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THE INGALLS-THOMAS BIJECTIONS

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Abstract. Given a finite acyclic quiver Q with path algebra Λ, Ingalls and Thomas have exhibited a bijection between the set of Morita equivalence classes of support-tilting modules and the set of thick subcategories of $\operatorname{mod}\Lambda$ with covers, and they have collected further bijections with these sets. We add some additional bijections and show that all these bijections hold for arbitrary hereditary artin algebras. The proofs presented here seem to be of interest also in the special case of the path algebra of a quiver.

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1. Introduction

1.1. Let Λ be a hereditary artin algebra (we recall that an artin algebra Λ is a k-algebra which is of finite length when considered as a k-module, where k is a commutative artinian ring; also, a ring is *hereditary* provided submodules of projective modules are projective). Since this means that the functors $\operatorname{Ext}^i_\Lambda$ vanish for $i \geq 2$, we write $\operatorname{Ext}(M, M')$ instead of $\operatorname{Ext}^1_\Lambda(M, M')$. A typical example of a hereditary artin algebra is the path algebra of a finite acyclic quiver. If k is an algebraically closed field, any hereditary artin k-algebra is Morita-equivalent to the path algebra of a finite acyclic quiver, but otherwise there are many other hereditary artin k-algebras.

We will consider left Λ-modules of finite length and call them just modules. The category of all modules will be denoted by $\operatorname{mod}\Lambda$.

Given a module M, we denote by $\Lambda(M)$ the *support algebra of* M; this is the factor algebra of Λ modulo the ideal generated by all idempotents e with $eM = 0$. The support algebra $\Lambda(M)$ is again a hereditary artin algebra (but usually not connected, even if Λ is connected). If M is a module, the set of simple modules S which occur as composition factors of M will be called the *support* of M. The module M is said to be *sincere* provided any simple module belongs to the support of M (thus provided the only idempotent $e \in \Lambda$ with $eM = 0$ is $e = 0$).

1.2. The subcategories of $\operatorname{mod}\Lambda$ which we will consider are full subcategories which are closed under finite direct sums and direct summands. Given a class $\mathcal{X}$ of modules, we denote by $\operatorname{add}\mathcal{X}$ the class of modules which are direct summands of finite direct sums of modules in $\mathcal{X}$. If $\mathcal{X} = \{X\}$ for a single module X, we write $\operatorname{add} X$ instead of $\operatorname{add}\{X\}$. The modules X, X' are said to be *Morita equivalent* provided $\operatorname{add} X = \operatorname{add} X'$. A module is said to be *multiplicity-free* provided it is a direct sum of pairwise non-isomorphic indecomposable modules. Multiplicity-free modules which are Morita equivalent are actually isomorphic. On the other hand, every module is Morita equivalent to a multiplicity-free module.

1.3. Support-tilting modules. Following earlier considerations of Brenner and Butler, tilting modules have been defined in [9]. We say that a module M has *no self-extensions,* provided $\operatorname{Ext}(M, M) = 0$. In the present setting, a module T without self-extensions is said to be a *tilting module* provided the number of isomorphism classes of indecomposable direct summands of T is equal to the number of isomorphism classes of simple Λ-modules, or, equivalently, provided ${}_{\Lambda}\Lambda$ is the kernel of a surjective map in $\operatorname{add} T$ (or, again equivalently, provided an injective cogenerator of $\operatorname{mod}\Lambda$ is the cokernel of an injective map in $\operatorname{add} T$). A module M is said to be *support-tilting* provided M considered as a $\Lambda(M)$-module is a tilting module.

Here is one of the sets we are interested in: *the set of Morita equivalence classes of support-tilting modules.*

1.4. Thick subcategories with a cover. A subcategory $\mathcal{A}$ of $\operatorname{mod}\Lambda$ is called a *thick* (or *wide*) subcategory provided it is closed under kernels, cokernels and extensions. Note that a thick subcategory is an abelian category, and the inclusion functor $\mathcal{A} \to \operatorname{mod}\Lambda$ is exact.

A module X is said to *generate* a module Y provided Y is a factor module of a direct sum of copies of X. Dually, a module X *cogenerates* a module Y provided Y is a submodule of a direct sum of copies of X (since the modules considered here are of finite length, it is sufficient to look at direct sums of copies of X; for general modules one would have to use products). Given a class $\mathcal{X}$ of modules, let $\mathcal{G}(\mathcal{X})$ be the subcategory of all modules which are generated by modules in $\operatorname{add}\mathcal{X}$, and let $\mathcal{H}(\mathcal{X})$ be the subcategory of all modules which are cogenerated by modules in $\operatorname{add}\mathcal{X}$. If $\mathcal{C}$ is a subcategory and $C \in \mathcal{C}$, then C is said to be a *cover* of $\mathcal{C}$ provided $\mathcal{C} \subseteq \mathcal{G}(C)$, and C is said to be a *cocover* of $\mathcal{C}$ provided $\mathcal{C} \subseteq \mathcal{H}(C)$.

This is the second set of interest: *the set of thick subcategories of* $\operatorname{mod}\Lambda$ *with covers.*

1.5. If Λ is the path algebra of a finite acyclic quiver, Ingalls and Thomas have exhibited a bijection between the set of Morita equivalence classes of support-tilting modules and the set of thick subcategories of $\operatorname{mod}\Lambda$ with covers. The aim of this paper is to provide a proof of the Ingalls-Thomas bijection for arbitrary hereditary artin algebras. Our proof draws attention to three additional sets which are in bijection with the set of Morita equivalence classes of support-tilting modules: the set of isomorphism classes of exceptional antichains in $\operatorname{mod}\Lambda$, as well as the set of isomorphism classes of normal or of conormal modules without self-extensions. Here are the definitions.

1.6. Exceptional antichains. Given an additive category $\mathcal{C}$, a *brick* is $\mathcal{C}$ is an object whose endomorphism ring is a division ring. Bricks A_1, A_2 are said to be *orthogonal,* provided $\operatorname{Hom}(A_1, A_2) = 0 = \operatorname{Hom}(A_2, A_1)$. An *antichain* $A = \{A_1, \ldots, A_t\}$ in $\mathcal{C}$ is a set of pairwise orthogonal bricks (antichains are called discrete subsets in [8] and Hom-free subsets in [10], see also the remark 7.3). Antichains $A = \{A_1, \ldots, A_t\}$ and $A' = \{A'_1, \ldots, A'_{t'}\}$ are said to be *isomorphic,* provided the objects $\bigoplus_i A_i$ and $\bigoplus_j A'_j$ are isomorphic.

Given an antichain $A = \{A_1, \ldots, A_t\}$ in $\operatorname{mod}\Lambda$, its Ext-quiver Q_A has as vertices the elements A_i and there is an arrow $A_i \to A_j$ provided $\operatorname{Ext}(A_i, A_j) \neq 0$ (one may endow this quiver with a valuation, taking into account the size of the Ext-groups, but this is not needed in the main parts of the paper). We say that an antichain A is *exceptional,* provided its Ext-quiver Q_A is acyclic, thus provided we may index the elements of A in such a way that $\operatorname{Ext}(A_i, A_j) = 0$ for all pairs $i \geq j$.

1.7. Normal (or conormal) modules without self-extensions. A module M is said to be *normal* provided given a direct decomposition $M = M' \oplus M''$ such that M' generates M'', we have $M'' = 0$. And M is *conormal* provided given a direct decomposition $M = M' \oplus M''$ such that M' cogenerates M'', we have $M'' = 0$.

There is the following well-known fact (see, for example [19]): *A sincere module without self-extensions is faithful,* thus any module M without self-extensions is a faithful $\Lambda(M)$-module.

1.8. Since its introduction, tilting theory concerns the study of suitable torsion pairs in $\operatorname{mod}\Lambda$. It seems worthwhile to include this aspect in our considerations. Recall that a *torsion class* in $\operatorname{mod}\Lambda$ is a class of modules which is closed under factor modules and extensions. A *torsionfree class* in $\operatorname{mod}\Lambda$ is a class of modules which is closed under submodules and extensions.

It was the decisive idea of Ingalls and Thomas [11] to relate the support-tilting modules to thick subcategories and to exhibit in this way a number of bijections. They were dealing with path algebras of finite acyclic quivers, here we consider the case of an arbitrary hereditary artin algebra.

Theorem 1.1. *Let Λ be a hereditary artin algebra. There are bijections between the following data:*

- (1) *Isomorphism classes of exceptional antichains.*
- (2) *Thick subcategories with a cover.*
- (3) *Isomorphism classes of normal modules without self-extensions.*

- (4) *Morita equivalence classes of support-tilting modules.*
- (5) *Torsion classes with a cover.*

If Λ is in addition representation-finite, then

- ($1'$) *All antichains are exceptional.*
- ($2'$) *All thick subcategories have a cover.*
- ($5'$) *All torsion classes have a cover.*

We have separated the five sets in Theorem 1.1 into two groups, since there is a great affinity between (1), (2) and (3) on the one hand, and (4) and (5) on the other hand. The essential bijection concerns the sets (2) and (4). As we have mentioned, such a bijection was exhibited by Ingalls-Thomas [11] in case Λ is the path algebra of a finite acyclic quiver. A bijection between (4) and (5) has been known for a long time. A bijection between (1) and (2) was exhibited already in 1976, see [18]. For a bijection between (1) and (3), one may refer to [7], as we will see below.

Whereas the sets of the form (1), (2) and (4) are preserved under duality, this is not the case for the sets (3) and (5), thus, using duality, we obtain bijections with two further sets: the set (6) of *isomorphism classes of conormal modules without self-extensions,* and the set (7) of the *torsionfree classes with a cocover.*

As a supplement to the theorem, we have mentioned that for Λ representation-finite, certain conditions are always satisfied. First of all, if Λ is representation-finite, then any subcategory of $\operatorname{mod}\Lambda$ has both a cover and a cocover. And second, it is well-known that for an antichain A which is not exceptional, the class $\mathcal{F}(A)$ of all modules with a filtration with factors in A contains infinitely many isomorphism classes of indecomposable Λ-modules, thus Λ cannot be representation-finite.

1.9. Outline of the paper. Sections 2 to 4 considers the Ingalls-Thomas bijections in detail, and provide the corresponding proofs. Section 5 is devoted to duality. In section 6 we deal with the support of the various modules and subcategories. In section 7, we discuss some possible generalizations and explain in which way antichains in additive categories correspond to antichains in posets. There is an appendix which concerns abelian categories with covers.

1.10. The case of Λ being representation-finite is studied in more detail in our paper [15]. Such an artin algebra Λ is called a *Dynkin algebra*, since the underlying graph of its valuated quiver is the disjoint union of Dynkin diagrams (and called the type of Λ). There, we discuss the number of tilting and support-tilting modules for these algebras. For the Dynkin cases $\mathbb{A}$, we obtain the Catalan triangle, for the cases $\mathbb{B}$ and $\mathbb{C}$ we obtain the increasing part of the Pascal triangle, and finally for the cases $\mathbb{D}$ we obtain an expansion of the increasing part of the Lucas triangle. For a further study of the Ingalls-Thomas bijections in general, we also may refer to the forthcoming survey [21].

2. The bijections between (1), (2) and (3)

From (1) to (2): *If A is an antichain, take $\mathcal{F}(A)$,* this is the set of all Λ-modules with a filtration with factors in A. The full subcategory $\mathcal{F}(A)$ is an abelian category with exact embedding functor and obviously closed under extensions, thus it is a thick subcategory of $\operatorname{mod}\Lambda$. The simple objects in $\mathcal{F}(A)$ are just the elements of A. The process of considering the elements of A as objects in $\mathcal{F}(A)$ is called *simplification* in [18].

If the antichain A is exceptional, the category $\mathcal{F}(A)$ is known to be equivalent to the module category of an artin algebra. For a proof, we may refer to [7]. Namely, an exceptional antichain A is a standardizable set as considered in [7] and the proof of Theorem 2 in [7] asserts that there is a quasi-hereditary artin algebra B such that the subcategory $\mathcal{F}(A)$ is equivalent to the category of Δ-filtered B-modules. Since the standardizable set A consists of pairwise orthogonal modules, the same is true for the Δ-modules of B, and consequently the Δ-modules of B are just the simple B-modules. This shows that the category of Δ-filtered B-modules is the whole category $\operatorname{mod} B$. Thus we see that $\mathcal{F}(A)$ is equivalent to $\operatorname{mod} B$. The category $\operatorname{mod} B$ has a progenerator, thus also $\mathcal{F}(A)$ has a progenerator, and every progenerator of $\mathcal{F}(A)$ is a cover for $\mathcal{F}(A)$.

From (2) to (1): *If $\mathcal{A}$ is a thick subcategory with a cover, let $\mathcal{S}(\mathcal{A})$ be the set of simple objects in $\mathcal{A}$,* one from each isomorphism class. Then $\mathcal{S}(\mathcal{A})$ is an exceptional antichain in $\operatorname{mod}\Lambda$.

Namely, for Λ a hereditary artin algebra, a thick subcategory of $\operatorname{mod}\Lambda$ with a cover is equivalent, as a category, to the module category $\operatorname{mod}\Lambda'$ of a hereditary artin algebra Λ'. This is well-known, but somewhat hidden in the literature. We include an appendix in order to outline a proof, see Proposition 8.5. Such an equivalence identifies the quiver $Q_{\mathcal{S}(\mathcal{A})}$ with the quiver of the artin algebra Λ' (the quiver of an artin algebra is just the Ext-quiver of the simple Λ'-modules). It is well-known (and easy to see) that the quiver of a hereditary artin algebra is acyclic.

From (2) to (3): *If $\mathcal{A}$ is a thick subcategory with a cover, let P be a minimal projective generator of $\mathcal{A}$.* Then P is a normal module without self-extensions.

If we start with (1), say with an exceptional antichain A, and use [7] in order to find an equivalence $\eta\colon \mathcal{F}(A) \to \operatorname{mod} B$, the proof of Theorem 2 in [7] first constructs indecomposable objects in $\mathcal{F}(A)$ which correspond under η to the indecomposable projective B-modules. In this way, one constructs a minimal projective generator for the abelian category $\mathcal{F}(A)$.

From (3) to (1). Let N be a normal module without self-extensions. Write $N = \bigoplus_i N_i$ with indecomposable modules N_i. For any i, let $u_i : U_i \to N_i$ be a minimal right $\mathcal{N}_i$-approximation of N_i, where $\mathcal{N}_i = \operatorname{add}(\{N_j \mid j \neq i\}$. Since N is normal, the map u_i cannot be surjective. Since Λ is hereditary, it follows that u_i is injective and we denote by $p_i : N_i \to \Delta(i)$ the cokernel of u_i. Since u_i is not surjective, we see that $\Delta(i) \neq 0$. We claim that the modules $\Delta(i)$ are pairwise orthogonal bricks. Let $h : N_j \to \Delta(i)$ be a map, and form the induced exact sequence

$$\begin{array}{ccccccccc}
0 & \longrightarrow & U_i & \longrightarrow & M & \longrightarrow & N_j & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow h & & \\
0 & \longrightarrow & U_i & \longrightarrow & N_i & \longrightarrow & \Delta(i) & \longrightarrow & 0
\end{array}$$

Since U_i belongs to $\mathcal{N}_i$ and N has no self-extensions, we have $\operatorname{Ext}(N_j, U_i) = 0$, thus the upper sequence splits. It follows that there is a map $h' : N_j \to N_i$ such that $h = p_i h'$. This has two consequences.

First of all, consider the case $j = i$. Let g be any endomorphism of $\Delta(i)$ and look at the map $h = gp_i\colon N_i \to \Delta(i)$. We see that there is an endomorphism $g'\colon N_i \to N_i$ with $gp_i = p_i g'$. Since all non-zero endomorphisms of N_i are invertible, the same is true for $\Delta(i)$. In this way, we see that $\Delta(i)$ is a brick.

Second, let $g : \Delta(j) \to \Delta(i)$ be a homomorphism with $j \neq i$ and consider $h = gp_j : N_j \to \Delta(i)$. There is $g' : N_j \to N_i$ such that $gp_j = p_i g'$. Since u_i is a left $\mathcal{N}_i$-approximation, it follows that $g' = u_i g''$ for some $g'' : N_j \to U_i$. But then $gp_j = p_i g' = p_i u_i g'' = 0$ and therefore $g = 0$.

In this way, we have shown that $\Delta = \{\Delta(i) \mid i\}$ is an antichain. Using induction on the length $|N_i|$ of N_i, we see that N_i belongs to $\mathcal{F}(\Delta)$. Namely, if N_i is of length 1, then $U_i = 0$ since $\Delta(i) \neq 0$. If $|N_i| \geq 2$, then U_i is a direct sum of modules of the form N_j with $|N_j| < |N_i|$, thus by induction U_i belongs to $\mathcal{F}(\Delta)$ and therefore also N_i belongs to $\mathcal{F}(\Delta)$.

The surjective map $p_i : N_i \to \Delta(i)$ yields a surjective map $\mathrm{Ext}(N, N_i) \to \mathrm{Ext}(N, \Delta(i))$, thus $\mathrm{Ext}(N, \Delta(i)) = 0$ for all i, and therefore $\mathrm{Ext}(N, M) = 0$ for all $M \in \mathcal{F}(\Delta)$. This shows that the objects N_i are indecomposable projective objects in $\mathcal{F}(\Delta)$; actually, N_i is the projective cover of $\Delta(i)$ in $\mathcal{F}(\Delta)$. As usual, one sees now that $\mathrm{Ext}(\Delta(i), \Delta(j)) \neq 0$ if and only if N_j is a direct summand of U_i. If N_j is a direct summand of U_i, then, in particular, $|N_j| < |N_i|$. This shows that the Ext-quiver of Δ is acyclic.

Starting with an exceptional antichain A in (1), and going via (2) to (3), we obtain a minimal projective generator P of $\mathcal{F}(A)$. Going from (3) to (1), we attach to P the antichain Δ whose elements are just the simple objects in $\mathcal{F}(A)$, but these are just the elements of A. Conversely, starting in (3) say with a normal module N without self-extensions, then going to (1), we attach to it the antichain Δ. Going via (2) to (3), we form a minimal projective generator in $\mathcal{F}(\Delta)$. But N is up to isomorphism the only minimal projective generator in $\mathcal{F}(\Delta)$.

3. The bijection between (3) and (4)

From (4) to (3): *If T is a support-tilting module, let $\nu(T)$ be its normalization.* This clearly is a normal module without self-extensions. Here we use that any module M can be written in the form $M = M' \oplus M''$ where M' is normal and generates M'' (this of course is trivial), and that such a decomposition is unique up to isomorphism (this is not so obvious); the module M' is called a *normalization* of the module M. The uniqueness was first shown by Roiter [22] and then also by Auslander-Smalø [6], see also [20]. The uniqueness shows that the map ν going from (4) to (3) is well-defined.

Let us show that ν is injective when we are dealing with support-tilting modules. We claim the following: if T, T' are support-tilting modules with $\nu(T) = \nu(T')$, then T and T' are Morita equivalent. For the proof, we may replace Λ by the support algebra $\Lambda(T) = \Lambda(T')$, thus we may assume that T, T' are tilting modules. Now, T' is generated by $\nu(T') = \nu(T)$, thus by T. Since T generates T', it follows from $\operatorname{Ext}(T,T) = 0$ that $\operatorname{Ext}(T,T') = 0$. Similarly, T' generates T and therefore $\operatorname{Ext}(T',T) = 0$. Altogether we see that $\operatorname{Ext}(T \oplus T', T \oplus T') = 0$. Since T is a tilting module, this implies that T' belongs to $\operatorname{add} T$. Similarly, since T' is a tilting module, we see that T belongs to $\operatorname{add} T'$.

In order to see that ν is also surjective, we need to find for any normal module N without self-extensions a support-tilting module T with $\nu(T) = N$. This we will show next.

From (3) to (4): If N is a module without self-extensions, there is a module Y, with the following properties: first, Y is generated by N, and second, $N \oplus Y$ is a support-tilting module; we call Y a *factor complement* for N (this is the dual version of forming a Bongartz complement for a sincere module without self-extensions, see for example [19]).

Here is the construction of a factor complement Y of a module without self-extensions (we follow [19]). Let $\Lambda(N)$ be the support algebra for N and Z an injective cogenerator for $\operatorname{mod} \Lambda(N)$. We claim that *there exists an epimorphism* $Y \to Z$ *with kernel in* $\operatorname{add} N$ *such that* $\operatorname{Ext}(Y,N) = 0$. Such an epimorphism may be called s *universal foundation of Z by N* (or a universal extension of Z by N from below): take exact sequences $0 \to N \to Y_i \to Z \to 0$ such that the corresponding elements in $\operatorname{Ext}(Z,N)$ generate $\operatorname{Ext}/Z,N)$ as a k-module, and form the direct sum of these sequences. Consider the induced sequence with respect to the diagonal inclusion $u : Z \to \bigoplus_i Z$, thus there is the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \bigoplus_i N \arrow[r] & \bigoplus_i Y_i \arrow[r] & \bigoplus_i Z \arrow[r] & 0 \\
0 \arrow[r] & \bigoplus_i N \arrow[r] \arrow[u, equal] & Y \arrow[r, "g"] \arrow[u] & Z \arrow[r] \arrow[u, "u"'] & 0
\end{tikzcd}$$

We obtain in this way an epimorphism $g : Y \to Z$ with kernel in $\operatorname{add} N$. If we apply $\operatorname{Hom}(-,N)$ to the lower exact sequence, we get a long exact sequence. Here is the part which is essential for us:

$$\operatorname{Hom}(\textstyle\bigoplus_i N, N) \xrightarrow{\delta} \operatorname{Ext}(Z,N) \longrightarrow \operatorname{Ext}(Y,N) \longrightarrow \operatorname{Ext}(\textstyle\bigoplus_i N, N)$$

with connecting homomorphism δ. By construction, δ is surjective. Since N has no self-extensions, the last term of the displayed sequence is zero. It follows that $\operatorname{Ext}(Y,N) = 0$. This shows that $g : Y \to Z$ is a universal foundation of Z by N.

In general, *given a universal foundation $g\colon Y \to Z$ of Z by N, say with kernel N', the module Y is generated by N.* Namely, since N has no self-extensions, it is a faithful $\Lambda(N)$-module, thus Z is generated by N. An epimorphism $h\colon N^t \to Z$ yields a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & N' \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0 \\
0 \arrow[r] & N' \arrow[r] \arrow[u, equal] & N'' \arrow[r] \arrow[u, "h'"] & N^t \arrow[r] \arrow[u, "h"] & 0
\end{tikzcd}$$

Since $\operatorname{Ext}(N,N) = 0$, the lower sequence splits, thus N'' belongs to $\operatorname{add} N$. Since h is surjective, also h' is surjective, thus Y is generated by N.

It remains to be seen that $N \oplus Y$ is support-tilting. Since N generates Y, it follows from $\operatorname{Ext}(N \oplus Y, N) = 0$ that $\operatorname{Ext}(N \oplus Y, Y) = 0$. In this way, we see that $N \oplus Y$ has no self-extensions. The exact sequence $0 \to \bigoplus_i N \to Y \to Z \to 0$ shows that Z is the cokernel of an injective map in $\operatorname{add}(N \oplus Y)$, thus $N \oplus Y$ is a support-tilting module. This completes the proof that Y is a factor complement for N.

If we choose a minimal direct summand $\phi(N)$ of Y such that $N \oplus \phi(N)$ is a support-tilting module, then $\phi(N)$ is uniquely determined by N and may be called a *minimal factor complement* for N. Thus, going from (3) to (4), we may attach to a normal module N without self-extension the multiplicity-free support-tilting module $N \oplus \phi(N)$.

Of course, if N is normal, then N is the normalization of $N \oplus Y$. Thus starting with a normal module N without self-extensions, then going from (3) to (4) and back to (3), we obtain N. On the other hand, let T be support-tilting. From (4) to (3) we take $\nu(T)$. From (3) to (4), we add to $\nu(T)$ a factor complement, say N'. But T and $T' = \nu(T) \oplus N'$ both are support-tilting modules with $\nu(T) = \nu(T')$ and generated by this module $\nu(T)$, thus they are Morita equivalent.

4. The bijection between (4) and (5)

First, we show the following: If T is a support-tilting module and $\mathcal{G} = \mathcal{G}(T)$, then $\operatorname{add} T$ is the class of the Ext-projective modules in $\mathcal{G}$. Tilting theory asserts that $\mathcal{G}$ is the class of $\Lambda(T)$-modules M such that $\operatorname{Ext}(T,M) = 0$. Let M be in $\mathcal{G}$ and $g : T' \to M$ be a right T-approximation of M. Then g is surjective and the kernel M' of g satisfies $\operatorname{Ext}(T, M') = 0$, thus belongs to $\mathcal{G}$. If M is Ext-projective, then the exact sequence $0 \to M' \to T' \to M \to 0$ splits, thus M is in $\operatorname{add} T$. This shows that the Ext-projective modules in $\mathcal{G}$ are just the modules in $\operatorname{add} T$.

From (4) to (5): If T is a module without self-extensions, let $\mathcal{G}(T)$ be the class of modules generated by T. Then it is well-known (and easy to see) that T is a torsion class. Of course, T is a cover for $\mathcal{G}(T)$.

From (5) to (4): If $\mathcal{C}$ is a torsion class with a cover C, then we attach to it a module T such that $\operatorname{add} T$ is the class of Ext-projective modules in $\mathcal{G}$. In order to do so, we need to know that the class $\mathcal{E}$ of Ext-projective modules in $\mathcal{C}$ is finite, say $\mathcal{E} = \operatorname{add} T$ for some module T. We also have to show that T is support-tilting.

Along with C, its normalization $\nu(C)$ is also a cover. A normal cover of a torsion class has no self-extension (see Proposition 1 of [20]). Let B be a factor complement for $\nu(C)$. As we have seen, $T = \nu(C) \oplus B$ is a support-tilting module. Since B is generated by $\nu(C)$, we have $\mathcal{G}(T) = \mathcal{G}(\nu(C)) = \mathcal{G}(C) = \mathcal{C}$. But we have shown already that $\operatorname{add} T$ is the class of Ext-projective modules in $\mathcal{G}(T)$.

From (4) to (5) to (4): Let us start with a support-tilting module T and attach to it $\mathcal{G} = \mathcal{G}(T)$. As we have seen, the class of Ext-projectives in $\mathcal{G}$ is $\operatorname{add} T$. We choose T' with $\operatorname{add} T' = \operatorname{add} T$. But this just means that T, T' are Morita equivalent.

From (5) to (4) to (5). We start with a torsion class $\mathcal{C}$ with a cover, we choose a support-tilting module T with $\mathcal{C} = \mathcal{G}(T)$, thus we are back at $\mathcal{C}$.

5. Duality

By definition, given an artin algebra Λ, there is a commutative artinian ring k such that Λ is a k-algebra and is of finite length when considered as a k-module. If Λ is an artin algebra, also the opposite algebra Λ^{op} is an artin algebra. If we denote by E a minimal injective cogenerator for $\operatorname{mod} k$, the functor $D = \operatorname{Hom}_k(-, E)$ provides an equivalence between $\operatorname{mod}\Lambda$ and $(\operatorname{mod}\Lambda^{\mathrm{op}})^{\mathrm{op}}$. We can use this duality in order to exhibit further bijections.

Using duality, the sets (1), (2) and (4) are preserved. Of course, the dual concept of a thick subcategory with a cover is a thick subcategory with a cocover. An abelian k-category with finitely many simple objects and such that the Hom and Ext-groups are k-modules of finite length, has a cover if and only if it has a cocover.

Dualizing (3) we get:

- (6) *The isomorphism classes of conormal modules without self-extensions.*

Dualizing (5) we get:

- (7) *The torsionfree classes with a cocover.*

The sets defined in (6) *and* (7) *correspond bijectively to the sets (1),...,(5).*

Remark. The bijections between the set (2) of thick subcategories $\mathcal{A}$ and the sets (1), (3) and (6) of isomorphism classes of suitable modules can be reformulated as follows: In an abelian category we may look at the semi-simple, the projective and the injective objects: the set of simple objects in $\mathcal{A}$ is an antichain in $\operatorname{mod}\Lambda$, a minimal projective generator in $\mathcal{A}$ is a normal module without self-extensions, a minimal injective cogenerator is a conormal module without self-extensions. These are the procedures to obtain from a thick subcategory the corresponding antichain, as well as a normal or conormal module without self-extensions.

Conversely, let us start with (1), (3) or (6). It has been mentioned already that starting with an antichain A, we take the full subcategory $\mathcal{F}(A)$ of all modules with a filtration with factors in A. Starting with a normal module P without self-extensions, the corresponding thick subcategory $\mathcal{A}$ consists of all modules which arise as the cokernel of a map in $\operatorname{add} P$ (in this way, we specify projective presentations of the objects in $\mathcal{A}$). Dually, starting with a conormal module I without self-extensions, the corresponding thick subcategory $\mathcal{A}$ consists of all modules which arise as the kernel of a map in $\operatorname{add} I$ (in this way, we specify injective copresentations of the objects in $\mathcal{A}$).

6. The support of a module, sincere modules and subcategories

Proposition 6.1. *The bijections which we have constructed preserve the support.*

Proof. This follows directly from the constrcutions. □

Specializing the Ingalls-Thomas bijections to sincere modules, it follows from the proposition that we get *bijections between:*

- (1) *Isomorphism classes of exceptional sincere antichains.*
- (2) *Thick subcategories with a sincere generator.*
- (3) *Isomorphism classes of normal sincere modules without self-extensions.*
- (4) *Morita equivalence classes of tilting modules.*
- (5) *Torsion classes with a sincere generator.*
- (6) *Isomorphism classes of conormal sincere modules without self-extensions.*
- (7) *Torsionfree classes with a sincere cogenerator.*

Of course, conversely this special case implies the general case.

7. Final remarks

7.1. The aim of our discussion was to extend results of Ingalls and Thomas which were established for path algebras of finite acyclic quivers to arbitrary hereditary artin algebras. Experts may not be surprised that results concerning path algebras of finite acyclic quivers can be extended in this way: after all, there is a general feeling that such generalizations are always possible. But the paper [17] may serve as a warning. The paper provides a description of the cofinite quotient-closed subcategories of $\operatorname{mod}\Lambda$, where Λ is the path algebra of a finite acyclic quiver. In section 9 of [17], the author discuss the problem of extending the result to finite-dimensional hereditary k-algebras, but they are able to provide a solution only in the case of k being a finite field.

On the other hand, one may ask whether the setting may be further enlarged to deal with hereditary artinian or even hereditary semi-primary rings, and not just with hereditary artin algebras. Note that our considerations use duality arguments and finiteness conditions which rely on the artin algebra assumption.

7.2. A further possible generalization has been stressed by the referee: to drop the condition on Λ to be hereditary, thus to deal with an arbitrary artin algebra. For any finite-dimensional k-algebra Λ, with k an algebraically closed field, the paper [2] by Adachi, Iyama and Reiten provides a bijection between support τ-tilting modules in $\operatorname{mod}\Lambda$ and torsion classes with covers, extending in this way the corresponding Ingalls-Thomas bijection (for a hereditary artin algebra, the τ-tilting modules are just the tilting modules). Also, let us remark that the relationship between torsion classes and thick subcategories in $\operatorname{mod}\Lambda$ has been discussed by Marks and Stovicek [12].

7.3. Our presentation of the Ingalls-Thomas bijections is centered around the notion of antichains in additive categories. Let us motivate the definition. Given a poset P, a *chain* in P is a subset of pairwise comparable elements, whereas an *antichain* in P is a subset of pairwise incomparable elements. Now consider the linearization kP of P, where k is a field: this is an additive k-category whose indecomposable objects are the elements of P such that $\operatorname{Hom}_{kP}(x,y) = k$ provided $x \le y$ in P and $\operatorname{Hom}_{kP}(x,y) = 0$ otherwise, such that the composition of maps in kP is given by the multiplication in k, and, finally, such that any object in kP is a finite direct sum of indecomposable objects. Of course, a subset A of P is an antichain in P if and only if A (considered as a set of objects in kP) consists of pairwise orthogonal bricks (thus, is an antichain in the additive category kP). As we see, antichains in additive categories have to be considered as a direct generalization of antichains in posets.

The reader should be aware that starting with a Dynkin diagram Δ and its set $\Phi_+(\Delta)$ of positive roots, several kinds of (different, but related) antichains have to be distinguished: First of all, $\Phi_+(\Delta)$ is in an intrinsic way a poset, called the *root poset* of type Δ, and we may consider the set $\mathbf{A}(\Delta)$ of antichains in this root poset $\Phi_+(\Delta)$. Second, choosing an orientation Ω of the Dynkin diagram (or, equivalently, a Coxeter element in the corresponding Weyl group), we may identify the elements of $\Phi_+(\Delta)$ with the indecomposable Λ-modules, where Λ is a hereditary artin algebra of type Δ, thus with the indecomposable objects in the additive category $\operatorname{mod}\Lambda$. The set of antichains in $\operatorname{mod}\Lambda$ only depends on Δ and Ω (and not on the choice of Λ), thus we may denote it by $\mathbf{A}(\Delta,\Omega)$. It is known for a long time that the set $\mathbf{A}(\Delta)$ of antichains in the root poset $\Phi_+(\Delta)$ and the set $\mathbf{A}(\Delta,\Omega)$ of antichains in $\operatorname{mod}\Lambda$ have the same enumeration (for a uniform proof, see [5]), but a fully satisfactory explanation is still missing. In the case of the quiver $\mathbb{A}_n$ with linear orientation, this concerns the quite obvious bijection between non-nesting and non-crossing partitions. Note that if Ω and Ω' are orientations of Δ, it is easy to construct a natural bijection between $\mathbf{A}(\Delta,\Omega)$ and $\mathbf{A}(\Delta,\Omega')$. For a detailed discussion of the sets $\mathbf{A}(\Delta)$ and $\mathbf{A}(\Delta,\Omega)$, we may refer to [21].

8. Appendix. Abelian categories with projective covers.

We have used in section 2 that given a hereditary artin algebra Λ, a thick subcategory $\mathcal{A}$ of $\operatorname{mod}\Lambda$ with a cover is equivalent to $\operatorname{mod}\Lambda'$ for some hereditary artin algebra Λ'. This is a well-known fact, however the proof seems to be somewhat hidden in the literature. Here we outline the main ingredients for a proof.

We start with a more general setting and require some additional definitions. If R is a ring, let us denote by $\operatorname{fp}(R)$ the category of all finitely presented R-modules (recall that an R-module M is *finitely presented* provided it is the cokernel of a map ${}_RR^s \to {}_RR^t$ for some natural numbers s,t).

Let $\mathcal{B}$ be an arbitrary abelian category. Given objects A,B in $\mathcal{B}$, we say that B is A-*static* provided there is an exact sequence of the form

$$A^s \to A^t \to B \to 0$$

which remains exact when we apply $\operatorname{Hom}(A,-)$. (It is easy to see that this condition is equivalent to the existence of an exact sequence $A'' \to A' \to B \to 0$ with $A',A'' \in \operatorname{add} A$ such that the sequence remains exact when we apply $\operatorname{Hom}(A,-)$; thus this concept corresponds to the usual concept of A-static modules as considered in [3,1,13,14,23], see also [16].) As an example, given a ring R, an R-module M is ${}_RR$-static if and only if it is finitely presented, thus $\operatorname{fp}(R) = \operatorname{stat}({}_RR)$.

Proposition 8.1. *Let $\mathcal{B}$ be an abelian category. Let A be an object in $\mathcal{B}$ and let $R = \mathrm{End}(A)^{\mathrm{op}}$. Then $\mathrm{stat}(A)$ is equivalent, as a category, to $\mathrm{fp}(R)$.*

Proof. See Lemma 2 of [3] or Lemma 3.1, Lemma 3.2 and Theorem 3.3 of [1]. $\square$

An object C in $\mathcal{B}$ will be said to be a *cover* provided every object B of $\mathcal{B}$ is a factor object of some finite direct sum of copies of C (since now we consider arbitrary abelian categories, we should stress that we ask for a **finite** direct sum).

Proposition 8.2. *Let C be an object in the abelian category $\mathcal{B}$ and assume that C is projective and a cover. Let $R = \mathrm{End}(C)^{\mathrm{op}}$. Then $\mathcal{B}$ is equivalent, as a category, to $\mathrm{fp}(R)$.*

Proof. This is an immediate consequence of 8.1, since for C a projective cover, any object of $\mathcal{B}$ is C-static, thus $\mathrm{stat}(C) = \mathcal{B}$. Namely, if C is a cover, and B is an object in $\mathcal{B}$, there is an exact sequence of the form

$$C^s \to C^t \to B \to 0.$$

If C is projective, then this sequence stays exact when we apply $\mathrm{Hom}(C, -)$. $\square$

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Proposition 8.3. *Let Λ be an artin algebra. A thick subcategory of $\mathrm{mod}\,\Lambda$ with a cover D has a cover C which is, in addition, projective, namely the normalization C of D.*

Proof. Let C be a normalization of D (see section 3). Since D is a cover of $\mathrm{mod}\,\Lambda$, also C is a cover for $\mathrm{mod}\,\Lambda$.

Let X be an indecomposable direct summand of C, say $C = X \oplus X'$. In oder to show that X is projective, we show that any epimorphism $\epsilon : B \to X$ in $\mathrm{mod}\,\Lambda$ splits. We will use that the endomorphism ring of X is a local ring with nilpotent radical.

Since C is a cover of $\mathrm{mod}\,\Lambda$, there is an epimorphism $C^t \to B$ for some natural number t, thus there is an epimorphism $(\phi_1, \ldots, \phi_t, \psi) : X^t \oplus Y \to B$ with $Y = (X')^t$. We compose it with ϵ and obtain an epimorphism

$$(\epsilon\phi_1, \ldots, \epsilon\phi_t, \epsilon\psi) : X^t \oplus Y \to X.$$

According to Lemma 1 (b) of [20], either one of the maps $\epsilon\phi_i$ is an automorphism of X or else $\epsilon\psi : Y \to X$ is an epimorphism. But if $\epsilon\psi$ is an epimorphism, then X' generates X, in contrast to the fact that C is normal. Thus we see that there is an index i such that $\epsilon\phi_i$ is an automorphism of X, and this implies that ϵ is a split epimorphism. $\square$

Proposition 8.4. *If* Λ *is an artin algebra, a thick subcategory of* $\operatorname{mod}\Lambda$ *with a cover is equivalent to the module category* $\operatorname{mod}\Lambda'$*, where* Λ' *is again an artin algebra.*

Proof. Let $\mathcal{A}$ be a thick subcategory of $\operatorname{mod}\Lambda$ with a cover D. Let C be the normalization of D. According to Proposition 8.3, C is also a cover and, in addition, projective. Let $\Lambda' = \operatorname{End}(C)^{\mathrm{op}}$, this is again an artin algebra. According to Proposition 8.2, $\mathcal{A}$ is equivalent to $\operatorname{fp}(\Lambda')$. But for an artin algebra R, $\operatorname{fp}(R) = \operatorname{mod} R$. $\square$

Proposition 8.5. *If* Λ *is a hereditary artin algebra, a thick subcategory of* $\operatorname{mod}\Lambda$ *with a cover is equivalent to the module category* $\operatorname{mod}\Lambda'$*, where* Λ' *is again a hereditary artin algebra.*

Proof. Let us say that an abelian category $\mathcal{B}$ is *hereditary* provided $\operatorname{Ext}^2_{\mathcal{B}}$ vanishes, thus provided $\operatorname{Ext}^1_{\mathcal{B}}(B, -)$ is a right exact functor for all objects B in $\mathcal{B}$. Clearly, the category $\operatorname{mod}\Lambda$ for an artin algebra Λ is hereditary as a category if and only if Λ is hereditary as a ring. For a thick subcategory $\mathcal{A}$ of an abelian category $\mathcal{B}$, one has $\operatorname{Ext}^1_{\mathcal{A}}(A, A') = \operatorname{Ext}^1_{\mathcal{B}}(A, A')$ for all objects A, A' in $\mathcal{A}$. Thus, if $\mathcal{B}$ is hereditary, also $\mathcal{A}$ is hereditary.

Altogether we see: If Λ is a hereditary artin algebra, $\operatorname{mod}\Lambda$ is hereditary as a category, thus any thick subcategory $\mathcal{A}$ of $\operatorname{mod}\Lambda$ is hereditary. But if $\mathcal{A}$ has a cover, then it is equivalent to $\operatorname{mod}\Lambda'$ for some hereditary artin algebra Λ'. Since $\operatorname{mod}\Lambda'$ is a hereditary category, Λ' is a hereditary ring. $\square$

Remark. If Λ, Λ' are artin algebras such that $\operatorname{mod}\Lambda'$ is equivalent to a thick subcategory of $\operatorname{mod}\Lambda$, and Λ has global dimension at least 2, then Λ' may have arbitrary large global dimension. Here is an example. Consider the following quiver Q (with three vertices and three arrows) and let Λ be the path algebra of Q modulo the ideal generated by $\alpha\beta$.

$$\begin{tikzcd} & 2 \arrow[dl, "\alpha"'] & \\ 1 & & 3 \arrow[ul, "\beta"'] \arrow[ll, "\gamma"] \end{tikzcd}$$

The algebra Λ has global dimension equal to 2. Let $\mathcal{A}$ be the smallest thick subcategory of $\operatorname{mod}\Lambda$ containing the simple representation $S(2)$ and the 2-dimensional indecomposable representation with composition factors $S(1)$ and $S(3)$. Then $\mathcal{A}$ is equivalent to the module category $\operatorname{mod}\Lambda'$ of an artin algebra of infinite global dimension.

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