Representations of quivers over the algebra of dual numbers.

Dedicated to the memory of J. A. Green

Claus Michael Ringel and Pu Zhang

Abstract: The representations of a quiver Q over a field k (the kQ-modules, where kQ is the path algebra of Q over k) have been studied for a long time, and one knows quite well the structure of the module category $\operatorname{mod} kQ$. It seems to be worthwhile to consider also representations of Q over arbitrary finite-dimensional k-algebras A. Here we draw the attention to the case when $A = k[\epsilon]$ is the algebra of dual numbers (the factor algebra of the polynomial ring $k[T]$ in one variable T modulo the ideal generated by T^2), thus to the Λ-modules, where $\Lambda = kQ[\epsilon] = kQ[T]/\langle T^2\rangle$. The algebra Λ is a 1-Gorenstein algebra, thus the torsionless Λ-modules are known to be of special interest (as the Gorenstein-projective or maximal Cohen-Macaulay modules). They form a Frobenius category $\mathcal{L}$, thus the corresponding stable category $\underline{\mathcal{L}}$ is a triangulated category. As we will see, the category $\mathcal{L}$ is the category of perfect differential kQ-modules and $\underline{\mathcal{L}}$ is the corresponding homotopy category. The category $\underline{\mathcal{L}}$ is triangle equivalent to the orbit category of the derived category $D^b(\operatorname{mod} kQ)$ modulo the shift and the homology functor $H\colon \operatorname{mod}\Lambda \to \operatorname{mod} kQ$ yields a bijection between the indecomposables in $\underline{\mathcal{L}}$ and those in $\operatorname{mod} kQ$. Our main interest lies in the inverse, it is given by the minimal $\mathcal{L}$-approximation. Also, we will determine the kernel of the restriction of the functor H to $\mathcal{L}$ and describe the Auslander-Reiten quivers of $\mathcal{L}$ and $\underline{\mathcal{L}}$.

Throughout the paper, k will be a field and Q will be a finite connected acyclic quiver. The starting point for the considerations of this paper is the following result which concerns the structure of the homotopy category of perfect differential kQ-modules. This assertion should be well-known, but we could not find a reference.

Let us recall that given a ring R, a differential R-module is by definition a pair (N, ϵ) where N is an R-module and ϵ an endomorphism of N such that $\epsilon^2 = 0$. If (N, ϵ) and (N', ϵ') are differential R-modules, a morphism $f\colon (N, \epsilon) \to (N', \epsilon')$ is given by an R-linear map $f\colon N \to N'$ such that $\epsilon' f = f\epsilon$. The morphism $f\colon (N, \epsilon) \to (N', \epsilon')$ is said to be *homotopic to zero* provided there exists an R-linear map $h\colon N \to N'$ such that $f = h\epsilon + \epsilon' h$. A differential R-module (N, ϵ) is said to be *perfect* provided N is a finitely generated projective R-module. We denote by $\operatorname{diff}_{\mathrm{perf}}(R)$ the category of perfect differential R-modules, and by $\underline{\operatorname{diff}}_{\mathrm{perf}}(R)$ the corresponding homotopy category. Let us denote by H the homology functor: it attaches to a differential R-module (N, ϵ) the R-module $H(N, \epsilon) = \operatorname{Ker}\epsilon/\operatorname{Im}\epsilon$. It is well-known that H vanishes on the maps which are homotopic to zero.

2010 *Mathematics Subject Classification.* Primary 16G10, 16G50, 16E65. Secondary 16G70, 18G25.

If R is noetherian, let us denote by $D^b(\operatorname{mod} R)$ the bounded derived category of finitely generated R-modules. This is a triangulated category and its shift functor will be denoted by [1].

Theorem 1. (a) *The category* $\operatorname{diff}_{\mathrm{perf}}(kQ)$ *of perfect differential* kQ*-module is a Frobenius category. The corresponding stable category* $\underline{\operatorname{diff}}_{\mathrm{perf}}(R)$ *is the homotopy category of perfect differential* kQ*-modules. This category* $\underline{\operatorname{diff}}_{\mathrm{perf}}(R)$ *is the orbit category* $D^b(\operatorname{mod} kQ)/[1]$.

(b) *The homology functor* $H\colon \operatorname{diff}_{\mathrm{perf}}(kQ) \to \operatorname{mod} kQ$ *is a full and dense functor which furnishes a bijection between the indecomposables in the homotopy category* $\underline{\operatorname{diff}}_{\mathrm{perf}}(kQ)$ *and those in* $\operatorname{mod} kQ$. *It yields a quiver embedding* ι *of the Auslander-Reiten quiver of* $\operatorname{mod} kQ$ *into the Auslander-Reiten quiver of the homotopy category* $\underline{\operatorname{diff}}_{\mathrm{perf}}(kQ)$.

We should remark that the study of differential modules themselves may have been neglected by the algebraists, however it is clear that the graded version, namely complexes, play an important role in many parts of mathematics. Theorem 1 is an immediate consequence of well-known results concerning perfect complexes over kQ: the category of perfect complexes is a Frobenius category, thus the corresponding stable category is $D^b(\operatorname{mod} kQ)$; the homology functor H_0 from the category of perfect complexes to $\operatorname{mod} kQ$ is full and dense and it furnishes a bijection between the shift orbits of the indecomposables in the homotopy category of perfect complexes and the indecomposables in $\operatorname{mod} kQ$; also, it yields an embedding ι of the Auslander-Reiten quiver of $\operatorname{mod} kQ$ into the Auslander-Reiten quiver of the homotopy category of the perfect complexes.

Theorem 1 follows from these assertions, using the covering theory as developed by Gabriel and his school [G] or the equivalent theory of group graded algebras by Gordon and Green [GG]; namely one just looks at the forgetful functor from the category of perfect complexes of kQ-modules to the category of perfect differential kQ-modules, and uses that indecomposable perfect complexes have bounded width. For the fact that the orbit category $D^b(\operatorname{mod} kQ)/[1]$ is triangulated, we may refer to the general criterion given by Keller [Ke], however one also may use directly the Frobenius category structure of the category of perfect differential kQ-modules. Some further comments on this proof will be given in section 3.

The proper framework for Theorem 1 seems to be the category of **all** differential kQ-modules, this category may be interpreted in several ways. It is the category of Λ-modules, where $\Lambda = kQ[\epsilon] = kQ[T]/\langle T^2\rangle$ (with T a central variable). Note that $\Lambda = AQ$ may be considered as the path algebra of the quiver Q over the 2-dimensional local algebra $A = k[\epsilon] = k[T]/\langle T^2\rangle$ of dual numbers, thus the Λ-modules are just the representations of Q over the ring A. Also, we may write Λ as the tensor product of kQ with A over k.

The aim of this paper to analyze Theorem 1 as dealing with two subcategories of the module category $\operatorname{mod}\Lambda$, the subcategories of interest are on the one hand the category $\operatorname{mod} kQ$ (these are the Λ-modules annihilated by ϵ), and the category of perfect differential kQ-modules on the other hand.

The decisive property which we will use is the fact that Λ is a 1-Gorenstein algebra. For any 1-Gorenstein algebra Λ, the category $\mathcal{L} = \mathcal{L}_\Lambda$ of the torsionless Λ-modules is of interest, these are the Λ-modules which are submodules of projective modules, but they also can be characterized differently: the torsionless Λ-modules are just the Gorenstein-projective or maximal Cohen-Macaulay modules as considered in [EJ,B], and the modules of G-dimension 0 in the sense [AB]. Note that $\mathcal{L}$ is a Frobenius category, thus the corresponding stable category $\underline{\mathcal{L}}$ (obtained from $\mathcal{L}$ by factoring out all the maps which factor through the subcategory $\mathcal{P}$ of the projective Λ-modules) is a triangulated category. In our case $\Lambda = kQ[\epsilon]$, the category $\mathcal{L}$ is precisely the category of perfect differential kQ-modules,

$$\mathcal{L} = \operatorname{diff}_{\mathrm{perf}}(kQ) \quad \text{and} \quad \underline{\mathcal{L}} = \underline{\operatorname{diff}}_{\mathrm{perf}}(R),$$

and every module in $\mathcal{L}$ is even strongly Gorenstein-projective, see 4.11.

The basic functor to be considered is the homology functor $H\colon \operatorname{mod}\Lambda \to \operatorname{mod} kQ$, it sends a representation $M = (M_i, M_\alpha)_{i\in Q_0, \alpha\in Q_1}$ of the quiver Q over $k[\epsilon]$ to the homology with respect to the action of ϵ, thus, $H(M) = (H(M_i), H(M_\alpha))_{i,\alpha}$. Besides the functor $H\colon \operatorname{mod}\Lambda \to \operatorname{mod} kQ$ we will consider a reverse construction η which is not functorial (but of course stably functorial), the minimal right $\mathcal{L}$-approximation.

Theorem 2. *The algebra $\Lambda = kQ[\epsilon]$ is 1-Gorenstein. The functor $H : \mathcal{L} \to \operatorname{mod} kQ$ is full and induces a bijection between the indecomposable Λ-modules in $\mathcal{L} \setminus \mathcal{P}$ and the indecomposable kQ-modules. The inverse bijection is given by taking the minimal right $\mathcal{L}$–approximation of an indecomposable kQ-module.*

We obtain an embedding ι of the Auslander-Reiten quiver $\Gamma(\operatorname{mod} kQ)$ of $\operatorname{mod} kQ$ into the Auslander-Reiten quiver $\Gamma(\underline{\mathcal{L}})$ of $\underline{\mathcal{L}}$ by sending an indecomposable kQ-module N to ηN. The only arrows which are not obtained in this may are the following "ghost maps": For any vertex y of Q, let $P_0(y)$ and $I_0(y)$ be the corresponding indecomposable projective or injective kQ-module, respectively. For any y, we construct a homomorphism $c(y)\colon \eta(I_0(y)/S(y)) \to P_0(y)$ (where $S(y)$ is the simple module corresponding to the vertex y). These homomorphisms $c(y)$ yield the arrows in $\Gamma(\underline{\mathcal{L}})$ which are not in the image of ι. In addition, we show that $\tau_{\mathcal{L}} P_0(y) = \eta I_0(y)$, where $\tau_{\mathcal{L}}$ is the Auslander-Reiten translation of $\mathcal{L}$. In this way we see how to extend the translation map of $\Gamma(\operatorname{mod} kQ)$ in order to get the translation map for $\Gamma(\underline{\mathcal{L}})$. Of course, in order to obtain the Auslander-Reiten quiver of $\mathcal{L}$ itself, we have to add the indecomposable projective Λ-modules $P(y) = P_0(y) \otimes_k A$. But this is easy: $\operatorname{rad} P(y)$ belongs to $\mathcal{L}$ and is indecomposable, actually $H(\operatorname{rad} P(y))$ is just the simple module $S(y)$, see Proposition 7.1.

Theorem 3. *The kernel of the functor $H\colon \mathcal{L} \to \operatorname{mod} kQ$ is a finitely generated ideal of $\mathcal{L}$, it is generated by the identity morphisms of the indecomposable projective Λ-modules and the maps $c(y)\colon \eta(I_0(y)/S(y)) \to P_0(y)$ for the vertices y of Q.*

Instead of taking the maps $c(y)$ one may also take all the non-invertible homomorphisms $\eta I_0(x) \to P_0(y)$, where x, y are vertices of Q. Note that the homomorphisms $\eta I_0(x) \to P_0(y)$ are nearly always non-invertible, the only exception occurs in case Q is of type $\mathbb{A}_n$ with linear orientation, with y the source and x the sink of Q (so that we have $P_0(y) = I_0(x)$ and therefore $I_0(x) = \eta I_0(x)$).

Here is an outline of the paper. The first section describes the context of this investigation. In section 2, we show that the perfect differential kQ-modules are precisely the torsionless $kQ[\epsilon]$-modules, thus the Gorenstein-projective $kQ[\epsilon]$-modules. Section 3 provides some details for the covering approach. Section 4 is the central part, here we discuss in which way the homology functor H and the $\mathcal{L}$-approximation η are inverse to each other. We should stress that all the considerations in this paper rely on the fact that the subcategories $\operatorname{mod} kQ$ and $\mathcal{L}$ of $\operatorname{mod}\Lambda$ are strongly related to each other, via the functor $H\colon \mathcal{L} \to \operatorname{mod} kQ$ and the inverse construction η.

Sections 5 and 6 deal with the ghost maps (these are the maps which vanish under H), section 7 with the position of the indecomposable projective Λ-modules in the Auslander-Reiten sequences of $\mathcal{L}$. Sections 8 to 10 are devoted to examples and further remarks.

1. The context.

1.1. An explicit description of the category of Gorenstein-projective modules is known only for very few algebras. In a recent paper Luo and Zhang [LZ] gave a characterization of the Gorenstein-projective AQ-modules, where Q is a finite acyclic quiver and A any k-algebra, thus one may try to use this result in order to construct these modules explicitly. The present paper deals with the very special case of the algebra $A = k[\epsilon]$ of dual numbers, this may be considered as an interesting test case.

Let us stress that the class of 1-Gorenstein algebras is a class of algebras which includes both the hereditary and the self-injective algebras — two classes of algebras whose representations have been investigated very thoroughly and have been shown to be strongly related to Lie theory. Thus one might hope that all the 1-Gorenstein algebras have such a property. Keller and Reiten [KR] have identified another class of 1-Gorenstein algebras, namely the cluster tilted algebras, and this again is a class of algebras related to Lie theory. Of course, the result of the present paper also supports the hope.

As we have mentioned, an explicit description of the category of Gorenstein-projective modules is known only in few cases. Chen ([C], see also [RX]) has shown that for Λ of Loewy length at most 2, the stable category of Gorenstein-projective Λ-modules is a union of categories with Auslander-Reiten quiver of tree type $\mathbb{A}_1$, thus not very exciting. Now, the next case of interest are the artin algebras of Loewy length 3, and for this case we provide a wealth of examples. Namely, if Q is a finite bipartite quiver (i.e., all vertices are sinks or sources), then the algebra $kQ[\epsilon]$ is of Loewy length at most 3.

1.2. The theorems allow to transfer a lot of results known for the module category $\operatorname{mod} kQ$ to the category $\mathcal{L}$ of Gorenstein-projective Λ-modules, where $\Lambda = kQ[\epsilon]$. For example, the Kac Theorem [Ka] yields:

The homology dimension vector $\mathbf{dim}\, H(-)$ *maps any indecomposable object in* $\mathcal{L}\setminus\mathcal{P}$ *to a positive root of the corresponding Kac-Moody algebra* $\mathbf{g}$. *For any positive real root* $\mathbf{r}$, *there is a unique isomorphism class of indecomposable modules* M *in* $\mathcal{L}$ *with* $\mathbf{dim}\, H(M) = \mathbf{r}$; *if* k *is an infinite field, then for every positive imaginary root* $\mathbf{r}$ *of* $\mathbf{g}$, *there are infinitely many isomorphism classes of indecomposable modules* M *in* $\mathcal{L}$ *with* $\mathbf{dim}\, H(M) = \mathbf{r}$.

1.3. The relationship between abelian categories and triangulated categories has always been considered as fascinating, but also mysterious. It was clear from the beginning that starting with a suitable abelian category $\mathcal{A}$ (namely the module category $\mathcal{A}$ of a self-injective algebra) one may obtain a triangulated categories by factoring out some finitely generated ideal (namely the ideal of all maps which factor through a projective module), see [H1]. Only quite recently, it was observed that there are also examples in the reverse direction: if one starts with a cluster category (a triangulated category, according to [Ke], Theorem 1) and factors out the ideal of all maps which factor through the additive category generated by a fixed cluster-tilting object, then one obtains an abelian category, namely the module category of the corresponding cluster-tilted algebra (thus an abelian category), see Buan-Marsh-Reineke-Reiten-Todorov [BMRRT]. For generalizations, see Koenig-Zhu [KZ] and Demonet-Liu [DL].

The results of the present paper should be seen in this context. As in the case of the cluster categories we start with a triangulated category $\mathcal{T}$ and factor out a finitely generated ideal of $\mathcal{T}$ in order to obtain an abelian category. But whereas in the cluster category case the ideal is generated by some identity maps, here we deal with an ideal I which lies inside the radical of $\mathcal{T}$. According to 6.4, we even know that $I^2 = 0$.

It seem to be of interest that actually we deal with two related subcategories of a module category, one is a Frobenius category $\mathcal{F}$, the other an abelian category $\mathcal{A}$, such that there is a canonical bijection between the indecomposable objects in the stable category $\underline{\mathcal{F}}$ and the indecomposable objects in $\mathcal{A}$. Of course, if we consider for a self-injective algebra R with socle I the subcategories $\mathcal{F} = \operatorname{mod} R$ and $\mathcal{A} = \operatorname{mod} R/I$, then there is such a canonical bijection, namely the identity. The examples which we consider are more intricate: here, $\mathcal{F} = \mathcal{L}$ is the category of Gorenstein-projective $kQ[\epsilon]$-modules, $\mathcal{A}$ is the category of kQ-modules, and the bijection is given by the functor H and the $\mathcal{L}$-approximation η.

1.4. A long time ago, it has been shown by Buchweitz [B] that given a Gorenstein algebra Λ, the Verdier quotient of the bounded derived category $D^b(\operatorname{mod}\Lambda)$ modulo the subcategory of perfect complexes can be identified with the stable category of Gorenstein-projective Λ-modules, and Orlov [O] proposed the name *triangulated category of singularities* for this Verdier quotient. In our case $\Lambda = kQ[\epsilon]$, we show that the triangulated category of singularities is just $D^b(\operatorname{mod} kQ)/[1]$.

2. The basic observation.

Let R be an arbitrary ring. As above, we define $R[\epsilon] = R[T]/\langle T^2\rangle$, where T is a variable which is supposed to commute with all the elements of R. The $R[\epsilon]$-modules are just the *differential* R-modules, they may be written as (N, f), where N is an R-module and $f\colon N \to N$ is an R-endomorphism with $f^2 = 0$ (namely, if such a pair (N, f) is given, then N can be considered as an $R[\epsilon]$-module by defining the action of ϵ on N as being given by f); by abuse of notation, we sometimes will write ϵ instead of f. A differential R-module (N, f) will be said to be *perfect* provided N is a finitely generated projective R-module.

For an R-module N, let $N[\epsilon] = (N \oplus N; \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right])$, this is an $R[\epsilon]$-module (note that if we take $N = R$, then the module $R[\epsilon]$ is just the regular representation of the ring $R[\epsilon]$, thus there is no conflict of notation). Of course, $N[\epsilon]$ has the submodule $N \oplus 0 = \epsilon(N[\epsilon])$, and both $N \oplus 0$ and $N[\epsilon]/(N \oplus 0)$ are isomorphic to N. If N is a finitely generated R-module, then $N[\epsilon]$ is a finitely generated $R[\epsilon]$-module; if N is a projective R-module, then $N[\epsilon]$ is a projective $R[\epsilon]$-module. In particular, if N is finitely generated and projective, then $N[\epsilon]$ is a perfect $R[\epsilon]$-module. But a perfect $R[\epsilon]$-module may not be of the form $N[\epsilon]$. Also note that a finitely generated projective $R[\epsilon]$ module is perfect, but the converse is not true.

Let us recall that an R-module is said to be *torsionless* provided it is a submodule of a projective R-module. Thus, a differential R-module is torsionless if it is a submodule of a projective $R[\epsilon]$-module. Also recall that a ring R is *left hereditary* provided any torsionless left module is projective.

2.1. Lemma. *Let R be left noetherian and left hereditary. A differential R-module is finitely generated and torsionless if and only if it is perfect.*

Proof. Let (N, f) be a differential R-module.

First, assume that (N, f) is finitely generated and torsionless. Since (N, f) is torsionless, we know that (N, f) is a submodule of a free $R[\epsilon]$-module, and since (N, f) is finitely generated, it is even a submodule of a free $R[\epsilon]$-module of finite rank. Thus (N, f) can be embedded into a module of the form $(R^t \oplus R^t, \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right])$ for some natural number t. In particular, we see that N is a submodule of R^{2t}. Since R is left noetherian and hereditary, we conclude that N is a finitely generated projective R-module.

Conversely, assume that N is a finitely generated projective R-module. We denote by N' the kernel, by N'' the image of f; let $u'\colon N'' \to N'$ and $u\colon N' \to N$ be the inclusion maps. Note that f induces an epimorphism $f'\colon N \to N''$ with $f = uu'f'$. Since R is left hereditary and N is a projective R-module, we see that N'' is also projective. It follows that there is a homomorphism $s\colon N'' \to N$ such that $f's = 1_{N''}$, and, as a consequence the map

$$[\,u \quad s\,] : N' \oplus N'' \to N$$

is an isomorphism. But $uu' = uu'f's = fs$ shows that

$$[\,u \quad s\,] \begin{bmatrix} 0 & u' \\ 0 & 0 \end{bmatrix} = f\,[\,u \quad s\,],$$

therefore $[\,u \quad s\,]$ is an isomorphism between $(N' \oplus N'', \left[\begin{smallmatrix} 0 & u' \\ 0 & 0 \end{smallmatrix}\right])$ and (N, f). It remains to observe that there is the following embedding of differential R-modules:

$$\begin{bmatrix} 1 & 0 \\ 0 & u' \end{bmatrix} : (N' \oplus N'', \begin{bmatrix} 0 & u' \\ 0 & 0 \end{bmatrix}) \to (N' \oplus N', \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}),$$

and that $(N' \oplus N', \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right])$ is a projective Λ-module. This shows that $(N' \oplus N'', \left[\begin{smallmatrix} 0 & u' \\ 0 & 0 \end{smallmatrix}\right])$ and therefore (N, f) is torsionless. Since N is a finitely generated R-module, we see that (N, f) is a finitely generated Λ-module. □

Let us consider now finite-dimensional k-algebras, where k is a field. Recall [AR1] that such an algebra R is called a *Gorenstein algebra of Gorenstein dimension* 1 or just a 1-*Gorenstein algebra* provided the injective dimension of ${}_RR$ as well as R_R is equal to 1. Since Λ is the tensor product of a hereditary and a self-injective algebra, one sees immediately that Λ is 1-Gorenstein.

Given any finite-dimensional k-algebra R, an R-module M is said to be *Gorenstein-projective* [EJ] provided there exists an exact (usually unbounded) complex $P_\bullet = (P_i, \delta_i)_i$ of finitely generated projective R-modules such that also $\mathrm{Hom}_\Lambda(P_\bullet, {}_RR)$ is exact and such that M is the image of δ_0. If R is Gorenstein, then an R-module M is Gorenstein-projective if and only if $\operatorname{Ext}^i_R(M,R)=0$ for all $i\geq 1$.

The following proposition is immediate.

2.2. Proposition. *Let R be a finite-dimensional k-algebra which is hereditary. Then $\Lambda=R[\epsilon]$ is a Gorenstein algebra of Gorenstein dimension 1. If M is a Λ-module, then the following conditions are equivalent:*
(i) *M is Gorenstein-projective,*
(ii) *M is torsionless,*
(iii) $\operatorname{Ext}^1_\Lambda(M,\Lambda)=0$.

In the following, we usually will assume that $\Lambda=R[\epsilon]$ where $R=kQ$ is the path algebra of the finite connected acyclic quiver Q and then $\mathcal{L}=\mathcal{L}(\Lambda)$ will denote the category of Gorenstein-projective Λ-modules (see also Section 10).

2.3. *Given any Λ-module M, there is an exact sequence*

$$0\to P\to \eta M \xrightarrow{g} M\to 0,$$

such that P is projective, ηM belongs to $\mathcal{L}$ and g is a right minimal map (see for example [EJ], Theorem 11.5.1).

The map g (or also the module ηM) is called the minimal right $\mathcal{L}$-approximation, since any map $h\colon L\to M$ with $L\in\mathcal{L}$, can be factorized as $h=h'g$ with $h'\colon L\to \eta M$. Of course, this factorization property and the minimality of g implies that ηM is uniquely determined by M, up to an isomorphism, but there there is not necessarily a canonical isomorphism. Also, one should note that ηM is the universal extension of M from below, using projective modules.

Section 4 will be devoted to a detailed study of the minimal right $\mathcal{L}$-approximations $\eta N\to N$, where N is a Λ-module which is annihilated by ϵ (thus a kQ-module).

2.4. In our setting $\Lambda=kQ[\epsilon]$ and $\mathcal{L}=\mathcal{L}(\Lambda)$, we always will consider $\mathcal{L}$ as an exact category with its standard exact structure given by its embedding into $\operatorname{mod}\Lambda$ (thus, using the exact sequences of $\operatorname{mod}\Lambda$ with $X,Y,Z\in\mathcal{L}$).

Lemma. *The exact category $\mathcal{L}$ is a Frobenius category.*

Proof. First, let us note the following: A sequence $\zeta=(0\to X\xrightarrow{f} Y\xrightarrow{g} Z\to 0)$ in $\mathcal{L}$ yields an exact sequences when we apply $\operatorname{Hom}(\Lambda,-)$, if and only if it is an exact sequence. If the sequence $\zeta=(0\to X\xrightarrow{f} Y\xrightarrow{g} Z\to 0)$ in $\mathcal{L}$ is exact, then it stays exact when we apply $\operatorname{Hom}(-,\Lambda)$, since $\operatorname{Ext}^1_\Lambda(\mathcal{L},\Lambda)=0$. This shows that Λ (and therefore any projective Λ-module) is both projective and injective with respect to the exact structure of $\mathcal{L}$.

It remains to be seen that the exact category $\mathcal{L}$ has enough projective and enough injective objects. If Z belongs to $\mathcal{L}$, a projective cover $PZ\to Z$ yields an exact sequence $0\to \Omega Z\to PZ\to Z\to 0$ in $\mathcal{L}$ with PZ being projective. On the other hand, assume that X belongs to $\mathcal{L}$. Let $u\colon X\to Y$ be a minimal left Λ-approximation of X. Since X is torsionless, u is a monomorphism, and we obtain an exact sequence $0\to X\xrightarrow{u} Y\to$

$\mathrm{Cok}(u) \to 0$. Since u is a left Λ-approximation, the sequence stays exact when we apply $\mathrm{Hom}(-,\Lambda)$ and therefore $\mathrm{Ext}^1(\mathrm{Cok}(u),\Lambda) = 0$, thus Z is in $\mathcal{L}$. Altogether we hve obtained in this way an exact sequence $0 \to X \to Y \to \mathrm{Cok}(u) \to 0$ in $\mathcal{L}$ with Y being projective. □

Let us add the following observations: If $\zeta = (0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0)$ is an exact sequence in $\mathcal{L}$, and Y is projective, then f is a left Λ-approximation. And, given an exact sequence $\zeta = (0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0)$ in $\mathrm{mod}\,\Lambda$ with $Z \in \mathcal{L}$, then ζ considered as an exact sequence of kQ-modules splits (since $Z \in \mathcal{L}$ means that Z is a perfect differential kQ-module, thus that Z is kQ-projective).

Proposition. *The functor $H\colon \underline{\mathcal{L}} \to \mathrm{mod}\,kQ$ is a cohomological functor.*

Proof: We have to show that any triangle $X \to Y \to Z \to$ in $\underline{\mathcal{L}}$ yields under H an exact sequence $H(X) \to H(Y) \to H(Z)$. Recall that such a triangle in $\underline{\mathcal{L}}$ starting with a homomorphism $f\colon X \to Y$ in $\mathcal{L}$ can be constructed as follows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & X & \xrightarrow{u} & P & \longrightarrow & \Sigma X & \longrightarrow & 0 \\ & & {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow & & \| & & \\ 0 & \longrightarrow & Y & \xrightarrow{g} & Z & \longrightarrow & \Sigma X & \longrightarrow & 0, \end{array}$$

this is a commutative diagram in $\mathrm{mod}\,\Lambda$ with exact rows such that u is a left Λ-approximation (therefore, P is a projective Λ-module and $\Sigma X \in \mathcal{L}$). Thus $Z = (Y \oplus P)/U$, where $U = \{(f(x), -u(x)) \mid x \in X\}$ and $g(y) = \overline{(y,0)}$, $f'(p) = \overline{(0,p)}$, for $y \in Y$ and $p \in P$. The exact sequence

$$0 \to X \xrightarrow{\left[\begin{smallmatrix} f \\ -u \end{smallmatrix}\right]} Y \oplus P \xrightarrow{[g\ f']} Z \to 0$$

yields under H a sequence

$$H(X) \xrightarrow{\left[\begin{smallmatrix} H(f) \\ -H(u) \end{smallmatrix}\right]} H(Y) \oplus H(P) \xrightarrow{[H(g)\ H(f')]} H(Z)$$

with zero composition. But $H(P) = 0$, thus $H(g)H(f) = 0$. It remains to be seen that any element of the kernel of $H(g)$ is in the image of $H(f)$. An element of $H(Y)$ is the residue class $\overline{y}$ modulo ϵY of an element $y \in Y$ with $\epsilon y = 0$, and $\overline{y}$ belongs to the kernel of $H(g)$ iff $g(y)$ belongs to ϵZ. Thus assume that $g(y) = \epsilon z$ with $z \in Z$, say $z = (y', p) + U$ for some $y' \in Y$ and $p \in P$. Thus $(y, 0) - (\epsilon y', \epsilon p)$ belongs to U, this means that there is $x \in X$ with $y - \epsilon y' = f(x)$ and $0 - \epsilon p = -u(x)$. Note that $u(\epsilon x) = \epsilon u(x) = \epsilon^2(p) = 0$. Since u is injective, $\epsilon x = 0$, thus x belongs to the kernel of ϵ. The equality $y = f(x) + \epsilon y'$ shows that $\overline{y} = \overline{f(x)} = H(f)(\overline{x})$ in $H(Y)$, thus $\overline{y}$ is in the image of $H(f)$. □

2.5. The main aim of the present investigation is to present the Auslander-Reiten quiver $\Gamma(\mathcal{L})$ where $\mathcal{L}$ is the category of torsionless Λ-modules and $\Lambda = kQ[\epsilon]$. It was shown by Bautista and Martinez [BM] that for any 1-Gorenstein algebra Λ, the category $\mathcal{L}$ of torsionless modules has Auslander-Reiten sequences (after all, $\mathcal{L}$ is functorially finite in $\operatorname{mod}\Lambda$, thus we also may refer to [AS]). If we denote by τ the Auslander-Reiten translation in $\operatorname{mod}\Lambda$ and by $\tau_{\mathcal{L}}$ the Auslander-Reiten translation in $\mathcal{L}$, then, for any indecomposable module M in $\mathcal{L}$, the module $\tau_{\mathcal{L}}M$ is a direct summand of a minimal right $\mathcal{L}$-approximation of τM.

3. Proof of theorem 1.

Given any ring R, we denote by $\mathcal{P} = \mathcal{P}_R$ the category of finitely generated projective R-modules.

We are interested in complexes of R-modules, such complexes may be considered as differential graded R-modules. In particular, we will consider perfect complexes (or perfect differential graded R-modules), these are the bounded complexes which use only finitely generated projective R-modules. We denote by $C^b(\mathcal{P}_R)$ the category of perfect complexes, and by $K^b(\mathcal{P}_R)$ the corresponding homotopy category. Let us stress that $C^b(\mathcal{P}_R)$ is a Frobenius category and $K^b(\mathcal{P}_R)$ is just the corresponding stable category, say with stabilization functor $\pi\colon C^b(\mathcal{P}_R) \to K^b(\mathcal{P}_R)$, which sends a map to its homotopy class.

Note that in case $R = kQ$, where Q is a finite acyclic quiver, the ring R has finite global dimension, and $K^b(\mathcal{P}_R)$ can be identified with the bounded derived category $D^b(\operatorname{mod} R)$. In dealing with categories of complexes, the shift functor will be denoted by [1].

There is the following commutative diagram

$$\begin{array}{ccc} C^b(\mathcal{P}_R) & \xrightarrow{\ \pi\ } & K^b(\mathcal{P}_R) \\ {\scriptstyle\gamma}\big\downarrow & & \big\downarrow{\scriptstyle\gamma} \\ \operatorname{diff}_{\mathrm{perf}}(R) & \xrightarrow{\ \pi\ } & \underline{\operatorname{diff}}_{\mathrm{perf}}(R) \end{array}$$

where the horizontal functors π are just the stabilization functors, whereas the vertical functors γ are obtained by forgetting the grading (such a forgetful functor is sometimes called a compression functor or, in the covering theory of the Gabriel school, the corresponding pushdown functor).

What is important here, is the fact that for $R = kQ$ the indecomposable objects in $C^b(\mathcal{P}_R)$ have bounded width (by definition, the *width* of a complex $P_\bullet$ is the maximal difference $i - j$ where $P_i \neq 0, P_j \neq 0$) — in fact, it is well-known and easy to see that for $R = kQ$, the width of any indecomposable object in $C^b(\mathcal{P}_R)$ is bounded by 1, see for example Happel [H1], page 49. It follows from the boundedness of the width that the functor $\gamma\colon C^b(\mathcal{P}_R) \to \operatorname{diff}_{\mathrm{perf}}(R)$ is dense. [This is one of the main assertions of covering theory. For the convenience of the reader let us sketch the argument: Let (N, ϵ) where N is a perfect differential R-module, thus N is an R-module and ϵ is an endomorphism of N with $\epsilon^2 = 0$. Let $N_\bullet = (N_i, \delta_i)$ be the complex with $N_i = N$ and $\delta_i = \epsilon$ for all $i \in \mathbb{Z}$. Of course, $N_\bullet$ is usually not bounded, but it can be written as an (infinite) direct sum of indecomposable perfect complexes $N'_\bullet$ (here we use the boundedness of the widths of indecomposable perfect complexes). Each indecomposable direct summand $N'_\bullet$ occurs with finite multiplicity, and, of course, it has local endomorphism ring, so we are in the setting of the Kull-Remak-Schmidt-Azumaya theorem. Since $N_\bullet$ is invariant under the shift, we can write $N_\bullet$ as a (finite) direct sum $N_\bullet = \bigoplus_{j=1}^m N_\bullet^{(j)}$, where each $N_\bullet^{(j)}$ is the direct sum of all the members of the shift orbit of some indecomposable perfect complex $X_\bullet^{(j)}$. It follows that $N_\bullet = \gamma(\bigoplus_{j=1}^m X_\bullet^{(j)})$.]

Since the functor $\gamma\colon C^b(\mathcal{P}_R) \to \mathrm{diff}_{\mathrm{perf}}(R)$ is dense, also the induced functor $\gamma\colon K^b(\mathcal{P}_R) \to \underline{\mathrm{diff}}_{\mathrm{perf}}(R)$ is dense. These functors γ provide a bijection between the shift-orbits of the indecomposable objects in $C^b(\mathcal{P}_R)$ and the indecomposable objects in $\mathrm{diff}_{\mathrm{perf}}(R)$ and similarly, between the shift-orbits of the indecomposable objects in $K^b(\mathcal{P}_R)$ and the indecomposable objects in $\underline{\mathrm{diff}}_{\mathrm{perf}}(R)$.

Let us denote by $H_0\colon C^b(\mathcal{P}_R) \to \mathrm{mod}\, R$ the functor which attaches to a complex $P_\bullet = (P_i, \delta_i)$ the homology at the position 0. Then H_0 provides a bijection between the shift orbits of the indecomposable objects in $K^b(\mathcal{P}_{kQ})$ and the indecomposable kQ-modules.

Altogether, we see: The functor $\gamma\colon K^b(\mathcal{P}_R) \to \underline{\mathrm{diff}}_{\mathrm{perf}}(R)$ induces a fully faithful functor $D^b(\mathrm{mod}\, kQ)/[1] = K^b(\mathcal{P}_R) \to \underline{\mathrm{diff}}_{\mathrm{perf}}(R)$ and the density of γ is needed for the last statement of (a). By (a), the Auslander-Reiten quiver of $\underline{\mathrm{diff}}_{\mathrm{perf}}(R)$ is the orbit quiver of the Auslander-Reiten quiver of $D^b(\mathrm{mod}\, kQ)$ by [1], and hence contains the Auslander-Reiten quiver of $\mathrm{mod}\, kQ$ as a subquiver. This completes the proof. □

For our case $R = kQ$, let us exhibit the diagram above again, but now using the notation $\mathcal{L} = \mathrm{diff}_{\mathrm{perf}}(kQ)$ and $\underline{\mathcal{L}} = \underline{\mathrm{diff}}_{\mathrm{perf}}(kQ)$:

$$\begin{array}{ccc} C^b(\mathcal{P}_{kQ}) & \xrightarrow{\ \pi\ } & K^b(\mathcal{P}_{kQ}) \\ \gamma\big\downarrow & & \big\downarrow\gamma \\ \mathcal{L} & \xrightarrow{\ \pi\ } & \underline{\mathcal{L}} \end{array}$$

and we recall that the horizontal functors π are the stabilization functors, whereas the vertical functors γ are obtained by forgetting the grading.

4. Explicit bijections.

Let Q be a finite acyclic quiver, kQ its path algebra and $\Lambda = kQ[\epsilon]$. Recall that we denote by $\mathcal{L}$ the category of (finitely generated) torsionless modules. Given any Λ-module M, let $\eta M \to M$ be a minimal right $\mathcal{L}$-approximation.

We consider $\mathrm{mod}\, kQ$ as the full subcategory of $\mathrm{mod}\, \Lambda$ given by all the Λ-modules N which are annihilated by ϵ.

4.1. *The Λ-modules in $\mathcal{L} \cap \mathrm{mod}\, kQ$ are just the projective kQ-modules.*

Proof. Let M be a module in $\mathcal{L} \cap \mathrm{mod}\, kQ$. Since M is in $\mathcal{L}$, we may consider M as a submodule of $({}_\Lambda\Lambda)^t$ for some t. Since M is annihilated by ϵ, it must lie inside the submodule $(\epsilon\Lambda)^t$ (since $\{x \in \Lambda \mid \epsilon x = 0\} = \epsilon\Lambda$). But $\epsilon\Lambda \simeq kQ$ as Λ-modules and thus also as kQ-modules. Now M is a submodule of $(kQ)^t$, thus it is a projective kQ-module. Of course, conversely, a projective kQ-module belongs to $\mathcal{L}$. □

4.2. *Let M be in $\mathcal{L}$, let $M' = \{x \in M \mid \epsilon x = 0\}$ and $M'' = \epsilon M$. Then $M'' \subseteq M'$, $H(M) = M'/M''$ and the exact sequence*

$$0 \to M'' \to M' \to H(M) \to 0$$

is a projective kQ-resolution of $H(M)$.

Proof. Since M' is a submodule of M, it belongs to $\mathcal{L}$. Since M' is annihilated by ϵ, it belongs to $\operatorname{mod} kQ$, thus M' is a projective kQ-module by 4.1. □

4.3. *Let M be a Λ-module. If M is projective, then $H(M) = 0$ and ϵM is a projective kQ-module. Conversely, if $H(M) = 0$ and $N = \epsilon M$ is a projective kQ-module, then M is isomorphic to $N[\epsilon]$, thus M is a projective Λ-module.*

Proof. If $M = {}_\Lambda\Lambda$, then clearly $H(M) = 0$ and $\epsilon M = \epsilon\Lambda = kQ$ is a projective kQ-module. Both properties carry over to direct sums and direct summands, thus to all projective Λ-modules.

Conversely, assume that $H(M) = 0$ and let $N = \epsilon M$. The multiplication with ϵ yields an isomorphism from $M/\epsilon M$ onto ϵM, thus we can identify $M/\epsilon M$ with N. Let $P = N[\epsilon]$ and consider the canonical map $p\colon P \to P/\epsilon P = N$. We can lift this map to a map $p'\colon P \to M$ which again has to be surjective (since ϵM lies in the radical of M). Now $\dim P = 2\dim N = \dim M/\epsilon M + \dim \epsilon M = \dim M$. This shows that p' is an isomorphism. □

4.4. *Any M in $\mathcal{L}$ has a projective submodule U such that $\epsilon M \subseteq U$ and such that M/U can be identified with $H(M)$.*

Proof. Let $M \in \mathcal{L}$. Let $M' = \{x \in M \mid \epsilon x = 0\}$ and $M'' = \epsilon M$. Then $M'' \subseteq M'$, since $\epsilon^2 = 0$ and the multiplication by ϵ yields an isomorphism $M/M' \to M''$. Obviously, M/M' is annihilated by ϵ, thus a kQ-module. On the other hand, M'' as a submodule of M is torsionless. Thus the module $M/M' \simeq M''$ belongs to $\mathcal{L} \cap \operatorname{mod} kQ$, and therefore is a projective kQ-module. But this means that the inclusion map $M'/M'' \to M/M''$ is a split monomorphism (of Λ-modules), since these are kQ-modules and the cokernel is a projective kQ-module. In this way, we obtain a Λ-submodule U of M such that $U \cap M' = M''$ and $U + M' = M$. Note that $M/U \simeq M'/M'' = H(M)$.

It remains to see that U is a projective Λ-module. In order to use 4.3, we have to show that $H(U) = 0$ and that ϵU is a projective kQ-module. Let $U' = \{x \in U \mid \epsilon x = 0\}$, then $U' = U \cap M'$, thus $U' = M''$. Let us show that $\epsilon U = M''$. On the one hand $\epsilon U \subseteq \epsilon M = M''$. On the other hand, any element in $M'' = \epsilon M$ has the form ϵx with $x \in M = U + M'$, thus $x = u + x'$ with $u \in U$ and $x' \in M'$, thus $\epsilon x = \epsilon(u + x') = \epsilon u$ belongs to ϵU. It follows that $H(U) = U'/\epsilon U = M''/M'' = 0$. Also, we know that $\epsilon U = M''$ is a projective kQ-module. □

We see: If M belongs to $\mathcal{L}$, then there is an exact sequence

$$0 \to U \to M \xrightarrow{g} H(M) \to 0$$

with U a projective Λ-module. By 2.2, this means that we have obtained a right $\mathcal{L}$-approximation g of $H(M)$. But note that we do not obtain a canonical such map, since the submodule U is not uniquely determined. Also observe that this is a minimal right $\mathcal{L}$-approximation if and only if M has no non-zero projective direct summand.

4.5. *If M is in $\mathcal{L}$, indecomposable and not projective, then $H(M)$ is indecomposable and $\eta H(M)$ is isomorphic to M.*

Proof. If there is a direct decomposition $H(M) = N_1 \oplus N_2$, let $M_i \to N_i$ be a minimal right $\mathcal{L}$-approximation of N_i, for $i = 1, 2$. Then $M_1 \oplus M_2 \to N_1 \oplus N_2$ is a minimal right $\mathcal{L}$-approximation. The uniqueness of minimal right $\mathcal{L}$-approximations yields that M is isomorphic to $M_1 \oplus M_2$, thus one of the M_i is zero, say $M_2 = 0$. But then also $N_2 = 0$.

Since $M \to H(M)$ is a minimal right $\mathcal{L}$-approximation, we see that $\eta H(M)$ and M are isomorphic. □

4.6. We have seen in 4.4 that for any module $M \in \mathcal{L}$ there is a projective submodule U of M such that $\epsilon M \subseteq U$ and such that M/U can be identified with $H(M)$. We should stress that U is unique up to isomorphism, namely $U \simeq (\epsilon M)[\epsilon]$. Let us show that for any projective submodule U' with $\epsilon M \subseteq U'$, the projection map $p\colon M \to M/U'$ induces an isomorphism $H(p)$.

Let M be a module in $\mathcal{L}$. Let U be a projective submodule of M such that $\epsilon M \subseteq U$. Then the map $H(M) \to H(M/U) = M/U$ induced by the projection is an isomorphism.

Proof. First, let us show that $\epsilon M = \epsilon U$. In order to prove this, we may consider M as an A-module (recall that $A = k[\epsilon]$) and U as an A-submodule of M. Since U is projective as a Λ-module, U is also projective as an A-module (since $kQ[\epsilon]$ is projective as an A-module). But A is self-injective, thus any projective A-module is also injective as an A-module. This shows that the embedding $U \to M$ splits as an embedding of A-modules. Thus there is an A-submodule U' of M such that $M = U \oplus U'$. But U' is isomorphic to M/U as an A-module, thus annihilated by ϵ. This shows that $\epsilon M = \epsilon U \oplus \epsilon U' = \epsilon U$.

Let $M' = \{x \in M \mid \epsilon x = 0\}$. Then $M' \cap U = \{x \in U \mid \epsilon x = 0\} = \epsilon U = \epsilon M$.

Also, $M' + U = M$. For the proof, consider an element $x \in M$. Since $\epsilon M = \epsilon U$, there is $u \in U$ such that $\epsilon x = \epsilon u$, thus $\epsilon(x - u) = 0$. This shows that $x - u \in M'$ and therefore $x = (x - u) + u \in M' + U$.

There is the canonical (Noether-) isomorphism

$$M'/(M' \cap U) \to (M' + U)/U.$$

It yields the required isomorphism

$$H(M) = M'/\epsilon M = M'/(M' \cap U) \simeq (M' + U)/U = M/U = H(M/U).$$

□

If we start with a module $N \in \operatorname{mod} kQ$ and form ηN, then by 2.3 there is an exact sequence

$$0 \to U \xrightarrow{u} \eta N \xrightarrow{g} N \to 0,$$

such that U is projective and ηN belongs to $\mathcal{L}$. Thus we can apply 4.6 in order to see that the map $H(\eta N) \to H(N) = N$ induced by g is an isomorphism. This shows:

4.7. *If $N \in \operatorname{mod} kQ$, then $H(\eta N)$ is isomorphic to N.* □

It remains to study the minimal right $\mathcal{L}$-approximation for $N \in \operatorname{mod} kQ$. Start with a minimal projective kQ-resolution of N, say

$$0 \to \Omega_0 N \to P_0 N \to N \to 0$$

and embed $\Omega_0 N$ into $(\Omega_0 N)[\epsilon]$. Since $\Omega_0 N$ is a projective kQ-module, we know that $(\Omega_0 N)[\epsilon]$ is a projective Λ-module. Forming the pushout of the given embeddings of $\Omega_0 N$ into $P_0 N$ and $(\Omega_0 N)[\epsilon]$, we obtain the following commutative diagram of Λ-modules with exact rows and columns

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \Omega_0 N & \longrightarrow & P_0 N & \longrightarrow & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & (\Omega_0 N)[\epsilon] & \longrightarrow & X & \xrightarrow{g} & N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \Omega_0 N & = & \Omega_0 N & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

4.8. *The module X belongs to $\mathcal{L}$ and has no non-zero projective direct summand. As a consequence, the map $g\colon X \to N$ is a minimal right $\mathcal{L}$-approximation of N.*

Thus, we can identify X with ηN.

Proof. The vertical sequence in the middle shows that X is an extension of $P_0 N$ and $\Omega_0 N$. Both $P_0 N$ and $\Omega_0 N$ belong to $\mathcal{L}$ and $\mathcal{L}$ is closed under extensions, thus X belongs to $\mathcal{L}$. On the other hand, we have already mentioned that $(\Omega_0 N)[\epsilon]$ is a projective Λ-module, thus g is a right $\mathcal{L}$-approximation.

Let us show that any projective direct summand P of X is zero. According to 4.6, we know that $H(X) = N$. Now assume that there is a direct decomposition of Λ-modules $X = M \oplus P$, with P projective. Then $H(M) = H(X)$. Let $M' = \{x \in M \mid \epsilon x = 0\}$ and $M'' = \epsilon M$. According to 4.2 the following exact sequence

$$0 \to M'' \to M' \to H(M) \to 0$$

is a projective kQ-resolution of $H(M)$. Since we are using a minimal projective kQ-resolution of $N = H(X) = H(M)$, we see that there is a projective kQ-module C with $C \oplus \Omega_0 N = M''$ and $C \oplus P_0 N = M'$. It remains to compare the (Jordan-Hölder) length of the modules. We denote the length of a module M by $|M|$ (let us stress that the simple Λ-modules are the simple kQ-modules).

Since $H(M) = N$, we have $|M| = 2|M''| + |N|$, thus

$$|M| = 2|M''| + |N| = 2|C| + 2|\Omega_0 N| + |N| = 2|C| + |X| = 2|C| + |M| + |P|.$$

This shows that P (and C) have to be zero.

This implies that g is minimal, since otherwise there is a non-trivial direct decomposition $X = X' \oplus X''$ with say X'' contained in the kernel of g. But then X'' is isomorphic to a direct summand of the kernel of g, thus projective. □

4.9. Here is a reformulation: *Let N be a kQ-module and $0 \to \Omega_0 N \to P_0 N \to N \to 0$ a minimal projective resolution in* $\operatorname{mod} kQ$. *Then there is an exact sequence*

$$0 \to (\Omega_0 N)[\epsilon] \to \eta N \xrightarrow{q} N \to 0$$

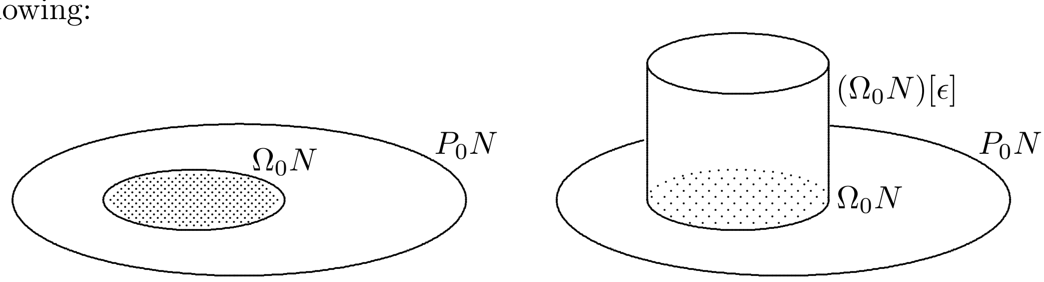
in $\operatorname{mod}\Lambda$ *such that q is a minimal left $\mathcal{L}$-approximation.* □

4.10. Proof of Theorem 2. Let M be an indecomposable module in $\mathcal{L}\backslash\mathcal{P}$. We attach to it $H(M)$, this is a kQ-module. According to 4.5, we know that $H(M)$ is indecomposable and also that $\eta H(M) = M$.

Conversely, let us start with an indecomposable kQ-module N. By construction, ηN belongs to $\mathcal{L}$. We know by 4.8 that ηN has no non-zero projective direct summands. And we know by 4.7 that $H(\eta N) = N$. We still have to show that ηN is indecomposable. Thus assume that there is a non-trivial direct decomposition $\eta N = M_1 \oplus M_2$ of Λ-modules. Both modules M_1, M_2 belong to $\mathcal{L} \setminus \mathcal{P}$, thus according to the first part of the proof, $H(M_1)$ and $H(M_2)$ are non-zero. As a consequence, $N = H(\eta N) = H(M_1) \oplus H(M_2)$ is a non-trivial direct decomposition. This contradicts the assumption that N is indecomposable.

It remains to be shown that $H\colon \mathcal{L} \to \operatorname{mod} kQ$ is full. Let M_1, M_2 be modules in $\mathcal{L}$. We have to show that H yields a surjection $\operatorname{Hom}_\Lambda(M_1, M_2) \to \operatorname{Hom}_{kQ}(H(M_1), H(M_2))$. Of course, we can assume that M_1 and M_2 both are indecomposable. If one of the modules M_i is projective, then $H(M_i) = 0$ and nothing has to be shown. Thus we can assume that both modules belong to $\mathcal{L}\setminus\mathcal{P}$. It follows that there are $\mathcal{L}$-approximations $g_i\colon M_i \to H(M_i)$ for $i = 1, 2$. But then any homomorphism $f\colon H(M_1) \to H(M_2)$ can be lifted to a map $\widetilde{f}\colon M_1 \to M_2$ with $g_2\widetilde{f} = f g_1$ and $H(\widetilde{f}) = f$. □

Illustration. The picture to have in mind when dealing with ηN for $N \in \operatorname{mod} kQ$ is the following:

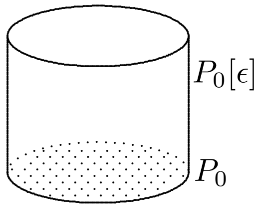


On the left, we show $\Omega_0 N$ as a submodule of $P_0 N$, note that N is obtained from $P_0 N$ by factoring out $\Omega_0 N$. The right pictures depicts ηN as an amalgamation of $P_0 N$ and

$(\Omega_0 N)[\epsilon]$ along $\Omega_0 N$. Note: if we write $M = \eta N$, and set $M' = \{x \in M \mid \epsilon x = 0\}$ and $M'' = \epsilon M$, then $M' = P_0 N$ and $M'' = \Omega_0 N$.

The right picture looks similar to the usual depiction of a mapping cylinder in topology, but better it should be compared with a mapping cone. After all, what here looks like a cylinder is a projective Λ-module, thus something that one should consider as a contractible object.

Of course, the same kind of pictures can be used also to depict the projective Λ-modules, stressing that they are of the form $P_0[\epsilon]$, with P_0 a projective kQ-module.



Remark. If one starts with an indecomposable non-projective kQ-module N and compares the Λ-modules N and ηN, the decisive difference is that ηN has $\Omega_0 N$ as a factor module, and this is a non-zero projective kQ-module. Thus $\mathrm{Hom}_\Lambda(\eta N, kQ) \neq 0$, whereas, of course, $\mathrm{Hom}_\Lambda(N, kQ) = 0$.

In section 3, we have defined the covering map $\gamma\colon C^b(\mathcal{P}_R) \to \mathrm{diff}_{\mathrm{perf}}(R)$, for any ring R. If P, P' are finitely projective R-modules and $f\colon P' \to P$ is a homomorphism, we may consider f as part of a perfect complex. Thus, the $R[\epsilon]$-module $\gamma(f) = (P \oplus P', \left[\begin{smallmatrix} 0 & f \\ 0 & 0 \end{smallmatrix}\right])$ is defined. The previous considerations may be reformulated as follows:

4.11. Proposition. *Let $0 \to \Omega_0 N \xrightarrow{f} P_0 N \to N \to 0$ be a minimal projective presentation of the kQ-module N. Then $\eta N = \gamma(f)$.*

Let us insert an interesting property of the torsionless Λ-modules. Recall that a module M is said to be *strongly Gorenstein-projective* [BM] provided there exists an exact sequence

$$0 \to M \xrightarrow{u} P \xrightarrow{p} M \to 0$$

such that u is a left Λ-approximation. This means, that M is the kernel of a map $f\colon P \to P$, where

$$\cdots \to P \xrightarrow{f} P \xrightarrow{f} P \to \cdots$$

is a complete projective resolution (namely, $f = up$).

4.12. Proposition. *Any torsionless $kQ[\epsilon]$-module is strongly Gorenstein-projective.*

Proof: Let M be a torsionless Λ-module, where $\Lambda = kQ[\epsilon]$. It is sufficient to construct an exact sequence of the form

$$0 \to M \xrightarrow{u} P \xrightarrow{p} M \to 0$$

with P projective. Namely, since Λ is 1-Gorenstein, any such injective map $M \to P$ is a left Λ-approximation. Alternatively, we construct an endomorphism $f\colon P \to P$ of a projective Λ-module P with $\mathrm{Im}(f) = \mathrm{Ker}(f) = M$. Of course, we can assume that M is indecomposable, but also that M is not projective. Thus $M = \eta N$ for some kQ-module N.

The construction of ηN starts with a projective presentation

$$0 \to \Omega_0 N \xrightarrow{u} P_0 N \to N \to 0,$$

and the map u gives rise to a map $u[\epsilon]\colon (\Omega_0 N)[\epsilon] \to (P_0 N)[\epsilon]$, thus we may consider the map

$$f = \begin{bmatrix} -\epsilon & 0 \\ u[\epsilon] & \epsilon \end{bmatrix} : P \longrightarrow P, \qquad \text{where} \quad P = (\Omega_0 N)[\epsilon] \oplus (P_0 N)[\epsilon].$$

First of all, one easily verifies that $f^2 = 0$ (since $\epsilon^2 = 0$ and ϵ commutes with all maps), thus $\mathrm{Im}(f) \subseteq \mathrm{Ker}(f)$. We claim that the cokernel $\mathrm{Cok}(f)$ maps onto ηN.

The module ηN was constructed as the following pushout

$$\begin{array}{ccc} \Omega_0 N & \xrightarrow{\quad u \quad} & P_0 N \\ {\scriptstyle \mu}\big\downarrow & & \big\downarrow{\scriptstyle \mu'} \\ (\Omega_0 N)[\epsilon] & \xrightarrow{\quad u' \quad} & \eta N \end{array}$$

Let us denote by $\pi\colon (\Omega_0 N)[\epsilon] \to \Omega_0 N$ and $\pi'\colon (P_0 N)[\epsilon] \to P_0 N$ the canonical projection maps (thus $\mu\pi$ is the multiplication by ϵ on $(\Omega_0 N)[\epsilon]$ and $\pi' \cdot u[\epsilon] = u\pi$). Let us consider the map

$$g = [\, u' \quad \mu'\pi' \,] : (\Omega_0 N)[\epsilon] \oplus (P_0 N)[\epsilon] \longrightarrow \eta N.$$

The composition of f and g is

$$gf = [\, u' \quad \mu'\pi' \,] \begin{bmatrix} -\epsilon & 0 \\ u[\epsilon] & \epsilon \end{bmatrix} = [\, -u'\epsilon + \mu'\pi' u[\epsilon] \quad \mu'\pi'\epsilon \,],$$

but this is the zero map since $u'\epsilon = u'\mu\pi = \mu' u\pi = \mu'\pi' u[\epsilon]$ and $\pi'\epsilon = 0$. Therefore g factors through the cokernel of f, thus g is of the form $P \to \mathrm{Cok}(f) \xrightarrow{\zeta} \eta N$ for some map $\zeta\colon \mathrm{Cok}(f) \to \eta N$. Since $[u', \mu']$ is surjective, also g is surjective, and therefore ζ is surjective.

It remains to look at the length of ηN. We have obtained a surjective map $\zeta\colon \mathrm{Cok}(f) \to \eta N$, thus, in particular, $|\eta N| \le |\mathrm{Cok}(f)|$. Since ηN is an extension of $P_0 N$ by $\Omega_0 N$, we see that $|\eta N| = |P_0 N| + |\Omega_0 N|$ and therefore

$$|P| = |(P_0 N)[\epsilon]| + |(\Omega_0 N)[\epsilon]| = 2|\eta N|.$$

Finally, we use that $f^2 = 0$, therefore $2|\mathrm{Cok}(f)| \le |P|$. Combining the inequalities, we see that

$$|P| = 2|\eta N| \le 2|\mathrm{Cok}(f)| \le |P|.$$

Consequently, we must have both $|\eta N| = |\operatorname{Cok}(f)|$ and $2|\operatorname{Cok}(f)| = |P|$. The first equality means that ζ is an isomorphism, thus the cokernel of f is isomorphic to ηN. The second equality means that $\operatorname{Im}(f) = \operatorname{Ker}(f)$. Altogether, we conclude that $f\colon P \to P$ is an endomorphism of the projective Λ-module P with $\operatorname{Im}(f) = \operatorname{Ker}(f) = M$. □

5. The connecting sequences.

For any vertex y of Q, we want to construct the Auslander-Reiten sequence in $\mathcal{L}$ which ends in the corresponding indecomposable projective kQ-module $P_0(y)$. We call these sequences the *connecting sequences.* We will have to distinguish whether the vertex y of Q is a source or not.

There is the corresponding indecomposable projective Λ-module $P(y) = (P_0(y))[\epsilon] = P_0(y) \otimes_k A$. We identify its submodule $P_0(y) \otimes k\epsilon$ with $P_0(y)$ and we have $P(y)/P_0(y) \simeq P_0(y)$. Similarly, we denote by $I_0(y)$ the indecomposable injective kQ-module with socle $S(y)$. The Λ-module $I(y) = (I_0(y))[\epsilon] = I_0(y) \otimes_k A$ is the indecomposable injective Λ-module with socle $S(y)$. Again, we identify $I_0(y) \otimes k\epsilon$ with $I_0(y)$ and we have $I(y)/I_0(y) \simeq I_0(y)$.

Recall that τ denotes the Auslander-Reiten translation in $\operatorname{mod}\Lambda$ and $\tau_{\mathcal{L}}$ the Auslander-Reiten translation in $\mathcal{L}$.

5.1. Lemma. *For any vertex y of Q, we have $\tau P_0(y) = I_0(y)$ and $\tau_{\mathcal{L}} P_0(y) = \eta I_0(y)$.*

Proof. In order to see that $\tau P_0(y) = I_0(y)$, one notes that $P_0(y)$ is the cokernel of the multiplication map $\epsilon\colon P(y) \to P(y)$, thus $\tau P_0(y)$ has to be the kernel of the multiplication map $\epsilon\colon I(y) \to I(y)$, and this is $I_0(y)$. For any indecomposable module M in $\mathcal{L} \setminus \mathcal{P}$ the Auslander-Reiten translate $\tau_{\mathcal{L}} M$ of M in $\mathcal{L}$ is a non-zero direct summand of $\eta\tau M$. Since $\eta I_0(y)$ is indecomposable, we conclude that $\tau_{\mathcal{L}} P_0(y) = \eta I_0(y)$. □

5.2. Proposition. *Let y be a vertex of Q.*

(a) *If y is a source, then the Auslander-Reiten sequence in $\mathcal{L}$ ending in $P_0(y)$ is of the form*

$$0 \to \eta I_0(y) \to P(y) \oplus \operatorname{rad} P_0(y) \to P_0(y) \to 0.$$

(b) *If y is not a source, then the Auslander-Reiten sequence in $\mathcal{L}$ ending in $P_0(y)$ is of the form*

$$0 \to \eta I_0(y) \to \eta(I_0(y)/S(y)) \oplus \operatorname{rad} P_0(y) \to P_0(y) \to 0.$$

5.3. Proof of (a). Let us construct a non-split exact sequence with end terms $\eta I_0(y)$ and $P_0(y)$.

We assume that y is a source, thus $I_0(y) = S(y)$. We denote by $\iota\colon \operatorname{rad} P_0(y) \to P_0(y)$ the inclusion map and by $\epsilon'\colon P(y) \to P_0(y)$ the surjective map induced by the multiplication with ϵ. It is easy to see that the kernel K of the map $[\,\epsilon' \quad \iota\,]\colon P(y) \oplus \operatorname{rad} P_0(y) \to P_0(y)$ is isomorphic to $\gamma(\iota)$, thus to $\eta S(y) = \eta I_0(y)$, according to Proposition 4.11. Of course, a sequence of the form

$$0 \to \eta S(y) \to P(y) \oplus \operatorname{rad} P_0(y) \to P_0(y) \to 0$$

does not split, since $P_0(y)$ is not a direct summand of the middle term $P(y) \oplus \operatorname{rad} P_0(y)$ (the module $P(y)$ is indecomposable and not annihilated by ϵ, the indecomposable direct summands N of $\operatorname{rad} P_0(y)$ satisfy $N_y = 0$).

Let $0 \to \eta S(y) \to Y \to P_0(y) \to 0$ be an Auslander-Reiten sequence. Since the sequence displayed above does not split, we obtain a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \eta S(y) & \longrightarrow & Y & \longrightarrow & P_0(y) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} & & \\ 0 & \longrightarrow & \eta S(y) & \longrightarrow & P(y) \oplus \operatorname{rad} P_0(y) & \longrightarrow & P_0(y) & \longrightarrow & 0 \end{array} .$$

Of course, the map ϕ' cannot be zero. But the endomorphism ring of $P_0(k)$ is equal to k, thus ϕ' is invertible and therefore also the lower sequence is an Auslander-Reiten sequence. This completes the proof. □

Remark. One also may show directly that the map $[\,\epsilon' \quad \iota\,] : P(y) \oplus \operatorname{rad} P_0(y) \to P_0(y)$ is minimal right almost split in $\mathcal{L}$. Here is the proof.

Let L be an indecomposable module in $\mathcal{L}$ and consider a map $f \colon L \to P_0(y)$. Now either f maps into the radical $\operatorname{rad} P_0(y)$ of $P_0(y)$ and we have a lifting. Otherwise, f maps onto $P_0(y)$. But then there is a map $f' \colon P(y) \to L$ such that $ff' = \epsilon'$. If the image of f' is annihilated by ϵ, then f' factors through ϵ', say $f' = f''\epsilon'$ and then $ff''\epsilon' = ff' = \epsilon'$ and therefore $ff'' = 1$, thus f is a split epimorphism. Otherwise we use an embedding of L into a projective Λ-module in order to see that f' has to be an isomorphism (here we use that y is a source). But then $f = \epsilon'(f')^{-1}$ is the required factorization. This shows that the map $[\,\epsilon' \quad \iota\,]$ is right almost split in $\mathcal{L}$.

In order to show that the map is minimal, one notes that $P(y)$ cannot be deleted, since the map ϵ' is surjective, whereas the map ι is not surjective. Also, if N is an indecomposable direct summand of $\operatorname{rad} P_0(y)$, say $\operatorname{rad} P_0(y) = N \oplus N'$, then the inclusion map $N \to P_0(y)$ cannot be factored through the restriction of $[\,\epsilon' \quad \iota\,]$ to $P(y) \oplus N'$, since the composition of any map $N \to P(y)$ with ϵ' is zero. □

For any vertex y of Q, we consider the kQ-module $Y = Y(y) = I_0(y)/S(y)$. Note that Y is a direct sum of indecomposable injective modules of the form $I_0(x)$, where x is a vertex of Q with an arrow $y \leftarrow x$.

5.4. Lemma. $\operatorname{Ext}^1_\Lambda(\eta Y, \operatorname{rad} P_0(y)) = 0$.

Proof. According to 4.9, there is an exact sequence

$$0 \to (\Omega_0 Y)[\epsilon] \to \eta Y \to Y \to 0,$$

which yields an exact sequence

$$\operatorname{Ext}^1_\Lambda(Y, \operatorname{rad} P_0(y)) \to \operatorname{Ext}^1_\Lambda(\eta Y, \operatorname{rad} P_0(y)) \to \operatorname{Ext}^1_\Lambda((\Omega_0 Y)[\epsilon], \operatorname{rad} P_0(y)).$$

The Λ-module $(\Omega_0 Y)[\epsilon]$ is projective, thus it is sufficient to show that $\operatorname{Ext}^1_\Lambda(Y, \operatorname{rad} P_0(y)) = 0$. Since Y is the direct sum of indecomposable injective kQ-modules of the form $I_0(x)$, with an arrow $y \leftarrow x$ in Q, the elements in the support of Y are proper predecessors of y. On the other hand, the elements in the support of $\operatorname{rad} P_0(y)$ are proper successors of y. thus the support of Y and of $\operatorname{rad} P_0(y)$ are separated by the vertex y. □

5.5. *Let y be a vertex of Q which is not a source. There is an exact sequence*

$$0 \to \eta(I_0(y)) \to \eta Y \to S(y) \to 0.$$

which does not split.

Proof. Let

$$0 \to \Omega_0(I_0(y)) \to P_0(I_0(y)) \to I_0(y) \to 0$$

be a projective presentation of $I_0(y)$ as a kQ-module. Since we assume that y is not a source, the projective cover $P_0(I_0(y))$ of $I_0(y)$ is also the projective cover of $Y = I_0(y)/S(y)$. Thus, there is a commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_0(I_0(y)) & \longrightarrow & P_0(I_0(y)) & \longrightarrow & I_0(y) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle p'} & & \Big\| & & \Big\downarrow{\scriptstyle p} & & \\ 0 & \longrightarrow & \Omega_0(Y) & \longrightarrow & P_0(I_0(y)) & \longrightarrow & Y & \longrightarrow & 0 \end{array}$$

and the snake lemma shows that p' is injective and that its cokernel is isomorphic to the kernel of p, thus to $S(y)$. In the category $\operatorname{mod}\Lambda$, we obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Omega_0(I_0(y))[\epsilon] & \longrightarrow & \eta(I_0(y)) & \longrightarrow & I_0(y) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle p'[\epsilon]} & & \Big\downarrow{\scriptstyle u} & & \Big\downarrow{\scriptstyle p} & & \\ 0 & \longrightarrow & \Omega_0(Y)[\epsilon] & \longrightarrow & \eta Y & \longrightarrow & Y & \longrightarrow & 0. \end{array}$$

The snake lemma yields the following exact sequence

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & S(y) & \longrightarrow & S(y)[\epsilon] & \longrightarrow & \operatorname{Cok} u & \longrightarrow & 0 \\ {\scriptstyle \operatorname{Ker} p'[\epsilon]} & & {\scriptstyle \operatorname{Ker} u} & & {\scriptstyle \operatorname{Ker} p} & & {\scriptstyle \operatorname{Cok} p'[\epsilon]} & & {\scriptstyle \operatorname{Cok} u} & & {\scriptstyle \operatorname{Cok} p} \end{array}$$

which shows that $\operatorname{Cok} u$ is equal to S.

It remains to be seen that the exact sequence

$$0 \to \eta(I_0(y)) \xrightarrow{u} \eta Y \to S(y) \to 0$$

does not split. Now $Y = I_0(y)/S(y)$ is the direct sum of indecomposable injective modules of the form $I_0(x)$, where x is a vertex of Q with an arrow $y \leftarrow x$. Thus ηY is the direct sum of Λ-modules of the form $\eta(I_0(x))$, and $H\eta(I_0(x)) = I_0(x)$. In particular, y is not in the support of $I_0(x)$. On the other hand, $H(S(y)) = S(y)$ has support $\{y\}$. This shows that $S(y)$ is not a direct summand of ηY. □

5.6. Proof of (b). We fix a vertex y of Q which is not a source and consider an Auslander-Reiten sequence

$$0 \to \eta(I_0(y)) \xrightarrow{f} M \xrightarrow{g} P_0(y) \to 0.$$

We consider in addition the exact sequence

$$0 \to \eta(I_0(y)) \xrightarrow{u} \eta Y \xrightarrow{q} S(y) \to 0$$

exhibited in 5.4. Since u is not split mono, we can factor u through f, thus we obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \eta(I_0(y)) & \xrightarrow{f} & M & \xrightarrow{g} & P_0(y) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u''} & & \\ 0 & \longrightarrow & \eta(I_0(y)) & \xrightarrow{u} & \eta Y & \xrightarrow{q} & S(y) & \longrightarrow & 0 \end{array}$$

We claim that $u'' \neq 0$. Namely, if $u'' = 0$, then $qu' = 0$ implies that u' factors through u, say $u' = ur$ for some $r\colon M \to \eta(I_0(y))$. But then $u = u'f = urf$ implies that $rf = 1$. However f is not split mono, a contradiction.

The non-zero map $u''\colon P_0(y) \to S(y)$ has to be surjective and its kernel has to be $\operatorname{rad} P_0(y)$. Of course, since u'' is surjective, also u' has to be surjective. The snake lemma asserts that the kernel of u' has to be isomorphic to the kernel of u'', thus isomorphic to $\operatorname{rad} P_0(y)$. It follows that there is an exact sequence

$$0 \to \operatorname{rad} P_0(y) \xrightarrow{u} M \xrightarrow{u'} \eta Y \to 0.$$

According to 5.4, we know that this sequence splits. Thus M is isomorphic to $\eta Y \oplus \operatorname{rad} P_0(y)$. This completes the proof of (b). □

6. The ghost maps.

A *ghost map* is by definition a homomorphism f such that $H(f) = 0$. Let us start with a characterization of the ghost maps between indecomposable Λ-modules in $\mathcal{L}$.

6.1. Proposition. *Let X, Y be indecomposable Λ-modules in $\mathcal{L}$ and let $f\colon X \to Y$ be a homomorphism. Let X' be the kernel of the multiplication map $\epsilon\colon X \to X$, let Y_P be a projective submodule of Y such that $\epsilon Y \subseteq Y_P$. Then $H(f) = 0$ if and only if f can be written as the sum of two homomorphisms $f_0, f_1\colon X \to Y$ such that f_1 vanishes on X', whereas the image of f_0 is contained in Y_P.*

Proof. First let us show that maps of the form f_0 and f_1 as mentioned in the assertion belong to the kernel of H. Now $H(f_1)$ is induced by the restriction of f_1 to X', thus if this restriction is zero, then $H(f_1) = 0$. And, if the image of f_0 is contained in Y_P, then f_0 factors through Y_P, but Y_P is a projective Λ-module, thus $H(Y_P) = 0$ and therefore $H(f_0) = 0$.

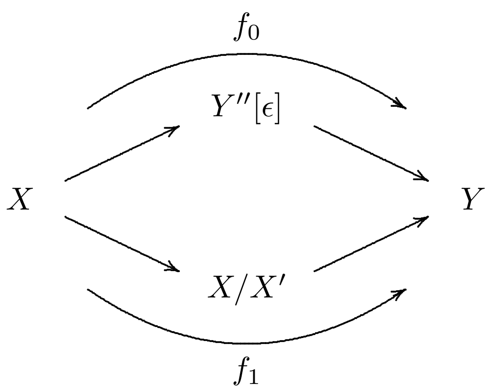
Conversely, let $f\colon X \to Y$ be a homomorphism such that $H(f) = 0$. Let $X'' = \epsilon X$, $Y'' = \epsilon Y$, and let Y' be the kernel of the multiplication map $\epsilon\colon Y \to Y$. We can write X and Y as pushouts according to the following diagrams (where the maps u', u'', v', v'' are the canonical inclusion maps).

$$\begin{array}{ccc} X'' & \xrightarrow{u'} & X' \\ {\scriptstyle u''}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ X''[\epsilon] & \xrightarrow[u''']{} & X \end{array} \qquad \text{and} \qquad \begin{array}{ccc} Y'' & \xrightarrow{v'} & Y' \\ {\scriptstyle v''}\big\downarrow & & \big\downarrow{\scriptstyle v} \\ Y''[\epsilon] & \xrightarrow[v''']{} & Y \end{array}$$

Note that we may identify $Y''[\epsilon]$ with the submodule Y_P of Y.

Now $H(f) = 0$ means that $f(X') \subseteq \epsilon Y$. We denote by $f'\colon X' \to Y''$ the restriction of f to X', it satisfies $vv'f' = fu$. Also, let $f''\colon X'' \to Y''$ be the restriction of f to X'', thus $f'' = f'u'$. We can extend the map $f''\colon X'' \to Y''$ to a map $f''[\epsilon]\colon X''[\epsilon] \to Y''[\epsilon]$ such that $v''f'' = f''[\epsilon]u''$. Since X is the pushout of the maps u' and u'', and $v''f'u' = v''f'' = f''[\epsilon]u''$, we obtain a map $h\colon X \to Y''[\epsilon]$ such that $hu = v''f'$ and $hu''' = f''[\epsilon]$. Let $f_0 = v'''h$. Then $(f - f_0)u = fu - v'''hu = vv'f' - v'''v''f' = (vv' - v'''v'')f' = 0$. This shows that $f_1 = f - f_0$ vanishes on X'. Altogether we see that $f = f_0 + f_1$, that f_1 vanishes on X_1 and that the image of f_0 is contained in the image of h, thus in Y_P. □

If we look at the two maps f_0 and f_1,



we see that they are of completely different nature: Namely, f_0 factors through a projective Λ-module (namely $Y_P = Y''[\epsilon]$), thus through an object which vanishes under H, whereas f_1 is a factorization inside the stable category $\underline{\mathcal{L}}$. The following special case is of interest:

6.2. Lemma. *Let M be indecomposable in $\mathcal{L} \setminus \mathcal{P}$ and let M' be the kernel of the multiplication map ϵ. Let Y be a projective kQ-module. Then, any homomorphism $f\colon M \to Y$ vanishes on M', thus is a ghost map.*

Proof. Let $M'' = \epsilon M$. Since Y is a kQ-module, we see that $M'' \subseteq \operatorname{Ker}(f)$. Using the Noether isomorphism, we obtain an embedding

$$M'/(M' \cap \operatorname{Ker}(f)) \to (M' + \operatorname{Ker}(f))/\operatorname{Ker}(f) \subseteq Y.$$

Now Y is a projective kQ-module, thus also $M'/(M' \cap \mathrm{Ker}(f))$ is a projective kQ-module. It is a factor module of $M'/M'' = H(M)$, thus a direct summand of $H(M)$. But M is indecomposable and in $\mathcal{L} \setminus \mathcal{P}$, thus we know that $H(M)$ has no non-zero projective direct summands. This shows that $M'/(M' \cap \mathrm{Ker}(f) = 0$, thus $M' \subseteq \mathrm{Ker}(f)$. □

6.3. Given a vertex y of Q which is not a source, we have exhibited in 5.3 an Auslander-Reiten sequence of $\mathcal{L}$ ending in $P(y)$. Let us denote by $u(y)\colon \mathrm{rad}\, P_0(y) \to P_0(y)$ the inclusion map and choose some map $c(y)\colon \eta(I_0(y)/S(y)) \to P_0(y)$ such that

$$[c(y), u(y)]\colon \eta(I_0(y)/S(y)) \oplus \mathrm{rad}\, P_0(y) \to P_0(y)$$

is a minimal right almost split map. If the vertex y is a source, then $I_0(y) = S(y)$, thus $I_0(y)/S(y) = 0$, and we may denote by $c(y)$ the zero map $0 = \eta(I_0(y)/S(y)) \to P_0(y)$.

Then we have:

Theorem 3. *The ideal of ghost maps in $\mathcal{L}$ is generated by the identity maps of the indecomposable projective Λ-modules as well as the maps $c(y)$ for the vertices y of Q.*

Proof. Let $\mathcal{I}$ be the ideal in $\mathcal{L}$ generated by the the identity maps of the indecomposable projective Λ-modules as well as the maps $c(y)$ with $y \in Q_0$. Of course, all the maps in $\mathcal{I}$ are ghost maps.

(a) First, let us show that *all the maps $M \to Y$, where M is indecomposable in $\mathcal{L} \setminus \mathcal{P}$ and not a projective kQ-module, whereas Y is a projective kQ-module, belong to $\mathcal{I}$.* For the proof, we can assume that Y is also indecomposable, say $Y = P_0(y)$ for some vertex y. We use induction on $l(y)$, where $l(y)$ is the maximal of a path in Q starting in y. If $(y) = 0$, then y is a sink, and therefore $\mathrm{rad}\, P_0(y) = 0$. According to 5.3, the Auslander-Reiten sequence in $\mathcal{L}$ ending in y shows that the minimal right almost split map for $P_0(y)$ is of the form

$$c(y)\colon \eta(I_0(y)/S(y)) \to P_0(y).$$

Since we assume that M is not projective, we can factor f through $c(y)$, but $c(y)$ belongs to $\mathcal{I}$, therefore f belongs to $\mathcal{I}$.

Next, assume that $l(y) > 0$. If y is not a source, then the right almost split map ending in $P_0(y)$ is of the form

$$g = [c(y), u(y)]\colon \eta(I_0(y)/soc) \oplus \mathrm{rad}\, P_0(y) \to P_0(y),$$

again according to 5.3. We factor f through g, thus we can write f as a sum of maps, where one factors through the map $c(y)$, thus belongs to $\mathcal{I}$, whereas each of the other maps factor through an indecomposable direct summand of $\mathrm{rad}\, P_0(y)$. But all the indecomposable direct summands of $\mathrm{rad}\, P_0(y)$ are of the form $P_0(x)$ with $l(x) < l(y)$, thus by induction we know already that the maps $M \to P_0(x)$ belong to $\mathcal{I}$, and therefore $f \in \mathcal{I}$.

It remains to look at the case where y is a source, so that the right almost split map ending in $P_0(y)$ is of the form

$$g\colon P(y) \oplus \mathrm{rad}\, P_0(y) \to P_0(y),$$

now using 5.2. Again, we factor f through g, thus we can write f as a sum of maps, where one map factors through the projective module $P(y)$, and therefore belongs to $\mathcal{I}$, whereas the other maps factor through a direct summand $P_0(x)$ of $\operatorname{rad} P_0(y)$. Again, we must have $l(x) < l(y)$, thus by induction all the maps $M \to P_0(x)$ belong to $\mathcal{I}$. This shows again that f belongs to $\mathcal{I}$.

(b) Now let us consider arbitrary modules X, Y in $\mathcal{L}$ and let $f\colon X \to Y$ be a ghost map. We want to show that f belongs to $\mathcal{I}$. We can assume that both modules X and Y are indecomposable. Also we can assume that none of the modules X, Y belongs to $\mathcal{P}$.

Let us exclude the case that X is a projective kQ-module. In that case $H(f) = 0$ means that $f(X) \subseteq \epsilon Y$, but then we write f as the composition of the following three maps

$$X \to X[\epsilon] \xrightarrow{f[\epsilon]} (\epsilon Y)[\epsilon] \to Y$$

where the last map is some embedding (we know that such an embedding exists). This shows that f factors through a projective Λ-module.

(c) Thus it remains to consider the following setting: There is given a ghost map $f\colon X \to Y$, where X, Y are indecomposable modules in $\mathcal{L} \setminus \mathcal{P}$, and X is not a projective kQ-module. Let X' be the kernel of the multiplication map $\epsilon\colon X \to X$. According to 6.1, we can write $f = f_0 + f_1$ where f_0 factors through a projective Λ-module and where f_1 vanishes on X'. Now f_0 belongs to $\mathcal{I}$, thus it remains to be seen that f_1 is in $\mathcal{I}$. Since f_1 vanishes on X', we can factor f_1 as

$$X \to X/X' \to Y.$$

Now X is indecomposable and not a projective kQ-module, whereas X/X' is a projective kQ-module, thus we have seen in (a) that the map $X \to X/X'$ belongs to $\mathcal{I}$. This shows that also f_1 belongs to $\mathcal{I}$ and thus f is in $\mathcal{I}$. □

6.4. Corollary. *The ideal I in $\underline{\mathcal{L}}$ of all ghost maps is a finitely generated ideal with $I^2 = 0$.*

Proof: According to Theorem 3, the ideal I is generated by the residue classes classes of the maps $c(y)$, with $y \in Q_0$, thus it is finitely generated. It remains to be seen that $I^2 = 0$. Thus, let y, y' be vertices and consider a composition of maps

$$\eta(I_0(y)/S(y)) \to P_0(y) \xrightarrow{g} \eta(I_0(y')/S(y')) \xrightarrow{f} P_0(y').$$

Let $X = \eta(I_0(y')/S(y'))$ and X' the kernel of the multiplication map $\epsilon\colon X \to X$. Since ϵ vanishes on $P_0(y)$, it also vanishes on the image of g, thus the image of g is contained in X'. On the other hand, according to Lemma 6.2, f vanishes on X', thus $fg = 0$. □

6.5. Finally, let us describe the maps $c(y)$ in terms of the arrows of the quiver.

If $\alpha\colon x \to y$ is an arrow of the quiver Q, let $s(\alpha) = x, t(\alpha) = y$. Given a vertex y, we may decompose $\eta(I_0(y)/S(y))$ as the direct sum of the modules $I_0(s(\alpha))$, where α runs through the arrows with $t(\alpha) = y$. Thus the map $c(y)$ considered in 6.4 is given by maps $I_0(s(\alpha)) \to P_0(y)$, for the various arrows $\alpha\colon s(\alpha) \to y$. Let us construct such maps explicitly.

Thus, for every arrow $\alpha\colon i \to j$ in Q, we want to construct a map

$$c(\alpha)\colon \eta I_0(i) \to P_0(j)$$

which is irreducible in $\mathcal{L}$.

In order to define $\eta I_0(i)$, we start with a minimal projective kQ-presentation

$$0 \to \Omega_0 I_0(i) \to P_0 I_0(i) \to I_0(i) \to 0,$$

this yields a canonical map $\eta I_0(i) \to \Omega_0 I_0(i)$. Now consider any arrow $\alpha\colon i \to j$ in Q. There is up to isomorphism an

$$0 \to P_0(j) \to N(\alpha) \to I_0(i) \to 0.$$

(Since there is an arrow $i \to j$ and Q is acyclic, the supports of $P_0(j)$ and $I_0(j)$ do not intersect and $P_0(j)_j = k$, $I_0(i)_i = k$. Thus we define $N(\alpha)$ by using for $N(\alpha)_\alpha$ the identity map $k \to k$.)

Since $P_0 I_0(i)$ is projective, and $N(\alpha) \to I_0(i)$ is an epimorphism, we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_0 I_0(i) & \longrightarrow & P_0 I_0(i) & \longrightarrow & I_0(i) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle c'(\alpha)} & & \downarrow & & \| & & \\
0 & \longrightarrow & P_0(j) & \longrightarrow & N(\alpha) & \longrightarrow & I_0(i) & \longrightarrow & 0
\end{array}$$

Note that the map $c'(\alpha)$ has to be surjective (namely, consider the induced exact sequence using the projection map $\pi\colon P_0(j) \to S(j)$; by the construction of $N(\alpha)$, this sequence does not split, thus the composition of $c'(\alpha)$ and π cannot be the zero map).

The required map $c(\alpha)\colon \eta I_0(i) \to P_0(j)$ is the composition of the canonical map $\eta I_0(i) \to \Omega_0 I_0(i)$ and $c'(\alpha)$:

$$\eta I_0(i) \to \Omega_0 I_0(i) \xrightarrow{c'(\alpha)} P_0(j).$$

One may also use a combined construction in order to deal with the various arrows β starting in y at the same time, or also to deal with the various arrows α ending in y. Let $N(y)$ be obtained by identifying the modules $I_0(y)$ and $P_0(y)$ at the vertex y, thus there are the following two exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & P_0(y) & \longrightarrow & N(y) & \longrightarrow & I_0(y)/S(y) & \longrightarrow & 0, \\
0 & \longrightarrow & \operatorname{rad} P_0(y) & \longrightarrow & N(y) & \longrightarrow & I_0(y) & \longrightarrow & 0.
\end{array}$$

Of course, $I_0(y)/S(y)$ is the direct sum of the modules $I_0(s(\alpha))$ where α is an arrow with $t(\alpha) = y$, whereas $\operatorname{rad} P_0(y)$ is the direct sum of the modules $P_0(t(\beta))$ where β is an arrow with $s(\beta) = y$.

As above, we take a minimal projective resolution of $I_0(y)/S(y)$ or of $I_0(y)$ and obtain maps $c'(y)\colon \Omega_0(I_0(y)/S(y)) \to P_0(y)$ and $d'(y)\colon \Omega_0 I_0(y) \to \operatorname{rad} P_0(y)$ as follows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_0(I_0(y)/S(y)) & \longrightarrow & P_0(I_0(y)/S(y)) & \longrightarrow & I_0(y)/S(y) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle c'(y)} & & \Big\downarrow & & \Big\| & & \\
0 & \longrightarrow & P_0(y) & \longrightarrow & N(y) & \longrightarrow & I_0(y)/S(y) & \longrightarrow & 0
\end{array}$$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_0 I_0(y) & \longrightarrow & P_0 I_0(y) & \longrightarrow & I_0(y) & \longrightarrow & 0 \\
 & & \Big\downarrow{\scriptstyle d'(y)} & & \Big\downarrow & & \Big\| & & \\
0 & \longrightarrow & \operatorname{rad} P_0(y) & \longrightarrow & N(y) & \longrightarrow & I_0(y) & \longrightarrow & 0
\end{array}$$

The composition of $c'(y)$ with the canonical map $\eta(I_0(y)/S(y)) \to \Omega_0(I_0(y)/S(y))$ yields a map $c(y)\colon \eta(I_0(y)/S(y)) \to P_0(y)$. Similarly, we compose $d'(y)$ with the canonical map $\eta I_0(y) \to \Omega_0 I_0(y)$ and obtain $d(y)\colon \eta I_0(y) \to \operatorname{rad} P_0(y)$. These maps $c(y)$ and $d(y)$ can be used in the Auslander-Reiten sequences exhibited in 5.2 and 5.3.

7. The position of the indecomposable projective modules.

7.1. Proposition. *Let $P(x)$ be the indecomposable projective Λ-module corresponding to the vertex x. Then $H(\operatorname{rad} P(x)) = S(x)$.*

Proof. We write $P(x) = P_0(x)[\epsilon]$, thus there is an exact sequence of the following form

$$0 \to (\operatorname{rad} P_0(x))[\epsilon] \to P_0(x)[\epsilon] \to S(x)[\epsilon] \to 0.$$

This implies that we obtain for the radical of $P(x)$ an exact sequence of he form

$$0 \to (\operatorname{rad} P_0(x))[\epsilon] \to \operatorname{rad} P(x) \to S(x) \to 0,$$

therefore $H(\operatorname{rad} P(x)) = S(x)$. $\square$

Thus, the Auslander-Reiten sequence with $P(x)$ as a direct summand of the middle term starts with $\eta S(x)$. We distinguish whether x is a source or not.

7.2 If x is a source, then $I_0(x) = S(x)$, thus we deal with the Auslander-Reiten sequence exhibited in Lemma 5.2:

$$\begin{array}{ccccc}
 & & P(x) & & \\
 & \nearrow & & \searrow & \\
\eta S(x) & & & & P_0(x) \\
 & \searrow & & \nearrow & \\
 & & \operatorname{rad} P_0(x) & &
\end{array}$$

What is of interest and should be remembered is the fact that in this case the irreducible map starting in $P(x)$ is surjective with target $P_0(x)$.

It remains to consider the case that x is not a source. Let us denote by τ_0 the Auslander-Reiten translation for $\operatorname{mod} kQ$. Since x is not a source, $S(x)$ is not an injective kQ-module, thus $\tau_0^{-1}S(x)$ is defined.

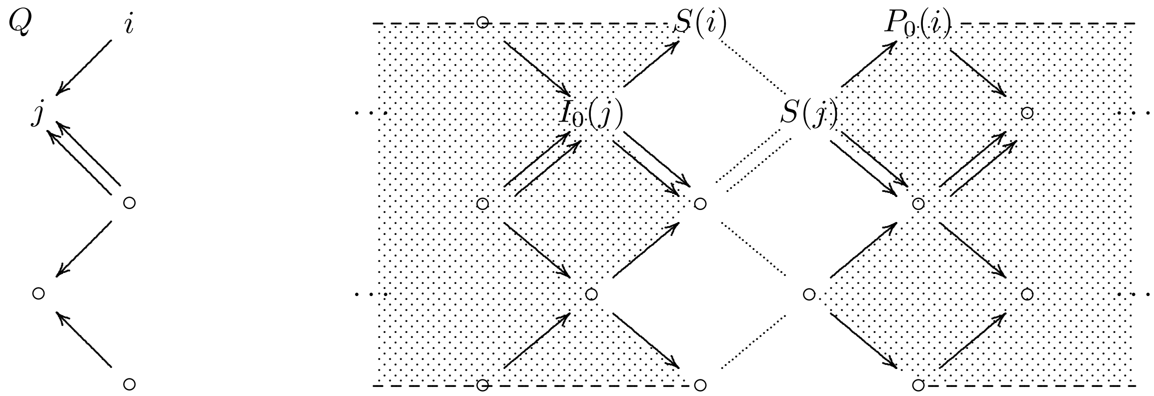
7.3. *If x is not a source, then the Auslander-Reiten sequence with $P(x)$ as a direct summand of the middle term starts with $\eta S(x)$ and ends in $\eta\tau_0^{-1}S(x)$.* □

8. Examples

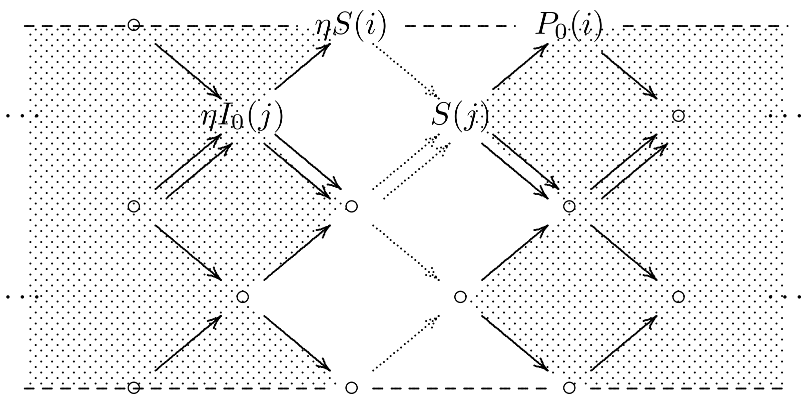
We include some pictures in order to illustrate in which way the Auslander-Reiten quiver $\Gamma(\mathcal{L})$ of $\mathcal{L}$ is obtained from the Auslander-Reiten quiver $\Gamma(\operatorname{mod} kQ)$ of $\operatorname{mod} kQ$. We use dotted arrows for arrows which correspond to ghost maps. Sometimes we will add dashed lines in order to accentuate the Auslander-Reiten translation.

8.1. First example. As a first example, we take a bipartite quiver Q (this means that all the vertices are sinks or sources). If $\alpha\colon i \to j$ is an arrow in Q, then $I_0(i) = S(i)$ and $P(j) = S(j)$ are simple kQ-modules.

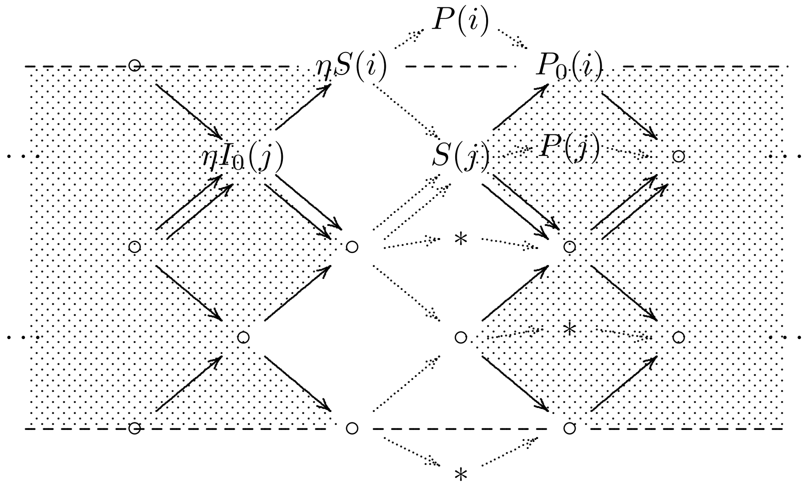
We consider the following quiver Q exhibited below on the left. On the right side, we present the decisive parts of the preinjective component and the preprojective component of $\Gamma(kQ)$, already as subquivers of the translation quiver $\mathbb{Z}(Q^{\mathrm{op}})$ separated only by some arrows (they are marked by dotted diagonal lines):



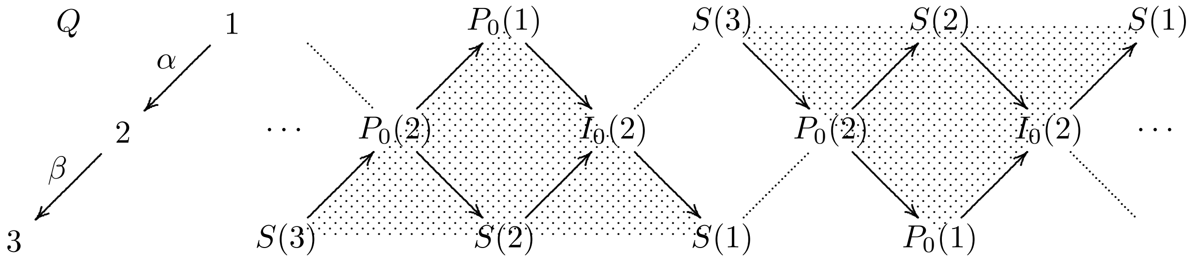
If we consider now $\underline{\mathcal{L}}$, we have to replace any kQ-module N by ηN, and we have to add an arrow $\eta S(i) \to S(j)$ for any arrow $i \to j$ in the quiver Q. These new arrows represent a k-basis of $\operatorname{Hom}(\eta S(i), S(j))$. Note that the new arrows represent ghost maps. Here is this part of $\Gamma(\underline{\mathcal{L}})$.



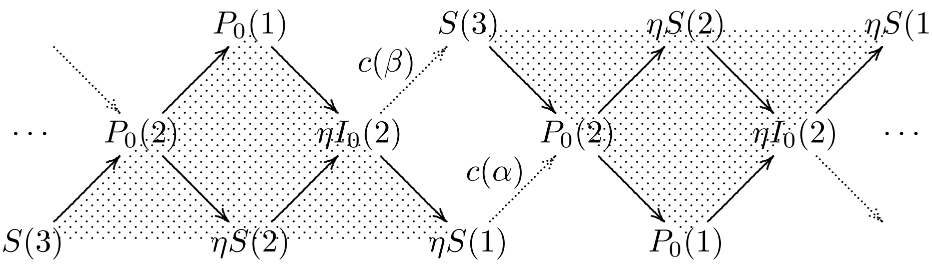
Finally, let us show the corresponding part of $\Gamma(\mathcal{L})$. Here, the indecomposable projective Λ-modules are added.



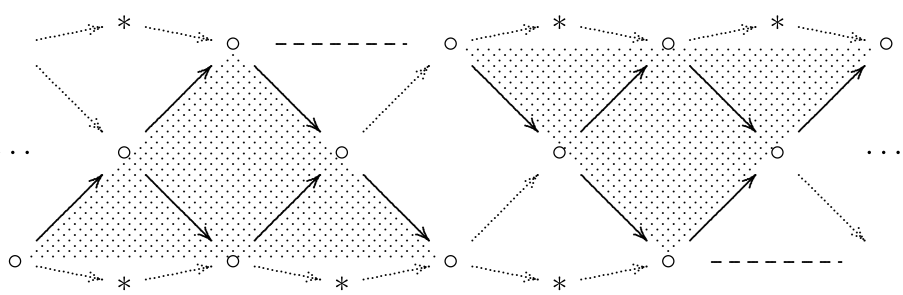
8.2. Second example. As a second example we take the quiver Q of type $\mathbb{A}_3$ with linear orientation. First, let us show two copies of $\Gamma(\operatorname{mod} kQ)$ appropriately embedded into the translation quiver $\mathbb{Z}\mathbb{A}_3$.



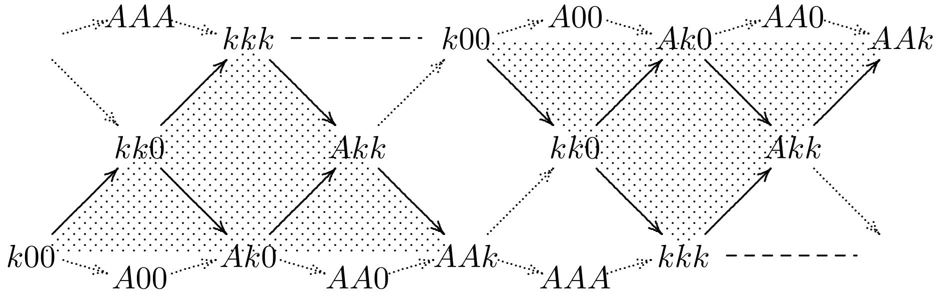
Next, we present the corresponding torsionless Λ-modules and insert the ghost arrows $c(\alpha)\colon \eta I_0(1) \to P(2)$ and $c(\beta)\colon \eta I_0(2) \to S(3)$.



The Auslander-Reiten quiver of $\mathcal{L}$ (or better its universal covering) looks as follows:



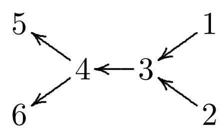
It may be helpful for the reader to use this example in order to write down all the Λ-modules as representations of Q over A (for example, the module $M = \eta Q(1) = \eta S(1)$ is written as AAk, since $M_3 = M_2 = A$ and $M_1 = k$).



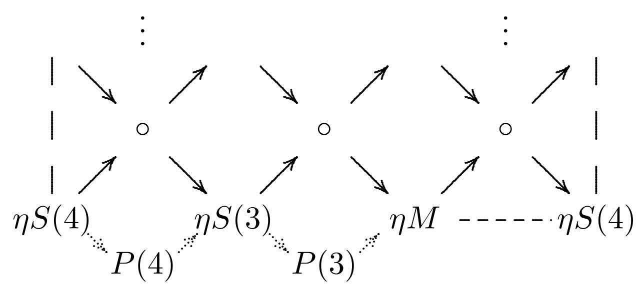
The structure of the Auslander-Reiten components of a connected hereditary artin algebra R are known, see [R]. There is a preprojective and a preinjective component (they coincide in case R is representation-finite), all other components are of the form $\mathbb{Z}\mathbb{A}_\infty$ in case R is wild, otherwise they are of the form $\mathbb{Z}\mathbb{A}_\infty/\tau^t$ for some natural number t.

Now assume that Q is a connected acyclic quiver and $\Lambda = kQ[\epsilon]$. According to section 5, the preinjective and the preprojective component of $\operatorname{mod} kQ$ yield a single component in the Auslander-Reiten quiver of the category $\mathcal{L}$ of the Gorenstein-projective Λ-modules. The aim of the previous examples was to illustrate the shape of such a component. The remaining components of the stable Auslander-Reiten quiver of $\mathcal{L}$ are of the form $\mathbb{Z}\mathbb{A}_\infty$ or $\mathbb{Z}\mathbb{A}_\infty/\tau^t$ for some natural number t. If we are interested in the Auslander-Reiten quiver of $\mathcal{L}$ itself, we have to take care of the position of the indecomposable projective Λ-modules, thus of the simple kQ-modules. A typical example will be exhibited next.

Third example. We consider the following quiver of type $\widetilde{\mathbb{D}}_5$



The component of $\mathcal{L}$ which contains $P(3)$ and $P(4)$ is a tube of rank 3:



Here, the boundaries on the left and on the right have to be identified, and M is the indecomposable kQ-module with dimension vector $\mathbf{dim}\, M = \begin{smallmatrix} 1 & & & 1 \\ & 1 & 1 & \\ 1 & & & 1 \end{smallmatrix}$.

9. Remarks on the behavior of the homology functor H on $\operatorname{mod}\Lambda$.

We have seen that the homology functor has nice properties when restricted to the subcategory $\mathcal{L} \setminus \mathcal{P}$. For example, it maps indecomposable modules to indecomposables, and non-isomorphic indecomposable modules to non-isomorphic ones. Also, for a fixed homology dimension vector $\mathbf{r}$, there is either one indecomposable object in $\mathcal{L} \setminus \mathcal{P}$ with $\mathbf{dim}\, H(M) = \mathbf{r}$ or at least a 1-parameter family. But all these assertions are only true for the restriction of H to $\mathcal{L} \setminus \mathcal{P}$. In general, one cannot expect such a pleasant behavior.

9.1. Example. Consider the quiver Q of type $\mathbb{A}_3$ with two sources and a sink. Then the AQ-module $M = k \to A \leftarrow k$ is indecomposable, but $H(M) = k \to 0 \leftarrow k$ is decomposable.

9.2. Example. Let Q be the Kronecker quiver with two arrows from 2 to 1. Then there is a $\mathbb{P}_1$-family of indecomposable AQ-modules M with $M_2 = k$ and $M_1 = A$. For all of them $H(M)$ is the simple module $S(2)$. Thus here we have many non-isomorphic AG-modules with isomorphic homology modules.

9.3. Example. Let Q be the quiver of type $\mathbb{D}_4$ with subspace orientation, let $1, 2, 3$ be the sources of Q and 0 the sink. Then there are precisely 2 isomorphism classes of indecomposable AQ-modules M with $H(M) = S(1)\oplus S(2)\oplus S(3)$, for one of them $M_0 = A$, for the other one, $M_0 = A^2$. (For the proof, one has to observe that in this case Λ is a tubular algebra, and one has to study the corresponding root system in detail.)

10. Generalization.

Instead of looking at the path algebra kQ of a quiver, one may start with an arbitrary finite-dimensional k-algebra H which is hereditary and take instead of $\Lambda = kQ[\epsilon]$ the corresponding algebra $H[\epsilon]$. Observe that the proofs of the theorems given here work in general. What is special in the quiver case is just the possibility to describe the maps $c(y)$ in terms of the arrows of the quiver, see 6.6.

Acknowledgments. The main parts of the paper were written in 2011 and presented in lectures at several universities. Section 2.4 was added in 2013, answering a question raised in corresponding discussions. Some results of the paper have been generalized in the meantime. In particular, let us mention J. Wei [W] who has shown that for any ring, a differential module is Gorenstein projective if and only if its underlying module is Gorenstein projective; and this result is generalized to n-th differential modules in [XYY].

We would like to thank referees and editors of four different journals for encouragement and valuable suggestions.

The collaboration was supported by grants from NSF China 11431010 and 11271251.

References.

[AB] M. Auslander, R. Buchweitz. The homological theory of maximal Cohen-Macaulay approximations. Soc. Math. France. Mem 38 (1989), 5–37.

[AR1] M. Auslander, I. Reiten. Applications of contravariantly finite subcategories. Adv. Math. 86 (1991), 111–152.

[AR2] M. Auslander, I. Reiten. Cohen-Macaulay and Gorenstein Artin algebras. In Representation theory of finite groups and finite-dimensional algebras, Birkhäuser (1991), 221–245.

[AS] M. Auslander, S. Smalø. Almost split sequences in subcategories. J. Algebra 69 (1981), 426–454.

[BM] R. Bautista, R. Martinez. Representations of partially ordered sets and l-Gorenstein Artin Algebras. In *Proceedings, Conference on Ring Theory, Antwerp, 1978,* Dekker, New York (1979). 385???-433.

[BMRRT] A. B. Buan, R. Marsh, M. Reineke, I. Reiten, G. Todorov. Tilting theory and cluster combinatorics. Adv. Math. 204 (2006), 572–618.

[B] R. Buchweitz. Maximal Cohen-Macaulay modules and Tate-cohomology. Ms. (unpublished)

[BM] D. Bennis, N. Mahdou. Strongly Gorenstein projective, injective, and flat modules. J. Pure Appl. Algebra 210 (2007), 437–445.

[BR] M. C. R. Butler, C. M. Ringel. Auslander-Reiten sequences with few middle terms, with applications to string algebras. Comm. Algebra 15 (1987), 145–179.

[C] X. W. Chen. Algebras with radical square zero are either self-injective of CM-free. Proc. Amer. Math. Soc. 140 (2012), 93–98.

[DL] L. Demonet, Y. Liu. Quotients of exact categories by cluster tilting subcategories as module categories J. Pure Appl. Algebra 217 (2013), 2282–2297.

[EJ] E. E. Enochs, O. M. G. Jenda. Relative homological algebra. De Gruyter Expo. Math. 30 (2000).

[G] P. Gabriel. The universal cover of a representation-finite algebra. In: Representations of Algebras, Lecture Notes in Mathematics 903 (1981), 68–105.

[GG] R. Gordon, E. L. Green. Representation theory of graded artin algebras. J. Algebra 76 (1982), 138–152.

[H1] D. Happel. Triangulated categories in the representation theory of finite dimensional algebras, LMS Lecture Notes Vo1.119, Cambridge University Press (1988).

[H2] D. Happel. On Gorenstein algebras. In Representation theory of finite groups and finite-dimensional algebras, Birkhäuser (1991) 389–404.

[Ka] V. Kac. Infinite root systems, representations of graphs and invariant theory, Invent. Math., 56 (1980), 57–92.

[Ke] B. Keller. On triangulated orbit categories. Doc. Math. 10 (2005), 551–581.

[KR] B. Keller, I. Reiten. Cluster-tilted algebras are Gorenstein and stably Calabi-Yau. Adv. Math. 211 (2007), 123–151.

[KZ] S. Koenig und B. Zhu. From triangulated categories to abelian categories - cluster tilting in a general framework. Math. Z. 258 (2008), 143–160.

[LZ] X. H. Luo, P. Zhang. Monic representations and Gorenstein-projective modules. Pacific Journal of Mathematics. 264 (2013), 163–194.

[O] D. Orlov. Triangulated categories of singularities and D-branes in Landau-Ginzburg models. Proc. Steklov Inst. Math. 246 (3) (2004), 227–248.

[R] C. M. Ringel. Finite-dimensional hereditary algebras of wild representation type. Math. Z. 161 (1978), 235–255.

[RX] C. M Ringel, B.-L. Xiong. On rings with radical square zero. Algebra and Discrete Mathematics. 14 (2012), 297–306.

[W] J. Wei. Gorenstein homological theory for differential modules Proceedings of the Royal Society of Edinburgh: Section A Mathematics 145 (2015) 639–655.

[XYY] H. B. Xu, S. L. Yang, H. L. Yao. Gorenstein theory for n-th differential modules. Period. Math. Hungar. 71(1) (2015), 112–124.

C. M. Ringel
Department of Mathematics, Shanghai Jiao Tong University
Shanghai 200240, P. R. China.
e-mail: ringel@math.uni-bielefeld.de

P. Zhang
Department of Mathematics, Shanghai Jiao Tong University
Shanghai 200240, P. R. China, and
e-mail: pzhang@sjtu.edu.cn