Artin algebras of dominant dimension at least 2.

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We consider artin algebras with duality functor D. We consider left modules (usually, we call them just modules) as well as right modules. Maps will act on the opposite side of the scalars. Thus, if M is an A-module and B is its endomorphism ring, then M is a right B-module, thus a (left) $B^{\rm opp}$-module. The module M is said to be *balanced* (or to satisfy the double centralizer condition), provided the canonical map from A into the endomorphism ring $\operatorname{End}(M_B)$ (which sends a onto the left-multiplication with a) is bijective, where $B = \operatorname{End}(M)$.

Let B be an artin algebra. Let $d \geq 1$. Definition: The (left) *dominance* (or the dominant dimension) of B is at least d (written $\operatorname{dom} B \geq d$) provided there is an exact sequence

$$0 \to {}_BB \to I_0 \to \cdots \to I_{d-1}$$

such that all the modules I_i with $0 \leq i \leq d-1$ are projective and injective (obviously, one only has to require that the corresponding modules I_i in a minimal injective coresolution of ${}_BB$ are projective). The right dominance of B is defined in the same way, but using right B-modules. Here, we are only interested in the cases $d = 1$ and $d = 2$. (Our deviation of speaking about dominance instead of dominant dimension is due to the fact that the dominant dimensions as introduced first by Nakayama (dealing with bimodules) and then by Tachikawa seem to be quite different when compared with the usual notions of dimensions.)

We have $\operatorname{dom} B \geq 1$ iff the injective envelope of ${}_BB$ is projective iff ${}_BB$ can be embedded into a module which is both projective and injective iff there exists a faithful module which is both projective and injective iff there exists a left ideal which is both faithful and injective (note that an injective left ideal is always also projective, since it is a direct summand); such rings are also called QF-3 rings, according to Thrall. We see: *The left dominance of B is at least* 1 *if and only if the right dominance of B is at least* 1 (the dual of a faithful projective-injective module is a faithful projective-injective right module). *If B is an artin algebra with* $\operatorname{dom} B \geq 1$, *then there is a multiplicity-free faithful module N which is both projective and injective, and this module is unique up to isomorphism;*

this module N is usually called the *minimal faithful* B-module (this terminology can be explained as follows: N is faithful, and is a direct summand of any faithful module).

Theorem (Morita-Tachikawa).
There is a bijection between

- *the (isomorphism classes of) pairs* (A, M) *where* A *is a basic artin algebra and* M *a multiplicity-free* A*-module which is a generator-cogenerator, and*
- *the (isomorphism classes of) pairs* (B, N) *where* B *is a basic artin algebra with* $\operatorname{dom} B \geq 2$*, and* N *a minimal faithful* B*-module,*

defined as follows:

Given a multiplicity-free A*-module which is a generator-cogenerator, attach to the pair* (A, M) *the pair* (B, M) *where* $B = \operatorname{End}_A(M)^{\mathrm{opp}}$*.*

Conversely, given an artin algebra B *with* $\operatorname{dom} B \geq 2$ *and minimal faithful module* N*, attach to the pair* (B, N) *the pair* (A, N) *where* $A = \operatorname{End}(N)^{\mathrm{opp}}$*.*

Remarks: (1) Let us stress that under this correspondence, the second entry of the pairs in question remains untouched, at least set-theoretically: the second entry is a bimodule and the bijection yields a mutual change of the module action to be considered.

(2) In particular, the theorem asserts that the modules M and N considered are balanced.

(3) Looking at the pairs (B, N), we should stress that the module N is determined by B; thus instead of dealing with the pairs (B, N), we may delete N and consider just the isomorphism classes of artin algebras B with $\operatorname{dom} B \geq 2$.

Direction 1. *Let* M *be an* A*-module which is a generator and a cogenerator, let* $B = \operatorname{End}(M)$*. Then the module* M *is balanced, and* M_B *is a faithful injective right ideal of* B*, whereas* $D(M_B)$ *is (isomorphic to) a faithful injective left ideal. The ring* B *has left dominance at least* 2 *and right dominance at least* 2*.*

Proof: We can assume that A is basic; the general case then follows using Morita equivalences. In general, one knows that generators are balanced. The remaining assertions are shown as follows:

First, M_B is obviously faithful. Second, in order to show that M_B is projective, write ${}_AM = {}_AA \oplus M'$, thus

$$\begin{aligned} B_B &= \operatorname{Hom}_A({}_AM, {}_AM_B) \\ &= \operatorname{Hom}_A({}_AA \oplus M', {}_AM_B) \\ &= \operatorname{Hom}_A({}_AA, {}_AM_B) \oplus \operatorname{Hom}_A(M', {}_AM_B) \\ &= M_B \oplus \operatorname{Hom}_A(M', {}_AM_B). \end{aligned}$$

This shows that M_B can be considered as direct summand of B_B, in particular, M_B is projective. Dualizing this, we see that $D(M_B)$ is a direct summand of $D(B_B)$, thus injective.

Third, we show that M_B is injective. Equivalently, we show that $D(M_B)$ is a projective left B-module. Write $M = DA \oplus M''$, then

$$\begin{aligned} {}_BB &= \mathrm{Hom}_A({}_AM_B, {}_AM) \\ &= \mathrm{Hom}_A({}_AM_B, {}_A(DA) \oplus {}_AM'') \\ &= \mathrm{Hom}_A({}_AM_B, {}_A(DA)) \oplus \mathrm{Hom}_A({}_AM_B, {}_AM'') \\ &= D(M_B) \oplus \mathrm{Hom}_A({}_AM_B, {}_AM''). \end{aligned}$$

Recall that the assertions 2 and 3 imply: If P is a projective A-module, then $\mathrm{Hom}(P, M)$ is a projective-injective right B-module (it is sufficient to show this for P indecomposable projective, but then $\mathrm{Hom}_A({}_AP, {}_AM_B)$ is a direct summand of M_B).

Now take a projective presentation $P_1 \to P_0 \to M \to 0$ and apply $\mathrm{Hom}_A(-, {}_AM_B)$. We get an exact sequence

$$0 \to \mathrm{Hom}_A(M, M) \to \mathrm{Hom}_A({}_AP_0, {}_AM_B) \to \mathrm{Hom}_A({}_AP_1, {}_AM_B),$$

where the right two terms are projective-injective. This shows that the right dominance of B is at least 2.

Similarly, the assertions 2 and 3 imply: If Q is an injective A-module, then $\mathrm{Hom}(M, Q)$ is a projective-injective left B-module (again, it is sufficient to show this for Q indecomposable injective, but then $\mathrm{Hom}_A({}_AM_B, {}_AQ)$ is a direct summand of $D(M_B)$ and thus both projective and injective).

Now take an injective copresentation $0 \to M \to Q_0 \to Q_1$ and apply $\mathrm{Hom}_A({}_AM_B, -)$. We get an exact sequence

$$0 \to \mathrm{Hom}_A(M, M) \to \mathrm{Hom}_A({}_AM_B, {}_AQ_0) \to \mathrm{Hom}_A({}_AM_B, {}_AQ_1),$$

where the right two terms are projective-injective. This shows that also the left dominance of B is at least 2.

Now the converse:

Direction 2. *Assume that the left dominance of B is at least 1. Let fB be a right ideal which is faithful and injective. Then the fBf-module ${}_{fBf}fB$ is both a generator and a cogenerator. In case the left dominance of B is at least 2, one has ${}_{fBf}fB = B$, canonically.*

For the proof, we need a quite general result:

Proposition 1. *Let R be any ring and $f \in R$ an idempotent. Then the natural transformation*

$$\eta\colon fR \otimes_R \mathrm{Hom}_{fRf}(fR, -) \to \mathrm{id} \quad \textit{with} \quad fr \otimes \phi \mapsto (fr)\phi$$

for $r \in R, \phi \in \mathrm{Hom}_{fRf}(fR, X)$ where X is a left fRf-module, is an equivalence, thus the composition of functors

$$\mathrm{mod}\, fRf \xrightarrow{\mathrm{Hom}_{fRf}(fR,-)} \mathrm{mod}\, R \xrightarrow{fR\otimes -} \mathrm{mod}\, fRf$$

is an equivalence of categories. Note that $(fR \otimes -) = \mathrm{Hom}_R(Rf, -)$, these functors send the R-module X to fX.

Proof, well-known, for example:

$$\begin{aligned} fR \otimes_R \mathrm{Hom}_{fRf}(fR, X) &= f\,\mathrm{Hom}(fR, X) = \mathrm{Hom}_R(Rf, \mathrm{Hom}_{fRf}(fR, X)) \\ &= \mathrm{Hom}_{fRf}(fR \otimes_R Rf, X) = \mathrm{Hom}_{fRf}(fRf, X) = X, \end{aligned}$$

as left fRf-modules.

Proposition 2. *Let R be any ring and let e, f be idempotents of R such that Re is an injective left module and fR is a faithful right module. Then*

(a) *The canonical map $\rho\colon Re \to \mathrm{Hom}_{fRf}({}_{fRf}fR, fRe)$ defined by $y(x\rho) = yx$ for $x \in Re, y \in fR$ is an isomorphism of left R-modules.*

(b) $\mathrm{End}({}_{fRf}fRe) = eRe$ (where $ere \in eRe$ corresponds to the endomorphism of ${}_{fRf}fRe$ given by right multiplication with ere).

(c) *The fRf-module ${}_{fRf}fRe$ is injective.*

Proof of (a). The map ρ is an R-homomorphism: the left R-module structure on the Hom-set is given by the right R-structure of fR_R, this means that for a homomorphism α we have $y(r\alpha) = (yr)\alpha$. Thus $y[(r(x\rho)] = (yr)(x\rho) = (yr)x = y(rx) = y[(rx)\rho]$ and therefore $r(x\rho) = (rx)\rho$. The map is injective, since fR_R is faithful. We show that the map is an essential embedding. Thus, let $0 \neq \alpha\colon fR \to fRe$ be a homomorphism of left fRf-modules. There is $r \in R$ such that $(fr)\alpha \neq 0$. Note that $(fr)\alpha \in fRe \subseteq Re$, thus we can apply ρ and $((fr)\alpha))\rho \neq 0$. Claim: $((fr)\alpha)\rho = (fr)\alpha$. Namely, apply it to $y \in fR$ we get

$$y[((fr)\alpha)\rho] = y \cdot ((fr)\alpha) = y \cdot ((ffr)\alpha) = y \cdot f \cdot ((fr)\alpha) = (yffr)\alpha = (yfr)\alpha y((fr)\alpha).$$

This shows that $(Re)\rho \cap R\alpha \neq 0$. Since ${}_RRe$ is injective, we see that ρ is also surjective.

Proof of (b): Under ρ, the subset eRe of Re is mapped into $\mathrm{Hom}_{fRf}({}_{fRf}fRe, fRe)$, the subset $eR(1-e)$ of Re is mapped into $\mathrm{Hom}_{fRf}({}_{fRf}fR(1-e), fRe)$. Altogether we deal with the following situation:

$$\begin{array}{ccc} Re & \xrightarrow{\ \rho\ } & \mathrm{Hom}_{fRf}({}_{fRf}fR, fRe) \\ \| & & \| \\ eRe \oplus (1-e)Re & \longrightarrow & \mathrm{Hom}_{fRf}({}_{fRf}fRe, fRe) \oplus \mathrm{Hom}_{fRf}({}_{fRf}fR(1-e), fRe) \end{array}$$

Thus, we see that $\rho(eRe) = \mathrm{Hom}_{fRf}({}_{fRf}fRe, fRe)$.

Proof of (c). Since fR_R is projective, the functor $fR \otimes_R -$ sends injective R-modules to injective fRf-modules. Since ${}_RRe$ is injective, it follows that the fRf-module $fRe = fR \otimes_R Re$ is injective.

Proposition 3. *Let B be an artin algebra of dominance at least 1, let e, f be idempotents in B such that the left module Be and the right module fB both are faithful and injective. Let $U = {}_{fBf}fBe_{eBe}$. Then U is balanced, ${}_{fBf}U$ is an injective cogenerator, U_{eBe} is also an injective cogenerator.*

Proof: Before we start with the proof, let us introduce the following notation: If A is an artin algebra, let $s(A)$ be the number of simple A-modules. If M is an A-module, let $s(A)$ be the number of isomorphism classes of indecomposable direct summands of M. Thus $s(A) = s({}_AA) = s(A_A)$, and $s(M) = s(\mathrm{End}(M))$ for any artin algebra A and any module M. Consider the fBf-eBe-bimodule $U = fBe$. We know that the fBf-module U is injective (by Proposition 2 (c)) thus $s(fBf) \geq s({}_{fBf}U)$. By Proposition 2 (b), the endomorphism ring of U is eBe, thus $s({}_{fBf}U) = s(eBe)$. This shows that $s(fBf) \geq s(eBe)$. By left-right symmetry, we also see $s(eBe) \geq s(fBf)$, thus $s(fBf) = s(eBe)$, and therefore $s(fBf) = s({}_{fBf}U)$. This shows, that any indecomposable injective fBf-module occurs as a direct summand of U, thus U is a cogenerator.

Proof of the first assertion in Direction 2: *The module ${}_{fBf}fB$ is a generator and a cogenerator.* Of course, ${}_{fBf}fB$ is a generator, since

$$ {}_{fBf}fB = {}_{fBf}fBf \oplus {}_{fBf}fB(1-f). $$

But we have also the decomposition

$$ {}_{fBf}fB = {}_{fBf}fBe \oplus {}_{fBf}fB(1-e), $$

and according to Proposition 2 (c), ${}_{fBf}fBe$ is an injective cogenerator, thus ${}_{fBf}fB$ is a cogenerator.

Proposition 4. *Let B be an artin algebra of dominance at least 1, let e, f be idempotents in B such that the left module Be and the right module fB both are faithful and injective. The functor $\mathrm{Hom}_{fBf}(fB, -)$ sends $\mathrm{mod}\, fBf$ onto the full category of left B-modules which have a Be-copresentation.*

Proof. Let $0 \to Y \to U_0 \to U_1$ be a injective copresentation of the fBf-module Y. Apply $\mathrm{Hom}_{fBf}(fB, -)$, we obtain the exact sequence

$$ 0 \to \mathrm{Hom}_{fBf}(fB, Y) \to \mathrm{Hom}_{fBf}(fB, U_0) \to \mathrm{Hom}_{fBf}(fB, U_1) $$

Since U_i is an injective fBf-module, it is in $\mathrm{add}\,{}_{fBf}fBe$, thus $\mathrm{Hom}_{fBf}(fB, U_i)$ is in $\mathrm{add}\,\mathrm{Hom}_{fBf}(fB, fBe) = \mathrm{add}\, Be$.

Conversely, assume that the B-module M has a Be-copresentation, thus there is an exact sequence of B-modules

$$ 0 \to M \to N_0 \to N_1 $$

where N_0, N_1 are in $\mathrm{add}\,{}_BBe$. Multiplying from the left with f, we obtain an exact sequence

$$ 0 \to fM \to fN_0 \to fN_1 $$

with fN_i in add fBe, thus this is an injective copresentation of fM. Apply $\mathrm{Hom}_{fBf}(fB,-)$ we obtain the exact sequence

$$0 \to \mathrm{Hom}_{fBf}(fB, fM) \to \mathrm{Hom}_{fBf}(fB, fN_0) \to \mathrm{Hom}_{fBf}(fB, fN_1)$$

There is the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M & \longrightarrow & N_0 & \longrightarrow & N_1 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathrm{Hom}_{fBf}(fB, fM) & \longrightarrow & \mathrm{Hom}_{fBf}(fB, fN_0) & \longrightarrow & \mathrm{Hom}_{fBf}(fB, fN_1) \end{array}$$

with exact rows. Since the right vertical maps are isomorphisms, also the left one is an isomorphism. This shows that any B-module M with a Be-copresentation is in the image of the functor $\mathrm{Hom}_{fBf}(fB,-)$.

Proof of the second assertion of Direction 2: *If the dominance of B is at least 2, then the module ${}_{fBf}fB_B$ is balanced.* If the dominance of B is at least 2, the module ${}_BB$ has a Be-copresentation, thus it corresponds under the categorical equivalence mentioned above to fB and the endomorphism ring of ${}_{fBf}fB$ is B.

Corollary. *The left dominance of B is at least 2 if and only if the right dominance of B is at least 2*

Proof: First, assume only that the left dominance of B is at least 1. Let $A = fBf$, and $M = fB$. Then by Direction 2, ${}_AM$ is a generator and a cogenerator. According to Direction 1, the endomorphism ring $\mathrm{End}_A(M)$ has right dominance at least 2. If the left dominance of A is at least 2, then by Direction 2, $B = \mathrm{End}_A(M)$, thus B has right dominance at least 2.

References.

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Remarks: (1) The result presented has its roots in Morita's treatment [M] of what now is called *Morita duality*: As we have seen, the bimodule $U = {}_{fBf}fBe_{eBe}$ is balanced and is an injective cogenerator on either side, this is the initial condition for the Morita duality given by the functors $\mathrm{Hom}_{fBf}(-,U)$ and $\mathrm{Hom}_{eBe}(-,U)$ (see for example [T2], Theorem

(3.3). The notion of the dominant dimension (as defined above) is due to Tachikawa [T1]. A full treatment of the result is given in the joint paper [MT] of Morita and Tachikawa (which never was published), see also the lecture notes by [T]: theorem (5.3) together with (7.1) and (7.7). The formulation of Direction 1 corresponds to the Queen Mary Notes by Auslander, p.135.

(2) There are several papers which extend the result to larger classes of rings (already the Morita-Tachikawa paper [MT] dealt with semi-primary rings).

(3) In general, the dominant dimension of R is the same as the dominant dimension of R^{opp}. For $\operatorname{dom} R = 1$, this is easy to see, and we have shown that the characterization of algebras with $\operatorname{dom} R \leq 2$ proves this assertion also for algebras with dominant dimension 2. For a proof in general, see [T], theorem (7.7).

(4) Proposal: Let A be an artin algebra. We propose to call a ring B a *propagation* of A provided $B = \operatorname{End}({}_AM)^{\mathrm{opp}}$, where $\_AM$ is a generator-cogenerator.