Example: Y the simple injective Module of the n-Kronecker quiver.

Let Λ = K(n) be the n-Kronecker quiver

\[ \circ \rightarrow \circ \rightarrow \circ \]

with \( n \) arrows and \( Y = T \) the simple injective module. Then it is reasonable to consider only the epimorphisms, thus we have to take \( C = P_2 = (n^2 - 1, n) \) (the indecomposable non-projective module which generates all indecomposable non-projective modules).

The elements in \( C \rightarrow Y \) are just the epimorphisms \( X \rightarrow Y \), where \( X \) is a local module \( X = P(T)/U \) where \( U \) is a \( t \)-dimensional subspace of the socle of \( P(T) \), thus \( \text{dim} \ X = (n - t, 1) \).

Now the dimension of \( \text{Hom}(C, Y) \) is \( n \), thus \( \mathcal{S} \text{Hom}(C, Y) \) is the \( (n - 1) \)-dimensional projective space. Say for \( n = 4 \), we get the following lattice

\[ \begin{align*}
\mathcal{S} \text{Hom}(C, Y) & \quad \text{Hom}(C, Y) & \quad 4 \\
& & 3 \\
& & 2 \\
& & 1 \\
0 & & 0
\end{align*} \]

On the right, we have listed the (affine) dimension of the various subspaces.

Correspondly, we have the diagram of (right equivalence classes of) maps

\[ \begin{align*}
C \rightarrow Y & \quad 0 \rightarrow Y \rightarrow Y & \quad t = 4 \\
& \quad \ldots \quad K \rightarrow \ast \rightarrow Y & \quad 3 \\
& \quad \ldots \quad \ldots \quad K^2 \rightarrow \ast \rightarrow Y & \quad 2 \\
& \quad \ldots \quad \ldots \quad K^3 \rightarrow \ast \rightarrow Y & \quad 1 \\
& \quad \ldots \quad \ldots \quad K^4 \rightarrow \ast \rightarrow Y & \quad 0
\end{align*} \]

The smallest element is the exact sequence \( K^4 \rightarrow P_1 \rightarrow Y \), altogether we see here (looking at paths going upwards) are the factorisations of the canonical map \( P_1 \rightarrow Y \) using only epimorphisms.

Let us analyze the Auslander bijection

\[ C \rightarrow Y \leftrightarrow \mathcal{S} \text{Hom} \ P(C, Y) \]
in this case.

We start with a submodule (subspace!) of $\text{Hom}(C, Y)$ say of dimension $t$; using a basis of this subset, we get a (surjective) map $g: C^t \to Y$, thus an exact sequence

$$0 \to W \to C^t \xrightarrow{g} Y \to 0$$

with $g$ being right minimal. We have to determine the structure of the kernel $W$. Now $W$ is cogenerated by $C$, thus it is a direct sum of copies of $P_0, P_1, P_2 (= C)$. But since $g$ is right minimal, there cannot be any copy of $P_2$, therefore $W$ is projective. Since the dimension vector is $\dim W = t(n^2 - 1, n) - (0, 1) = (tn^2 - t, tn - 1)$, it follows that $W$ is isomorphic to the direct sum of $tn - 1$ copies of $P_1$ and $n - t$ copies of $P_0$. For the proof, we only have to check that the dimensions are right:

$$(tn - 1) \dim P_1 + (n - t) \dim P_0 = (tn - 1)(n, 1) + (n - t)(1, 0)$$

$$= (tn^2 - n + n - t, tn - 1) = (tn^2 - t, tn - 1)$$

$$= \dim W.$$

Note that the direct sum $W'$ of copies of $P_1$ is uniquely determined inside $W$, it is the sum of all images $P(T) \to W$. It follows that $W \to W/W' = \bigoplus_{i=1}^{n-t} P_0$ is a minimal left $P_0$-approximation of $W$, thus we have to form the following induced exact sequence

$$0 \longrightarrow W \longrightarrow C^t \xrightarrow{g} Y \longrightarrow 0$$

$$0 \longrightarrow W/W' \longrightarrow X \xrightarrow{f} Y \longrightarrow 0$$

and the lower sequence is just the required element inside $C[\to Y]$.

Remark: The bijection discussed here is nothing spectacular, it just corresponds to the Auslander-Reiten formula

$$D \text{Ext}^1(Y, \tau C) \leftrightarrow \text{Hom}(C, Y)$$

(since we have $\mathcal{P}(C, Y) = 0$, thus $\text{Hom}(C, Y) = \text{Hom}(C, Y)$); on the left, the linear forms on $\text{Ext}^1(Y, \tau C)$ are related to corresponding exact sequences $0 \to K \to X \to Y \to 0$ with $K \in \text{add } \tau C$, or better such a linear form corresponds bijectively to the right-equivalence class of the map $X \to Y$.

In which way is the Auslander bijection better than the Auslander-Reiten formula?

1) We do not only deal with the set $\text{Hom}(C, Y)$ but with all of $\text{Hom}(C, Y)$. To extend such a bijection to a larger setting should always be interesting. But also note that the set $\text{Hom}(C, Y)$ depends on the module category which we consider, not just on the modules themselves.
2) To specify more details: we get a bijection

\[ M \leftrightarrow S(\mathcal{P}(C,Y)) \quad \text{for some subset } M \subseteq C[\to Y] \]

We should determine this subset! Is this the set of right equivalence classes of epimorphisms?

2) The duality is replaced by a covariant bijection. This correspondence

\[ S(D\text{Ext}^1(Y,\tau C)) \]

\[ \downarrow \]

\[ M \]

\[ \cap \]

\[ C[\to Y] \]

should be described! XXXXXXXXXXXX
Lecture I (Shanghai)

0) References: Auslander, Auslander, ARS, Ringel, see also Krause

1.1. The object of investigation: \( \mathcal{M}(\to Y) \), the poset of right equivalence classes of right minimal maps to \( Y \).

To describe arbitrary maps ending in \( Y \). Define a preorder relation. Any equivalence class contains a right-minimal map. And right minimal maps are in the same equivalence class iff they are right equivalent (this means XXX). This defines \( \to Y \).

Gelfand: The main data are groups and posets (usually lattices).

1.2. Auslander’s theorems.

Let \( C,Y \) be objects. Define \( \eta_{\mathcal{C}Y} : [\to Y] \to S(\text{Hom}(C,Y)) \) (the set of \( \Gamma \)-submodules of \( \text{Hom}(C,Y) \), where \( \Gamma = \text{End}(C)^{\text{op}} \) by \( \eta_{\mathcal{C}Y}(f) = f \cdot \text{Hom}(C,X) \) for \( f : X \to Y \). (Note that \( f \cdot \text{Hom}(C,X) \) is a \( \Gamma \)-subspace.)

Def: Call \( f \) right-determined by \( C \) provided... Let \( \mathcal{M}_{\mathcal{C}}(\to Y) \) be XXXXX

“right-determined” is a technical notion, introduced to be able to specify \( \mathcal{M}_{\mathcal{C}}(\to Y) \). Often, we can characterize \( \mathcal{M}_{\mathcal{C}}(\to Y) \) much easier! For example, if \( C \) is a generator, then it turns out ... In particular, if \( C = \Lambda \), then ...

**Theorem 1.** Any \( f \) is right-determined by some \( C \), for example by \( \tau^{-1}\ker(f) \oplus P(\text{soc Cok}(f)) \).

**Theorem 2.** \( \eta_{\mathcal{C}Y}|_{\mathcal{M}_{\mathcal{C}}(\to Y)} \) is bijective, it is a poset isomorphism.

**Lemma.** Injectivity.

See Lemma below!

Some examples for \( S(\text{Hom}(C,Y)) \).

- If \( \text{End}(C) = k \), we deal with the projective space \( \mathbb{P} \text{Hom}(C,Y) \).
- If \( C = \Lambda \), we deal with the Grassmanian of all submodules of \( Y \). (Note a recent result of Reineke).

We will see: If \( C \) is a generator, then \( \mathcal{M}_{\mathcal{C}}(\to Y) \) is easy to describe: These are just the right-minimal maps to \( Y \) (not necessarily surjective) with kernel in \( \text{add} \tau C \).

**Comparison with the usual AR-theory.**

For \( C = Y \) indecomposable, and the subspace \( \text{rad}(Y,Y) \) of \( \text{Hom}(Y,Y) \) we get the right almost split map ending in \( Y \).

Here, we allow \( C \neq Y \), we allow that these modules are decomposable and we allow that we deal with an arbitrary submodule of \( \text{Hom}(C,Y) \).

**Auslander’s thinking.**
He was used to work with modules over commutative rings, this could explain the mistake in the determination paper.

His problem with non-commutative phenomena can be seen best looking at various questions raised in the Queen-Mary-Notes, where the $A_2$-quiver yields (counter-)examples.

He was not a geometer, otherwise he would have insisted more to look at $\mathcal{S} \text{Hom}(C, Y)$ as a variety. Note that he stressed the importance of Grassmanians in the Queen-Mary-Notes, but he did not continue in this direction.
Lecture 2. (Shanghai)
I. Any morphism \( \alpha \) is right determined by some explicit module \( C(\alpha) \).
II. This module \( C(\alpha) \) is the minimal right determiner of \( \alpha \).
III. Maps determined by a projective-free module.

I. The module \( C(\alpha) \) right determines \( \alpha \).

Recall: A set \( C \) of modules right determines \( \alpha: X \to Y \), provided any map \( \alpha': X' \to Y \) with \( \alpha' \text{Hom}(C, X') \subseteq \alpha \text{Hom}(C, X) \) for all \( C \in C \) factors through \( \alpha \) (instead of for all \( C \in C \), we also may ask for all \( C \in \text{add} C \), or for all indecomposable \( C \) in \( \text{add} C \)).

Note: \( \alpha' \text{Hom}(C_1, X') \subseteq \alpha \text{Hom}(C_1, X) \) and \( \alpha' \text{Hom}(C_2, X') \subseteq \alpha \text{Hom}(C_2, X) \) is equivalent to \( \alpha' \text{Hom}(C_1 \oplus C_2, X') \subseteq \alpha \text{Hom}(C_1 \oplus C_2, X) \), thus we always may assume that the modules in \( C \) are indecomposable. Or else, if \( C \) is finite, we may replace it by just a single module, namely the direct sum of the modules in \( C \). Also, we see that we may assume that all the modules in \( C \) are multiplicity-free.

New definition. Let \( C \) be an indecomposable module. One says that a map \( \eta: C \to Y \) almost factors through \( \alpha \) provided it does not factor through \( \alpha \), but the composition \( g\eta \) factors through \( \alpha \), where \( g \) is the right minimal almost split map ending in \( C \).

Construction of \( C(\alpha) \).

Let \( C(\alpha) \) be the set of all \( \tau L \) with \( L \) an indecomposable direct summand of the intrinsic kernel of \( \alpha \) and of all indecomposable projective modules \( P \) with a map \( P \to Y \) which almost factors through \( \alpha \). Note that this is a finite set.

Theorem 1. \( C(\alpha) \) right determines \( \alpha \).

Proof. We need the following Lemma.

Lemma 1. = Lemma 2 of the preprint.

XXXXX(3 induced sequences)XXXXXXX

Proof of Theorem 1. 

XXXXX(a large diagram)XXXXXXX

Lemma 2. Let \( P(S) \) be the projective cover of the simple module \( S \) and assume that \( P(S) \) almost factors through \( \alpha \). Then \( S \) is a submodule of \( \text{Cok}(\alpha) \).

Proof: Easy XXXXXXXX

Corollary of Theorem 1: The module \( \tau \text{Ker}(\alpha) \oplus P(\text{soc} \text{Cok}(\alpha)) \) right determines \( \alpha \).

Corollary of Theorem 2: The module \( \tau \text{Ker}(\alpha) \oplus \Lambda \) right determines \( \alpha \).

II. If \( C \) right determines \( \alpha \), then \( C(\alpha) \) belongs to \( \text{add} C \).
(a) If there is a map \( C \rightarrow Y \) which almost factors through \( \alpha \), and \( C \) right determines \( \alpha \), then \( C \) belongs to \( C \).

Proof: Assume \( C \) does not belong to \( C \).

See ARS Lemma 2.1

(b) Let \( L \) be an indecomposable direct summand of the intrinsic kernel of \( \alpha: X \rightarrow Y \). Then there is a map \( \text{Tr} \ D(L) \rightarrow Y \) which almost factors through \( \alpha \).

See Lemma 5 or the Bielefeld presentation.

Theorem. \( C(\alpha) \) is the minimal right determiner of \( \alpha \) (it right determines \( \alpha \) and is contained in \( \text{add} \ C \) for any class \( C \) which right determines \( \alpha \)). Also, \( C(\alpha) \) is the set of all indecomposables \( C \) with a map \( C \rightarrow Y \) which almost factors through \( \alpha \).

Proof: Theorem 1 asserts that \( C(\alpha) \) right determines \( \alpha \). Conversely, all modules in \( C(\alpha) \) almost factor through \( \alpha \), according to (b). If \( C \) is any class which right determines \( \alpha \), then according to (a), all the modules in \( C(\alpha) \) belong to \( C \).

According to (b) all the modules \( C \) in \( C(\alpha) \) have a map \( C \rightarrow Y \) which almost factors through \( \alpha \). Conversely, if \( U \rightarrow Y \) almost factors through \( \alpha \), then \( U \) is a direct summand of \( C(\alpha) \) by (a) and Theorem 1.

We see: The set using \( P(\text{Cok}(\alpha)) \) instead of \( P(\text{soc Cok}(\alpha)) \) may not right-determine \( \alpha \).

Example: Take \( \alpha \) with serial cokernel of length 2, with two different composition factors.

III. Maps determined by a module without an indecomposable projective direct summand,

Typical example. Let \( S \) be a simple module and \( \alpha: P_1 \rightarrow P_0 \) a minimal projective presentation of \( S \). Then: \( P(S) \) (or better 1: \( P(S) \rightarrow P_0 \)) almost factors through \( \alpha \) iff \( \alpha \) is injective iff the projective dimension of \( S \) is at most 1.

Proof: XXXXXXXX (direct proof)

Formula:

\[
C(\alpha) = \begin{cases} 
P(S) & \text{proj.dim.} \leq 1, \\
\tau^{-}\Omega_2(S) & \text{proj.dim.} \geq 2 
\end{cases}
\]

(and in the second case \( \Omega_2(S) \) is not injective, thus \( \tau^{-}\Omega_2(S) \) is not projective).

We see: If \( \Lambda \) is not hereditary, there are non-injective maps \( \alpha \) such that \( C(\alpha) \) has no indecomposable projective direct summand.

Proposition. Let \( \Lambda \) be hereditary and \( \alpha: X \rightarrow Y \) right minimal. If \( C(\alpha) \) has no projective direct summand, then \( \alpha \) is injective.
Proof: Let $S$ be a submodule of $\text{Cok}(\alpha)$ and lift the map $P(S) \to S$ to $Y$. Then we see that $P(S)$ almost factors through $\alpha$, thus $P(S)$ is a direct summand of $C(\alpha)$.

Now Preprint, section 5. In particular Theorem 3. We need
The small envelope $\overline{N}$
For any epimorphism $\epsilon: X \to N$, the submodule $I_\epsilon(N) \subseteq \overline{N}$.
(uniqely defined, see Lemma 9).
The summary.
Example 6

**IV. Maps determined by a generator $C$.**
Lecture III (Shanghai)

(1) Recall our outline:

(1) \[ \rightarrow Y = \bigcup_C C \rightarrow Y \]

(2) \[ C \rightarrow Y \leftrightarrow S \text{Hom}(C, Y) \]

Everyone admits that the concept of being determined is not very intuitive, however in the special case when \( C \) is a generator (and this is often the only important case, one knows: The maps in \( C \rightarrow Y \) can be described by the exact sequences

\[
0 \rightarrow K \rightarrow X \xrightarrow{\ell} Y' \rightarrow 0
\]

where \( K \) is in \( \tau C \) and \( m: Y' \rightarrow Y \) is an inclusion map (thus \( Y' \) is just a submodule of \( Y \)): here, the map in \( C \rightarrow Y \) in question is the composition \( me \).

This means: for \( C \) a generator, the set \( C \rightarrow Y \) can be visualized very well. Unfortunately, in general not, but the notion of determination just allows to have the bijection \( C \rightarrow Y \leftrightarrow S \text{Hom}(C, Y) \).

(2) The notion of ’determination’ is actually very helpful: A lot of proofs needed just work without working.

(3) The set \( \rightarrow Y \) is very large, in contrast to the subsets \( C \rightarrow Y \).

\( \rightarrow Y \) is neither artinian, nor noetherian: XXXXXXXXXXXX

(4) The only non-trivial assertion for showing (2) is that \( \eta_{CY} \) is surjective.

Proof: XXXXXXXXXXXXXXXXXXXX

The meaning of the bijection: Examples.

XXXXXXXXXXXXXXXXXX
Morphisms determined by objects: Geometry

Claus Michael Ringel

We deal with an artin algebra Λ, and the Λ-modules to be considered are always finitely generated left Λ modules. The following survey is based on investigations of Maurice Auslander; it addresses the question in which way the morphisms \( X \to Y \) with \( Y \) a fixed module can be classified. Obviously, it is justified to restrict to the right minimal morphisms, and to deal with the equivalence classes with respect to the following equivalence relation: Call \( \alpha : X \to Y \) and \( \alpha' : X' \to Y \) right equivalent provided there is an isomorphism \( \gamma : X \to X' \) with \( \alpha' = \alpha \gamma \).

Let us recall that a morphism \( \alpha : X \to Y \) is said to be right minimal provided \( \alpha \beta = \alpha \) for any \( \beta : X \to X \) implies that \( \beta \) is an automorphism. Also, if \( \alpha : X \to Y \) is arbitrary, then there is a direct decomposition \( X = X_0 \oplus X_1 \), such that \( \alpha |_{X_1} : X_1 \to Y \) is right minimal. If \( X = X'_0 \oplus X'_1 \) and \( \alpha |_{X'_0} = \alpha \) is the zero map, whereas \( \alpha |_{X'_1} \) is right minimal, then \( X_0 \) and \( X_1 \) are isomorphic modules and the maps \( \alpha |_{X_1} \) and \( \alpha |_{X'_1} \) are right equivalent.

In his famous Philadelphia notes, Auslander used the formulation that every morphism in \( \text{mod} \Lambda \) is right determined by a \( \Lambda \)-module in a precise sense, so that one obtains a convenient classification of all the (right minimal) morphisms. One even may strengthen the assertion by saying every morphism in \( \text{mod} \Lambda \) is right determined by the isomorphism class of a multiplicityfree \( \Lambda \)-module. Clearly, such a formulation is irritating, since the set of isomorphism classes of multiplicityfree modules may outnumber the set of right equivalence classes of right minimal morphisms by far: In case \( \Lambda \) is representation finite and uncountable, then there are only finitely many isomorphism classes of multiplicityfree modules, but nearly always many right equivalence classes of right minimal morphisms. So how should it be possible that finitely many modules \( C \) determine uncountably many morphisms \( \alpha \)? The solution is rather simple: It is not just the module \( C \) which is needed to recover \( \alpha : X \to Y \) but one actually needs a submodule of \( \text{Hom}(C,Y) \), with \( \text{Hom}(C,Y) \) being considered as an \( \text{End}(C)^\text{op} \)-module. In the setting where \( \Lambda \) is representation-finite and uncountable, one should be aware that usually the modules \( \text{Hom}(C,Y) \) will have uncountably many submodules, thus we are no longer in trouble.

So when \( C \) is called a determiner for \( \alpha : X \to Y \) one should keep in mind that \( C \) is only part of the data which are required to recover \( \alpha \); in addition to \( C \) one will need a submodule of \( \text{Hom}(C,Y) \).

The aim of the survey is to present and to discuss Auslander’s bijection between the submodules of \( \text{Hom}(C,Y) \) on the one hand, and a set \( C\rightarrow Y \) of right equivalence classes of some right minimal maps \( \alpha : X \to Y \) (those which are said to be right determined by \( C \)), and as we have mentioned already, any right minimal map \( \alpha : X \to Y \) belongs to some \( C\rightarrow Y \). We should stress that not only the set of submodules of a module is a lattice, but also \( C\rightarrow Y \) carries in a natural way a lattice structure, and \( \eta_{C\rightarrow Y} \) will be shown to be an isomorphism of lattices. Whereas the description of the set \( C\rightarrow Y \) looks slightly technical in general, this set is easy to describe in case \( C \) is a generator: in this case \( C\rightarrow Y \) is the set of right equivalence classes of maps \( X \to Y \) with kernel in \( \text{add} \tau C \).
This bijection is given as follows: for any map $\alpha : X \to Y$ and any module $C$, we may define

$$\eta_{CY}(f) = \{ f\phi \mid \phi : C \to X \}.$$ 

**Lemma.** If $X = X_0 \oplus X_1$ and $f(X_0) = 0$, then $\eta_{CY}(f) = \eta_{CY}(f|X_1)$.

In particular: If $f_1$ is a right minimal version of $f$, then $\eta_{CY}(f) = \eta_{CY}(f_1)$.

**Proof:** Let $X = X_0 \oplus X_1$ and write $f = [f_0 \ f_1]$ with $f_i : X_i \to Y$. Then, for $\phi_i : C \to X_i$, we have $[f_0 \ f_1] \begin{bmatrix} \phi_0 \\ \phi_1 \end{bmatrix} = f_1 \phi_1$.

Let us consider the map

$$\eta_{CY} : C[\to Y] \to S \text{Hom}(C, Y)$$

**Claim:** This is a bijection!

The bijection yields a bijection

$$C[\to Z]_{\text{epi}} \leftrightarrow S(\text{Hom}(C, Z)/\mathcal{P}(C, Z))$$

**Note:** the special case of the bijection dealing with $C = \Lambda$ is trivial

The dimension of $\text{Hom}(C, Y)$ is just the length of a maximal chain

$$X_0 \to X_1 \to \cdots \to X_n = Y$$

such that all the compositions $X_i \to \cdots \to X_n = Y$ are right minimal right $C$-determined.

There is a Jordan-Hoelder Theorem

If $C$ is a brick, then we deal just with a projective space. Also note that in case the maps in $C[\to Y]$ are all epimorphisms, we just deal with all the extensions of $Y$ by $\tau C$ from below. — But note the $\text{Ext}^2$-phenomenon: not all maps have to be epimorphisms. See the one-point extension of the Kronecker quiver using a regular module of length 2.
Typical examples for the bijection to be considered:
(a) $C = \Lambda$. Then $S\text{Hom}(C, Y)$ is just the Grassmanian of all submodules of $Y$.
(b) $C$ without indecomposable projective direct summands. Then $C[\to Y]$ consists of the epimorphisms with kernel in $\text{add } \tau C$.
(c) $C$ a generator. Then $C[\to Y]$ consists of all maps $X \to Y$ with kernel in $\text{add } \tau C$.

More about using generators.

Let $C$ be a generator and $Z' \subseteq Z$. Then

\[
\begin{align*}
C[\to Z] & \hookrightarrow S\text{Hom}(C, Z) \\
\uparrow & \\
C[\to Z'] & \hookrightarrow S\text{Hom}(C, Z')
\end{align*}
\]

where the vertical maps are inclusion maps (and on the right, $S\text{Hom}(C, Z')$ is a lattice ideal in $S\text{Hom}(C, Z)$).

In order to have inclusion maps, we need that $C$ is a generator: XXXXXXXXXXXXXXX

For $C$ a generator, the submodule 0 of $\text{Hom}(C, Z)$ just corresponds to the map $0 \to Z$.

On the other hand, if $P(\text{soc } Z)$ is not in $\text{add } C$, then (and only then) the submodule 0 of $\text{Hom}(C, Z)$ yields a non-trivial element in $[\to Z]$, namely an exact sequence $* \to * \to Z'$ with $Z'$ the largest submodule of $Z$ which belongs to the Serre subcategory generated by all simple modules $S$ with $P(S)$ not in $\text{add } C$. (and the sequence itself is a universal extension from below...)

Power posets.

Given a poset $H$, let $\mathcal{P}(H)$ be the set of antichains with the following ordering XXXXXXXXXX - we call it the power poset of $M$. Better: Let $\mathcal{P}(H)$ be the poset of cuts of $H$; a cut is by definition a pair $(I, J)$ where $I$ is an ideal and $J$ its complement, thus a coideal. The set of cuts carries a natural partial ordering. (We can describe both $I$ and $J$ is being generated (as ideal, or as coideal, respectively) by an antichain, in this way we get different orderings on the set of antichains — the orderings are the same, namely the ordering of the cuts, but the labeling of the cuts using antichains is completely different.)

In case $H$ is a poset with no comparable elements, thus just a set, then $\mathcal{P}(H)$ is the usual power set of $H$ with the usual partial ordering. We use the elements of $H$ as labels for
the cover pairs. Here is the chain with three elements, the poset of cuts, and the labelling using generators of the ideal, or generators of the coideal:

\[ \begin{array}{ccc}
\bullet c & \rightarrow & \bullet (c, \emptyset) \\
\bullet b & \rightarrow & \bullet (b, c) \\
\bullet a & \rightarrow & \bullet (a, b) \\
\end{array} \]

Note that the edges in the Hasse diagram of \( \mathcal{P}(M) \) are labelled by the elements of \( M \). Namely, they correspond just to the exchange of a single element: such an edges is between two cuts, say

\[ \begin{array}{c}
(I', J') \\
(I, J) \\
\end{array} \]

where \( I \subset I' \) and \( I' \) differs from \( I \) by a unique element, say \( x \). Then \( J' \subset J \) and \( x \) is the unique element in \( J \) which does not belong to \( J' \). Of course, then \( x \) is a maximal element of \( I' \) (thus belongs to the antichain which generates the ideal \( I' \)) and is a minimal element of \( J \) (thus belongs to the antichain which generates the coideal \( J \). We write in this case

\[ \begin{array}{c}
(I', J') \\
x \\
(I, J) \\
\end{array} \]

Let us return to our chain with 3 elements. Here are the labels of the edges:

\[ \begin{array}{ccc}
\bullet c & \rightarrow & \bullet (c, \emptyset) \\
\bullet b & \rightarrow & \bullet (b, c) \\
\bullet a & \rightarrow & \bullet (a, b) \\
\end{array} \]

Conversely, we may consider the set of irreducible elements in a poset.

Brainstorming: We need a word for this process (and the inverse process).
Powering
Magnification (and demagnification)
Amplification

Idealization (since antichains are just the generators of ideals, and also of coideal).

NOTE: Via this correspondence, we get natural orderings (but different ones) on the set of antichains!
The parametrized families may comprize modules with varying dimension vectors.

Of interest is the following: The families which we obtain via the Auslander bijection may comprize modules with varying dimension vectors. Example: Take the one-point extension of the Kronecker quiver using a regular module of length 2, say with vertices 1, 2, 3 with two vertices from 2 to 1 and one vertex from 3 to 2 (and one relation). Let \( R = R(\infty) = \text{rad } P(3) \).

\[
\begin{array}{ccc}
1 & \xleftarrow{\approx} & 2 \\
\end{array}
\]

Let \( K = S(1) \) and \( C = \tau^{-1}K = P_2 \) (the indecomposable module with dimension vector \((3, 2, 1)\)). Let \( Y = P(3)/S(1) \), thus its dimension vector is \((0, 1, 1)\). We have \( \dim \text{Hom}(C, Y) = 2 \). Since \( \text{End}(C) = k \), the submodule lattice \( S \text{Hom}(C, Y) \) is just the projective line. Under \( \eta_{SY} \) we obtain a \( \mathbb{P}_1 \)-family of right minimal maps ending in \( Y \), namely those with the following short exact sequences:

\[
\begin{align*}
K & \to P(3) \to Y \\
K & \to R(\lambda) \to S(2) \quad \text{for } \lambda \in k
\end{align*}
\]

Here is \( S \text{Hom}(C, Y) \)

\[\text{It should be noted that the short exact sequence}
\]
\[
K \to R(\infty) \to S(2)
\]

yields the map \( R(\infty) \to S(2) \subset Y \) (denote it by \( f \)) which is not \( C \)-determined: There is the following commutative diagram

\[
\begin{array}{ccc}
\text{rad } P(3) & \longrightarrow & P(3) \\
\downarrow & & \downarrow^p \\
R(\infty) & \longrightarrow & Y
\end{array}
\]

with \( p \) the canonical projection. We see that \( p \) almost factorizes through \( f \), thus the theory asserts that \( P(3) \) has to belong to any determiner of \( f \).

Thus, let us add \( P(3) \) to \( C \) and consider the Auslander bijection for \( C \oplus P(3) \) and \( Y \). We have \( \dim \text{Hom}(C \oplus P(3), Y) = 3 \). Note that the endomorphism ring of \( C \oplus P(3) \) is
hereditary of type $A_2$ and the submodule structure of $\text{Hom}(C \oplus P(3), Y)$ looks as follows:

For the proof, we only have to verify that a non-trivial map $C \to P(3)$ does not annihilate the module $\text{Hom}(C \oplus P(3), Y)$.

The corresponding diagram in $C\oplus P(3)[\to Y]$ looks as follows; here, we write its elements as short exact sequences ending in a submodule of $Y$:

$$
\begin{align*}
0 & \to Y \to Y \\
1 & \to P(3) \to Y \\
1 & \to R(\infty) \to 2 \\
1 & \to R(\lambda) \to 2 \\
1^2 & \to P_1 \to 2
\end{align*}
$$

We may label the lines of the Hasse diagram of $S\text{Hom}(C \oplus P(3))$ by the corresponding indecomposable direct summand, thus either by $C$ or by $P(3)$:

We should add that we obtain $S\text{Hom}(C, Y)$ from $S\text{Hom}(C \oplus P(3), Y)$ by deleting the shaded part:
The poset \([\rightarrow Y]\).

General convention: The elements of \([\rightarrow Y]\) will be written as short exact sequences
\[U \rightarrow V \rightarrow W,\]
where \(W\) is a submodule of \(Y\), so that \(V \rightarrow W \subseteq Y\) is the map in question.

**Lemma.** \([\rightarrow Y]\) is a modular lattice with zero \(0 \rightarrow 0 \rightarrow 0\) and one: \(0 \rightarrow Y \rightarrow Y\), the
intersection is given by the pullback, the union by XXXXXXX.

XXX

Remark: An important special case concerns the case of \(C\) being a generator. Starting
with a right-minimal morphism \(f: X \rightarrow Y\), with kernel \(K\), we may consider \(f\) as element
in \(C[\rightarrow Y]\) with \(C = \tau^{-1}K \oplus \Lambda\), thus, by the Auslander bijection, it belongs to a submodule
of \(\text{Hom}(C,Y)\). That means: for every right minimal morphism, we get a submodule of a
module as an invariant. For example, this submodule could be a “waist” - this explains
the interest of Auslander et. al. in waists.

Remark: The following feature seems to be of interest: We deal with a question
which concerns the category of maps with fixed target \(Y\), we want to know whether two
given objects are comparable. We find a test set of modules (the indecomposable direct
summands of \(T[\alpha]\)). For the testing procedure, they are just modules, but any such object
\(L\) comes equipped with a non-zero (thus right minimal) map \(L \rightarrow Y\). Isn’t this funny?

We deal with \([\rightarrow Y]\). In case \(Y\) is indecomposable injective, say the injective envelope
of \(S\), then \([\rightarrow Y]\) is just the hammock for \(S\).

Question. Since the hammocks are special cases of \(C[\rightarrow Z]\), namely take for \(C\) an
additive generator of the module category (it has to be representation-finite), and \(Z = I(S)\)
for some simple module \(S\). Are the hammock results special cases of more general results???

The poset \([\rightarrow Y]\) is the datum needed to describe the 1-point coextension using \(Y\).
Or, dually (as I am used to): if one constructs the 1-point extension using the module
\(M\), then one needs to know the category \(\text{Hom}(M \rightarrow)\). The easiest case is the one where
dim \(\text{Hom}(M,N) \leq 1\) for all indecomposable modules \(N\). Then \(\text{Hom}(M \rightarrow)\) is described
by a poset.

**After lecture I.**

We know: Theorem 2 asserts that any \(f\) is right determined by

\[
\tau^{-} \text{Ker}(f) \oplus \text{P(soc Cok}(f)
\]

17
One may ask when one of the summands is sufficient. Note that the first summand is a module without an indecomposable projective summand, the second is a projective module.

The separate cases.
(a) If $C$ is projective, then $M_C(\to Z)$ is a set of submodules of $Z$. WHICH ONES?
(b) If $C$ has no indec proj direct summand, then $M_C(\to Z)$ contains the (right equivalence classes of the) right minimal epimorphism ending in $Z$, but may be larger.

Reformulation:
(a) A map is a monomorphism iff it is right determined by a projective module.
(b) Any epimorphism is right determined by a module without indec proj direct summands, but the converse is not true.

Finiteness conditions.
Note that $M_A(\to Z)$ is the set of submodules of $Z$, thus a lattice of finite length and this means that it satisfies the ascending chain condition as well as the descending chain condition.
In general, $[\to Z]$ is not of finite length.

Failure of the descending chain condition. Example that $[\to Z]$ does not satisfy the descending chain condition. Take the Kronecker quiver with arrows from $b$ to $a$. Let $Z = I_0 = S(b)$ be the simple injective module. Let $R$ be indecomposable of length 2. Then there is a chain of epimorphisms (inside the preinjective component)

$$
\cdots \to I(2) \to I(1) \to I(0)
$$

such that all the kernels $I(n + 1) \to I(n)$ are equal to $R$.

Also, there is such a chain of epimorphisms (in the preinjective component) such that all the kernels are pairwise different and of length 2 (if the ground field is infinite).

In the first case, the kernels of the maps to $Z$ are all indecomposable (namely $R[n]$ for $n \in \mathbb{N}$), in the second, they are direct sums of pairwise non-isomorphic modules of length 2.

Failure of the ascending chain condition. This time take the one point extension of the Kronecker quiver by $R$ with extension vertex $\omega$. Let $M[n]$ be the amalgamation of $R[n]$ and $P(\omega)$ via $R$. Then there is the following chain of maps:

$$
M[1] \to M[2] \to M[3] \to \cdots \to S(\omega),
$$

here all the maps $M[n] \to S(\omega)$ are right minimal epimorphisms, and the kernel is $R[n]$. 
One of the interesting features of Auslander’s bijection is that for any \( C, Z \) the poset \( \mathcal{M}_C(\rightarrow Z) \) has finite length, since it is embedded into \( \mathcal{S}\Hom(C, Z) \).

But note: the embedding is the trivial part!

Here is the precise statement of the trivial part:

**Lemma.** Let \( C, Z \) be modules. Consider the map

\[
\eta_{CZ} : \mathcal{M}_C(\rightarrow Z) \to \mathcal{S}\Hom(C, Z).
\]

Then:

(a) \( \eta_{CZ} \) is injective.

(b) \( \eta_{CZ} \) respects the ordering.

(c) \( \eta_{CZ} \) reflects the ordering.

Proof: Let \( f : X \to Z \) and \( f' : X \to Z \) be given. (b) is trivial: Suppose that there is \( h : X \to X' \) with \( f = f'h \). Then

\[
f \cdot \Hom(C, X) = f'h \cdot \Hom(C, X) = f' \cdot (h \cdot \Hom(C, X)) \subseteq f' \Hom(C, X').
\]

For (c) we need that \( f \) is right-determined by \( C \). Namely, assume that \( f \cdot \Hom(C, X) \subseteq f'\Hom(C, X') \). Take any map \( \phi' : C \to X' \) Then \( f'\phi' \) belongs to \( f \cdot \Hom(C, X) \), thus to \( f'\Hom(C, X') \), thus there is \( \phi : C \to X \) with \( f'\phi = f\phi \). In the following picture, all the diagrams on the left can be completed, thus also the right diagram can be completed (since \( f \) is right determined by \( C \)):

\[
\begin{array}{ccc}
C & \xrightarrow{\phi'} & X \\
\downarrow{\phi} & & \downarrow{f} \\
X' & \xrightarrow{f'} & Y
\end{array} \quad \text{and} \quad \begin{array}{ccc}
X & \xrightarrow{f} & X' \\
\downarrow{f'} & & \downarrow{\phi'} \\
Y
\end{array}
\]

This shows that \( f' \leq f \).

The assertion (a) follows from the assertion (c): If \( f \cdot \Hom(C, X) = f'\Hom(C, X') \), then (c) asserts that \( f \leq f' \) and \( f' \leq f \), thus \( f, f' \) are right equivalent. This completes the proof.

**Properties of morphisms which are right determined by \( C \).**

(1) Let \( \alpha : X \to Y \) be right determined by \( C \). Then one recover the image of \( \alpha \) as the largest submodule \( Y' \) of \( Y \) (with inclusion map \( u : Y' \to Y \)) such that \( u\mathcal{P}(C, Y') \subseteq f\Hom(C, X) \).

Proof. Let \( Y' \) be the image of \( \alpha \) with inclusion map \( u \) and \( uu' = \alpha \) (with \( \alpha' \) surjective). First of all, we show that \( u\mathcal{P}(C, Y') \subseteq f\Hom(C, X) \). Let \( \phi' : C \to \Lambda \) and \( \phi'' : \Lambda \to Y' \) (the maps \( \phi''\phi' \) obtained in this way generate \( \mathcal{P}(C, Y') \)) additively). We want to show that \( u\phi''\phi \) factors through \( \alpha \). Since \( \alpha' : X \to Y' \) is surjective, there is \( \psi : \Lambda \to X \) such that \( \phi'' = \alpha'\psi \) (since \( \Lambda \) is projective). Thus \( u\phi''\phi = u\alpha'\psi\phi' = \alpha\psi\phi' \). Thus \( u\phi''\phi \) factors through \( \alpha \).
On the other hand, let \( u'' : Y'' \to Y \) be a submodule of \( Y \) such that \( u'' \mathcal{P}(C, Y'') \subseteq \text{f Hom}(C, X) \). Let \( p : P(Y'') \to Y'' \) be a projective cover. Consider the map \( \alpha' = u''p : P(Y'') \to Y \). It has the property that for all maps \( \phi : C \to P(Y'') \) the composition \( \alpha' \phi \) factors through \( \alpha \) (namely \( \alpha' \phi = u''p\phi \) belongs to \( u'' \mathcal{P}(C, Y'') \subseteq \text{f Hom}(C, X) \)). But \( \alpha \) is right determined by \( C \), thus we conclude that \( \alpha' \) factors through \( \alpha \), say \( \alpha' = \alpha \phi' \) for some \( \phi' : C \to P(Y'') \). Thus the image \( Y'' \) of \( \alpha' \) is contained in the image \( Y' \) of \( \alpha \). This is what we wanted to prove.

\[(1') \text{ Corollary.} \quad \text{If } \alpha \text{ is right determined by } C, \text{ then } \alpha \text{ is surjective if and only if } \mathcal{P}(C, Y) \subseteq \alpha \text{ Hom}(C, X).\]

\[(1'') \text{ Corollary.} \quad \text{If } \alpha \text{ is right determined by } C, \text{ and } \alpha \text{ is not surjective, then } \mathcal{P}(C, Y) \neq 0.\]

Here is a direct proof: Assume that \( \alpha \to X \to Y \) is not an epimorphism. A composition factor of the socle of the cokernel of \( \alpha \) yields a commutative diagram

\[
\begin{array}{ccc}
\text{rad } P(S) & \xrightarrow{u} & P(S) \\
\eta' \downarrow & & \downarrow \eta \\
\alpha(X) & \xrightarrow{\alpha_2} & Y
\end{array}
\]

such that \( P(S) \to Y \) does not factor through \( \alpha \). If we could lift \( \eta' \) to \( X \), the module \( P(S) \) would almost factor through \( \alpha \), but this is not possible. Thus \( \alpha_2 \eta' \) does not factor through \( \alpha \) and therefore there is a map \( \phi : C \to \text{rad } P(S) \) such that \( \alpha_2 \eta' \phi \) does not factor through \( \alpha \). In particular, this is a non-zero map. Since \( \alpha_2 \eta' \phi = \eta u \phi \), this is a non-zero map in \( \mathcal{P}(C, Y) \).

In general we have: All maps in \( C[\to Y] \) are epimorphisms if and only of \( \mathcal{P}(C, Y) = 0 \).
Examples.

**Example.** Take the Kronecker quiver, let $C$ be indecomposable preprojective $P_2$ (we denote by $P_n$ the indecomposable preprojective module with dim vector $(n + 1, n)$). Let $Y$ be the indecomposable injective module of length 3. Note that $\tau C = S = P_0$ is simple projective.

$$\begin{array}{cccc}
\mathcal{S} \text{Hom}(C,Y) & \text{add } \tau C & C[\to Y] & SY \\
\text{Hom}(C,Y) & 0 & \longrightarrow & Y & \longrightarrow & Y \\
\bigcup & \uparrow & \uparrow & \uparrow & \uparrow \\
2\text{-dim subspaces} & S & \longrightarrow & M & \longrightarrow & Y \\
\bigcup & \uparrow & \uparrow & \uparrow & \uparrow \\
1\text{-dim subspaces} & S^2 & \longrightarrow & P_2 & \longrightarrow & Y \\
\bigcup & \uparrow & \uparrow & \uparrow & \uparrow \\
0 & S^3 & \longrightarrow & P_1^2 & \longrightarrow & Y
\end{array}$$

Here the modules $M$ are regular of length 4, without multiplicities, thus of regular rank 1 or 2.

Note that here $\text{End}(P_2) = k$, thus $\mathcal{S} \text{Hom}(P_2, Y)$ is just the projective space $\mathbb{P} \text{Hom}(P_2, Y)$. 

21
Examples: The case $Y = S$ simple.

**Lemma.** If $P(S)|C$ then $0 \to 0 \to 0$ is in $C[\to S]$ and If $P(S)$ is not a direct summand of $C$, then $0 \to 0 \to 0$ is not in $C[\to S]$, thus all the maps which are right determined by $C$ are epimorphisms (and $\mathcal{P}(C, S) = 0$).

Proof: Assume that $0 \to 0 \to 0$ is in $C[\to S]$, thus $0 \to S$ is right $C$-determined. But the canonical projection $P(S) \to S$ almost factors through $0 \to S$, thus $P(S)$ belongs to the minimal determiner, thus $P(S)|C$. (Second proof: If $0 \to 0 \to 0$ is in $C[\to S]$, then this is a map in $C[\to S]$ which is not an epimorphism, therefore $\mathcal{P}(C, S) \neq 0$. Take a non-zero map $\phi: C \to S$ which factors through a projective module $P$, then $\phi$ factors through the canonical projection $\pi: P(S) \to S$, say $\phi = \pi \alpha$ with $\alpha: C \to P(S)$. But since $\pi \alpha \neq 0$, we see that $\alpha$ does not map into the radical of $P(S)$, thus $\alpha$ is surjective and therefore a split epimorphism. This shows that $P(S)|C$.)

On the other hand, assume that $P(S)|C$. Then clearly $0 \to S$ is right determined by $C$.

Since the map $0 \to S$ is of no interest, we will assume that $P(S)$ is not a direct summand of $C$, when dealing with $C[\to S]$. Also the other indecomposable projective modules $P$ do not play a role as summand of $C$ (since Hom($P, S) = 0$ for these $P$), thus we can assume that $C$ is projective-free.

**Special case of the Auslander bijection.** Let $C$ be a projective-free module and $S$ simple. Then $C[\to S]$ consists of all right minimal epimorphisms with kernel in add $\tau C$ and $C[\to S]$ is indexed by $S$ Hom($C, S$).

In particular: Let $C$ be the direct sum of all indecomposable non-projective modules, then $C[\to S]$ is the set of all right minimal maps to $S$, and this set is indexed by $S$ Hom($C, S$).

Now take for $C$ the direct sum of all indecomposable non-projective modules such that $S$ is not a composition factor. (For example, if we deal with a one-point extension of $\Lambda'$ with extension vertex $\omega$ and $S = S(\omega)$, then $C$ will be the class of non-zero modules of the form $\tau^{-1}M$, where $M$ is an indecomposable $\Lambda'$-module.)

In this case, $C[\to S]$ is the set of right minimal maps onto $S$ such that the kernel has no composition factor $S$.

**HOW DOES THIS RELATE TO THE THEORY OF ONE-POINT EXTENSIONS**
(dealing with posets and vectorspace categories...)?

**Example.** Consider the path algebra of the $A_3$-quiver $Q'$ with one source and two sinks, and $\Lambda$ the one-point extension of $\Lambda'$ using the indecomposable projective module of length 3, thus $\Lambda$ is the path algebra of the quiver $Q$ with one source and two sinks.

Let $Y = S(\omega)$ the simple $\Lambda$-module corresponding to the extension vertex, let $K$ be the direct sum of all indecomposable $\Lambda'$-modules $K_i$ with Ext$^1(S, K_i) \neq 0$, thus $K$ looks
as follows (it is a subcategory of mod $\Lambda'$):

Let $C = \tau^{-1}_\Lambda K$, thus we get a corresponding category in mod $\Lambda$:

Here is the poset $C[\rightarrow Y]$ (drawn from left to right):

To be precise: The elements of the poset are the paths ending in $Y$, or better the (right equivalence classes of the) compositions of the corresponding can.

Note that the new poset $C[\rightarrow Y]$ is the poset of antichains of the poset $H$ on the left:

$H$

$C[\rightarrow Y]$
**Example.** Let \( \Lambda \) be a uniserial ring of length 6, the indecomposable module of length \( n \) will be denoted just be \( n \). Let \( C = Y = 4 \).

\[
\begin{array}{c|ccc}
S \text{ Hom}(C, Y) & \text{add } \tau C & C[\to Y] & SY \\
\hline
\text{Hom}(C, C) & 0 & \rightarrow & 4 & \rightarrow & 4 \\
\cup & & & \uparrow [e m] & & \uparrow \|
\text{rad}(C, C) & 4 & \rightarrow & 5 \oplus 3 & \rightarrow & 4 & \text{AR-sequence}
\cup & & & \uparrow [e m] & & \uparrow \|
\mathcal{P}(C, C) = \text{rad}(C, C)^2 & 4 & \rightarrow & 6 \oplus 2 & \rightarrow & 4 & \text{max extensions}
\cup & & & \uparrow [e m] & & \uparrow \|
\text{rad}(C, C)^3 & 4 & \rightarrow & 6 \oplus 1 & \rightarrow & 3 \\
\cup & & & \uparrow [1] & & \uparrow \\
0 & 4 & \rightarrow & m & \rightarrow & e & 2
\end{array}
\]

here, we have denoted by \( m \) the canonical inclusion maps, by \( e \) the canonical projections and \( \phi \) is a radical generator for a uniserial ring (the endomorphism ring of some \( n \)).

**Example 2.** Let \( \Lambda \) be uniserial of length 4. Let \( C = 1 \oplus 2 \) and \( Y = 3 \). Let \( \Gamma = \text{End}(C)^{op} \), this is the Nakayama algebra with Kupish series 2, 3. The \( \Gamma \)-module \( \text{Hom}(C, Y) \) is of length 3, it is indecomposable projective module of length 3.

\[
\begin{array}{c|ccc}
S \text{ Hom}(C, Y) & \text{add } \tau C & C[\to Y] & SY \\
\hline
\text{Hom}(C, Y) & 0 & \rightarrow & 3 & \rightarrow & 3 \\
\cup & & & \uparrow [e m] & & \uparrow \|
* & 2 & \rightarrow & 4 \oplus 1 & \rightarrow & 3 \\
\cup & & & \uparrow [1] & & \uparrow \\
\mathcal{P}(C, Y) = * & 1 & \rightarrow & m & \rightarrow & e & 3 \\
\cup & & & \uparrow \phi & & \uparrow m \\
0 & 2 & \rightarrow & m & \rightarrow & e & 2
\end{array}
\]
Note the role of $\mathcal{P}(C, Y)$. If we work over a uniserial ring of length at least 5, so that $\mathcal{P}(C, Y) = 0$, then the situation is as follows:

| $S\text{Hom}(C, Y)$ | $\text{add } \tau C$ | $C[\to Y]$ | $SY$ |
|---------------------|-----------------------|-----------|
| $\text{Hom}(C, Y)$ | 0 $\longrightarrow$ 3 | $\overline{\equiv}$ $\equiv$ 3 |
| $U$                 | $\uparrow$ $[m]_{-e}$ | $\uparrow[e_m]$ | $\equiv$ |
| $*$                 | 2 $\longrightarrow$ 4 $\oplus$ 1 | $\uparrow[e_m]$ | $\equiv$ |
| $U$                 | $\uparrow$ $[1]$ | $\uparrow[e_m]$ | $\equiv$ |
| $*$                 | 1 $\longrightarrow$ 4 | $\uparrow[e_m]$ | $\equiv$ |
| $U$                 | $\uparrow$ $[m]$ | $\uparrow[e_m]$ | $\equiv$ |
| $\mathcal{P}(C, Y) = 0$ | 2 $\longrightarrow$ 5 $\oplus$ 1 | $\uparrow[e_m]$ | $\equiv$ |

On the other hand, if we consider the factor ring $\Lambda/I$ of length 3, so that $C = 1 \oplus 2$ and $Y = 3$ are still $\Lambda/I$-modules, then

<table>
<thead>
<tr>
<th>$S\text{Hom}(C, Y)$</th>
<th>$\text{add } \tau_{\Lambda/I} C$</th>
<th>$\mathcal{M}_{\Lambda/I}(C, Y)$</th>
<th>$SY$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{P}_{\Lambda/I}(C, Y) = \text{Hom}(C, Y)$</td>
<td>0 $\longrightarrow$ 3</td>
<td>$\overline{\equiv}$ $\equiv$ 3</td>
<td></td>
</tr>
<tr>
<td>$U$</td>
<td>$\uparrow$ $[m]_{-e}$</td>
<td>$\uparrow[\phi m]$</td>
<td>$\uparrow m$</td>
</tr>
<tr>
<td>$*$</td>
<td>2 $\longrightarrow$ 3 $\oplus$ 1</td>
<td>$\uparrow[e_m]$</td>
<td>$\equiv$</td>
</tr>
<tr>
<td>$U$</td>
<td>$\uparrow$ $[1]$</td>
<td>$\uparrow[e_m]$</td>
<td>$\equiv$</td>
</tr>
<tr>
<td>$*$</td>
<td>1 $\longrightarrow$ 3</td>
<td>$\uparrow[e_m]$</td>
<td>$\equiv$</td>
</tr>
<tr>
<td>$U$</td>
<td>$\uparrow\phi$</td>
<td>$\uparrow m$</td>
<td>$\equiv$</td>
</tr>
<tr>
<td>0</td>
<td>2 $\longrightarrow$ 3</td>
<td>$\uparrow[e]$</td>
<td>$\equiv$</td>
</tr>
</tbody>
</table>

**An easier example:** Take for $\Lambda$ the path algebra of the quiver $z \leftarrow a \leftarrow b$, let $C = a$ and $Z$ indecomposable with support $a, b$. Finally, let $I = \langle z \rangle$. (We are interested in $\Lambda/I$ and we have added the vertex $z$ so that the simple module $a$ is no longer projective.)
For \( \Lambda \), we deal with

\[
\begin{array}{cccc}
\mathcal{S} \text{Hom}(C, Z) & \text{add } \tau C & C \rightarrow Z & \mathcal{S}Z \\
\text{Hom}(C, Z) & 0 & \begin{array}{cc}
\{& b \\
\{& a \\
\{& b \\
\end{array} & \begin{array}{cc}
\{& a \\
\}
\end{array} \\
\cup & \begin{array}{c}
\uparrow \\
\uparrow \\
\uparrow
\end{array} & \begin{array}{c}
\uparrow \\
\uparrow \\
\uparrow
\end{array} & \\
0 & z & \begin{array}{cc}
\{& a \\
\{& z \\
\end{array} & \begin{array}{cc}
\{& b \\
\{& a \\
\end{array}
\end{array}
\]

whereas for \( \Lambda / I \), we have

\[
\begin{array}{cccc}
\mathcal{S} \text{Hom}(C, Z) & \text{add } \tau_{\Lambda / I} C & \mathcal{M}_{\Lambda / I}(C, Z) & \mathcal{S}Z \\
\text{Hom}(C, Z) & 0 & \begin{array}{cc}
\{& b \\
\{& a \\
\{& b \\
\end{array} & \begin{array}{cc}
\{& a \\
\}
\end{array} \\
\cup & \begin{array}{c}
\uparrow \\
\uparrow \\
\uparrow
\end{array} & \begin{array}{c}
\uparrow \\
\uparrow \\
\uparrow
\end{array} & \\
0 & 0 & \begin{array}{cc}
\{& 0 \\
\{& 0 \\
\end{array} & \begin{array}{cc}
\{& 0 \\
\}
\end{array}
\end{array}
\]

An example with \( Y \) decomposable. Let \( \Lambda \) be uniserial of length at least 5, let \( C = 2 \) and \( Z = 3 \oplus 1 \). Thus \( \text{Hom}(C, Z) \) is of length 3, namely the \( \Gamma \)-module \( 2 \oplus 1 \). Note that the submodule lattice of such a \( \Gamma \)-module is

\[
\begin{array}{c}
\circ \\
\circ \\
\circ \\
\bullet \\
\bullet \\
\circ \\
\circ \\
\circ \\
\bullet \\
\bullet \\
\circ \\
\circ \\
\circ
\end{array}
\]
Here is the correspondence

\[
\begin{array}{cccc}
S \text{Hom}(C, Z) & \text{add } \tau C & C[\to Z] & SZ \\
\end{array}
\]

\[
\begin{array}{cccc}
\text{Hom}(C, Z) & 0 & 3 \oplus 1 & 3 \oplus 1 \\
\circ & 2 & 3 \oplus 3 & 3 \oplus 1 \\
\bullet & & 4 \oplus 1 \oplus 1 & 3 \oplus 1 \\
\bullet & 2 \oplus 2 & 4 \oplus 1 \oplus 3 & 3 \oplus 1 \\
\circ & & 5 \oplus 1 & 3 \oplus 1 \\
0 & 2 \oplus 2 & 5 \oplus 3 & 3 \oplus 1 \\
\end{array}
\]

There are two possible steps corresponding to two kinds of composition factors of \( \text{Hom}(C, Y) \). Either we deal with exact sequences with the same right term \( Y' \) (a submodule of \( Y \), then we deal with an induced sequence, induced by some map in \( \text{add } \tau C \). Or else we work with a step involving a proper inclusion of submodules of \( Y \).

**Change of rings.**

Consider modules \( C, Z \) with \( \mathcal{P}(C, Z) = 0 \), thus \( C[\to Z] \) consists of epimorphisms. Let \( I \) be an ideal of \( \Lambda \) which annihilates both \( C \) and \( Z \). Thus, \( C, Z \) both are \( \Lambda/I \)-modules and we have

\[
\text{Hom}_{\Lambda/I}(C, Z) = \text{Hom}_{\Lambda}(C, Z).
\]

We want to see how to obtain \( \mathcal{M}_{\Lambda/I}(C, Z) \) from \( \mathcal{M}_{\Lambda}(C, Z) \).

There SHOULD BE a commutative diagram with exact rows, as follows:

\[
\begin{array}{cccc}
in \text{mod } \Lambda & 0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
in \text{mod } \Lambda/I & 0 & \longrightarrow & {}_I X & \longrightarrow & {}_I Y & \longrightarrow & Z' & \longrightarrow & 0 \\
\end{array}
\]

with all vertical maps being inclusion maps. Here, \( {}_I Y \) COULD BE (a direct summand of) the maximal \( \Lambda/I \) submodule of \( Y \).

Of special interest should be the lowest sequence, it is given by the zero submodule of \( \text{Hom}(C, Y) \), thus is uniquely determined by \( C \) and \( Y \). To be precise:
Observation. Given modules $C$ and $Y$, there is a unique submodule $Y'$ which is minimal with respect of having an exact sequence $0 \to K \to X \to Y' \to 0$ such that $X \to Y' \to Y$ is right determined by $C$. This submodule is the largest submodule $Y'$ of $Y$ such that $P(C, Y') = 0$.

Note: Assume that $P(C, Y_1) = 0$ and $P(C, Y_2) = 0$. Let $\psi: C \to Y_1 + Y_2$ belong to $\mathcal{P}(C, Y_1 + Y_2)$, thus there is a projective module $P$ and maps $\psi': C \to P$, $\psi'': P \to Y_1 + Y_2$ with $\psi = \psi''\psi'$. Now lift $\psi''$ to a map $\beta: P \to Y_1 + Y_2$ (thus $\psi''$ is the composition of $\beta$ and the projection map $Y_1 + Y_2 \to Y_1$ and $\psi$ is the composition of $\beta\psi'$ and the projection map $Y_1 + Y_2 \to Y_1 + Y_2$. But the two components of $\beta\psi'$ are maps in $\mathcal{P}(C, Y_i)$, thus zero. Therefore $\psi = 0$.

We are interested in $\mathcal{S}Hom(C, Y)$, these are the $\Gamma$-submodules of $Hom(C, Y)$, where $\Gamma = End(C)^{op}$. Auslander asserts that this set is in bijection to the set of right minimal morphisms ending in $Y$ which are right determined by $C$, thus these are certain maps with kernel in $\text{add} \tau C$ and image a submodule of $Y$. (But note that any right minimal map ending in $Y$ occurs in this way for some module $C$; here is a QUESTION: Given a map ending in $Y$. Can we know the possible $C$?)

(0) There is a largest submodule (namely $Hom(C, Y)$), it corresponds to the identity map of $Y$. And here is a smallest submodule, the zero submodule, it corresponds to the following map: Take for $Y'$ the maximal submodule of $Y$ such that $P(C, Y') = 0$, and form the universal extension of $Y'$ from below by $\tau C$.

The set of submodules is a lattice of finite height (= the length of the module $Hom(C, Y)$ as a $\Gamma$-module). WHAT DOES THIS MEAN?
INTERSECTION?
SUM?
WHAT IS THE MEANING OF THE COMPOSITION LENGTH AND THE COMPOSITION FACTORS OF $Hom(C, Y)$?

Next: Modules may be indecomposable, semisimple, uniserial, and so on. What does this mean for the morphisms corresponding to the submodules of $Hom(C, Y)$?

Next: Submodules of $Hom(C, Y)$ are used as indices. What is the meaning of a corresponding factor module?

(1) It is sufficient to consider these sets for multiplicity-free modules $C$.

(2) For $C = \Lambda$, this set is just the usual Grassmanian of all submodules of $Y$. WHAT IF WE TAKE A PROJECTIVE MODULE $C = P$?

(3) For $C$ projective-free and $\Lambda$ hereditary, we only deal with the possible extensions of $Y$ from below using modules in $\tau C$.

Remarks Here we deal with a geometrical interpretation of Auslander’s bijection. Why did he himself not provide it? He was never a geometer (and also not combinatorially minded). Auslander’s work is the basis for both a combinatorial as well as a geometrical study of module categories, but this was not of interest for him.
The book ARS ends with the bijection theorem, but it does not make proper use of it. This really is the final piece of Auslander’s consideration!

A submodule lattice is always of interest. But ARS does not even mention that we deal with lattices! Theorem XI.3.9 (c) asserts that there is a poset isomorphism between the submodule lattice of \( \text{Hom}(C,Y) \) and the poset of right-minimal, right \( C \)-determined morphisms ending in \( Y \).

**The Auslander-Reiten formula** \( \text{Hom}(C,Y) \simeq D \text{Ext}^1(Y, \tau C) \)

The Auslander bijection provides a bijection between the submodules of \( \text{Hom}(C,Y) \) which contain \( \mathcal{P}(C,Y) \) with the right equivalence classes of surjective maps \( X \to Y \) with kernel in \( \text{add}(\tau C) \). But the submodules of \( \text{Hom}(C,Y) \) which contain \( \mathcal{P}(C,Y) \) correspond bijectively to the submodules of \( \text{Hom}(C,Y) \). Thus we only should note that there is a bijection between the surjective maps \( X \to Y \) with kernel in \( \text{add}(\tau C) \) and the elements of \( D \text{Ext}^1(Y, \tau C) \).

The interest in the Auslander bijection comes from the fact that the connection given by the Auslander-Reiten formula is extended to deal with \( \text{Hom}(C,Y) \) itself, not just the factor space \( \text{Hom}(C,Y) \).

**The usual AR-picture** concerns indecomposable modules, and almost split sequences, thus indecomposable modules and irreducible maps.

In the language of the Auslander bijection, we only deal with \( C = Y \) indecomposable and only with the submodule \( \text{rad}(C,C) \subset \text{Hom}(C,C) \), whereas we should not restrict to indecomposable modules, not to the condition \( C = Y \) and to all submodules of \( \text{Hom}(C,Y) \), not just the radical subspace.

Concerning the classical AR-theory, there is an essential difference whether \( C \) is projective or not. If \( C \) is projective, we obtain an inclusion map, if \( C \) is not projective, then an extension. — This feature dominates the whole Auslander bijection: the extreme cases are: \( C \) is projective, then we consider submodules (and we consider arbitrarily ones, not just the radicals of the indecomposable projective modules. If \( C \) has no indecomposable projective direct summand, then we deal with extensions ending in \( Y \); and we deal with all possible extension of \( Y \) from below using modules in \( \text{add} \tau C \).

QUESTION: What is the relationship between the elements of \( \text{Ext}^1(Y, \tau C) \) and the extension of \( Y \) from below using modules in \( \text{add} \tau C \). TO BE DONE.
**Proof of the Auslander bijection.**

The claim is: Given any $\text{End}(C)^{\text{op}}$-submodule of $\text{Hom}(C, Y)$, there is a map $\alpha: X \to Y$ which is right determined by $C$ and such that $H = \text{f Hom}(C, X)$.

First, one consider the case that $\mathcal{P}(C, Y) \subseteq H$. In this case one wants to obtain such an $\alpha$ which is in addition an epimorphism.

There is the following construction: Take a generating set $g_1, \ldots, g_n$ of $H$ and an epimorphism $g_0: \Lambda^m \to Y$, thus we get an exact sequence

$$0 \to K \to \Lambda^m \oplus C^n \xrightarrow{g} Y \to 0$$

with $g = [g_0, g_1, \ldots, g_n]$ and now form the induced exact sequence with respect to a left $\tau C$-approximation $K \to K'$, thus we get

$$0 \to K' \to \Lambda^m \oplus C^n \xrightarrow{\alpha} Y \to 0$$

and we need to show: For any map $\phi: C \to X$ there is a map $\phi': C \to \Lambda^m \oplus C^n$ such that $\alpha \phi = g \phi'$.

As ARS show, one can use Auslander’s defect formula in order to obtain a proof.

One deals with the following setting:

There is given an exact sequence

$$\delta: 0 \to U \to V \to W \to 0$$

and one looks at $\delta^*(V)$ and $\delta_*(\tau V)$. They are dual to each other (this is the defect formula, and the $\tau V$-approximation of $A$ should be related to the endomorphisms of $V$).

**Krause’s proof of the Defect Theorem.**

Genaugenommen wird das Bild des verbindenden Homomorphismus mit dem Kokern der Kernabbildungen $\text{Coker Hom}(X, \psi)$ bzw. dem Kern der Kokernabbildungen $\text{Ker}(\phi \otimes \text{Tr} X)$ identifiziert.

The usual Snake Lemma (as in your Wikipedia reference) deals with:

```
\begin{array}{c}
\text{Ker} \\
A \longrightarrow B \longrightarrow C \longrightarrow 0 \\
\downarrow \quad \downarrow \quad \downarrow \\
0 \longrightarrow A' \longrightarrow B' \longrightarrow C' \longrightarrow \text{Cok}
\end{array}
```
whereas your diagram is of the form

\[
\begin{array}{ccccccc}
0 & \rightarrow & (X, A) & \rightarrow & (P_0, A) & \rightarrow & (P_1, A) & \rightarrow & A \otimes \text{Tr} X & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \phi \otimes \text{Tr} X \\
0 & \rightarrow & (X, B) & \rightarrow & (P_0, B) & \rightarrow & (P_1, B) & \rightarrow & B \otimes \text{Tr} X & \rightarrow & 0 \\
\downarrow^{(X, \psi)} & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \rightarrow & (X, C) & \rightarrow & (P_0, C) & \rightarrow & (P_1, C) & \rightarrow & C \otimes \text{Tr} X & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\text{Cok}(X, \psi) & 0 & & & & & & & & \\
\end{array}
\]
An old example.

We mark $X$ by a bullet, $Y$ by a black square, and the direct summands of $C$ by stars. The hammock ending in $Y$ is shaded.

First, let us take a Brenner hammock $(P(x) \to I(x))$, for a hereditary algebra:

In this case, we just take all the direct successors of $P(x)$ (here, $x = 3$) namely $\tau^- P(w)$ for the arrows $w \leftarrow x$, and $P(z)$ for the arrows $x \leftarrow z$. (Note that this corresponds to Henning’s rule $S^{-1}(\text{cone}(\alpha))$.)

Next, the example in the last section of the paper: