\mathcal{F}$ the category of torsionfree abelian groups F of finite rank such that such that $pF = F$ for almost all prime numbers p. If $F \in \mathcal{F}$, then the finite rank of F means that we may embed F into a finite direct sum $\mathbb{Q}^n$ of copies of $\mathbb{Q}$, and we can do this in such a way that the corresponding factor module $\mathbb{Q}^n/F$ is a torsion group, and thus a direct sum of Prüfer groups (since a factor group of a divisible group such as $\mathbb{Q}^n$ is divisible again). Clearly, in $\mathbb{Q}^n/F$ any Prüfer group $P(p)$ occurs with multiplicity at most n, and $P(p)$ appears as a direct summand only in case $pF \neq F$. Thus our assumption on $F \in \mathcal{F}$ that $pF = F$ for almost all p implies that $\mathbb{Q}^n/F$ is a finite direct sum of Prüfer groups, and therefore artinian. Thus, we deal with exact sequences of abelian groups of the following form

$$0 \longrightarrow F \longrightarrow \mathbb{Q}^n \longrightarrow A \longrightarrow 0,$$

where A is artinian. Such an exact sequence is a minimal injective resolution of F, and thus any map $\alpha\colon F \to F'$ in $\mathcal{F}$ gives rise to a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & \mathbb{Q}^n & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha''} & & \\ 0 & \longrightarrow & F' & \longrightarrow & \mathbb{Q}^{n'} & \longrightarrow & A' & \longrightarrow & 0 \end{array}$$

and α', α'' both are uniquely determined by α. This shows that *the category $\mathcal{F}$ is equivalent to the category $\mathcal{A}$ of surjective $\mathbb{Z}$-linear maps $\gamma\colon B \to A$, where B is torsionfree divisible of finite rank and A is an artinian $\mathbb{Z}$-module.* Note that the torsionfree divisible abelian groups are just the $\mathbb{Q}$-vector spaces, and the $\mathbb{Z}$-linear maps between $\mathbb{Q}$-vector spaces are always $\mathbb{Q}$-linear. The morphisms in $\mathcal{A}$ are pairs of maps such as α', α'' which yield a commutative square

$$\begin{array}{ccc} B & \xrightarrow{\gamma} & A \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha''} \\ B' & \xrightarrow{\gamma'} & A'. \end{array}$$

As we have seen, a functor $\mathcal{F} \to \mathcal{A}$ is given by forming a minimal injective resolution, the corresponding reverse functor is even easier to describe: just take the kernel of γ. The category $\mathcal{A}$ is a full subcategory of the category $\mathcal{A}'$ of all $\mathbb{Z}$-linear maps $\gamma\colon B \to A$, where A, B are abelian groups with B torsionfree divisible. The category $\mathcal{A}'$ may be interpreted as the category of all R-modules, where

$$R = \begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{bmatrix},$$

[4] We follow here the forthcoming paper of Pimonov and Yakovlev [PY].

this is a subring of the ring of all 2×2 matrices with coefficients in $\mathbb{Q}$, there is an isomorphism[5] of categories $\eta\colon \mathcal{A}' \to \operatorname{Mod} R$. To be more precise, an object of $\mathcal{A}'$ is a map $\gamma\colon B \to A$, thus really a triple (B, A, γ), where A, B are abelian groups with B torsionfree divisible and $\gamma\colon B \to A$ is $\mathbb{Z}$-linear. To such a triple (B, A, γ), the functor η attaches the column module $M = \left[\begin{smallmatrix} B \\ A \end{smallmatrix}\right]$ with the obvious matrix multiplication. Note that the notation $\left[\begin{smallmatrix} B \\ A \end{smallmatrix}\right]$ hides the given map $\gamma\colon B \to A$; however, to consider $M = \left[\begin{smallmatrix} B \\ A \end{smallmatrix}\right]$ as a left R-module requires to be able to multiply the elements of M by the matrices $\left[\begin{smallmatrix} x & 0 \\ y & z \end{smallmatrix}\right]$ with $x, y \in \mathbb{Q}$ and $z \in \mathbb{Z}$. On the one hand, for the matrices of the form $\left[\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & 0 \\ 0 & z \end{smallmatrix}\right]$ with $x \in \mathbb{Q}$ and $z \in \mathbb{Z}$, this is achieved via the $\mathbb{Q}$-module structure of B and the $\mathbb{Z}$-module structure of A, respectively. On the other hand, the multiplication with the matrix $\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ just requires to deal with a $\mathbb{Z}$-linear map $\gamma\colon B \to A$ as follows: $\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} b \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ \gamma(b) \end{smallmatrix}\right]$, the general rule being $\left[\begin{smallmatrix} 0 & 0 \\ y & 0 \end{smallmatrix}\right] \cdot \left[\begin{smallmatrix} b \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ \gamma(yb) \end{smallmatrix}\right]$. Finally, let us remark in which way an arbitrary R-module N can be written in the form $\left[\begin{smallmatrix} B \\ A \end{smallmatrix}\right]$. Let $B = e_1 N$ and $A = e_2 N$, where $e_1 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ and $e_2 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Since this pair e_1, e_2 is a pair of orthogonal idempotents with $e_1 + e_2 = 1$, we have $N = B \oplus A$, and the multiplication on N with the matrix $\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ yields the required map $B \to A$ so that $\eta(B, A, \gamma) = N$.

Under η , the objects in $\mathcal{A}$ correspond to **artinian** R-modules. Indeed, nearly all artinian R-modules occur in this way: given an R-module M, say $M = \left[\begin{smallmatrix} B \\ A \end{smallmatrix}\right]$, we obviously have the following submodule $M' = \left[\begin{smallmatrix} 0 \\ A \end{smallmatrix}\right]$ which is annihilated by the twosided ideal Re_1R, and $R/Re_1R = \mathbb{Z}$. Similarly, the factor module M/M' is annihilated by the twosided ideal Re_2R and $R/Re_2R = \mathbb{Q}$. We consider M as an extension of the submodule M' by the module M/M', where M' is nothing else than the $\mathbb{Z}$-module A and M/M' is just the $\mathbb{Q}$-module B. Note that M is an artinian R-module if and only if both M' and M/M' are artinian R-modules, thus if and only if A is an artinian abelian group and B is a finite dimensional $\mathbb{Q}$-vector space.

Recall that we know that the category $\mathcal{F}$ does not satisfy KRS. Using the categorical equivalence $\mathcal{F} \simeq \mathcal{A}$, we see that also the category $\mathcal{A}$ does not satisfy KRS. Thus we deal with artinian modules over some ring R which do not satisfy KRS.

There are several conclusions which we want to stress.

(a) The negative solution to Krull's problem is provided by considering a really innocent ring: a ring R of 2×2 matrices with coefficients being just rational

[5] This is a general observation: Given two rings S, T and a T-S-bimodule U, one may form the ring $\left[\begin{smallmatrix} S & 0 \\ U & T \end{smallmatrix}\right]$ with matrix multiplication. Indeed, in order to see that the usual matrix multiplication makes sense and yields a ring, one just needs the ring structure on S and T and the T-S-bimodule structure on U. The case considered by us concerns the canonical $\mathbb{Z}$-$\mathbb{Q}$-bimodule structure on $U = \mathbb{Q}$. In the same way, as we are going to show that the $\left[\begin{smallmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{smallmatrix}\right]$-modules are just triples (B, A, γ) with $\gamma\colon B \to A$ $\mathbb{Z}$-linear, one can identify in general the $\left[\begin{smallmatrix} S & 0 \\ U & T \end{smallmatrix}\right]$-modules with the triples (B, A, γ) where B is an S-module, A a T-module and $\gamma\colon U \otimes_S B \to A$ is T-linear.

numbers[6] and also R-modules which are easy to visualize[7] Apparently, until 1998 no one had studied artinian modules over a ring such as $\left[\begin{smallmatrix}\mathbb{Q} & 0\\ \mathbb{Q} & \mathbb{Z}\end{smallmatrix}\right]$. To get inspiration from examples should be one of the most important endeavour of mathematicians. But in contrast, to look at examples was considered quite obsolete by many algebraists until very recent times.[8]

(b) Let us look at typical examples of torsionfree abelian groups of finite rank which do not satisfy KRS. The first such example was presented by Jónsson [J] in 1945, later ones were given by Butler, Corner, Fuchs and others. The usual construction goes as follows: One starts with $\mathbb{Q}^n$ and its subgroup $\mathbb{Z}^n$, and adds some elements in order to get F with

$$\mathbb{Z}^n \subset F \subset \mathbb{Q}^n.$$

But to specify such a subgroup F of $\mathbb{Q}^n$ is nothing else then to define an exact sequence

$$0 \longrightarrow F \longrightarrow \mathbb{Q}^n \xrightarrow{\gamma} \mathbb{Q}^n/F \longrightarrow 0,$$

and this is a minimal injective resolution of F. By working inside $\mathbb{Q}^n$, the old constructions produced already the needed triples $(\mathbb{Q}^n, \mathbb{Q}^n/F, \gamma)$, the only point which was not realized at that time was the identification of such triples $(\mathbb{Q}^n, \mathbb{Q}^n/F, \gamma)$ with $\left[\begin{smallmatrix}\mathbb{Q} & 0\\ \mathbb{Q} & \mathbb{Z}\end{smallmatrix}\right]$-modules. In some sense, Krull's problem has been solved already 55 years ago, but no-one did notice it!

(c) As Butler has shown (see [A]), for every prime number $p \geq 5$, there do exist subgroups F of $\mathbb{Q}^n$ which do not satisfy KRS and such that $\mathbb{Q}^n/F$ is even a p-group. This means that we may consider instead of $\left[\begin{smallmatrix}\mathbb{Q} & 0\\ \mathbb{Q} & \mathbb{Z}\end{smallmatrix}\right]$ the ring

$$R(p) = \begin{bmatrix}\mathbb{Q} & 0\\ \mathbb{Q} & \mathbb{Z}_{(p)}\end{bmatrix},$$

and still get a ring with artinian modules which do not satisfy KRS. Note that this ring is both left serial and right serial (this means that the submodule lattice of any the indecomposable projective (left or right) module is a chain). We have shown in [R2] how to modify this example in order to get also examples of artinian modules over local rings which do not satisfy KRS.

(d) The existence of "complicated" artinian modules is a typical phenomenon of non-commutative algebra. If we deal with a commutative ring R, then any artinian R-module M is the union of an ascending chain of finite length modules M_i

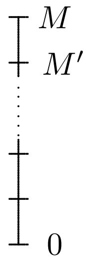
$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \subseteq \bigcup_{i\in\mathbb{N}} M_i = M.$$

[6] Of course, the ring in question is not the ring $M(2\times 2,\mathbb{Q})$ of **all** 2×2 matrices with coefficients in $\mathbb{Q}$, but a proper subring. In fact, the ring $M(2\times 2,\mathbb{Q})$ is simple artinian, thus all its finitely generated representations satisfy KRS.

[7] Explicite examples of such modules are given in [R1] and [R2]; see also our further discussion which gives more details.

[8] A so called "Handbook of Ring and Module Theory" manages to avoid any nontrivial examples whatsoever. It should not be surprising that it claims to present a proof that all artinian modules have local endomorphism rings, an assertion which would imply the validity of KRS for artinian modules.

This follows from the fact that for R commutative, any cyclic artinian R-module is of finite length. Indeed, let M be a cyclic artinian module, then, as a left R-module, $M \simeq R/I$ where I is a left ideal. Since we assume that R is commutative, I actually is a twosided ideal, thus R/I is a ring. But an artinian ring is always of finite length, thus also M is of finite length. — In contrast, a non-commutative ring R may have artinian modules whose submodule structure is much more complicated, in particular, there may exist cyclic artinian modules which are not of finite length. A typical example is provided already by our ring $R = \left[\begin{smallmatrix}\mathbb{Q} & 0\\ \mathbb{Q} & \mathbb{Z}\end{smallmatrix}\right]$ or also $R(p)$. Let $M = \left[\begin{smallmatrix}\mathbb{Q}\\ \mathbb{Q}\end{smallmatrix}\right]/\left[\begin{smallmatrix}0\\ \mathbb{Z}_{(p)}\end{smallmatrix}\right]$. Note that M is a cyclic R-module, and that its submodule lattice is of the form



Here, the submodule M' is annihilated by Re_1R and is, as a module over $R/Re_1R = \mathbb{Z}$, just a Prüfer group $P(p)$, whereas M/M' is the simple R-module $\left[\begin{smallmatrix}\mathbb{Q}\\ \mathbb{Q}\end{smallmatrix}\right]/\left[\begin{smallmatrix}0\\ \mathbb{Q}\end{smallmatrix}\right]$. As we see, the usual intuition concerning the structure of modules which is derived from commutative algebra can be misleading.

(e) It is necessary to focus the attention to non-commutative phenomena which deviate from well-known commutative standards. But one may still bear in mind that there are intimate relations between commutative and non-commutative algebra, and actually that a better understanding of non-commutative phenomena may also shed light on problems in commutative algebra. Indeed, our problem of relating the category $\mathcal{F}$ and the subcategory $\mathcal{A}$ of the category of all $\left[\begin{smallmatrix}\mathbb{Q} & 0\\ \mathbb{Q} & \mathbb{Z}\end{smallmatrix}\right]$-modules is a splendid example.

3. The mystery of matrix multiplication. The use of matrices is a very old technique for efficient calculations. It is a scheme to store data which depend on two parameters coming from finite sets, but the important theme is the handling of these data, the multiplication of rows or columns with a fixed scalar and the addition of some row to another, or of some column to another, thus the use of the matrix multiplication. When referring to these elementary matrix transformations, mathematicians in the west often refer to Gauss-elimination, but the method is much older, see the Jiù zhāng suàn shù which records traditional Chinese methods of calculations invented more than 2000 years ago. Matrix calculations have always played an important role for practical problems, the examples mentioned in the Jiù zhāng suàn shù dealing with economical questions show this very clearly. And nowadays, matrix calculations are indispensable in physics, in economy and elsewhere!

But matrices also have been used inside of mathematics in various ways, in particular as source for clever proofs, see for example the Quillen-Suslin solution of the Serre conjecture [La]: *Every projective $k[T_1,\dots,T_n]$-module is free, or, equivalently, every vector bundle over the affine space $\mathbb{A}(k)^n$ is trivial.* Matrix rings always have served as an important source of examples, let us mention Small who apparently noted for the first time that there do exist rings which are noetherian on one side but not on the other (just take $\left[\begin{smallmatrix}\mathbb{Q} & 0\\ \mathbb{Q} & \mathbb{Z}\end{smallmatrix}\right]$). Such examples often have been called "counterexamples", they were considered as very surprising. But it seems now that such a behavior should be rated as quite usual.

In our discussion of artinian modules, we worked with a 2×2 matrix ring. Let us present some other topics where again examples of matrix rings will be helpful[9].

Let R be a ring. Recall that an R-module M is said to be discrete provided every non-zero direct summand of M has an indecomposable direct summand. On the other hand, M is said to be said to be superdecomposable provided no direct summand of M is indecomposable. The theorem of Gabriel and Oberst asserts that *any injective module I is the direct sum of a discrete module I_1 and a superdecomposable module I_2.* There also is a strong unicity assertion: If $I = I_1'\oplus I_2'$ is a second decomposition with I_1' discrete and I_2' superdecomposable, then $I = I_1\oplus I_2' = I_1'\oplus I_2$. However, in contrast to claims in the literature, neither I_1 nor I_2 are really unique! To construct examples, let us denote by $S = k\langle X,Y\rangle$, the free k-algebra in two generators, let E be th injective envelope of the regular representation ${}_SS$. On the one hand, consider the subring

$$R' = \begin{bmatrix} S & 0\\ S & k\end{bmatrix} \quad\subseteq\quad M(2\times 2, S),$$

and the R'-modules $I_1 = \left[\begin{smallmatrix}E\\ E\end{smallmatrix}\right]$ and $I_2 = \left[\begin{smallmatrix}E\\ E\end{smallmatrix}\right]/\left[\begin{smallmatrix}0\\ E\end{smallmatrix}\right]$. Then both modules I_1, I_2 are injective, I_1 is discrete, whereas I_2 is superdecomposable. Of course $\mathrm{Hom}(I_1,I_2)\neq 0$. Let I_1' be the graph of a non-zero homomorphism $f\colon I_1\to I_2$, then also I_1' is discrete and $I_1\oplus I_2 = I_1'\oplus I_2$, but $I_1'\neq I_1$. On the other hand, consider the subring

$$R'' = \begin{bmatrix} k & 0\\ S & S\end{bmatrix} \quad\subseteq\quad M(2\times 2, S),$$

the R''-module $I_1 = \left[\begin{smallmatrix}k\\ S\end{smallmatrix}\right]/\left[\begin{smallmatrix}0\\ S\end{smallmatrix}\right]$, and the injective envelope I_2 of $\left[\begin{smallmatrix}k\\ S\end{smallmatrix}\right]$. Then again I_1 is discrete (it is even simple), and I_2 is superdecomposable, and this time $\mathrm{Hom}(I_2,I_1)\neq 0$. If we denote by I_2' the graph of a non-zero homomorphism $I_2\to I_1$, then also I_2' is superdecomposable and $I_1\oplus I_2 = I_1\oplus I_2'$, but $I_2'\neq I_2$.

It is often interesting to know whether a given module M may be isomorphic to a proper submodule or a proper factor module of itself, or even to a proper subfactor M'/M'', where $0\subset M''\subset M'\subset M$. In case R is commutative and M is cyclic, then it is quite usual that M is isomorphic to a proper submodule of itself, but clearly M could not be isomorphic to a proper factor module of itself. A typical non-commutative example is the ring $R(p)$ considered above and the cyclic $R(p)$-module $M = \left[\begin{smallmatrix}\mathbb{Q}\\ \mathbb{Q}\end{smallmatrix}\right]/\left[\begin{smallmatrix}0\\ \mathbb{Z}_{(p)}\end{smallmatrix}\right]$. Note that M has a simple socle U and M/U is isomorphic to M. Of course, it is also easy to construct similar examples where M is isomorphic to a proper subfactor. Such kind of examples have been used by Facchini [Fc] in order to see that KRS does not hold for serial modules, in this way

[9] In order to show some important facets of the behavior of matrix multiplication, we try to exhibit examples of $n\times n$ matrix rings with n as small as possible since we feel that small examples are easier to grasp, but the reader may verify immediately that the same effects are present also for arbitrarily large matrices. Of course, there do exist also some anomalies of the matrix multiplication which do occur only for $n\times n$ matrices with small n, but this is not our theme.

solving a problem raised by Warfield. An explicit example can be constructed over the following matrix ring

$$\begin{bmatrix} \mathbb{Z}_{(p)} & p\mathbb{Z}_{(p)} & 0 & 0 \\ \mathbb{Z}_{(p)} & \mathbb{Z}_{(p)} & 0 & 0 \\ \mathbb{Q} & \mathbb{Q} & \mathbb{Z}_{(q)} & q\mathbb{Z}_{(q)} \\ \mathbb{Q} & \mathbb{Q} & \mathbb{Z}_{(q)} & \mathbb{Z}_{(q)} \end{bmatrix}$$

where p, q are different primes.

A famous open problem concerns the finiteness of the finitistic dimension of a finite dimensional algebra A: Is there a finite bound on the projective dimension of all A-modules M with finite projective dimension; the minimal upper bound is called the finitistic dimension of A. Actually, one may consider only the finite dimensional modules M, then one denotes this dimension by $\operatorname{fdim}(A)$, whereas $\operatorname{Fdim}(A)$ is the minimal upper bound when dealing with all A-modules. The general believe that $\operatorname{fdim}(A) = \operatorname{Fdim}(A)$ for every finite dimensional algebra A was disproved by Zimmermann-Huisgen in 1992. In 1998 Smalø exhibited a quite lucid example A with $\operatorname{fdim}(A) \neq \operatorname{Fdim}(A)$ which can be written as a sort of 3×3 matrix ring

$$A = \begin{bmatrix} k & 0 & 0 \\ U_1 & k & 0 \\ U_2 & U_3 & S \end{bmatrix}$$

where $S = k[X,Y]/(X,Y)^2$; and where U_1, U_2, U_3 denote suitable bimodules (with an additional bilinear map $U_3 \otimes U_1 \to U_2$ being given) [Sm].

Many other problems not only in algebra but in many parts of mathematics can be rewritten as questions which concern matrix rings. A very intriguing problem concerns the classification of the vector bundles over the projective n-space $\mathbb{P}_n$. As Beilinson [Be] and Bernstein-Gelfand-Gelfand [BGG] have shown, the category of all such vector bundles is equivalent to a (nice) subcategory of the module category of some algebra of $(n+1) \times (n+1)$-matrices of the form

$$\begin{bmatrix} k & & 0 \\ & \ddots & \\ * & & k \end{bmatrix},$$

again the $*$ part is filled by suitable bimodules. Already the case $n = 2$, thus a 3×3 matrix ring, is interesting.

4. As we have outlined, the development of algebra in the 19-th century was stopped by three obstacles: the difficulties of calculation, of visualization and the missing theoretical foundation, but all these difficulties have vanished. We have used examples of matrix rings in order to show in which way the structural approach provides new insight. Questions concerning non-commutative operations are still mysterious, but they can be handled. And it should be stressed that a wide range of computer algorithms is now available both for calculations as well as for visualization. This concerns, in particular, the use of combinatorial data which provide a wealth of information on algebraic objects. For example, the new representation theory of finite dimensional algebras as it has been developed in the last 30 years reduces many classification problems to questions concerning posets, directs graphs ("quivers") and root systems. Several schemes for visualization have turned out to be very fruitful, in particular the so called Auslander-Reiten quivers. And computer algebra packages such as CREP [DN] provide handy tools both for calculation as well as visualization.

Our discussion was concerned with the past, after all, Europeans just are nostalgic and old-fashioned. But we hope that the material selected also sheds some light on possible directions of future research. Being in East Asia, a European has to be very reluctant to try to envision the speed of further development.

In the final section, let me try to formulate some maybe provocative postulates concerning the prospects of algebra in the new century.

- It is the non-commutative algebra which deserves full interest.
- Non-commutative algebra is still at its beginning. (We do not even understand completely the multiplication of 2×2 matrices.)
- Many new phenomena should be discovered when studying non-commutative structures in greater detail. (The preoccupation with the development of "theories" has neglected up to now the careful study of examples.)
- A forceful development of non-commutative algebra will be helpful for many parts of mathematics, even for the study of commutative situations. (But the usual predominance of commutative thinking should be regarded as an obvious source for misdirection.)

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