An introduction to the representation dimension of artin algebras.

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0. Introduction.

Let Λ be an artin algebra (this means that Λ is a module-finite k-algebra, where k is an artinian commutative ring). The modules to be considered will be left Λ-modules of finite length. Given a class $\mathcal{M}$ of modules, we denote by $\operatorname{add}\mathcal{M}$ the modules which are (isomorphic to) direct summands of direct sums of modules in $\mathcal{M}$. We say that $\mathcal{M}$ is *finite* provided there are only finitely many isomorphism classes of indecomposable modules in $\operatorname{add}\mathcal{M}$, thus provided there exists a module M with $\operatorname{add}\mathcal{M} = \operatorname{add} M$.

The representation dimension of artin algebras was introduced by M.Auslander in his famous Queen Mary Notes [A] but remained a hidden treasure for a long time. Only very recently some basic questions concerning the representation dimension have been solved by Iyama and Rouquier, and now there is a steadily increasing interest in this dimension (in particular, see papers by Oppermann, and also Krause-Kussin, Avramov-Iyengar, and Bergh). This introduction will recall the basic setting and outline a general scheme in order to understand some of the artin algebras with representation dimension at most 3. But we should stress that the main focus at present lies on the artin algebras with representation dimension greater than 3.

0.1. Some results.

A module M is called a *generator* if any projective module belongs to $\operatorname{add} M$; it is called a *cogenerator* if any injective module belongs to $\operatorname{add} M$. It was Auslander who stressed the importance of the global dimension d of the endomorphism rings $\operatorname{End}(M)$, where M is both a generator and a cogenerator. Note that d is either 0 (this happens precisely when Λ is semisimple) or greater or equal to 2 (of course, it may be infinite). The representation dimension of an artin algebra Λ which is not semisimple is the smallest possible such value d; whereas the representation dimension of a semisimple artin algebra is defined to be 1. Here is a list of some basic results:

The main tool for calculating the representation dimension is the following criterion due to Auslander (implicit in [A]). Given modules M, X, denote by $\Omega_M(X)$ the kernel of a minimal right $(\operatorname{add} M)$-approximation $M' \to X$. By definition, the *M-dimension* $M\text{-}\dim X$ is the minimal value i such that $\Omega_M^i(X)$ belongs to $\operatorname{add} M$.

(A) **Theorem** (Auslander). *Let M be a Λ-module which is a both a generator and a cogenerator and let $d \geq 2$. The global dimension of* $\mathrm{End}(M)$ *is less or equal to d if and only if M*-dim *$X \leq d-2$ for all Λ-modules X.*

(B) **Theorem** (Auslander). *An artin algebra Λ is of finite representation type if and only if* rep. dim. $\Lambda \leq 2$. This result was the starting observation and indicates that the representation dimension may be considered as a measure for the distance of being representation-finite.

(C) **Theorem** (Morita-Tachikawa). *If M is a Λ-module which is a generator and cogenerator, then* $\mathrm{End}(M)$ *is an artin algebra with dominant dimension at least* 2 *and any artin algebra with dominant dimension at least* 2 *arises in this way.*

(D) **Theorem** (Iyama). *The representation dimension is always finite.* This asserts, in particular, that any artin algebra Λ can be written in the form $\Lambda = e\Lambda' e$, where Λ' is an artin algebra with finite global dimension; thus many homological questions concerning Λ-modules can be handled by dealing with modules for an algebra with finite global dimension.

(E) **Theorem** (Igusa-Todorov). *If* rep. dim. $\Lambda \leq 3$, *then Λ has finite finitistic dimension.*

Until 2001, for all artin algebras Λ where the representation dimension was calculated, it turned out that rep. dim. $\Lambda \leq 3$. But most of these algebras were torsionless-finite: Λ is *torsionless-finite* if there are only finitely many isomorphism classes of indecomposable submodules of projective Λ-modules.

(F) **Theorem** (Auslander) *If Λ is torsionless-finite, then* rep. dim. $\Lambda \leq 3$.

Thus, there was a strong feeling that all artin algebras Λ could satisfy the condition that rep. dim. $\Lambda \leq 3$. this property. If this would have been true, thefinitistic dimension conjecture and therefore a lot of other homological conjectures would have been proven by (E).

(G) **Example** (Rouquier). *Let V be a finite-dimensional k-space, where k is a field, and $\Lambda(V)$ the corresponding exterior algebra. Then* rep. dim. $\Lambda(V) = -1 + \dim V$.

The aim of the first four lectures is to present the results (A), (B), (C), (D) and (F) with full proofs, and to outline applications as well as some connections to related topics. We also will exhibit some examples which we hope will be of interest. The last lecture intends to survey some of the recent examples of artin algebras with representation dimension at least 4. The series of lectures follows the report

`http://www.math.uni-bielefeld.de/birep/2008/survey.html`

which has been written as an introduction for a workshop to be held at Bielefeld in May 2008, we also want to refer to the corresponding list of references:

`http://www.math.uni-bielefeld.de/birep/2008/literature.pdf`

collected by Kussin and Oppermann.

1. The global dimension of endomorphism rings.

1.1. Minimal right M-approximations. Given modules M, X, denote by $\Omega_M(X)$ the kernel of a minimal right $(\operatorname{add} M)$-approximation $M' \to X$. By definition, the *M-dimension* M-dim X is the minimal value i such that $\Omega^i_M(X)$ belongs to $\operatorname{add} M$.

Lemma. *Let M be a Λ-module with endomorphism ring Γ. Then*

$$\operatorname{proj.dim} \operatorname{Hom}(M, X) \le M\text{-}\dim X$$

for all Λ-modules X. If M is a generator, then equality holds.

Remark: If M is not a generator, then equality may not hold: For example, consider the quiver of type A_2, let M be indecomposable projective-injective. Then Γ has global dimension zero, thus $\operatorname{proj.dim} \operatorname{Hom}(M, X) = 0$ for all X, but the simple injective module has M-dimension 1.

Proof of Lemma. Let M-dim $X = d$. For $d = 0$, we have that X belongs to $\operatorname{add} M$, thus $\operatorname{Hom}(M, X)$ is projective. Assume now $d \ge 1$. For $0 \le i < d$ there is are exact sequences

$$0 \longrightarrow X_{i+1} \longrightarrow M_i \xrightarrow{p_i} X_i$$

with minimal right M-approximations p_i, where $X_0 = X$, thus $X_i = \Omega^i_M(X)$ for all i. By asumption, $X_d = \Omega^d_M(X)$ is in $\operatorname{add} M$. It induces under $\operatorname{Hom}(M, -)$ an exact sequence

$$0 \longrightarrow \operatorname{Hom}(M, X_{i+1}) \longrightarrow \operatorname{Hom}(M, M_i) \xrightarrow{p_i} \operatorname{Hom}(M, X_i) \to 0,$$

and these sequences combine to a projective resolution

$$0 \longrightarrow \operatorname{Hom}(M, X_d) \longrightarrow \operatorname{Hom}(M, M_{d-1}) \longrightarrow \cdots \longrightarrow \operatorname{Hom}(M, M_0) \longrightarrow \operatorname{Hom}(M, X) \longrightarrow 0$$

Thus the projective dimension of $\operatorname{Hom}(M, X)$ is at most d.

Remark. Yoneda philosophy.

Let M be a Λ-module with endomorphism ring $\Gamma = \operatorname{End}(M)$. We consider the functor

$$\operatorname{Hom}(M, -)\colon \operatorname{mod} \Lambda \to \operatorname{mod} \Gamma.$$

(a) Yoneda: The functor $\operatorname{Hom}(M, -)$ provides a bijection

$$\operatorname{Hom}_\Lambda(M', X) \to \operatorname{Hom}_\Gamma(\operatorname{Hom}(M, M'), \operatorname{Hom}(M, X))$$

Here, $g\colon M' \to X$ is send to $\operatorname{Hom}(M, g)$. It is the converse which is of interest. Assume that $f\colon \operatorname{Hom}(M, M') \to \operatorname{Hom}(M, X)$ is given and that $M = M' \oplus M''$ with inclusion map $u'\colon M' \to M$, then f is the image of fu' under $\operatorname{Hom}(M, -)$.

(b) The functor $\mathrm{Hom}(M,-)$ provides an equivalence from $\operatorname{add} M$ onto a dense subcategory of the full subcategory of projective Γ-modules.

(c) If g_{MX} is a minimal right M-approximation of X, then $\mathrm{Hom}(M, g_{MX})$ is a projective cover of $\mathrm{Hom}(M, X)$ (and conversely, if $f\colon Hom(M, M') \to \mathrm{Hom}(M, X)$ is a projective cover, then $f = \mathrm{Hom}(M, g)$ with g a minimal right M-approximation of X and g is uniquely determined by f).

(d) Assume that $0 \to X_1 \to X_2 \to X_3$ is exact in $\operatorname{mod}\Lambda$. Then $0 \to \mathrm{Hom}(M, X_1) \to \mathrm{Hom}(M, X_2) \to \mathrm{Hom}(M, X_3)$ is exact in $\operatorname{mod}\Gamma$. The converse is true in case M is a generator (and only in this case).

(e) Altogether we see: *If M is a generator, then the minimal* $(\operatorname{add} M)$*-resolutions of X correspond bijectively to the minimal projective resolutions of the* Γ*-modules* $\mathrm{Hom}(M, X)$.

1.2. The Auslander Criterion: The shift by 2.

Lemma. *Let M be a Λ-module with endomorphism ring Γ. For any Γ-module Y, there is a Λ-module X with*

$$\operatorname{proj.dim} Y \leq 2 + \operatorname{proj.dim} \mathrm{Hom}(M, X).$$

If M is both a generator and a cogenerator, then conversely for any non-injective Λ-module X there is a Γ-module Y with

$$2 + \operatorname{proj.dim} \mathrm{Hom}(M, X) = \operatorname{proj.dim} Y.$$

Remark: If M is not a generator or not a cogenerator, then the last inequality may not hold. Example: Again take the A_2-quiver, and its path algebra Λ. then there exists a non-injective Λ-module X, however if M is either the minimal generator or the minimal cogenerator, then $\Gamma = \mathrm{End}(M)$ is isomorphic to Λ, thus of global dimension 1.

Proof of Lemma. First, take a Γ-module Y and take a projective presentation of Y, say

$$\mathrm{Hom}(M, M') \xrightarrow{h} \mathrm{Hom}(M, M'') \to Y \to 0$$

with $M', M'' \in \operatorname{add} M$. Its kernel is $\mathrm{Hom}(M, X)$, where X is the kernel of h (since applying $\mathrm{Hom}(M,-)$ to the exact sequnce $0 \to X \to M' \to M''$ yields an exact sequence). We have

$$\operatorname{proj.dim} Y \leq 2 + \operatorname{proj.dim} \mathrm{Hom}(M, X).$$

For the second inequality, let X be a non-injective Λ-module. Take an injective copresentation

$$0 \to X \xrightarrow{u} I_0 \xrightarrow{v} I_1$$

and apply $\mathrm{Hom}(M,-)$. We obtain an exact sequence

$$0 \to \mathrm{Hom}(M,X) \xrightarrow{\mathrm{Hom}(M,u)} \mathrm{Hom}(M,I_0) \xrightarrow{\mathrm{Hom}(M,v)} \mathrm{Hom}(M,I_1)$$

Since I_0, I_1 are injective, thus in $\operatorname{add} M$, we see that $\mathrm{Hom}(M,I_0)$ and $\mathrm{Hom}(M,I_1)$ are projective Γ-modules.

We claim that the image of $\mathrm{Hom}(M,v)$ is not projective. Otherwise the embedding $\mathrm{Hom}(M,X) \to \mathrm{Hom}(M,I_0)$ would split, say by a map g. But according to Yoneda, we have $g = \mathrm{Hom}(M,f)$ for some $f\colon I_0 \to X$ and $\mathrm{Hom}(M,fu) = \mathrm{Hom}(M,\mathrm{id}_X)$. Since we assume that M is a generator, this implies that $fu = \mathrm{id}_X$, and therefore $u\colon X \to I_0$ splits. Thus X is injective, contrary to our assumption.

It follows that the cokernel Y of $\mathrm{Hom}(M,h)$ has projective dimension at least 2 and the second syzygy of Y is the direct sum of a projective module and $\mathrm{Hom}(M,X)$. Thus

$$\operatorname{proj.dim} Y \le 2 + \operatorname{proj.dim} \mathrm{Hom}(M,X).$$

Theorem (Auslander). *Let M be a Λ-module which is both a generator and a cogenerator let $d \ge 2$. The global dimension of* $\mathrm{End}(M)$ *is less or equal to d if and only if M-$\dim X \le d-2$ for all Λ-modules X.*

Corollary 1. *Let M be a Λ-module wich is both a generator and a cogenerator, let Γ be its endomorphism ring. Then the global dimension of Γ is at most* 2 *if and only if* $\operatorname{add} M = \operatorname{mod} \Lambda$.

In particular, in this case Λ is representation-finite.

Corollary 2. *Let M be a Λ-module wich is both a generator and a cogenerator, let Γ be its endomorphism ring. If the global dimension of Γ is less or equal to* 1, *then Λ is semi-simple (and then also Γ is semi-simple).*

Proof: If Λ is semi-simple, then also Γ is semi-simple. If Λ is not semisimple, there is a Λ-module X which is not injective. Now apply the second part of Lemma 1.2 in order to see that the global dimension of Γ is at least 2.

Thus:

- Global dimension of Γ equal to 0 arises only for Λ semi-simple.
- Global dimension equal to 1 is impossible.
- Global dimension of Γ equal to 2 arises only for Λ representation-finite, not semi-simple and $\operatorname{add} M = \operatorname{mod} \Lambda$.
- If $\operatorname{add} M \ne \operatorname{mod} \Lambda$, then the global dimension of Λ is at least 3.

Partial reformulation: *The representation dimension of Λ is at most* 2 *if and only if Λ is representation-finite.*

1.3. The case of Λ being hereditary.

In case Λ is hereditary, one can determine the set of all possible values of the global dimension of endomorphism rings of Λ-modules which are generator-cogenerators. Let τ_Λ denote the Auslander-Reiten translation for the category $\operatorname{mod}\Lambda$.

Theorem (Dlab-Ringel [DR]). *Let Λ be a hereditary artin algebra and let $d \geq 3$ be in $\mathbb{N} \cup \{\infty\}$. There exists a Λ-module M which is both a generator and a cogenerator such that the global dimension of* $\operatorname{End}(M)$ *is equal to d if and only if there is a τ_Λ-orbit of cardinality at least d.*

1.4. Auslander's finiteness theorem for self-injective algebras.

Proposition. *Let Λ be an artin algebra with radical J. Let $M = \bigoplus_{i\geq 1}^{t} M_i$, with $M_i = \Lambda/J^i$. Then M-*$\dim X < \operatorname{LL}(X)$ *for all modules X.*

Proof. Let $s = \operatorname{LL}(X)$. If $i > s$, then any map $f\colon M_i \to X$ factors through M_s. Namely, $M_i = \Lambda/J^i$ and f vanishes on J^s/J^i, since $J^sX = 0$. It follows that $\Omega_M(X) = \Omega_{M(s)}(X)$, where $M(s) = \bigoplus_{i=1}^{s} M_i$. We get a right $M(s)$-approximation of X as follows: On the one hand, take a projective cover $p\colon P_s \to X$ of X as a Λ/J^s-module, thus p is a right M_s-approximation of X. On the other hand, take a right $M(s-1)$-approximation $g\colon Y \to X$, and form the pullback

$$\begin{tikzcd} P_s \arrow[r, "p"] & X \\ U \arrow[u, "g'"] \arrow[r, "p'"'] & Y \arrow[u, "g"'] \end{tikzcd}$$

thus $\begin{bmatrix} p \\ g \end{bmatrix} : P_s \oplus Y \to X$ is a right $M(s)$-approximation, thus it is a right M-approximation of X. It follows that U is a direct summand of $\Omega_M(X)$.

Now U is a submodule of $P_s \oplus Y$, but even of $\operatorname{rad} P_s \oplus Y$ (namely, g maps into JX and f maps JP_s onto JX). Note that $\operatorname{rad} P_s$ as well as Y both have Loewy length at most $s-1$, thus U has Loewy length at most $s-1$.

Using induction, we see: the Loewy length of $\Omega^i_M(X)$ is at most $t-1-i$, thus $\Omega^{t-2}_M(X)$ has Loewy length at most $t-1-(t-2) = 1$, this means that $\Omega^{t-2}_M(X)$ is semisimple. However, all semisimple modules are in $\operatorname{add} M_1 = \Lambda/J$, thus in $\operatorname{add} M$.

Now M always is a generator, but usually not a cogenerator. If Λ is self-injective, then M is a cogenerator, thus in this case we obtain the following bound:

Corollary (Auslander-Reiten) *If Λ is a self-injective artin algebra with Loewy length* $\operatorname{LL}(\Lambda)$, *then* $\operatorname{rep.dim.}\Lambda \leq \operatorname{LL}(\Lambda)$.

Proof. If $\mathrm{LL}(\Lambda) = 1$, then Λ is semisimple, thus $\operatorname{rep.dim.} \Lambda = 1$.

We may assume that $t = \mathrm{LL}(\Lambda) \geq 2$. Let J be the radical of Λ. Let $M = \bigoplus_{i=1}^{t} M_i$, with $M_i = \Lambda/J^i$. Thus $M_1 = \Lambda/J$ so that $\operatorname{add} M_1$ are the semi-simple modules, whereas $M_t = \Lambda$ so that $\operatorname{add} M_t$ are the projective modules; in particular, M is a generator, and since Λ is self-injective, M is also a cogenerator.

If X is a projective Λ-module, then $\Omega_M(X) = 0$. Thus assume that X is indecomposable and not projective. It follows that $\mathrm{LL}(X) \leq t-1$, thus by the proposition

$$M\text{-}\dim X \leq \mathrm{LL}(\Omega_M(X)) < \mathrm{LL}(X).$$

Therefore $\operatorname{rep.dim.} \Lambda < LL(\Lambda)$.

Example of artin algebras A with $\operatorname{rep.dim.} A = 1 + \mathrm{LL}(A)$: *Any n-Kronecker algebra with $n \geq 3$ has representation dimension* 3 *and Loewy length 2.*

If Λ is not self-injective, then one only may hope that the following is true:

$$\operatorname{rep.dim.} \Lambda \leq 1 + \mathrm{LL}(\Lambda).$$

Example. Let V be an n-dimensional vectorspace, $\Lambda(V)$ its exterior algebra, and $A = \Lambda(V)/\operatorname{soc}\Lambda(V)$. Then $\mathrm{LL}(A) = n$ and the representation dimension of A is $n+1$.

1.5. Deletion of projective-injective modules.

The following is due to [EHIS]; but there only the case $\operatorname{rep.dim.} B \leq 3$ is treated.

Lemma. *Let P be indecomposable projective A-module, let $B = A/\operatorname{soc} P$. Either B is semisimple or else* $\operatorname{rep.dim.} A \leq \operatorname{rep.dim.} B$.

Proof: First assume that A is representation finite, thus $\operatorname{rep.dim.} A \leq 2$. Now B is also representation finite, and by assumption not semisimple, thus $\operatorname{rep.dim.} B = 2$. This yields the claim. (Actually, A cannot be semisimple, since otherwise also B semisimple, thus $\operatorname{rep.dim.} A = 2$ and therefore $\operatorname{rep.dim.} A = \operatorname{rep.dim.} B$.)

Now assume that A is not representation finite. Let $\mathcal{B}$ be an Auslander subset of $\operatorname{mod} B$. Let $\mathcal{A}$ be obtained from $\mathcal{B}$ by adding P. Let X be an indecomposable A-module. If $X = P$, then $\Omega_{\mathcal{A}} P = 0$. Now assume X is not isomorphic to P, thus X is a B-module. We show: the universal $\mathcal{B}$-approximation of X is a universal $\mathcal{A}$-approximation. Let $g\colon M \to X$ be the minimal $\mathcal{B}$-approximation of X. This is also a $\mathcal{A}$-approximation: namely, given a map $f\colon P \to X$, then f vanishes on $\operatorname{soc} P$ since otherwise it would be a monomorphism, thus an isomorphism (since P is injective), impossible. Thus f induces a map $\overline{f}\colon P/\operatorname{soc} P \to X$. Now $P/\operatorname{soc} P$ is indecomposable projective B-module, thus in $\mathcal{B}$, therefore $\overline{f}$ factors through g, thus also f factors through g. Of course, g is even a minimal approximation, therefore $\Omega_{\mathcal{A}}(X) = \Omega_{\mathcal{B}}(X)$. Finally, we note the following: Since $\Omega_{\mathcal{A}}(X)$ is a B-module, we can use induction.

1.6. Example: The Kronecker quiver.

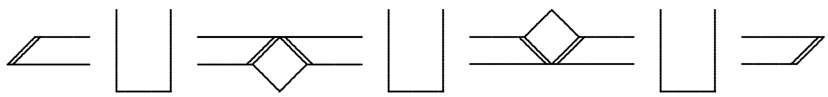
As we will see in section 4, any hereditary artin algebra has representation dimension at most 3, thus equal to 3 in case it is representation-infinite. More precisely we will see that the endomorphism ring of the minimal generator-cogenerator usually has global dimension 3, the only exceptions are the cases A_2, A_2, B_2, A_3 with no paths of length 2 (in these cases, the minimal generator-cogenerator is actually an additive generator for $\operatorname{mod}\Lambda$, thus the global dimension of its endomorphism ring is at most 2).

Let Λ be the path algebra of the Kronecker quiver and $M = \Lambda \oplus D\Lambda$ with endomorphism ring Γ. Then Γ has four verties $1, 2, 3, 4$ and for $1 \leq i \leq 3$ there are two arrows $i \leftarrow i{+}1$, say always labelled α_1, α_2, and the relations are $\alpha_i^2 = 0$ and $\alpha_1\alpha_2 = \alpha_2\alpha_1$. so that $P(3) = I(1),\ P(4) = I(2)$:

$$\begin{tikzcd} \overset{1}{\circ} & \overset{2}{\circ} \arrow[l, shift right, "\alpha_1"'] \arrow[l, shift left, "\alpha_2"] & \overset{3}{\circ} \arrow[l, shift right, "\alpha_1"'] \arrow[l, shift left, "\alpha_2"] & \overset{4}{\circ} \arrow[l, shift right, "\alpha_1"'] \arrow[l, shift left, "\alpha_2"] \end{tikzcd}$$

It is easy to calculate that the global dimension of Γ is equal to 3. (But we can argue also as follows: An algebra with directed quiver and 4 simple modules has global dimension at most 3. Since its dominant dimension is at least 2, it cannot have global dimension at most 2, otherwise it would be an Auslander algebra. Thus the global dimension of Γ is equal to 3.)

The category $\operatorname{mod}\Gamma$ consists of three Kronecker quiver categories which are joint together by identifying a simple injective module with a simple projective one, adding at the same time a projective-injective module:



The embedding of $\operatorname{mod}\Lambda$ into $\operatorname{mod}\Gamma$ is as follows: The non-injective indecomposable modules are sent to the corresponding Γ-modules wirh support in $\{1, 2\}$, the indecomposable injective Λ-modules are sent to the two projective-injective Γ-modules.

AUSBLICK.

- Finiteness conditions in representation theory
- Various dimensions of categories

2. Characterization of the endomorphism rings

We consider artin algebras with duality functor D. We consider left modules (usually, we call them just modules) as well as right modules. Maps will act on the opposite side of the scalars. Thus, if M is an Λ-module and Γ is its endomorphism ring, then M is a right Γ-module, thus a (left) Γ^{op}-module. The module M is said to be *balanced* (or to satisfy the double centralizer condition), provided the canonical map from Λ into the endomorphism ring $\mathrm{End}(M_\Gamma)$ (which sends a onto the left-multiplication with a) is surjective, where $\Gamma = \mathrm{End}(M)$.

Let Γ be an artin algebra. Let $d \geq 1$. Definition: The (left) *dominance* (or the dominant dimension) of Γ is at least d (written $\mathrm{dom}\,\Gamma \geq d$) provided there is an exact sequence

$$0 \to {}_\Gamma\Gamma \to I_0 \to \cdots \to I_{d-1}$$

such that all the modules I_i with $0 \leq i \leq d-1$ are projective and injective (obviously, one only has to require that the corresponding modules I_i in a minimal injective coresolution of ${}_\Gamma\Gamma$ are projective). The right dominance of Γ is defined in the same way, but using right Γ-modules. Here, we are only interested in the cases $d = 1$ and $d = 2$. (Our deviation of speaking about dominance instead of dominant dimension is due to the fact that the dominant dimensions as introduced first by Nakayama (dealing with bimodules) and then by Tachikawa seem to be quite different when compared with the usual notions of dimensions.)

We have $\mathrm{dom}\,\Gamma \geq 1$ iff the injective envelope of ${}_\Gamma\Gamma$ is projective iff ${}_\Gamma$ can be embedded into a module which is both projective and injective iff there exists a faithful module which is both projective and injective iff there exists a left ideal which is both faithful and injective (note that an injective left ideal is always also projective, since it is a direct summand); such rings are also called QF-3 rings, according to Thrall. We see: *The left dominance of* Γ *is at least* 1 *if and only if the right dominance of* Γ *is at least* 1 (the dual of a faithful projective-injective module is a faithful projective-injective right module). *If* Γ *is an artin algebra with* $\mathrm{dom}\,\Gamma \geq 1$*, then there is a multiplicity-free faithful module* N *which is both projective and injective, and this module is unique up to isomorphism;* this module N is usually called the *minimal faithful* Γ-module (this terminology can be explained as follows: N is faithful, and is a direct summand of any faithful module).

Theorem (Morita-Tachikawa).
There is a bijection between

- *the (isomorphism classes of) pairs* (Λ, M) *where* Λ *is a basic artin algebra and* M *a multiplicity-free* Λ*-module which is a generator-cogenerator, and*
- *the (isomorphism classes of) pairs* (Γ', N) *where* Γ' *is a basic artin algebra with* $\mathrm{dom}\,\Gamma' \geq 2$*, and* N *a minimal faithful* Γ'*-module,*

defined as follows:

Given a multiplicity-free Λ*-module which is a generator-cogenerator, attach to the pair* (Λ, M) *the pair* (Γ', M) *where* $\Gamma' = \mathrm{End}_\Lambda(M)^{\mathrm{op}}$.

Conversely, given an artin algebra Γ' *with* $\operatorname{dom}\Gamma' \geq 2$ *and minimal faithful module* N*, attach to the pair* (Γ', N) *the pair* (Λ, N) *where* $\Lambda = \operatorname{End}(N)^{\mathrm{op}}$.

Remarks: (1) Let us stress that under this correspondence, the second entry of the pairs in question remains untouched, at least set-theoretically: the second entry is a bimodule and the bijection yields a mutual change of the module action to be considered.

(2) In particular, the theorem asserts that the modules M and N considered are balanced.

(3) Looking at the pairs (Γ', N), we should stress that the module N is determined by Γ'; thus instead of dealing with the pairs (Γ', N), we may delete N and consider just the isomorphism classes of artin algebras Γ' with $\operatorname{dom}\Gamma' \geq 2$.

We prefer to work with $\Gamma = \operatorname{End}(M)$ instead of its opposite $\Gamma' = \Gamma^{\mathrm{op}}$.

2.1. From Λ to Γ.

Let M *be an* Λ*-module which is a generator and a cogenerator, let* $\Gamma = \operatorname{End}(M)$*. Then the module* M *is balanced, and* M_Γ *is a faithful injective right ideal of* Γ*, whereas* $D(M_\Gamma)$ *is (isomorphic to) a faithful injective left ideal. The ring* Γ *has left dominance at least* 2 *and right dominance at least* 2*.*

Proof: We can assume that Λ is basic; the general case then follows using Morita equivalences. In general, one knows that generators are balanced. The remaining assertions are shown as follows:

First, M_Γ is obviously faithful. Second, in order to show that M_Γ is projective, write ${}_\Lambda M = {}_\Lambda\Lambda \oplus M'$, thus

$$\begin{aligned}\Gamma_\Gamma &= \operatorname{Hom}_\Lambda({}_\Lambda M, {}_\Lambda M_\Gamma)\\ &= \operatorname{Hom}_\Lambda({}_\Lambda\Lambda \oplus M', {}_\Lambda M_\Gamma)\\ &= \operatorname{Hom}_\Lambda({}_\Lambda\Lambda, {}_\Lambda M_\Gamma) \oplus \operatorname{Hom}_\Lambda(M', {}_\Lambda M_\Gamma)\\ &= M_\Gamma \oplus \operatorname{Hom}_\Lambda(M', {}_\Lambda M_\Gamma).\end{aligned}$$

This shows that M_Γ can be considered as direct summand of Γ_Γ, in particular, M_Γ is projective. Dualizing this, we see that $D(M_\Gamma)$ is a direct summand of $D(\Gamma_\Gamma)$, thus injective.

Third, we show that M_Γ is injective. Equivalently, we show that $D(M_\Gamma)$ is a projective left Γ-module. Write $M = D\Lambda \oplus M''$, then

$$\begin{aligned}{}_\Gamma\Gamma &= \operatorname{Hom}_\Lambda({}_\Lambda M_\Gamma, {}_\Lambda M)\\ &= \operatorname{Hom}_\Lambda({}_\Lambda M_\Gamma, {}_\Lambda(D\Lambda) \oplus {}_\Lambda M'')\\ &= \operatorname{Hom}_\Lambda({}_\Lambda M_\Gamma, {}_\Lambda(D\Lambda)) \oplus \operatorname{Hom}_\Lambda({}_\Lambda M_B, {}_\Lambda M'')\\ &= D(M_\Gamma) \oplus \operatorname{Hom}_\Lambda({}_\Lambda M_\Gamma, {}_\Lambda M'').\end{aligned}$$

Recall that the assertions 2 and 3 imply: If P is a projective Λ-module, then $\operatorname{Hom}(P, M)$ is a projective-injective right Γ-module (it is sufficient to show this for P indecomposable projective, but then $\mathrm{Hom}_\Lambda({}_\Lambda P, {}_\Lambda M_\Gamma)$ is a direct summand of M_Γ).

Now take a projective presentation $P_1 \to P_0 \to M \to 0$. If we apply $\mathrm{Hom}_\Lambda(-, {}_A M_B)$, we get an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(M, M) \to \mathrm{Hom}_\Lambda({}_A P_0, {}_\Lambda M_\Gamma) \to \mathrm{Hom}_\Lambda({}_\Lambda P_1, {}_\Lambda M_\Gamma),$$

where the right two terms are projective-injective. This shows that the right dominance of Γ is at least 2.

Similarly, the assertions 2 and 3 imply: If Q is an injective Λ-module, then $\mathrm{Hom}(M, Q)$ is a projective-injective left Γ-module (again, it is sufficient to show this for Q indecomposable injective, but then $\mathrm{Hom}_\Lambda({}_\Lambda M_\Gamma, {}_\Lambda Q)$ is a direct summand of $D(M_\Gamma)$ and thus both projective and injective).

Take an injective copresentation $0 \to M \to Q_0 \to Q_1$, apply $\mathrm{Hom}_\Lambda({}_\Lambda M_\Gamma, -)$. We get an exact sequence

$$0 \to \mathrm{Hom}_\Lambda(M, M) \to \mathrm{Hom}_\Lambda({}_\Lambda M_\Gamma, {}_\Lambda Q_0) \to \mathrm{Hom}_\Lambda({}_\Lambda M_\Gamma, {}_\Lambda Q_1),$$

where the right two terms are projective-injective. This shows that also the left dominance of Γ is at least 2.

2.2. From Γ to Λ.

Now the converse!

Assume that the left dominance of Γ *is at least* 1. *Let* $f\Gamma$ *be a right ideal which is faithful and injective. Then the* $f\Gamma f$*-module* ${}_{f\Gamma f} f\Gamma$ *is both a generator and a cogenerator. In case the left dominance of* Γ *is at least* 2, *one has* ${}_{f\Gamma f} f\Gamma = \Gamma$, *canonically.*

For the proof, we need a quite general result:

Proposition 1. *Let R be any ring and $f \in R$ an idempotent. Then the natural transformation*

$$\eta\colon fR \otimes_R \mathrm{Hom}_{fRf}(fR, -) \to \mathrm{id} \quad \textit{with} \quad fr \otimes \phi \mapsto (fr)\phi$$

for $r \in R, \phi \in \mathrm{Hom}_{fRf}(fR, X)$ where X is a left fRf-module, is an equivalence, thus the composition of functors

$$\mathrm{mod}\, fRf \xrightarrow{\mathrm{Hom}_{fRf}(fR,-)} \mathrm{mod}\, R \xrightarrow{fR\otimes-} \mathrm{mod}\, fRf$$

is an equivalence of categories. Note that $(fR\otimes -) = \mathrm{Hom}_R(Rf, -)$, these functors send the R-module X to fX.

Proof, well-known, for example:

$$\begin{aligned} fR\otimes_R \operatorname{Hom}_{fRf}(fR,X) &= f\operatorname{Hom}(fR,X) = \operatorname{Hom}_R(Rf,\operatorname{Hom}_{fRf}(fR,X)) \\ &= \operatorname{Hom}_{fRf}(fR\otimes_R Rf,X) = \operatorname{Hom}_{fRf}(fRf,X) = X, \end{aligned}$$

as left fRf-modules.

Proposition 2. *Let R be any ring and let e, f be idempotents of R such that Re is an injective left module and fR is a faithful right module. Then*

(a) *The canonical map $\rho\colon Re \to \operatorname{Hom}_{fRf}({}_{fRf}fR, fRe)$ defined by $y(x\rho) = yx$ for $x \in Re, y \in fR$ is an isomorphism of left R-modules.*

(b) $\operatorname{End}({}_{fRf}fRe) = eRe$ (where $ere \in eRe$ corresponds to the endomorphism of ${}_{fRf}fRe$ given by right multiplication with ere).

(c) *The fRf-module ${}_{fRf}fRe$ is injective.*

Proof of (a). The map ρ is an R-homomorphism: the left R-module structure on the Hom-set is given by the right R-structure of fR_R, this means that for a homomorphism α we have $y(r\alpha) = (yr)\alpha$. Thus $y[(r(x\rho)] = (yr)(x\rho) = (yr)x = y(rx) = y[(rx)\rho]$ and therefore $r(x\rho) = (rx)\rho$. The map is injective, since fR_R is faithful. We show that the map is an essential embedding. Thus, let $0 \neq \alpha\colon fR \to fRe$ be a homomorphism of left fRf-modules. There is $r \in R$ such that $(fr)\alpha \neq 0$. Note that $(fr)\alpha \in fRe \subseteq Re$, thus we can apply ρ and $((fr)\alpha))\rho \neq 0$. Claim: $((fr)\alpha)\rho = (fr)\alpha$. Namely, apply it to $y \in fR$ we get

$$y[((fr)\alpha)\rho] = y{\cdot}((fr)\alpha) = y{\cdot}((ffr)\alpha) = y{\cdot}f{\cdot}((fr)\alpha) = (yffr)\alpha = (yfr)\alpha y((fr)\alpha).$$

This shows that $(Re)\rho \cap R\alpha \neq 0$. Since ${}_RRe$ is injective, we see that ρ is also surjective.

Proof of (b):
Under ρ, the subset eRe of Re is mapped into $\operatorname{Hom}_{fRf}({}_{fRf}fRe, fRe)$, the subset $eR(1-e)$ of Re is mapped into $\operatorname{Hom}_{fRf}({}_{fRf}fR(1-e), fRe)$. Altogether we deal with the following situation:

$$\begin{array}{ccc} Re & \xrightarrow{\ \rho\ } & \operatorname{Hom}_{fRf}({}_{fRf}fR, fRe) \\ \Vert & & \Vert \\ eRe \oplus (1-e)Re & \longrightarrow & \operatorname{Hom}_{fRf}({}_{fRf}fRe, fRe) \oplus \operatorname{Hom}_{fRf}({}_{fRf}fR(1-e), fRe) \end{array}$$

Thus, we see that $\rho(eRe) = \operatorname{Hom}_{fRf}({}_{fRf}fRe, fRe)$.

Proof of (c). Since fR_R is projective, the functor $fR \otimes_R -$ sends injective R-modules to injective fRf-modules. Since ${}_RRe$ is injective, it follows that the fRf-module $fRe = fR \otimes_R Re$ is injective.

Proposition 3. *Let Γ be an artin algebra of dominance at least 1, let e, f be idempotents in Γ such that the left module Γe and the right module $f\Gamma$ both are faithful and injective. Let $U = {}_{f\Gamma f}f\Gamma e_{e\Gamma e}$. Then U is balanced, ${}_{f\Gamma f}U$ is an injective cogenerator, $U_{e\Gamma e}$ is also an injective cogenerator.*

Proof: Before we start with the proof, let us introduce the following notation: If Λ is an artin algebra, let $s(\Lambda)$ be the number of simple Λ-modules. If M is an Λ-module, let $s(\Lambda)$ be the number of isomorphism classes of indecomposable direct summands of M. Thus $s(\Lambda) = s({}_\Lambda\Lambda) = s(\Lambda_\Lambda)$, and $s(M) = s(\mathrm{End}(M))$ for any artin algebra Λ and any module M. Consider the $f\Gamma f$-$e\Gamma e$-bimodule $U = f\Gamma e$. We know that the $f\Gamma f$-module U is injective (by Proposition 2 (c)) thus $s(f\Gamma f) \geq s({}_{f\Gamma f}U)$. By Proposition 2 (b), the endomorphism ring of U is $e\Gamma e$, thus $s({}_{f\Gamma f}U) = s(e\Gamma e)$. This shows that $s(f\Gamma f) \geq s(e\Gamma e)$. By left-right symmetry, we also see $s(e\Gamma e) \geq s(f\Gamma f)$, thus $s(f\Gamma f) = s(e\Gamma e)$, and therefore $s(f\Gamma f) = s({}_{f\Gamma f}U)$. This shows, that any indecomposable injective $f\Gamma f$-module occurs as a direct summand of U, thus U is a cogenerator.

Proof of the first assertion: *The module ${}_{f\Gamma f}f\Gamma$ is a generator and a cogenerator.* Of course, ${}_{f\Gamma f}f\Gamma$ is a generator, since

$${}_{f\Gamma f}f\Gamma = {}_{f\Gamma f}f\Gamma f \oplus {}_{f\Gamma f}f\Gamma(1-f).$$

But we have also the decomposition

$${}_{f\Gamma f}f\Gamma = {}_{f\Gamma f}f\Gamma e \oplus {}_{f\Gamma f}f\Gamma(1-e),$$

and according to Proposition 2 (c), ${}_{f\Gamma f}f\Gamma e$ is an injective cogenerator, thus ${}_{f\Gamma f}f\Gamma$ is a cogenerator.

Proposition 4. *Let Γ be an artin algebra of dominance at least 1, let e, f be idempotents in Γ such that the left module Γe and the right module $f\Gamma$ both are faithful and injective. The functor $\mathrm{Hom}_{f\Gamma f}(f\Gamma, -)$ sends $\mathrm{mod}\, f\Gamma f$ onto the full category of left Γ-modules which have a Γe-copresentation.*

Proof. Let $0 \to Y \to U_0 \to U_1$ be a injective copresentation of the fBf-module Y. Apply $\mathrm{Hom}_{f\Gamma f}(f\Gamma, -)$, we obtain the exact sequence

$$0 \to \mathrm{Hom}_{f\Gamma f}(f\Gamma, Y) \to \mathrm{Hom}_{f\Gamma f}(f\Gamma, U_0) \to \mathrm{Hom}_{f\Gamma f}(f\Gamma, U_1)$$

Since U_i is an injective $f\Gamma f$-module, it is in $\mathrm{add}\, {}_{f\Gamma f}f\Gamma e$, thus $\mathrm{Hom}_{f\Gamma f}(f\Gamma, U_i)$ is in $\mathrm{add}\,\mathrm{Hom}_{f\Gamma f}(f\Gamma, f\Gamma e) = \mathrm{add}\,\Gamma e$.

Conversely, assume that the Γ-module M has a Γe-copresentation, thus there is an exact sequence of Γ-modules

$$0 \to M \to N_0 \to N_1$$

where N_0, N_1 are in add ${}_\Gamma\Gamma e$. Multiplying from the left with f, we obtain an exact sequence

$$0 \to fM \to fN_0 \to fN_1$$

with fN_i in add $f\Gamma e$, thus this is an injective copresentation of fM. Apply $\mathrm{Hom}_{f\Gamma f}(f\Gamma, -)$ we obtain the exact sequence

$$0 \to \mathrm{Hom}_{f\Gamma f}(f\Gamma, fM) \to \mathrm{Hom}_{f\Gamma f}(f\Gamma, fN_0) \to \mathrm{Hom}_{f\Gamma f}(f\Gamma, fN_1)$$

There is the following commutative diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & M & \longrightarrow & N_0 & \longrightarrow & N_1 \\
 & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Hom}_{f\Gamma f}(f\Gamma, fM) & \longrightarrow & \mathrm{Hom}_{f\Gamma f}(f\Gamma, fN_0) & \longrightarrow & \mathrm{Hom}_{f\Gamma f}(f\Gamma, fN_1)
\end{array}$$

with exact rows. Since the right vertical maps are isomorphisms, also the left one is an isomorphism. This shows that any Γ-module M with a Γe-copresentation is in the image of the functor $\mathrm{Hom}_{f\Gamma f}(f\Gamma, -)$.

Proof of the second assertion: *If the dominance of* Γ *is at least* 2*, then the module* ${}_{f\Gamma f}f\Gamma_\Gamma$ *is balanced.* If the dominance of Γ is at least 2, the module ${}_\Gamma\Gamma$ has a Γe-copresentation, thus it corresponds under the categorical equivalence mentioned above to $f\Gamma$ and the endomorphism ring of ${}_{f\Gamma f}f\Gamma$ is Γ.

Corollary. *The left dominance of* Γ *is at least* 2 *if and only if the right dominance of* Γ *is at least* 2

Proof: First, assume only that the left dominance of Γ is at least 1. Let $\Lambda = f\Gamma f$, and $M = f\Gamma$. Then by Direction 2, ${}_\Lambda M$ is a generator and a cogenerator. According to Direction 1, the endomorphism ring $\mathrm{End}_\Lambda(M)$ has right dominance at least 2. If the left dominance of Λ is at least 2, then by Direction 2, $\Gamma = \mathrm{End}_\Lambda(M)$, thus Γ has right dominance at least 2.

References.

[M] K. Morita: Duality for modules and its applications in the theory of rings with minimum condition. Sci. Rep. Tokyo Daigaku A 6 (1958), 83-142.

[A] M. Auslander: Representation dimension of artin algebras. Queen Mary College Mathematics Notes (1971).

[T1] H. Tachikawa: On dominant dimension of QF-3 algebras. Trans. Amer. Math. Soc. 112 (1964), 249-266.

[T2] H. Tachikawa: Quasi-Frobenius Rings and Generalizations. QF-3 and QF-1 rings. Springer Lecture Notes in Mathematics 351 (1973).

[MT] K. Morita, H. Tachikawa: On QF-3 rings (unpublished ms).

Remarks: (1) The result presented has its roots in Morita's treatment [M] of what now is called *Morita duality*: As we have seen, the bimodule $U = {}_{f\Gamma f}f\Gamma e_{e\Gamma e}$ is balanced and is an injective cogenerator on either side, this is the initial condition for the Morita duality given by the functors $\mathrm{Hom}_{f\Gamma f}(-,U)$ and $\mathrm{Hom}_{e\Gamma e}(-,U)$ (see for example [T2], Theorem (3.3). The notion of the dominant dimension (as defined above) is due to Tachikawa [T1]. A full treatment of the result is given in the joint paper [MT] of Morita and Tachikawa (which never was published), see also the lecture notes by [T]: theorem (5.3) together with (7.1) and (7.7). The formulation of Direction 1 corresponds to the Queen Mary Notes by Auslander, p.135.

(2) There are several papers which extend the result to larger classes of rings (already the Morita-Tachikawa paper [MT] dealt with semi-primary rings).

(3) In general, the dominant dimension of R is the same as the dominant dimension of R^{op}. For $\operatorname{dom} R = 1$, this is easy to see, and we have shown that the characterization of algebras with $\operatorname{dom} R \leq 2$ proves this assertion also for algebras with dominant dimension 2. For a proof in general, see [T], theorem (7.7).

(4) Let Λ be an artin algebra. We propose to call a ring Γ a *propagation* of Λ provided $\Gamma = \mathrm{End}({}_\Lambda M)^{\mathrm{op}}$, where ${}_\Lambda M$ is a generator-cogenerator.

2.3. Consequences for $\operatorname{rep.dim.}\Lambda = 2$**.**

There is the following recovering theorem: *There is a bijection between the Morita classes of representation-finite artin algebras and the Morita classes of artin algebras of dominant dimension at least* 2 *and global dimension at most* 2*.*

This implies: *There is a bijection between the Morita classes of representation-finite artin algebras which are connected and not simple and the Morita classes of connected artin algebras of dominant dimension equal to* 2 *and global dimension rqual to* 2*.*

It just remains to be seen that in case Λ is not semisimple, the dominant dimension of Γ cannot be greater 2 (here, $\Gamma = \mathrm{End}(M)$, where $\operatorname{add} M = \operatorname{mod}\Lambda$). Namely, assume that there is given anexact sequence

$$0 \longrightarrow \Gamma \xrightarrow{u} I_0 \xrightarrow{f} I_1 \longrightarrow I_2$$

with I_0, I_1, I_2 projective and injective. Since the global dimension is (at most) 2, the image of f is projective, but then u splits and Γ is self-injective. But a self-injective algebra of finite global dimension is semi-simple.

2.4. The embedding $\mathrm{Hom}(M,-)\colon\ \operatorname{mod}\Lambda \to \operatorname{mod}\Gamma$**.**

We are interested in $\operatorname{mod}\Lambda$. The Λ-module M with endomorphism ring Γ is chosen in such a way that we obtain some insight into the structure of the category $\operatorname{mod}\Lambda$. Recall that Auslander's aim was to replace working inside the category mod Λ by working in a larger category with better homological properties: Note that Λ may have infinite global dimension and then to work with projective resolutions can be difficult (for example, when working with finite projective resolutions one may use induction along the projective dimension — this is impossible for modules with infinite projective dimension). Thus, Auslander's proposal is to use the embedding functor

$$\mathrm{Hom}(M,-)\colon \bmod\Lambda \to \bmod\Gamma$$

thus, being interested in the Λ-module X, we should instead consider the Γ-module $\mathrm{Hom}(M,X)$.

(1) $\mathrm{Hom}(M,-)$ is a full embedding (but not exact!) and the image is the full subcategory $\mathcal{D}(\Gamma)$ of all Γ-modules Y with dominance at least 2 (these are the kernels of maps between modules which are projective-injective).

$\mathrm{Hom}(M,-)$ is only exact in case M is projective, but a cogenerator is projective only in case Λ is semsimple.

The assertion (1) has been shown in Proposition 4.

(2) If the global dimension of Γ is $d \geq 2$, then $\mathcal{D}(\Gamma)$ is an extension closed subcategory of Γ-modules with projective dimension at most $d-2$.

Proof: Let $Y \in \mathcal{D}(\Gamma)$, thus there is an exact sequence

$$0 \longrightarrow Y \longrightarrow I_0 \xrightarrow{h} I_1,$$

let Y' be the cokernel of h. Then $Y = P \oplus \Omega^2(Y')$, thus the projective dimension of Y is at most $d-2$.

Why is $\mathcal{D}(\Gamma)$ closed under extension? Just use the usual horse-show arguement for injective copresentations.

Note. Already for $d=3$, $\mathcal{D}(\Gamma)$ may not contain all Γ-modules of projective dimension 1.

Example: Take the Auslander algebra for $k[T]/T^3$. Then all Δ-good modules have projective dimension at most 1, but only 6 indecomposables belong to $\mathcal{D}(\Gamma)$.

(3) One may characterize $\mathcal{D}(\Gamma)$ as those modules Y with $\mathrm{Hom}(S,Y) = 0 = \mathrm{Ext}^1(S,Y)$ for all the simple Γ-modules which cannot be embedded into Γ.

(4) The left adjoint to the embedding functor $\mathcal{D}(\Gamma) \to \bmod\Gamma$ can be described as follows: Start with Y, factor out its torsion submodule (that means: take the largest factor module cogenerated by Γ, or, what is the same, by the minimal faithful Γ-module. This of of course functorial. Now complete by forming the universal extension using inductively simple modules which do not embed into Γ. This is called the **Lambek-localisation** and is also functorial!

One obtains in this way a module over the algebra $\Lambda' = \mathrm{End}(I(\Gamma))$, of course this algebra Λ is Morita equivalent to the endomorphism ring Λ of the minimal faithful Γ-module.

(5) To consider an embedding of $\mathrm{mod}\,\Lambda$ into any abelian category means to endow $\mathrm{mod}\,\Lambda$ with a new exact structure!

Definition: A sequence $0 \to X_1 \to X_2 \to X_3 \to 0$ is M-exact, if the induced sequence

$$0 \to \mathrm{Hom}(M, X_1) \to \mathrm{Hom}(M, X_2) \to \mathrm{Hom}(M, X_3) \to 0$$

is exact. Since M is a generator, an M-exact sequence is an exact sequence (with the additional property that any map $M \to X_3$ factors through $X_2 \to X_3$. Thus the M-deflations are the surjective maps $X_2 \to X_3$ such that any map $M \to X_3$ factors through $X_2 \to X_3$. And the M-inflations are the injective maps $X_1 \to X_2$ such that its cokernel map is an M-deflation.

(6) The change of the Auslander-Reiten quiver involves only the deletion of finitely many translations, namely those ending in a module in $\mathrm{add}\, M$.

The only Auslander-Reiten sequences which are not M-exact are those ending in a module in $\mathrm{add}\, M$. Now the modules in $\mathrm{add}\, M$ are relative projective with resect to the M-exact structure, thus the M-translation is not defined for them. On the other hand, given an Auslander-Reiten sequence $0 \longrightarrow X_1 \xrightarrow{f} X_2 \xrightarrow{g} X_3 \longrightarrow 0$ which does not end in a module in $\mathrm{add}\, M$, then this sequence is M-exact, the map f is a source map, the map g is a sink map, thus this is what has to be called a relative Auslander-Reiten sequence with respect to the M-exact structure.

(7) What about representation dimension 3, or even: Consider a generator-cogenerator M such that the global dimension of $\Gamma = \mathrm{End}(M)$ is equal to 3. Then $\mathcal{D}(\Gamma)$ is a **hereditary** exact category.

2.5. Digression: The category $\mathcal{O}$

Let $\mathbf{g}$ be a finite-dimensional semi-simple complex Lie algebra with triangular decomposition $\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}$ and Weyl group W. Bernstein-Gelfand-Gelfand have introduced the corresponding category $\mathcal{O}$, it consists of the finitely generated $U(\mathbf{g})$-modules, which are semisimple as $U(\mathbf{h})$-modules and locally finite as $U(\mathbf{n})$-modules.

Any block $\mathcal{B}$ of $\mathcal{O}$ is equivalent to $\mathrm{mod}\, A(\mathcal{B})$ for some fin-dim algebra $A(\mathcal{B})$.

Theorem (König-Slungard-Xi 2001). The dominant dimension of the algebra $A(\mathcal{B})$ is at least 2.

In the case of a **regular** block, the minimal faithful $A(\mathcal{B})$-module P is indecomposable.

Theorem (Soergel 1990). The endomorphism ring of P is just the coinvariant algebra of corresponding Weyl group W acting on $\mathbf{h}^*$ (in particular a commutative self-injective algebra).

On the other hand, one knows:

Theorem (Cline-Parshall-Scott (1988). The algebra $A(\mathcal{B})$ is quasi-hereditary.

Let us repeat: We deal with a quasi-hereditary algebra of dominant dimension at least 2, and the endomorphism ring of the minimal faithful module is even a self-injective algebra!

The same combination of properties occurs also for many Schur algebras!

AUSBLICK.

- Dominant dimension is not a dimension.
- Bicentralizer assertions.
- Higher dimensional Auslander theory
- Morita duality
- Hammocks

3. Iyama's finiteness theorem.

Recall that the radical rad of the category $\operatorname{mod}\Lambda$ is defined as follows: If X, Y are Λ-modules and $f\colon X \to Y$, then f belongs to $\operatorname{rad}(X,Y)$ provided for any indecomposable direct summand X' of X with inclusion map $u\colon X' \to X$ and any indecomposable direct summand Y' of Y with projection map $p\colon Y \to Y'$, the composition $ufp\colon X' \to Y'$ is non-invertible.

3.1. Derivation. Let X be a (left) module, let $\mathbf{r}$ be the radical of the endomorphism ring of X. We put $\partial X = X\mathbf{r}$, this is a submodule of X.

Note that $\operatorname{rad} X$ and $X\mathbf{r}$ usually are incomposable. As an example, consider the Kronecker algebra Λ. Let $X = R[2]$ be a 4-dimensional indecomposable regular Kronecker module with a 2-dimensional regular submodule $R[1]$. Here, $\operatorname{rad} X = \operatorname{soc} X$ is semisimple and of length 2, whereas $X\mathbf{r}$ is also of length 2, but indecomposable.

(1) The module X generates the module ∂X. Proof: Let $\phi_1, \dots, \phi_m$ be a generating set of $\operatorname{rad}(X,X)$, say as a k-module. Then $\partial X = \sum_i \phi_i(X)$, thus the map $\phi = (\phi_i)_i\colon X^t \to \partial X$ is surjective.

(2) If X is non-zero, then ∂X is a proper submodule of X. Proof. The ring $\Gamma = \operatorname{End}(X)$ is again an artin algebra and the radical of a non-zero Γ-module is a proper submodule (it is enough to know that Γ is semi-primary).

(3) Any radical map $X \to X$ factors through ∂X. Proof: This follows directly from the construction.

(4) If $X = \bigoplus X_i$, then $\partial X = \bigoplus X_i \operatorname{rad}(X, X_i)$.

We define inductively $\partial^0 X = X, \partial^{t+1} X = \partial(\partial^t X)$. Note that $\partial^2 X$ usually is different from $X\mathbf{r}^2$, a typical example will be given by a serial modules with composition factors $1,1,2,1,1$ (in this order) such that the submodule of length 2 and the factor module of length 2 are isomorphic. Here, $X\mathbf{r}^2 = 0$, whereas $\partial^2 X$ is simple.

(1′) *If $i \le j$, then $\partial^i X$ generates $\partial^j X$.*

(2′) *if X has length m, then $\partial^m X = 0$.*

(3′) *Any radical map $\partial^{i-1} X \to \partial^{i-1} X$ factors through $\partial^i X$.*

(4′) If $\partial^t X = \bigoplus N_i$, then $\partial^{t+1} X = \bigoplus N_i \operatorname{rad}(\partial^t X, N_i)$

(4″) In particular, *an indecomposable summands N of any $\partial^t X$ is a submodule of some indecomposable summand of X.*

Let $\mathcal{C}_i = \mathcal{C}_i(X) = \operatorname{add}\{\partial^j X \mid i \le j\}$. Thus we obtain a filtration

$$\mathcal{C} = \mathcal{C}_0 \supseteq \mathcal{C}_1 \supseteq \cdots \supseteq \mathcal{C}_{n-1} \supseteq \mathcal{C}_n = \{0\}.$$

(5) **Main Lemma.** *If N is an indecomposable module in $\mathcal{C}_i \setminus \mathcal{C}_{i+1}$, let $\alpha(N) = N \operatorname{rad}(\partial^i X, N)$. Then $\alpha(N)$ is a proper submodule of N and the inclusion map $\alpha(N) \to N$ is a right $\mathcal{C}_{i+1}$-approximation (and of course minimal).*

Note that the assumption that N belongs to $\mathcal{C}_i \setminus \mathcal{C}_{i+1}$ implies that N is a direct summand of $\partial^{i-1}X$, but it means in addition that N does not occur as a direct summand of $\partial^i X$.

Proof: If $\alpha(N) = N$, then $(4')$ shows that N belongs to $\operatorname{add} \partial^{i+1}X$, thus to $\mathcal{C}_{i+1}$. But this is not the case. In order to see that the inclusion map $u\colon \alpha(N) \to N$ is a right $\mathcal{C}_{i+1}$-approximation, we have to show that any map $g\colon \partial^j X \to N$ with $j \geq i+1$ factors through u, thus that the image of g is contained in $\alpha(N)$. By $(1')$, there is a surjective map $\eta\colon \left(\partial^i X\right)^t \to \partial^j X$. We claim that the composition ηg is a radical map. Otherwise, there is an indecomposable direct summand U of $\left(\partial^i X\right)^t$ such that the composition

$$U \to \left(\partial^i X\right)^t \xrightarrow{\eta} \partial^j X \xrightarrow{g} N$$

is an isomorphism, but then N is a direct summand of $\partial^j X$, thus in $\mathcal{C}_{i+1}$, but this is not the case. Write $\eta = (\eta_s)_s$ with maps $\eta_s\colon \partial^i X \to \partial^j X$ for $1 \leq s \leq t$. Since ηg is a radical map, all the maps $\eta_s g$ are radical maps. The image of ηg is the sum of the images of the maps $\eta_s g$, and thus contained in $\alpha(N) = N \operatorname{rad}(\partial^i X, N)$. Since η is surjective, it follows that also the image of g itself is contained in $\alpha(N)$.

Theorem (Iyama). *Let X be a module. Write $\mathcal{C}_i = \mathcal{C}_i(X)$. Let $\mathcal{C}_0 = \operatorname{add} M$ and $\Gamma = \operatorname{End}(M)$. Let N be indecomposable in $\mathcal{C}_{i-1} \setminus \mathcal{C}_i$ for some $i \geq 1$. Then the minimal right $\mathcal{C}_i$-approximation $u\colon \alpha(N) \to N$ yields an exact sequence*

$$0 \to \operatorname{Hom}(M, \alpha(N)) \xrightarrow{\operatorname{Hom}(M,u)} \operatorname{Hom}(M, N) \to \operatorname{Hom}(M, N)/\langle C_i \rangle \to 0$$

of Γ-modules.

The module $R(N) = \operatorname{Hom}(M, \alpha(N))$ is projective, and the composition factors of $\operatorname{top} R(N)$ are of the form $S(N'')$ with $N'' \in \mathcal{C}_i$.

The endomorphism ring of $\Delta(N) = \operatorname{Hom}(M, N)/\langle C_i \rangle$ is a division ring and the composition factors of $\operatorname{rad} \Delta(N)$ are of the form $S(N')$ where N' is an indecomposable Λ-module in $\mathcal{C}_0 \setminus \mathcal{C}_{i-1}$.

Proof: Since u is injective, also $\operatorname{Hom}(u, -)$ is injective. Now $\alpha(N)$ belongs to $\mathcal{C}_i$, thus $\operatorname{Hom}(M, \alpha(N))$ is mapped unter u to a set of maps $f\colon M \to N$ which factor through $\mathcal{C}_i$. But since u is a right $\mathcal{C}_i$-approximation, we see that the converse also is true: any map $M \to N$ which factors through $\mathcal{C}_i$ factors through u. This shows that the cokernel of $\operatorname{Hom}(M, u)$ is $\operatorname{Hom}(M, N)/\langle C_i \rangle$.

Of course, $R(N)$ is projective. If we decompose $\alpha(N)$ as a direct sum of indecomposable modules N'', then $\operatorname{Hom}(M, \alpha(N))$ is a direct sum of the corresponding projective Γ-modules $\operatorname{Hom}(M, N'')$ with N'' indecomposable and in $\mathcal{C}_i$, and $\operatorname{top} R(N)$ is the direct sum of the corresponding simple Γ-modules $S(N'')$.

Now we consider $\Delta(N)$. Let N' be an indecomposable direct summand of M such that $S(N')$ is a compositon factor of $\Delta(N)$. This means that there is a map $f\colon N' \to N$ which does not factor through $\mathcal{C}_i$. In particular, N' itself does not belong to $\mathcal{C}_i$, thus N' is a direct summand of $\partial^{i-1}X$. Also N is a direct summand of $\partial^{i-1}X$. Now, according to (3$'$) any radical map $\partial^{i-1}X \to \partial^{i-1}X$ factors through $\partial^i X$, thus f has to be invertible. This shows that for $N' \in \mathcal{C}_i$, the only possibility is that N' is isomorphic to N and that the composition factor of $\Delta(N)$ given by the map f is the top composition factor. Thus, $S(N)$ appears exactly once as composition factor of $\Delta(N)$, namely at the top: this shows that the endomorphism ring of $\Delta(N)$ is a division ring. Also we have shown that the remaining composition factors of $\Delta(N)$, thus those of $\operatorname{rad}\Delta(N)$ are of the form $S(N')$ with N' indecomposable and in $\mathcal{C}_{i-1}$.

3.2. Strongly quasi-hereditary algebras.

Let Γ be an artin algebra. Let $\mathcal{S} = \mathcal{S}(\Gamma)$ be the set of isomorphism classes of simple Γ-modules. For any module M, let $P(M)$ be the projective cover of M.

We say that Γ is (left) *strongly quasihereditary with n layers* provided there is a funcion $l\colon \mathcal{S} \to \{1, 2, \dots, n\}$ (the layer function) such that for any $S \in \mathcal{S}(\Gamma)$, there is an exact sequence

$$0 \to R(S) \to P(S) \to \Delta(S) \to 0$$

with the following two properties: First of all, if S' is a composition factor of $\operatorname{rad}\Delta(S)$, then $l(S') < l(S)$. And second, $R(S)$ is a direct sum of projective modules $P(S'')$ with $l(S'') > l(S)$.

Proposition. *If Γ is strongly quasi-hereditary with n layers, then Γ is quasi-hereditary and the global dimension of Γ is at most n.*

Proof. The top of $R(S)$ is given by simple modules S' with $l(S') > l(S)$, thus $\Delta(S)$ is the maximal factor module with composition factors S' sich that $l(S') \le l(S)$. Since S does not occur as composition factor of $\operatorname{rad}\Delta(S)$, we see that the endomorphism ring of $\Delta(S)$ is a division ring.

It remains to be shown that $P(S)$ has a Δ-filtration for all S. This we show by decreasing induction on $l(S)$. If $l(S) = n$, then $P(S) = \Delta(S)$. Assume we know that all $P(S)$ with $l(S) > i$ have a Δ-filtration. Let $l(S) = i$. Then $R(S)$ is a direct sum of projective modules $P(S')$ with $l(S') > l(S)$, thus it has a Δ-filtration. Then also $P(S)$ has a Δ-filtration. This shows that Γ is quasi-hereditary (see for example [DR, London.Math.Soc.]).

Now we have to see that the global dimension of Γ is at most n. We show by induction on $l(S)$ that $\operatorname{proj.dim} S \le l(S)$. We start with $l(S) = 1$. In this case, $\Delta(S) = S$, thus there is the exact sequence $0 \to R(S) \to P(S) \to S \to 0$ with $R(S)$ projective. This shows that $\operatorname{proj.dim} S \le 1$. For the induction step, consider some $i \ge 2$ and assume that $\operatorname{proj.dim} S \le l(S)$ for all S with $l(S) < i$. Now there is the exact sequence

$$0 \to R(S) \to \operatorname{rad} P(S) \to \operatorname{rad}\Delta(S) \to S \to 0.$$

All the composition factors S' os $\operatorname{rad}\Delta(S)$ satisfy $l(S') < i$, thus $\operatorname{proj.dim} S' < i$. Also, $R(S)$ is projective, thus $\operatorname{proj.dim} R(S) = 0 < i$. This shows that $\operatorname{rad} P(S)$ has a filtration whose factors have projective dimension less than i, and therefore $\operatorname{proj.dim}\operatorname{rad} P(S) < i$. As a consequence, $\operatorname{proj.dim} S \le i$.

Since all the simple Γ-modules have layer at most n, it follows that all the imple modules have projective dimension at most n, thus the global dimension of Γ is bounded by n.

3.3. The finiteness Theorem.

Theorem (Iyama). *Let X be a Λ-module. Then there is a Λ-module Y such that $\Gamma = \operatorname{End}(X \oplus Y)$ is strongly quasi-hereditary with $|X|$ layers. In particular, the global dimension of Γ is at most $|X|$.*

The construction of A yields the following dditional information: Any indecomposable direct summand of the module Y is a submodule of an indecomposable direct summand of X.

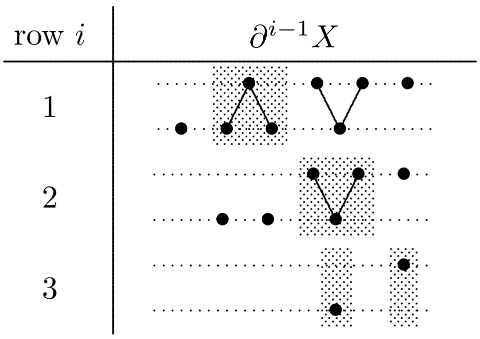
Proof: By $(2')$ of 2.1., we know that $\partial^n X = 0$ where $n = |X|$. Take $Y = \bigoplus_{i\ge 1} \partial^i X$, and $M = X \oplus Y$ with endomorphism ring $\Gamma = \operatorname{End}(M)$. Also, let $\mathcal{C}_i = \mathcal{C}_i(X)$. If N is an indecomposable module in $\mathcal{C}_i \setminus \mathcal{C}_{i+1}$, we define the layer $l(S(N)) = i + 1$. Thus we obtain a layer function with values in $\{1, 2, \ldots, n\}$. According to Iyama's theorem, Γ is left stongly quasi-hereditary with n layers, thus the global dimension of Γ is bounded by n, according to (3.2).

The additional information comes from $(4'')$ in 3.1.

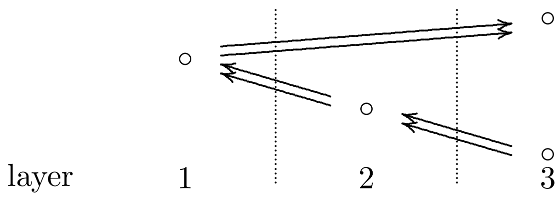
Corollary. *The representation dimension of Λ is at most $2|\Lambda|$.*

Proof: Consider the module $X = \Lambda \oplus D\Lambda$. Its length is $n = |\Lambda \oplus D\Lambda| = 2|\Lambda|$. Let $M = X \oplus Y$ as in Theorem. By construction, M is a generator-cogenerator, thus the representation dimension of Λ is bounded by n.

Example. Let us consider in detail the minimal generator-cogenerator $X = \Lambda \oplus D\Lambda$ for the Kronecker algebra Λ.



In row i ($1 \le i \le 3$) we have exhibited the indecomposable direct summands N of the module $\partial^{i-1}X$ by specifying a suitable basis of N using bullets; these bullets are connected by arrows ponting downwards (we draw just line segments) which indicate scalar multiplications by some elements of Λ. The modules in $\mathcal{C}_{i-1} \setminus \mathcal{C}_i$ are shaded. The quiver of Γ with its layer structure looks as follows:



AUSBLICK.

Theorem. *For any module X, there is an exact sequence*

$$X \to Y_0 \to Y_1 \to \cdots \to Y_n$$

which is a $\mathcal{C}$-coresolution (this means it remains exact when we apply $\mathrm{Hom}(-, C)$ *with $C \in \mathcal{C}$) such that Y_i belongs to $\mathcal{C}_i$ for all i.*

- Strongly quasihereditary algebras have been studied for example in [DR].
- Gabriel-Roiter measure

4. Torsionless finite artin algebras.

We call an artin algebra Λ *torsionless-finite* provided there are only finitely many isomorphism classes of indecomposable modules which are torsionless (i.e. submodules of projective modules). Let $\mathcal{L} = \mathcal{L}(\Lambda)$ be the class of torsionless Λ-modules.

4.1. The Auslander-Bridger equivalence

According to Auslander-Bridger [AB] a torsionless-finite artin algebra has also only finitely many isomorphism classes of indecomposable modules which are factor modules of injective modules.

Let $\mathcal{P} = \mathcal{P}(\Lambda)$ be the class of projective Λ-modules. We have $\mathcal{P}(\Lambda) \subseteq \mathcal{L}(\Lambda)$, and we denote by $\mathcal{L}(\Lambda)/\mathcal{P}(\Lambda)$ the factor category obtained from $\mathcal{L}(\Lambda)$ by factoring out the ideal of all maps which factor through a projective module.

Theorem 1. *There is a duality*

$$\eta\colon \mathcal{L}(\Lambda)/\mathcal{P}(\Lambda) \longrightarrow \mathcal{L}(\Lambda^{\mathrm{op}})/\mathcal{P}(\Lambda^{\mathrm{op}})$$

with the following property: If U is a torsionless module, and $f: P_1(U) \to P_0(U)$ is a projective presentation of U, then for $\eta(U)$ we can take the image of $\mathrm{Hom}(f, \Lambda)$.

Note that there also is a duality between $\mathcal{P}(\Lambda)$ and $\mathcal{P}(\Lambda^{\mathrm{op}})$, given by the functor $\mathrm{Hom}(-, \Lambda)$. Using these two dualities, we see:

Corollary 1. *There is a canonical bijection between the isomorphism classes of the indecomposable torsionless Λ-modules and the isomorphism classes of the indecomposable torsionless Λ^{op}-modules.*

Proof: $\mathrm{Hom}(-, \Lambda)$ provides a bijection between the isomorphism classes of the indecomposable projective Λ-modules and the isomorphism classes of the indecomposable projective Λ^{op}-modules. For the non-projective indecomposable torsionless modules, we use the duality η.

Remark. As we have seen, there are canonical bijections between the indecomposable projective Λ-modules and Λ^{op}-modules, as well between the indecomposable non-projective torsionless Λ-modules and Λ^{op}-modules, both being given by categorical dualities, but these bijections do not combine to a bijection with nice categorical properties. We will exhibit suitable examples below. There, we will use the duality D in order to replace the category $\mathcal{L}(\Lambda^{\mathrm{op}})$ of torsionless Λ^{op}-modules by Λ-modules, namely by the category $\mathcal{K}(\Lambda)$ of all factor modules of injective modules.

We call Λ *torsionless-finite* provided there are only finitely many isomorphism classes of indecomposable torsionless Λ-modules.

Remark. A related notion was introduced by N.Richmond [Rm]: Λ is said to be *subfinite* provided for any projective module P, there are only finitely many isomorphism classes of indecomposable submodules of P. Torsionless-finite artin algebras are of course subfinite. One may conjecture that for infinite artin algebras also the converse is true, but this requires a Brauer-Thrall-II-type assertion for the subcategory of torsionless modules.

Corollary 3. *If Λ is torsionless-finite, also $\Lambda^{\rm op}$ is torsionless-finite.*

Whereas corollaries 1 and 2 are of interest only for non-commutative artin algebras, the theorem itself is also of interest for Λ commutative.

Corollary 3. *For Λ a commutative artin algebra, the category $\mathcal{L}/\mathcal{P}$ has a self-duality.*

For example, consider the factor algebra $\Lambda = k[T]/\langle T^n\rangle$ of the polynomial ring $k[T]$ in one variable, with k is a field. Since Λ is self-injective, all the modules are torsionless. Note that in this case, η coincides with the syzygy functor Ω.

Proof of theorem 1. We call an exact sequence $P_1 \to P_0 \to P_{-1}$ with projective modules P_i *strongly exact* provided it remains exact when we apply $\operatorname{Hom}(-,\Lambda)$. Let $\mathcal{E}$ be the category of strongly exact sequences $P_1 \to P_0 \to P_{-1}$ with projective modules P_i (as a full subcategory of the category of complexes).

Lemma. *The exact sequence $P_1 \xrightarrow{f} P_0 \xrightarrow{g} P_{-1}$, with all P_i projective and epi-mono factorization $g = ue$ is strongly exact if and only if u is a left Λ-approximation.*

Proof: Under the functor $\operatorname{Hom}(-,\Lambda)$, we obtain

$$\operatorname{Hom}(P_{-1},\Lambda) \xrightarrow{g^*} \operatorname{Hom}(P_0,\Lambda) \xrightarrow{f^*} \operatorname{Hom}(P_1,\Lambda)$$

with zero composition. Assume that u is a left Λ-approximation. Given $\alpha \in \operatorname{Hom}(P_0,\Lambda)$ with $f^*(\alpha) = 0$, we rewrite $f^*(\alpha) = \alpha f$. Now e is a cokernel of f, thus there is α' with $\alpha = \alpha' e$. Since u is a left Λ-approximation, there is α'' with $\alpha' = \alpha'' u$. It follows that $\alpha = \alpha' e = \alpha'' ue = \alpha'' g = g^*(\alpha'')$.

Conversely, assume that the sequence $(*)$ is exact, let U be the image of g, thus $e\colon P_0 \to U, u\colon U \to P_{-1}$. Consider a map $\beta\colon U \to \Lambda$. Then $f^*(\beta e) = \beta e f = 0$, thus there is $\beta' \in \operatorname{Hom}(P_{-1},\Lambda$ with $g^*(\beta') = \beta e$. But $g^*(\beta) = \beta' g = \beta' ue$ and $\beta e = \beta' ue$ implies $\beta = \beta' u$, since e is an epimorphism.

Let $\mathcal{U}$ be the full subcategory of $\mathcal{E}$ of all sequences which are direct sums of sequences of the form

$$P \to 0 \to 0, \quad P \xrightarrow{1} P \to 0, \quad 0 \to P \xrightarrow{1} P, \quad 0 \to 0 \to P.$$

Define the functor $q\colon \mathcal{E} \to \mathcal{L}$ by $q(P_1 \xrightarrow{f} P_0 \xrightarrow{g} P_{-1}) = \operatorname{Im} g$. Clearly, q sends $\mathcal{U}$ onto $\mathcal{P}$, thus it induces a functor

$$\overline{q}\colon \mathcal{E}/\mathcal{U} \longrightarrow \mathcal{L}/\mathcal{P}.$$

Claim: *This functor $\overline{q}$ is an equivalence.*

First of all, the functor q is dense: starting with $U \in \mathcal{L}$, let

$$P_1 \xrightarrow{f} P_0 \xrightarrow{e} U \to 0$$

be a projective presentation of U, let $u\colon U \to P_{-1}$ be a left Λ-approximation of U, and $g = ue$.

Second, the functor q is full. This follows from the lifting properties of projective presentations and left Λ-approximations.

It remains to show that $\overline{q}$ is faithful. We will give the proof in detail (and it may look quite technical), however we should remark that all the arguments are standard; they are the usual ones dealing with homotopy categories of complexes. Looking at strongly exact sequences $P_1 \xrightarrow{f} P_0 \xrightarrow{g} P_{-1}$, one should observe that the image U of g has to be considered as the essential information: starting from U, one may attach to it a projective presentation (this means going from U to the left in order to obtain $P_1 \xrightarrow{f} P_0$) as well as a left Λ-approximation of U (this means going from U to the right in order to obtain P_{-1}).

In order to show that $\overline{q}$ is faithful, let us consider the following commutative diagram

$$\begin{tikzcd} P_1 \arrow[r, "f"] \arrow[d, "h_1"'] & P_0 \arrow[r, "g"] \arrow[d, "h_0"'] & P_{-1} \arrow[d, "h_{-1}"'] \\ P_1' \arrow[r, "f'"] & P_0' \arrow[r, "g'"] & P_{-1}' \end{tikzcd}$$

with strongly exact rows. We consider epi-mono factorizations $g = eu, g' = e'u'$ with $e\colon P_0 \to U, u\colon U \to P_{-1}, e'\colon P_0' \to U', u'\colon U' \to P_{-1}'$, thus $q(P_\bullet) = U, q(P_\bullet') = U'$. Assume that $q(h_\bullet) = ab$, where $a\colon U \to X, b\colon X \to U'$ with X projective. We have to show that $h_\bullet$ belongs to $\mathcal{U}$.

The factorizations $g = eu, g' = e'u', q(h_\bullet) = ab$ provide the following equalities:

$$eab = h_0e', \quad uh_1 = abu'.$$

Since $u\colon U \to P_{-1}$ is a left Λ-approximation and X is projective, there is $a'\colon P_{-1} \to X$ with $ua' = a$. Since $e'\colon P_0' \to U'$ is surjective and X is projective, there is $b'\colon X \to P_0'$ with $b'e' = b$.

Finally, we need $c\colon P_0 \to P_1'$ with $cf' = h_0 - eab'$. Write $f' = w'v'$ with w' epi and v' mono; in particular, v' is the kernel of g'. Note that $eab'g' = eab'e'u' = eabu' = h_0e'u' = h_0g'$, thus $(h_0 - eab')g' = h_0g' - eab'g' = h_0g' - h_0g' = 0$, thus $h_0 - eab'$ factors through the kernel v' of g', say $h_0 - eab' = c'v'$. Since P_0 is projective and w' is surjective, we find $c\colon P_0 \to P_1'$ with $cw' = c'$, thus $cf' = cw'v' = c'v' = h_0 - eab'$.

Altogether, we obtain the following commutative diagram

$$
\begin{array}{ccccc}
P_1 & \xrightarrow{f} & P_0 & \xrightarrow{g} & P_{-1} \\
{\scriptstyle [1\ f]}\downarrow & & \downarrow{\scriptstyle [1\ ea]} & & \downarrow{\scriptstyle [a'\ h_1 - a'bu']} \\
P_1 \oplus P_0 & \xrightarrow{\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]} & P_0 \oplus X & \xrightarrow{\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]} & X \oplus P'_{-1} \\
{\scriptstyle \left[\begin{smallmatrix} h_1 - fc \\ c \end{smallmatrix}\right]}\downarrow & & \downarrow{\scriptstyle \left[\begin{smallmatrix} h_0 - eab' \\ b' \end{smallmatrix}\right]} & & \downarrow{\scriptstyle \left[\begin{smallmatrix} bu' \\ 1 \end{smallmatrix}\right]} \\
P'_1 & \xrightarrow{f'} & P'_0 & \xrightarrow{g'} & P'_{-1}
\end{array}
$$

which is the required factorization of $h_\bullet$ (here, the commutativity of the four square has to be checked; in addition, one has to verify that the vertical compositions yield the maps h_i; all these calculations are straight forward).

Now consider the functor $\mathrm{Hom}(-, \Lambda)$, it yields a duality

$$\mathrm{Hom}(-, \Lambda) \colon \mathcal{E}(\Lambda) \longrightarrow \mathcal{E}(\Lambda^{\mathrm{op}})$$

which sends $\mathcal{U}(\Lambda)$ onto $\mathcal{U}(\Lambda^{\mathrm{op}})$. Thus, we obtain a duality

$$\mathcal{E}(\Lambda)/\mathcal{U}(\Lambda) \longrightarrow \mathcal{E}(\Lambda^{\mathrm{op}})/\mathcal{U}(\Lambda^{\mathrm{op}}).$$

Combining the functors considered, we obtain the following sequence

$$\mathcal{L}(\Lambda)/\mathcal{P}(\Lambda) \xleftarrow{\overline{q}} \mathcal{E}(\Lambda)/\mathcal{U}(\Lambda) \xrightarrow{\mathrm{Hom}(-,\Lambda)} \mathcal{E}(\Lambda^{\mathrm{op}})/\mathcal{U}(\Lambda^{\mathrm{op}}) \xrightarrow{\overline{q}} \mathcal{L}(\Lambda^{\mathrm{op}})/\mathcal{P}(\Lambda^{\mathrm{op}}),$$

this is duality, and we denote it by η.

It remains to show that η is given by the mentioned recipe. Thus, let U be a torsionless module. Take a projective presentation

$$P_1 \xrightarrow{f} P_0 \xrightarrow{e} U \to 0$$

of U, and let $m \colon U \to P_{-1}$ be a left $\mathcal{P}$-approximation of U and $g = eu$. Then

$$P_\bullet = (P_1 \xrightarrow{f} P_0 \xrightarrow{g} P_{-1})$$

belongs to $\mathcal{E}$ and $q(P_\bullet) = U$. The functor $\mathrm{Hom}(-, \Lambda)$ sends $P_\bullet$ to

$$\mathrm{Hom}(P_\bullet, \Lambda) = (\mathrm{Hom}(P_{-1}, \Lambda) \xrightarrow{\mathrm{Hom}(g,\Lambda)} \mathrm{Hom}(P_0, \Lambda) \xrightarrow{\mathrm{Hom}(f,\Lambda)} \mathrm{Hom}(P_1, \Lambda))$$

in $\mathcal{E}(\Lambda^{\mathrm{op}})$. Finally, the equivalence

$$\mathcal{E}(\Lambda^{\mathrm{op}})/\mathcal{U}(\Lambda^{\mathrm{op}}) \xrightarrow{\overline{q}} \mathcal{L}(\Lambda^{\mathrm{op}})/\mathcal{P}(\Lambda^{\mathrm{op}})$$

sends $\operatorname{Hom}(P_\bullet, \Lambda)$ to the image of $\operatorname{Hom}(f, \Lambda)$.

A module is said to be *co-torsionless* provided it is a factor module of an injective module. Let $\mathcal{K} = \mathcal{K}(\Lambda)$ be the class of co-torsionless Λ-modules. Of course, the duality functor D provides a bijection between the isomorphism classes of co-torsionless modules and the isomorphism classes of torsionless right modules.

If we denote by $\mathcal{Q} = \mathcal{Q}(\Lambda)$ the class of injective modules, then we see that D provides a duality

$$D\colon \mathcal{L}(\Lambda^{\mathrm{op}})/\mathcal{P}(\Lambda^{\mathrm{op}}) \longrightarrow \mathcal{K}(\Lambda)/\mathcal{Q}(\Lambda).$$

We get the following corollaries of Theorem 1.

Corollary 4. *The categories $\mathcal{L}(\Lambda)/\mathcal{P}(\Lambda)$ and $\mathcal{K}(\Lambda)/\mathcal{Q}(\Lambda)$ are equivalent under the functor $D\eta$.*

Note that $D\eta$ is is equal to $\Sigma\tau$ (restricted to $\Lambda/\mathcal{P}$), where τ is the Auslander-Reiten translation and Σ is the suspension functor (defined by $\Sigma(V) = I(V)/V$, where $I(V)$ is an injective envelope of V). Namely, in order to calculate $\tau(U)$, we start with a minimal projective presentation $f\colon P_1 \to P_0$ and take as $\tau(U)$ the kernel of

$$D\operatorname{Hom}(f, \Lambda)\colon D\operatorname{Hom}(P_1, \Lambda) \longrightarrow D\operatorname{Hom}(P_0, \Lambda).$$

Now the kernel inclusion $\tau(U) \subset D\operatorname{Hom}(P_1, \Lambda)$ is an injective envelope of $\tau(U)$; thus $\Sigma\tau(U)$ is the image of $D\operatorname{Hom}(f, \Lambda)$, but this image is $D\eta(U)$.

Corollary 5. *If Λ is torsionless-finite, the number of isomorphism classes of indecomposable factor modules of injective modules is equal to the number of isomorphism classes of indecomposable torsionless modules.*

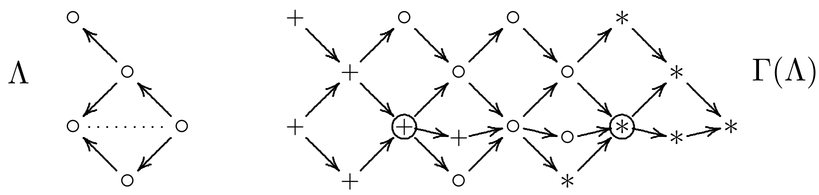
Examples: (1) The path algebra of a linearly oriented quiver of type A_3 modulo the square of its radical.

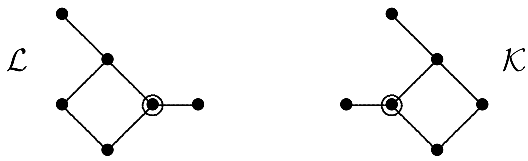
$$\circ \longleftarrow \circ \longleftarrow \circ$$

We present twice the Auslander-Reiten quiver. Left, we mark by $+$ the indecomposable torsionless modules and encircle the unique non-projective torsionless module. On the right, we mark by $*$ the indecomposable co-torsionless modules and encircle the unique non-injective co-torsionless module:

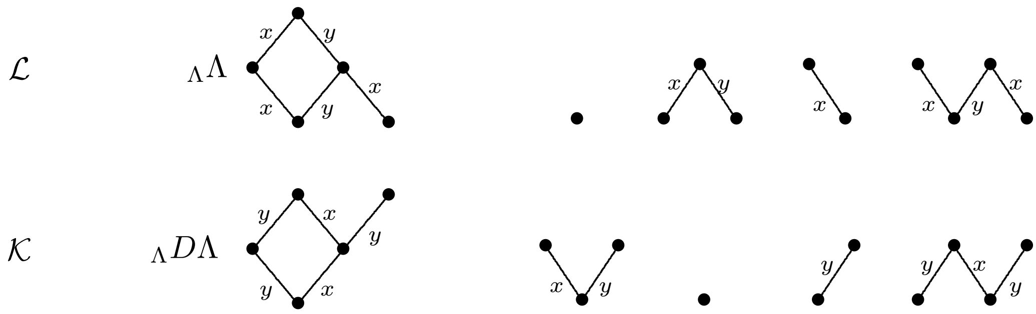
(2) Next, we look at the algebra Λ given by the following quiver with a commutative square; to the right, we present its Auslander-Reiten quiver $\Gamma(\Lambda)$ and

mark the torsionless and co-torsionless modules as in the previous example. Note that the subcategories $\mathcal{L}$ and $\mathcal{K}$ are linearizations of posets.





(3) The local algebra Λ with generators x, y and relations $x^2 = y^2$ and $xy = 0$. In order to present Λ-modules, we use the following convention: the bullets represent base vectors, the lines marked by x or y show that the multiplication by x or y, respectively, sends the upper base vector to the lower one (all other multiplications by x or y are supposed to be zero). The upper line shows all the indecomposable modules in $\mathcal{L}$, the lower one those in $\mathcal{K}$.



Let us stress the following: All the indecomposable modules in $\mathcal{L} \setminus \mathcal{P}$ as well as those in $\mathcal{K} \setminus \mathcal{Q}$ are Λ'-modules, where $\Lambda' = k[x, y]/\langle x, y\rangle^2$. Note that the category of Λ'-modules is stably equivalent to the category of Kronecker modules, thus all its regular components are homogeneous tubes. In $\mathcal{L}$ we find two indecomposable modules which belong to one tube, in $\mathcal{K}$ we find two indecomposable modules which belong to another tube. The algebra Λ' has an automorphism which exchanges these two tubes; this is an outer automorphism, and it **cannot** be lifted to an automorphism of Λ itself.

4.2. Torsionless-finite algebras have representation dimension at most 3.

The proof is implicit in the Queen Mary Notes [A]. Let L be an additive generator for the subcategory of all torsionless modules, and F an additive generator for the subcategory of all factor modules of injective modules. Given any

Λ-module X, let X' be the F-trace in X, thus the inclusion map $X' \to X$ is a right $(\operatorname{add} F)$-approximation of X. Let $p\colon X'' \to X$ be a right $(\operatorname{add} L)$-approximation of X. Then there is an exact sequence of the form $0 \to p^{-1}(X') \to X'' \oplus X' \to X \to 0$ which shows that $\Omega_{L\oplus F}(X)$ is a direct summand of $p^{-1}(X')$. Since $p^{-1}(X')$ is a submodule of X'', it follows that $\Omega_{L\oplus F}(X)$ is in $\operatorname{add} L$.

Theorem 2. *Assume that Λ is torsionless-finite (thus, $\mathcal{L}$ and $\mathcal{K}$ are finite). Let K, L be modules with $\operatorname{add} K = \mathcal{K}$, and $\operatorname{add} L = \mathcal{L}$. Then the endomorphism ring of $K \oplus L$ has global dimension at most 3.*

Note that L is a generator, K a cogenerator, thus $K \oplus L$ is a generator-cogenerator. By definition, the representation dimension of Λ is the minimum of the global dimension of the endomorphism rings of generator-cogenerators. Thus, the theorem implies the following:

Corollary 6. *The representation dimension of a torsionless-finite artin algebra is at most* 3.

Corollary 7. *Let Λ be an artin algebra with a faithful module M such that the subcategory of modules cogenerated by M is finite. Then the representation dimension of Λ is at most* 3.

Corollary 7 follows immediately from Corollary 6, since Λ itself is cogenerated by any faithful module M, thus all the torsionless modules are cogenerated by M.

Proof of Theorem. Let $M = K \oplus L$. In order to prove that the global dimension of $\operatorname{End}(M)$ is at most 3, we have to show that for any Λ-module X, the kernel $\Omega_M(X)$ of a minimal right M-approximation of X belongs to $\operatorname{add} M$ (Auslander-Lemma, see [E] or [CP]).

Let X be a Λ-module. Let U be the trace of $\mathcal{K}$ in X (this is the sum of the images of maps $K \to X$). Since $\mathcal{K}$ is closed under direct sums and factor modules, U belongs to $\mathcal{K}$ (it is the largest submodule of X which belongs to $\mathcal{K}$). Let $p\colon V \to X$ be a right $\mathcal{L}$-approximation of X (it exists, since we assume that $\mathcal{L}$ is finite). Since $\mathcal{L}$ contains all the projective modules, it follows that p is surjective. Now we form the pullback

$$\begin{tikzcd} V \arrow[r, "p"] & X \\ W \arrow[u, "u'"] \arrow[r, "p'"'] & U \arrow[u, "u"'] \end{tikzcd}$$

where $u\colon U \to X$ is the inclusion map. With u also u' is injective, thus W is a submodule of $V \in \mathcal{L}$. Since $\mathcal{L}$ is closed under submodules, we see that W belongs to $\mathcal{L}$. On the other hand, the pullback gives rise to the exact sequence

$$0 \to W \xrightarrow{[p' \;\; -u']} U \oplus V \xrightarrow{\begin{bmatrix} u \\ p \end{bmatrix}} X \to 0$$

(the right exactness is due to the fact that p is surjective). By construction, the map $\begin{bmatrix} u \\ p \end{bmatrix}$ is a right M-approximation, thus $\Omega_M(X)$ is a direct summand of W and therefore in $\mathcal{L} \subseteq \operatorname{add} M$. This completes the proof.

4.3. Examples of torsionless-finite algebras.

Many classes of artin algebras are known to be torsionless-finite: the hereditary algebras (Auslander), the algebras with $J^n = 0$ such that Λ/J^{n-1} is representation-finite, where J is the radical of Λ (Auslander), in particular: the algebras with $J^2 = 0$, but also the minimal representation-infinite algebras, then the artin algebras stably equivalent to hereditary algebras (Auslander-Reiten), the right glued algebras and the left glued algebras (Coelho, Platzeck; an artin algebra is right glued provided almost all indecomposable modules have projective dimension 1), as well as the special biserial (Schröer). Also, if Λ is a local algebra of quaternion type, then $\Lambda/\operatorname{soc}\Lambda$ is torsionless-finite, so that again its representation dimension is equal to 3 (Holm).

Call a finite set $\mathcal{A}$ of isomorphism classes of indecomposable modules an Auslander subset provided it contains all indecomposable projecive modules, all indecomposable injective modules and such that the global dimension d of the endomorphism ring of $\bigoplus_{M\in\mathcal{A}} M$ is at most $\max(3, \operatorname{rep.dim.} A)$ (thus either equal to $\operatorname{rep.dim.} A$ or else $d \le 3$ and $\operatorname{rep.dim.} A \le 2$).

Note that $\mathcal{A}$ is an Auslander subset iff any A-module has a universal $\mathcal{A}$-resolution of length at most $\max(-2 + \operatorname{rep.dim.} A, 1)$.

(1) Let $J = \operatorname{rad}\Lambda$ and assume that $J^n = 0$. Claim: *Any indecomposable torsionless module is either projective or annihilated by* J^{n-1}. Namely, let M be an indecomposable submodule of the projective module P, write $P = \bigoplus P_i$ with P_i indecomposable. Let $u\colon M \to P$ be the inclusion and $p_i\colon P \to P_i$ the canonical projections. If we assume that $J^{n-1}M \neq 0$, then $J^{n-1}(Mup_i) \neq 0$ for some i. But then Mup_i cannot be a submodule of JP_i, since $J^n = 0$. Since JP_i is the unique maximal submodule of P_i, it follows that up_i is surjective. Since P_i is projective, we see that up_i is a split epimorphism and thus an isomorphism (since M is indecomposable). Thus we see: if M is not annihilated by J^{n-1}, then M is projective. As a consequence, we see: *If* Λ/J^{n-1} *is representation-finite, then there are only finitely many isomorphism classes of indecomposable torsionless modules.* By left-right symmetry, we also see that there are only finitely many isomorphism classes of indecomposable torsionless right modules.

This implies: *If* Λ/J^{n-1} *is representation-finite, then the representation dimension of* Λ *is at most* 3. (Auslander [A], Proposition, p.143)

(2) Following Auslander (again [A], Proposition, p.143) the special case $J^2 = 0$ should be mentioned here. It is obvious that an indecomposable torsionless module is either projective or simple, an indecomposable co-torsionless module is either injective or simple, and any simple module is either torsionless or co-torsionless. Thus M is the direct sum of all indecomposable projective, all indecomposable injective, and all simple modules. Thus, *the representation dimension of an artin algebra with radical square zero is at most* 3.

(3) Another special case of (1) is of interest: We say that Λ is minimal representation-infinite provided Λ is representation-infinite, but any proper factor algebra is representation-finite. If Λ is minimal representation-infinite, and n is minimal with $J^n = 0$, then Λ/J^{n-1} is a proper factor algebra, thus representation-finite. It follows: *The representation dimension of a minimal representation-infinite algebra is at most* 3.

(4) If Λ is hereditary, then the only torsionless modules are the projective modules, the only co-torsionless modules are the injective ones, thus both classes $\mathcal{K}$ and $\mathcal{L}$ are finite. Thus we recover Auslander's result ([A], Proposition, p. 147) that *the representation dimension of a hereditary artin algebra is at most* 3.

(5) More generally, the classes $\mathcal{K}$ and $\mathcal{L}$ are finite in case Λ is stably equivalent to a hereditary artin algebra. Namely, an indecomposable torsionless module is either projective or simple ([AR1], Theorem 2.1), and dually, an indecomposable co-torsionless module is either injective or simple. Thus, *the representation dimension of an artin algebra which is stably equivalent to a hereditary artin algebra is at most* 3*;* (a result of Auslander-Reiten [AR2], Proposition 4.7).

Remarks concerning algebras which are stably equivalent to hereditary algebras: It follows that and indecomposable torsionless module is projective or simple (Dualizing III).

If $\mathcal{L}(\Lambda) = \operatorname{add} L$, then the global dimension of the endomorphism ring of L is at most 2 (dualizing V). More generally, if M is any generator, and $\operatorname{add} M$ is closed under submodules, then the global dimension of the endomorphism ring of M is at most 2 (dualizing V, Proposition 1.3).

In general, the question whether stably equivalent algebras have the same rep dim was asked in Dualizing V.

(6) Right glued algebras (and similarly left glued algebras): An artin algebra Λ is said to be *right glued*, provided the functor $\operatorname{Hom}(D\Lambda, -)$ is of finite length, or equivalently, provided almost all indecomposable modules have projective dimension equal to 1. The condition that $\operatorname{Hom}(D\Lambda, -)$ is of finite length implies that $\mathcal{K}(\Lambda)$ is finite. Also, the finiteness of the isomorphism classes of indecomposable modules of projective dimension greater than 1 implies that $\mathcal{L}(\Lambda)$ is finite. We see that *right glued algebras have representation dimension at most* 3 (a result of Coelho-Platzeck [CP]).

(7) The **special biserial** algebras without indecomposable projective-injective modules.

Proof: We need the following Lemma.

Lemma. Let A be special biserial and N an A-module. The following assertiongs are equivalent.
(i) M is a direct sum of local string modules.
(ii) $\alpha M \cap \beta M = 0$ for arrows $\alpha \neq \beta$.

Proof: (i) $\implies$ (ii). We can assume that M is indecomposable, thus $M = M(\alpha_1 \cdots \alpha_s \beta_1^{-1} \cdots \beta_r^{-1})$ with arrows α_i, β_j and $r, s \geq 0$. Condition (ii) is obviously satisfied.
(ii) $\implies$ (i): We can assume that M is indecomposable. For a band module, condition (ii) is not satisfied. For a string module, condition (ii) is not satisifed in case we deal with a word with a subword of the form $\alpha^{-1}\beta$.

Proof of Proposition: If there is no indecomposable projective-injective module, then all the indecomposable projective modules are string modules (and of course local). Thus ${}_AA$ satisfied the condition (ii). But then also any torsionless module satisfies the condition (ii). Thus torsionless indecomposable modules are local string modules, and the number of such modules is finite.

(8) $\Lambda/\operatorname{soc}\Lambda$, where Λ is a local algebra of dihedral, semidihedral or quaternion type.

Proof: Let us consider those of quaternion type. Here,

$$\Lambda/\operatorname{soc}\Lambda = k\langle x, y\rangle/(x^2 - (yx)^n y, y^2 - (xy)^n x).$$

Its radical M is a module over

$$\Lambda/\operatorname{soc}^2\Lambda = k\langle x, y\rangle/(x^2, (yx)^n, y^2, (xy)^n),$$

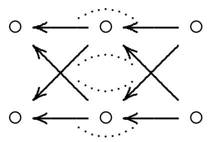
this is a special biserial algebra, and M is the band module for the cyclic word $w = (xy)^n x^{-1}(yx)^n y^{-1}$ and the eigenvalue $\lambda = 1$. Claim: there are only finitely many isomorphism classes of indecomposable modules cogenerated by M: all but M itself are string modules for strings of length at most $2n + 1$. This follows from the following lemma:

Lemma. Let A be special biserial. Let $M(w, \lambda, n)$ be a band module where w is minimal. Then $M(w, \lambda, n)$ is subfinite.

4.4. Algebras with representation dimension at most 3 which are not torsionless-finite.

It should be stressed that there are many classes of artin algebras with representation dimension 3 which are not necessarily torsionless-finite: for example the tilted algebras (Assem-Platzeck-Trepode), the trivial extensions of hereditary algebras (Coelho-Platzeck) as well as the canonical algebras (Oppermann).

(a) Some special biserial algebras. Example: $\Lambda = k[x,y]/(x^2,y^2)$; the radical of Λ is the indecomposable injective module for $k[x,y]/(x,y)^2$, thus it cogenerates all $k[x,y]/(x,y)^2$-modules. Here is an example of a special algebra Λ' without indecomposable projective-injective modules:



The radical of Λ' contains a one-parameter family of band modules.

Also, the algebra with quiver

$$\circ \underset{\alpha_2}{\overset{\alpha_1}{\leftleftarrows}} \circ \underset{\beta_2}{\overset{\beta_1}{\leftleftarrows}} \circ$$

and relations $\alpha_1\beta_1 = \alpha_2\beta_2 = \alpha_1\beta_2 - \alpha_2\beta_1$ is special biserial, but not torsionless-finite: all the indecomposable non-injective Kronecker modules for the subquiver with arrows α_1, β_1 are torsionless.

(b) Some tilted algebras. Example: Take the one-point extension of the Kronecker quiver by the indecomposable injective module of length 3. This is the algebra

$$\circ \underset{y}{\overset{x}{\leftleftarrows}} \circ \underset{y}{\overset{x}{\leftleftarrows}} \circ \qquad \text{with relations } xy = yx,\ x^2 = 0 = y^2$$

(c) Some canonical algebras. The torsionless-finite canonical algebras have been classfied by Barot and Crawley-Boevey [BC].

Here is an example: Consider the canonical algebra with 7 branches of length 2 (and parameters $\lambda_1 = \infty, \lambda_2 = 0, \lambda_3 = 1, \lambda_4, \ldots, \lambda_7$). Let $U = \operatorname{rad} P(\infty)$; this is given by the 2-dimensional vectorspace k^2 and seven 1-dimensional subspaces, namely

$$\begin{gathered} 0k,\ k0,\ \Gamma(\lambda_3) = \{(x,x) \mid x \in k\}, \\ \Gamma(\lambda_4) = \{(x,\lambda_4 x) \mid x \in k\}, \quad \Gamma(\lambda_5) = \{(x,\lambda_5 x) \mid x \in k\}, \\ \Gamma(\lambda_6) = \{(x,\lambda_6 x) \mid x \in k\}, \quad \Gamma(\lambda_7) = \{(x,\lambda_7 x) \mid x \in k\}, \end{gathered}$$

where $\Gamma(\lambda) = \{(x, \lambda x) \mid x \in k\}$ is the graph of the multiplication with λ.

In U^2, given by the vectorspace

$$UU = kkkk = \{(x_1, x_2; x_3, x_4) \mid x_1, \dots, x_4 \in k\},$$

we consider the subspaces

$$\{(0, x; 0, y) \mid x, y \in k\}, \quad \{(x, 0; y, 0) \mid x, y \in k\} \quad \{(x, x; y, y) \mid x, y \in k\},$$
$$\{(0, 0; x, \lambda_4 x) \mid x \in k\}, \quad \{(x, \lambda_5 x; 0, 0) \mid x \in k\},$$
$$\{(x, \lambda_6 x; x, \lambda_6 x) \mid x \in k\}, \quad \{(x, \lambda_7 x; \mu x, \mu\lambda_7 x) \mid x \in k\},$$

with $\mu \in k$.

(d) Some algebras A with a radical embedding $A \subset B$ such that B is representation-finite.

Let B be the path algebra with relations of a commutative square, say with arrows $\alpha, \beta, \gamma, \delta$ such that $\alpha\beta = \gamma\delta$. Let $A = k + \operatorname{rad} B$, this ais a local algebra with 5-dimensional radical with basis $\alpha, \beta, \gamma, \delta, \eta$, such that the only non-zero products are $\alpha\beta = \eta = \gamma\delta$. Then ${}_A \operatorname{rad} A$ is annihilated by β, δ, η, thus it is a module over the local algebra $A' = A/\langle \beta, \delta, \eta \rangle$ with basis $1, \alpha, \gamma$ such that α, γ is a basis of the radical and the radical square is zero. Note that ${}_A \operatorname{rad} A$ is the direct sum of the indecomposable injective A'-module and three simple modules.

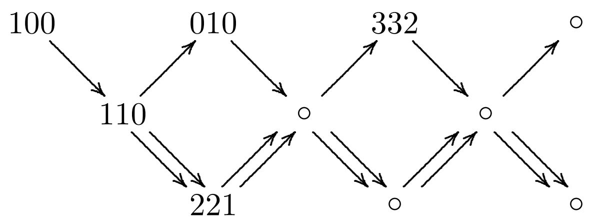
(f) Cluster tilted: Not all are torsionless-finite (but one conjectures that all have representation dimesnion at most 3). Example: Let $\Lambda = \Lambda'[I]$ be the one-point extension of the Kronecker algebra by the indecomposable injective module I of length 3 (this is the graded exterior algebra)! The algebra is tilted from the hereditary algebra H with quiver

$$\circ \longleftarrow \circ \leftleftarrows \circ$$

since it contains a slice module with this endomorphism ring. In order to construct the corresponding cluster tilted algebra, take the three indecomposable H-modules with dimension vectors

$$1\,0\,0, \quad 3\,3\,2, \quad 4\,4\,3.$$

The last module is regular, the first ones belong to the preprojective component:



(note that the module $100 \oplus 221 \oplus 332$ is a tilting module and mutation exchanges 221 by 443).

Now $\operatorname{Hom}(100, 332)$ is 3-dimensional, $\operatorname{Hom}(332, 443)$ is 2-dimensional, and

$$\begin{aligned}\operatorname{Hom}(443, \tau^{-1}100[1]) &= \operatorname{Ext}^1(443, \tau^{-1}100) = D\operatorname{Hom}(\tau^{-1}100, \tau 443)\\ &= D\operatorname{Hom}(100, \tau^2 443) = D\operatorname{Hom}(100, 223)\end{aligned}$$

is 2-dimension. This shows we obtain the quiver with vertices say a, b, c, with 3 arrows $a \to b$, 2 arrows $b \to c$ and 2 arrows $c \to a$.

On the other hand,

$$\begin{aligned}\operatorname{Hom}(332, \tau^{-1}100[1]) &= \operatorname{Ext}^1(332, \tau^{-1}100) = D\operatorname{Hom}(\tau^{-1}100, \tau 332)\\ &= D\operatorname{Hom}(100, \tau^2 332) = D\operatorname{Hom}(100, 100)\end{aligned}$$

is 1-dimensional, and one sees easily that the algebra given by the arrows $b \to c$ and $c \to a$ is just the graded exterior algebra.

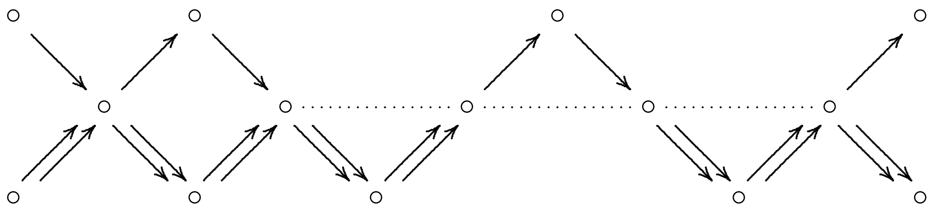
Now observe that

$$\operatorname{Hom}(332, \tau^{-1}332[1]) = 0$$

(this is a general fact for preprojective (or preinjective modules), this means that the projective module $P(332) = P(b)$ has just one composition factor b, and therefore $P(332)$ is just the 4-dimension projective Λ-module.

Remark: We get the commutativity relations by taking the endomorphism ring of the tilting module $100 \oplus 221 \oplus 332$; its quiver looks as follows: there are the vertices say a, b, c, there are tweo arrows $a \to b$ and 2 arrows $b \to c$. Thee cluster tilted algebra is obtained by adding one arrow $c \to a$ as wee as all possible zero relations $b \to c \to a$ and $c \to a \to b$.

The Auslander-Reiten quiver of the cluster tilted algebra looks as follows:



AUSBLICK.

- Stratification of module varieties
- Task: Given a torsionless-finite algebra, characterize the endomorphismring of torsionless $\oplus$ co-torsionless.
- For Λ hereditary, we get an Auslander subcategory by taking all projectives and all injectives, thus an additive generator of this subcategory is $M = \Lambda \oplus D\Lambda$. Note that

$$\operatorname{End}(M) = \begin{bmatrix} \Lambda & D\Lambda \\ 0 & \Lambda \end{bmatrix}$$

and this is part of the repetitive algebra $\widehat{\Lambda}$.

References

[A] M. Auslander: The representation dimension of artin algebras. Queen Mary College Mathematics Notes (1971)

[AB] M. Auslander, M. Bridger: Stable module theory. Mem. Amer. Math. Soc. 94 (1969).

[AR1] M. Auslander, I. Reiten: Stable equivalence of dualizing R-varieties: III. Dualizing R-varieties stably equivalent to hereditary dualizing R-varieties. Adv. in Math. 17 (1975), 122-142.

[AR2] M. Auslander, I. Reiten: Stable equivalence of dualizing R-varieties: V. Artin algebras stably equivalent to hereditary algebras. Adv. in Math. 17 (1975), 167-195.

[BC] Barot, Crawley-Boevey.

[CP] F. Coelho, M. I. Platzeck: On the representation dimension of some classes of algebras. J. Algebra 275 (2004), 615-628

[DR] V. Dlab, C.M. Ringel: The global dimension of the endomorphism ring of a generator-cogenerator for a hereditary artin algebra. Preprint (2008).

[EHIS] K. Erdmann, Th. Holm, O. Iyama, J. Schröer: Radical embeddings and representation dimension. Adv. in Math. 185 (2004), 159-177.

[IT] K. Igusa, G. Todorov: On the finitistic global dimension conjecture for artin algebras. In: Representations of algebras and related topics. Fields Inst. Commun. 45 (2005), 201-204.

[I] O. Iyama: Finiteness of representation dimension. Proc. Amer. Math. Soc. 131 (2003), 1011-1014.

[Rm] N.Richmond.

[Ro] R. Rouquier: Representation dimension of exterior algebras. Invent. Math. 168 (2006), 357-367.

[X] Ch. Xi: Representation dimension and quasi-hereditary algebras. Advances in Math. 168 (2002), 193-212.