Bautista and the Development of the Representation Theory or Artin Algebras

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On the occasion of the 60th birthday of Raymundo Bautista I want to outline[1] his contributions to the development of the representation theory of artin algebras in the years 1977 - 1985. I have to apologize, that in this way I only deal with a small part of his scientific activities, but I hope that this will still provide a vivid picture of his mathematical thoughts. A glance on the long list of publications of Raymundo Bautista reveals his interest in rather different types of questions. Here are the three main areas of his research:

- Representation Theory of Artin Algebras,
- BOCS Representations,
- Mathematical Physics,

and there are several other topics which he has considered, see for example his early papers concerning the cohomology of finite groups. However, I feel competent only to deal with the first area. As I have mentioned already, this report makes a further restriction by concentrating on the years 1977 - 1985. This is a period which can be covered in a unified way. But before I start, let me consider for a moment the relationship between the representation theory of artin algebras and the other two areas.

BOCS Representations.[2] Recall that the abbreviation BOCS stands for *bimodules over categories.* The BOCS representation theory was introduced by Kleiner and Roiter in 1984 in order to formalize matrix calculations which were introduced to deal with representations of finite dimensional algebras. Thus, there is a strong relationship between the first two topics. However, the reduction techniques used for BOCS representations are of a very different nature compared to those methods used elsewhere in the representation theory of algebras.

Mathematical Physics. The relationship to the third area of investigation, questions in mathematical physics, is much more loose: indeed, these investigations rely at least partly on methods in representation theory (however usually one

[1] This is the written version of a lecture presented at the XV Coloquio Latinoamericano de Álgebra, held at Cocoyoc, México, July 2003.

[2] During the conference, there had been an additional lecture by Rita Zuazua, a former student and present cooperator of Raymundo, on the development of BOCS representation theory and Raymundos contributions.

invokes representations of groups or Lie algebras, and not of artin algebras or BOCSes), but also on methods in differential geometry and other parts of mathematics. A corresponding review on Raymundos papers in mathematical physics would be appropriate. Let me draw your attention at least to some of the topics covered in these papers by quoting a few titles: *Kaluza-Klein model for the unification of the bosonic sector of the electroweak model with gravitation* (a joint paper with Rosenbaum, D'Olivo, Nahmad-Achar and Muciño, published in 1989 in the proceedings of a meeting on General Relativity), SU(2)*-multi-instantons over* $S^2 \times S^2$ and *Yang-Mills connections over homogeneous spaces*, both being joint work with Muciño and Rosenbaum, published in 1992, respectively 1993, and finally the 1996 paper in the Journal of Mathematical Physics with the title *Quantum Clifford algebras from spinor representations,* together with Criscuolo, Durdević, Rosenbaum and Vergara as coauthors. For the 1998 Mexico conference on *Interdisciplinary tendencies in mathematics*, Raymundo wrote a survey on *Algebra, geometry and physics* (published in volume 26 of Aportaciones Mat. Comun.) which provides a synopsis of the algebraic concepts of geometry viewed in the context of quantum groups. It should be noted that here some classical concepts of differential geometry, such as vector fields and differential forms, are considered from an algebraic point of view using bimodule structures.

ICRA III and ICRA VII. As already mentioned, the following report will focus the attention to the period 1977 - 1985. We want to describe Raymundo's impact on the development of the representation theory of artin algebras in these years. There are regular meetings called *International Conference on Representations of Algebras* (ICRA), which provide an overview of the latest developments in the subject and which aim to bring together all the specialists in the field. The first two ICRAs were both held at Carleton University, Ottawa, in 1974 and in 1979, as an initiative of Vlastimil Dlab. ICRA III was scheduled to take place 1980 (thus just one year later) at Puebla. It was organized by Raymundo Bautista who was in Puebla at that time. The international organizing board had a lengthy discussion about reasonable intervals for such meetings and there was quite a lot of opposition to have meetings in consecutive years. – However the 1980 meeting turned out to be very fruitful with a lot of decisive new ideas: for example covering theory and tilting theory were discussed at that conference. A further ICRA has been held in Mexico in 1994 (ICRA VII, in Cocoyoc)[3] Although this falls outside the proposed time slot, it should nevertheless be mentioned for one reason: it was the last ICRA, Maurice Auslander has participated, he died that year in November when visiting Trondheim. Note that the Mexican research group in the representation theory of algebras was initiated by Auslander and many of the topics investigated there were in the lines of ideas of Auslander, or have to be considered as complementing his views. In an obituary with the title *The influence of Auslander in Mexico* [B11], Raymundo writes: *Maurice's influence was very important in the development of our group. This influence was not only through the suggestion of specific mathematical problems but through more general ideas of how to look at mathematics.* He recalls that Auslander visited Mexico in the summer of 1975 and gave lectures on several subjects in the representation theory of algebras: *We were impressed mainly in the part of the lectures related to almost split sequences, then recently discovered*

[3] and there will be another one in Mexico, ICRA X, in August 2004.

by M. Auslander and I. Reiten. From 1976 to 1977 Bautista and Martinez spent nearly two years at Brandeis University: *There, we had the opportunity of knowing, and living in, an exciting atmosphere. We met many people through Maurice who were interested in the representation theory of algebras.* Thus, for example, I met in this way both of them in 1976 and since then we had frequent contacts and enjoyed the possibility to discuss all kinds of problems in representation theory.

My report will be divided into two parts. In the first part we will single out the year 1985 and two fulminant papers which appeared in that year: a joint one with Gabriel, Roiter and Salmerón, which shows the existence of multiplicative bases for representation-finite and for minimal representation-infinite algebras, the other one provides a solution of the second Brauer-Thrall conditions. In the second part, we will concentrate on the earlier years 1977 - 1984. In this period, the basic notions of the present representation theory of artin algebras have been elaborated, and it was Raymundo who has pushed forward these developments considerably. Let us mention some of the notions which will be discussed: irreducible maps, Auslander-Reiten-quiver, knitting of components, replication numbers, sectional paths, and the s-condition. In 1984, Raymundo gave a survey lecture at the XVIIIth national congress of the Mexican Mathematical Society at Mérida [B10] under the title *Incursions into the representation theory of algebras* (in Spanish) about his own contributions as well as those of other Mexican mathematicians, and we may just follow the paths which he has outlined there.

We will use the following conventions: we usually will consider an artin algebra Λ (this means that Λ is an associative ring with unit element, and of finite length when considered as a module over its center; note that this implies that the center itself is of finite length, thus the center is a commutative artinian ring). Typical examples of artin algebras are finite dimensional k-algebras, where k is a field. The modules to be considered will be unital left Λ-modules of finite length. Note that such a module can be written as the direct sum of indecomposable modules and the theorem of Krull-Remak-Schmidt asserts that such a decomposition is unique up to isomorphism. We say that Λ is representation-finite provided there are only finitely many isomorphism classes of indecomposable Λ-modules.

1985. The Culmination.

The Brauer-Thrall conjectures. Let us start with what always will be attached to Raymundo's name: his solution of the second Brauer-Thrall conjecture. First, I want to recall the statement of the two Brauer-Thrall conjectures: We assume that Λ is a finite dimensional k-algebra, where k is a field.

(I) If Λ is not representation-finite, then there are indecomposable Λ-modules of arbitrarily large length.

(II) If Λ is not representation-finite and k is an infinite field, then there are infinitely many natural numbers $d_1 < d_2 < ...$ such that Λ has infinitely many isomorphism classes of indecomposable modules of length d_i, for $i = 1, 2, ...$.

These conjectures were formulated in the late forties by Brauer and Thrall, with the first written record in a paper of Jans in 1957. Actually, Brauer once mentioned that he had posed these questions as a tutorial homework for his students in a course on modular representations of finite groups, and he was surprised that none of the students handed in a solution. Concerning Brauer-Thrall I, for a long time only some partial results (by Jans, Curtis and others) had been known. The solution by Roiter in 1968 marks the beginning of modern representation theory of algebras, and it had a strong influence on the further development. Roiter's paper contained a remark asserting that his proof would work not only for finite dimensional algebras but for arbitrary artinian rings. It soon became clear that this assertion was wrong and one had to wait until 1974 when Auslander published a proof which is valid in general.

We now turn the attention to Brauer-Thrall II, say under the assumption that k is an algebraically closed field. After having solved the first Brauer-Thrall conjecture, Roiter collaborated with Nazarova on Brauer-Thrall II, using matrix calculations. They published a rather long paper which claimed to provide a proof in general. However, the main part of the paper considered only the special case where $\mathrm{Ext}^1(S,S) = 0$ for any simple module S, and only insufficient hints were given how to deal with the general case. But even the proof of the special case was incomplete and inaccurate. When the solution of Bautista was reported at ICRA IV (Ottawa 1984), Roiter insisted that there also did exist a new Kiev proof - he gave two lectures during the workshop and handed in a corresponding manuscript for the proceedings. However this manuscript was withdrawn when it was pointed out that there are several counter-examples to intermediate steps. Bautista's proof is given in the paper [B8] published in Commentarii Mathematici Helvetici. The paper used the additional hypothesis that the characteristic of k is different from 2; this condition was removed by Bongartz in a subsequent paper. One easily can show that the positive solution of Brauer-Thrall II, for k algebraically closed, implies that Brauer-Thrall II is valid for any perfect field k. In case k is not perfect, the problem is still open!

Multiplicative Bases. Bautista's proof of Brauer-Thrall II relies on the existence of a special basis for minimal representation-infinite algebras: a multiplicative Cartan basis. The existence proof for such a basis is the main aim of the second paper which we have to mention here: the joint work of Bautista, Gabriel, Roiter and Salmerón with the title *Representation-finite algebras and multiplicative bases,* which has appeared in the journal *Inventiones Mathematicae* [B-S]. We assume again that Λ is a k-algebra with k an algebraically closed field. In addition, we require that Λ is basic: this means that the factor algebra $\underline{\Lambda}$ of Λ modulo its radical is a product of copies of k (this additional requirement can always be achieved by replacing Λ by a Morita equivalent algebra Λ_0; note that for Morita equivalent algebras, the module categories are equivalent). We say that a k-basis of Λ is a *multiplicative Cartan basis* provided the following three conditions are satisfied:

(1) If b_1, b_2 belong to B, then $b_1 b_2$ is either zero or belongs to B,
(2) B contains a complete set B' of orthogonal primitive idempotents.
(3) The non-idempotent elements of B generate the radical $\operatorname{rad}\Lambda$ of Λ.

If B is a multiplicative Cartan basis, let B'' be the set of all non-idempotent elements of B. Then it is easy to see that B is the disjoint union of B' and B'', that the non-zero elements in $(B'')^t$ form a basis of $(\operatorname{rad}\Lambda)^t$, for $t = 1, 2, \dots$, and that for any element $b \in B$, there are idempotents $e_1, e_2 \in B$ such that $b = e_1 b e_2$. The latter conditions just means that the basis B consists of homogeneous elements with respect to the Cartan decomposition $\Lambda = \bigoplus_{i,j\in B'} e_i \Lambda e_j$ (this is the reason why we prefer to call such a basis a Cartan basis, in contrast to the terminology of the paper which speaks of a "normed" multiplicative basis). The multiplicativity property (1) asserts that one deals with a combinatorially defined algebra; however in applications also the remaining properties turn out to be of great importance: after all, any group algebra has, for trivial reasons, a basis with the multiplicative property (1), but usually will not have a multiplicative Cartan basis.

The main result of [B-S] asserts that any representation-finite, as well as any minimal representation-infinite algebra, has a multiplicative basis. The paper is quite long, however it still is very condensed: the main method used is the so-called cleaving procedure, a sort of partial covering theory, and in the numerous applications of this method usually only the main ingredients are provided, whereas the actual verifications are left to the reader.

In the Mathematical Reviews, I wrote in 1987: *The result has a rather long history. ... The general result was announced by Roiter in 1981, however his proof was incomplete and partly incorrect. The first complete seems to have been given by Bautista in his lectures at U.N.A.M. (Mexico) in the spring of 1983, but was not published. Since Bautista's proof was based on the ideas of Roiter, the result may be referred to as a theorem of Roiter and Bautista.* Gabriel was furious about this comment and insisted that he also had a complete proof at the same time, thus I wrote a corresponding addendum for the Reviews (I may add that at that time he promised to send me a copy of his own draft, which however I never obtained). I remember very well the spring 1983: I was visiting U.N.A.M. at that time and listened to the lectures of Raymundo. What he presented was clearly his own work, of course (as mentioned before) based on the old draft of Roiter, using contributions of Salmerón and also suggestions of Gabriel[4]. And I know that he sent his proof to Gabriel, who then proposed many changes. Clearly, Gabriel made very remarkable contributions to the final paper which strongly differs from Raymundo's version: indeed the various structure theorems for representation-finite and, more generally, for "mild" algebras are due to a very fruitful collaboration of the authors of the paper. These structure theorems deal with small factor algebras of Λ which have to be considered as being nasty: there are three essentially different kinds, called penny-farthing, dumbbell and diamond, and the essential observation asserts that such factor algebras have only minor overlappings. We should also mention the topological considerations in sections 8 and 10, dealing with the simplicial complex of a ray category and showing the vanishing of a second cohomology group.

The structure theory presented in [B-S] has obviously scared away all other mathematicians: One would expect to find a big variety of papers which are based on this marvelleous investigation. But this is not the case! This is really a pity, for several reasons: one should try to squeeze the arguments in order to obtain a more comprehensive version; one needs corresponding results in the species case (thus working with k-algebras, where k is not algebraically closed); and one should try to understand the module theoretic behaviour in case one deals with slightly larger

[4] I should add that Bautista himself stresses the parallel streamlines and the progressive interrelation. He wrote to me: *I cannot claim priority on this work. It is true hat I had a preliminary version of a proof, but this was on the bases of joint work with Salmerón, on discussions in Kiev in May 1982 with Roiter and Ovsienko and on Gabriel's advice* (during his visit to Puebla at the end of 1982). *This version was never published, may-be in the future I will look again at this old manuscript.*

overlappings of critical parts: when do we still have tameness? when do we get wild clusters, just as islands which can still be separated from the surrounding.

It seems to be customary to hide priority fights (especially in birthday lectures) and to refrain from inquiring into the parallelity of arguments and ideas. But why? When Gabriel in his ICRA lectures 1979 tried to give some historical account about parallel streamlines, he stressed the following: *The interrelation between these various works may be difficult to track. Even the authors are not always aware of them, since ideas ooze away and resurge.* The only possibility to avoid priority fights seems to be to choose a research topic no-one else will be interested in and will take the pain to put his energy into! As long as one directs the attention to questions of general interest, one has to cope with competing teams in other research groups and with parallel results.

1977 - 1984. The Foundation of the Theory.

The radical of an additive category. As before, we denote by Λ an artin algebra. Let $\operatorname{mod}\Lambda$ be the category of all Λ-modules. Now $\operatorname{mod}\Lambda$ is an additive category and such a category is quite similar to a ring (there is an addition and the composition of maps is like a multiplication - both addition and multiplication are only partially defined, but they satisfy the usual ring axioms). The only difference is that in general there is no global identity element, but many idempotents (the identity elements of the various objects). Thus one often calls an additive category $\mathcal{A}$ a "ring with several objects". As a consequence, one can try to generalize the usual concepts of ring theory to additive categories, and often this works out very well. Under this analogy, Λ-modules correspond to additive functors from $\mathcal{A}$ (or its opposite category) to the category of abelian groups; thus the category $\operatorname{mod}\Lambda$ corresponds to a functor category. For example, the Jacobson radical of a ring is the intersection of the annihilator of the simple modules - thus the Jacobson radical of an additive category $\mathcal{A}$ should be the intersection of the annihilator of the simple (additive) functors. Let us apply these considerations to the additive category $\mathcal{A} = \operatorname{mod}\Lambda$. Any simple functor on this $\mathcal{A}$ is of the form $S = S_M$, indexed by an indecomposable Λ-module M; with $S_M(X) = 0$ for any indecomposable Λ-module X which is not isomorphic to M, whereas $S_M(M) = \underline{\operatorname{End}}(M)$ (the factor of $\operatorname{End}(M)$ modulo its radical). It turns out that the Jacobson radical $J = \operatorname{rad}\operatorname{mod}\Lambda$ of $\operatorname{mod}\Lambda$ can also be described as follows: it is an ideal of $\operatorname{mod}\Lambda$; thus we have to single out, for any pair X, Y of Λ-modules, the subgroup $J(X,Y) \subseteq \operatorname{Hom}(X,Y)$. A morphism $f : X \to Y$ belongs to $J(X,Y)$, provided for any indecomposable direct summand X_i of X with inclusion map $u_i \colon X_i \to X$, and for any indecomposable direct summand Y_j of Y with projection map $p_j \colon Y \to Y_j$, the composition $p_j f u_i$ is non-invertible.

Having defined the radical $J = \operatorname{rad}\operatorname{mod}\Lambda$, we may consider its powers J^t, where t is a natural number. Of particular interest is J^2 and the factor J/J^2. If X, Y are indecomposable modules, then the elements of $J(X,Y) \setminus J^2(X,Y)$ are just the irreducible maps $X \to Y$ in the sense of Auslander-Reiten: these are the non-invertible maps which have no proper product-factorizations. One may call the factor group $J(X,Y)/J^2(X,Y)$ the *bimodule of irreducible maps*; it is an $\underline{\operatorname{End}}(X)$-$\underline{\operatorname{End}}(Y)$-bimodule, it controls the structure of the Auslander-Reiten sequences of

$\operatorname{mod}\Lambda$, since for an Auslander-Reiten sequence

$$0 \to X \to \bigoplus_i Y_i \to Z \to 0$$

with indecomposable modules Y_i (of course X, Z also are indecomposable), all the maps $X \to Y_i$ and $Y_i \to Z$ involved in the sequence are irreducible, and any irreducible map between indecomposable modules arises in this way. More precisely, the dimension of $J(X,Y)/J^2(X,Y)$ as an $\underline{\operatorname{End}}(X)$-space or as an $\underline{\operatorname{End}}(Y)$-space measures the multiplicities of X or Y respectively, occurring as middle term of Auslander-Reiten sequences. It seems that Bautista was the first to notice that the notion of the Jacobson radical of an additive category may be used as a starting point for presenting the basic notions of the Auslander-Reiten theory; the corresponding paper [B7] was published quite late, but apparently was written during his time at Brandeis - it reflects his early interest in the Auslander-Reiten theory.

In case Λ is representation-finite, $J^n = 0$ for some n, and in this case the elements of $J \setminus J^2$ clearly generate J. Now dealing with the additive category $\operatorname{mod}\Lambda$, one is interested to obtain a description by generators and relations - as in any algebraic theory. It turns out that the basic relations to be considered are those which arise from the Auslander-Reiten sequences. Knowing generators as well as basic relations, one may try to invoke a corresponding combinatorial object, the Auslander-Reiten quiver $\Gamma(\Lambda)$ of Λ which is defined for an arbitrary artin algebra as follows: its vertices are the isomorphism classes of the indecomposable Λ-modules, there is an arrow $[X] \to [Y]$ (we denote the isomorphism class of a module Z by $[Z]$), provided there exists an irreducible map $X \to Y$, and one endows the arrow $[X] \to [Y]$ with the dimensions of $J(X,Y)/J^2(X,Y)$ as an $\underline{\operatorname{End}}(X)$-space and as an $\underline{\operatorname{End}}(Y)$-space. In addition, one fixes the pairs $([X],[Z])$, so that there exists an Auslander-Reiten sequence starting in X and ending in Z (and one writes $\tau[Z] = [X]$ and calls τ the Auslander-Reiten translation). One may wonder whether it is possible to recover the category $\operatorname{mod}\Lambda$ from these data. This question apparently was raised for the first time by Bautista. The final answer was given by Riedtmann: in case we deal with a field k of characteristic 2, then there are examples of representation-finite algebras Λ, such that $\operatorname{mod}\Lambda$ is **not** the mesh category of its Auslander-Reiten quiver $\Gamma(\Lambda)$, but if we deal with an algebraically closed field of characteristic different from 2, then $\operatorname{mod}\Lambda$ **is** the mesh category of $\Gamma(\Lambda)$. But even in characteristic 2, the question has a positive answer for all algebras with at least one indecomposable module which is faithful. Thus, in general, we look at all the factor algebras $\Lambda/I(M)$, where $I(M)$ is the annihilator of an indecomposable module M, and we may consider Λ as being built up from these local data $\Lambda/I(M)$.

The irreducible maps usually considered are morphisms $f\colon M \to N$ with M or N indecomposable. It should be stressed that Bautista's paper [B7] considered irreducible maps in general, without the indecomposablity condition. It is proved that one may restrict to study irreducible maps of the form $f\colon M_1^{m_1} \to M_2^{m_2}$ with both M_1, M_2 indecomposable, and that such irreducible maps are related to definite bilinear forms as introduced by Hopf. A further study of this relationship was done by Brenner, Butler and King[5].

[5] See the paper *Irreducible maps and bilinear forms,* Linear Algebra and Applications 365 (2003), 99-105.

The Auslander-Reiten quiver. The idea of dealing with what now are called Auslander-Reiten components can be found in Auslander's second Philadelphia paper[6]. When I visited Brandeis in 1976, I proposed to visualize these components, and as examples I presented the preprojective components of the generalized Kronecker quivers. At that time, all the known Auslander-Reiten sequences had middle terms which are direct sums of at most two indecomposable modules and Auslander was wondering whether this could be a general feature, in contrast to these components which showed that there is no bound on the number of indecomposable summands. The new examples made the problem of describing the possible shapes of the Auslander-Reiten components much more intriguing. The first one who took up this challenge was Raymundo. When he visited Bonn in 1977, he brought with him some examples which he had calculated: Auslander-Reiten quivers of quivers of finite representation type[7]. Let us recall that the path algebra $k\Delta$ of a connected finite quiver Δ is representation-finite if and only if the underlying graph $\overline{\Delta}$ is one of the Dynkin diagrams A_n, D_n, E_6, E_7, E_8, as Gabriel has shown. It was interesting to see that the Auslander-Reiten quiver $\Gamma(k\Delta)$ is a full translation subquiver of $\mathbb{Z}\Delta$; with the indecomposable projectives as the boundary on one side, the indecomposable injectives as the boundary on the other side. What seemed to be strange at the time was the fact that these two boundary parts are not always parallel[8].

The knitting procedure. There is an inductive procedure in order to construct the Auslander-Reiten quiver (or at least parts of it) for various algebras. Since the Auslander-Reiten sequences yield what are called "meshes", this algorithm is called the "knitting of components". There does not seem to exist a written hint about the origin. Apparently it was used at many different places as early as 1977 - definitely at U.N.A.M. as well as at Bonn, but may-be also at Liverpool and Trondheim. Let X be an indecomposable module, and assume we know all the indecomposable modules W_i with an irreducible map $W_i \to X$ (thus either X is projective and we know its radical, or else X is not projective and we know the Auslander-Reiten sequence ending in X). Actually, we are only interested in those modules W_i which are non-injective, say that these are the modules $W_1, \dots, W_t$. We suppose in addition that for all these modules W_i the Auslander-Reiten sequences $0 \to W_i \to X_i \to Y_i \to 0$ are known (observe that X occurs as a direct summand of X_i, for any i), and that we know all the indecomposable projective modules $P_1, \dots, P_s$, such that X is a direct summand of $\operatorname{rad} P_i$, for $1 \le i \le s$. Then all the irreducible maps starting in X are of the form $X \to Y_i$ and $X \to P_j$, and we can construct in this way the minimal left almost split map $f\colon X \to r(X)$, where $r(X)$ is a direct sum of a suitable number of copies of the modules Y_i and P_j. Note that in case f is not injective, X is an injective module (and then f is

[6] *Applications of morphisms determined by modules.* In: Representation Theory of Algebras. Proceedings of the Philadelphia Conference 1976. Marcel Dekker (1978), section 6.

[7] He included these examples in his Brandeis paper [B7] mentioned above; but note that one of the examples is odd.

[8] This still was considered a mystery when Gabriel presented his survey on the Auslander-Reiten theory at ICRA II, Ottawa 1979, but is explained in the written version of these lectures (Springer LNM 831): For the Dynkin types A_n, D_{2n-1} and E_6, one has to use the (unique) diagram automorphism of order 2.

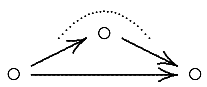
surjective having as kernel the socle of X). Of course, otherwise f and its cokernel yield an Auslander-Reiten sequence starting in X.

For any artin algebra Λ, the knitting algorithm constructs inductively the *reachable* modules, according to their distance to the simple projective modules. Those of distance 0 are the simple projective modules themselves; in case Z is indecomposable and not projective, and $\bigoplus_i Y_i \to Z$ is a minimal right almost split map with indecomposable modules Y_i, then Z is reachable with distance m provided all the modules Y_i are reachable and the maximum of the distances of these modules Y_i is $m-1$.

Note that the algorithm for constructing reachable modules may stop for two reasons: either we have obtained all the indecomposables, thus Λ is representation-finite and has a directed Auslander-Reiten quiver (Λ is then said to be *representation-directed*), or else we encounter an indecomposable direct summand X_i of the radical of an indecomposable projective module P such that some other indecomposable direct summand of $\operatorname{rad} P$ is not reachable.

The s-condition. The knitting procedure works well in case any indecomposable projective module P to be considered has an indecomposable radical $\operatorname{rad} P$ or, more generally, in case all the indecomposable direct summands of $\operatorname{rad} P$ are isomorphic. The so called s-condition of Bautista-Larrion-Salmerón [BLS] provides a criterion for dealing with the case when $\operatorname{rad} P$ has non-isomorphic indecomposable direct summands. Here, we assume that we deal with an algebra Λ given by a directed quiver Δ with relations. Note that the indecomposable projective modules $P(i)$ are indexed by the vertices i of Δ (here $P(i)$ is the projective cover of the simple module concentrated at the vertex i). The s-condition for $P(i)$ deals with the support of the indecomposable direct summands of $\operatorname{rad} P(i)$: one has to require that these subquivers are not only disjoint, but actually belong to different connected components of the quiver which is obtained from Δ by removing all the proper successors of i. If Λ is connected and representation-finite, then all the indecomposable projective modules satisfy the s-condition if and only if $\Gamma(\Lambda)$ is simply-connected (this explains the letter s).

Let us exhibit an easy example of what may happen in case the radical of an indecomposable projective module is the direct sum of two non-isomorphic modules. We start with the path algebra Λ_0 of a quiver of Dynkin type A_2. There are up to isomorphism precisely three indecomposable Λ_0-modules: a simple projective module S, a simple injective module T and a length 2 module I with socle S and top T. Let Λ be the "one-point-extension" of Λ_0 using $S \oplus T$, thus Λ is given by the following quiver with the indicated zero relation:



The algebra Λ is representation-finite (there are 9 isomorphism classes of indecomposables), however there are only two isomorphism classes of reachable modules, those of S and I. In particular, the indecomposable projective module P with radical $S \oplus T$ belongs to a cyclic path, since both $\operatorname{Ext}^1(P/T, T)$ and $\operatorname{Ext}^1(T, P/T)$ are non-zero.

The structure of Auslander-Reiten components. For several years, Raymundo's work was devoted to the problem of determining the possible structure of

Auslander-Reiten components. Some of these investigations deal with the global structure, others focus the attention to local properties. We have already mentioned his interest in knitting components. What one obtains in this way are the so-called preprojective components, and, using the dual procedure, the preinjective components. Algebras with preprojective or preinjective components are of special interest since the corresponding modules are given by combinatorial data and thus can be handled quite easily. On the other hand, many algebras which are needed in applications are of this kind, in particular, any hereditary algebra has a preprojective and a preinjective component.

Actually, in case Λ is a hereditary algebra, the structure of all the components is known. We may assume that Λ is connected and wild[9]. There is one preprojective and one preinjective component, the remaining components are of the form $\mathbb{Z}A_\infty$. This result is contained in a paper by Auslander, Bautista, Platzek, Reiten and Smalø [A-S], but also in a paper of mine: again one of the situations where different mathematicians were trying to compete. The question was raised at the 1977 Oberwolfach conference in an evening lecture, where I outlined such a result for a special class of hereditary algebras (those with large "growth number"). The long list of names involved in the subsequent investigations shows the great interest in this question.[10] After all, components of the form $\mathbb{Z}A_\infty$ yield only Auslander-Reiten sequences where the middle terms are direct sums of at most two indecomposable modules, in accordance with Auslander's expectation.

Sectional paths. Most of the advances of the representation theory of artin algebras in the last 30 years are based on combinatorial investigations, dealing with quivers and posets, with corresponding quadratic forms and root systems, the most decisive ones seem to be those concerning the structure of Auslander-Reiten components. The Auslander-Reiten quiver of an artin algebra is a translation quiver: this means, it is a locally finite quiver $Q = (Q_0, Q_1)$ with an injective map $\tau : Q_0' \to Q_0$ where Q_0' is a subset of Q_0 such that the number of arrows $y \to z$ is equal to the number of arrows $\tau z \to y$, for every vertex y; in a translation quiver, the vertices which do not belong to Q_0' are said to be projective, those which do not belong to $\tau(Q_0)$ are said to be injective. By now, there is a vast literature dealing with translation quivers which arise for artin algebras (but also for posets and vectorspace categories, for isolated singularities or orders), and all these investigations rely on the pioneering work of Bautista. One of the basic ideas which he introduced concerns the study of sectional behaviour [B3]: of sections and slices, of sectional paths and sectional cycles. The idea is to study subquivers of a translation quiver which contain precisely one, or only at most one, representative from each τ-orbit.

Let us start with the notions of sectional paths and sectional cycles. By definition, a path in a quiver $Q = (Q_0, Q_1)$ of length n is of the form $(x_0, x_1, \dots, x_n)$, where x_i are elements of Q_0, such that there is at least one arrow $x_{i-1} \to x_i$, for

[9] If Λ is connected and tame, then one has a similar statement: there is one preprojective and one preinjective component, the remaining components are so called tubes: they are obtained from translation quivers of the form $\mathbb{Z}A_\infty$ by factoring out some power of the translation.

[10] In his Ottawa lectures in 1979, Gabriel put oil into this priority fight by praising only my work: but actually my first attempt for a general proof was incomplete, as he himself has pointed out, and my final solution was completed only **after** the paper [A-S] was sent around!

every $1 \leq i \leq n$. Now, a path $(x_0, x_1, \ldots, x_n)$ in a translation quiver is said to be *sectional* provided $\tau x_{i+1} \neq x_{i-1}$, for all $0 < i < n$.

The translation quivers which arise as Auslander-Reiten quivers of artin algebras do not have loops (since one knows that irreducible maps are either monomorphisms or epimorphisms, but of course never isomorphisms), thus in this case a path is sectional if and only if any subpath of length 2 contains at most one representative from any τ-orbit. It is easy to exhibit sectional paths (x_0, x_1, x_2, x_3) with $x_0 = \tau x_3$; consider the path algebra of the quiver $\begin{tikzcd}[sep=small] & \circ \arrow[dr] & \\ \circ \arrow[ur] \arrow[rr] & & \circ \end{tikzcd}$, let $x_0 = (001), x_1 = (011), x_2 = (112), x_3 = (122)$ (here we specify isomorphism classes of indecomposable modules by noting the corresponding dimension vector, note that the modules mentioned are uniquely determined in this way).

A path $(x_0, x_1, \ldots, x_n)$ is called cyclic provided $x_n = x_0$. Such a cyclic path is said to be sectional, provided it is a sectional path and, in addition, we also have $\tau x_1 \neq x_{n-1}$ (or, equivalently, provided also the path $(x_1, \ldots, x_n, x_1)$ is sectional). Note that there do exist paths which are both cyclic and sectional, but are not sectional cyclic paths: Let Λ be a uniserial local k-algebra of length n, say $\Lambda = k[T]/\langle T^n \rangle$, where $k[T]$ is the polynomial ring in one variable T with coefficients in a field k and $n \geq 2$. The regular representation $P = {}_\Lambda\Lambda$ is an indecomposable module which is both projective and injective. Let R be its radical; this again is an indecomposable module and R is isomorphic to the factor module of P modulo its radical. Thus the inclusion map $R \to P$ as well as the projection $P \to R$ are both irreducible maps and $([P], [R], [P])$ is a sectional path in $\Gamma(\Lambda)$. On the other hand, τR is isomorphic to R, thus $([R], [P], [R])$ is not sectional.

As we have seen, there do exists paths which are both cyclic as well as sectional, but Bautista and Smalø [BS] have shown that an Auslander-Reiten quiver never contains sectional cyclic paths. This has several important consequences, let us note at least the following: If X, Y are indecomposable modules and if there exists irreducible maps $X \to Y$ and $Y \to X$, then X is isomorphic to τX or Y is isomorphic to τY (an indecomposable module Z which is isomorphic to τZ is said to be homogeneous, thus the claim is that at least one of the modules X, Y is homogeneous). For, if neither X nor Y is homogeneous, then $([X], [Y], [X])$ is a sectional cyclic path, but this is not possible.

Replication Numbers. We now turn to the question of finding properties of $\Gamma(\Lambda)$, where Λ is a representation-finite k-algebra and k is algebraically closed. The first such property is the famous four-in-the-middle theorem of Bautista and Brenner [BB1]: it asserts that for a representation-finite algebra, the middle term of any Auslander-Reiten sequence decomposes into at most 4 indecomposable modules, and in case it decomposes into 4 indecomposable modules, then precisely one of these summands is both projective and injective, whereas the remaining ones are neither projective nor injective.

This result turns out to be a special case of a very general result which sheds light on the number 4 and which incorporates also other observations of Raymundo. Namely, we may reformulate this result as follows: the "replication number" of the Euclidean diagram $\widetilde{D}_4$ is equal to 1. If Γ is a translation quiver and M is a subset of Γ_0, we may apply powers of τ and τ^{-1} to M and we say that the *replication number* of M is t provided there are natural numbers $p, q \geq 0$ with $n = p + q + 1$, such that the following properties are satisfied:

(1) The set $\tau^p M$ contains a projective vertex,

(2) The sets $\tau^i M$ for $0 \le i < p$ do not contain any projective vertices.
(3) The sets $\tau^{-i} M$ for $0 \le i < q$ do not contain any injective vertices.
(4) The set $\tau^{-q} M$ contains an injective vertex.
If M is a subset of Γ_0 which does not contain a projective vertex, we may say that the subsets M and τM are parallel, and we may extend this concept to an equivalence relation on the set of subsets of Γ_0. If the replication number of some subset M is n (where n is a natural number), then the parallelity class of M contains precisely n sets. The subsets M we are interested in are the supports of sectional subquivers with underlying graph a Euclidean diagram: here, a subquiver Δ is called sectional, provided any path in Δ is sectional, and the Euclidean diagrams are the graphs $\widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$ (also called "extended Dynkin diagrams" or "affine diagrams") which occur in Lie theory as simply laced Cartan data of affine Kac-Moody algebras.

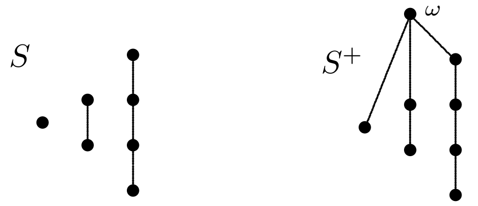
Now, Bautista and Brenner [BB2] have shown the following remarkable facts: If Γ is the Auslander-Reiten quiver of a representation-finite algebra over an algebraically closed field, and Δ is a sectional subquiver of Γ, then the replication number is finite and there is an effective bound b in terms of Lie theory. The bound is optimal, corresponding examples can be constructed using tilting theory. In fact, the combinatorial information used here may be expressed in terms of $\mathbb{Z}\Delta$: of interest are the pairs (x, y) of vertices in $\mathbb{Z}\Delta$ with a path from x to y, such that $\mathrm{Hom}(x, y) = 0$ in the mesh category $k\langle \mathbb{Z}\Delta \rangle$.

As we have mentioned, a special case is the four-in-the-middle theorem, here we consider the Euclidean diagram $\widetilde{D}_4$. The case $\widetilde{A}_1$ concerns the case when dealing with a pair of vertices x, y and two arrows $x \to y$. In this situation, the replication number is 0. This means: the existence of two indecomposable modules X, Y such that the bimodule of irreducible maps from X to Y is at least two-dimensional, implies that Λ is representation-infinite.

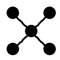
Finite Posets. When dealing with representations of quivers, one often encounters the problem to find, for a given vector space V_ω, a basis which is compatible with several subspaces V_i of V_ω. These problems are called subspace problems and one knows that already the 5-subspace problem is wild. However, one may be in the fortunate situation to know some inclusion relations between the given subspaces. If S is a finite poset, one calls $(V_\omega, V_i)_{i \in S}$ an S-space provided V_ω is a vector space, any V_i is a subspace of V_ω, and $i \le j$ in S implies $V_i \subseteq V_j$. A general theory of S-spaces was established parallel to the development of the representation theory of artin algebras, with substantial contributions by the Kiev school of Nazarova and Roiter, by Gelfand and Ponomarev, by Gabriel and Loupias, as well as Brenner and Butler. Many of the techniques which have been developed in the representation theory of algebras have analogies for the category of S-spaces of a given poset S, but there are also genuine methods which seem to work only for S-spaces, such as the Kiev differentiation algorithms. As we have mentioned, questions in the representation theory of artin algebras lead to problems on S-spaces, but there is also a transfer in the opposite direction: Let S^+ be obtained from S by adjoining a largest element (say ω), thus we may consider any S-space $(V_\omega, V_i)_{i \in S}$ as a representation of the incidence algebra $I(S^+)$ of S^+; thus we may consider the category of S-spaces as a full subcategory of $\mathrm{mod}\, I(S^+)$.

The study of S-spaces puts some old geometrical considerations into their proper context. For example, if $C_t = \{s_1 < s_2 < \cdots < s_t\}$ is a chain, then a C_t-

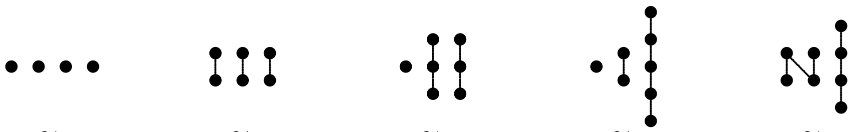
space is nothing else than a vector space with a filtration (or a "flag") consisting of t subspaces. Let us consider the special case when S is the disjoint union of three chains C_1, C_2 and C_4, as depicted to the left. Then S^+ is the partially ordered set shown to the right, note that $I(S^+)$ is just the path algebra of a quiver of Dynkin type E_8



(these drawings are the corresponding Hasse diagrams).

A poset S is said to be *subspace-finite*, provided there are only finitely many isomorphism classes of indecomposable S-spaces. In case the incidence algebra $I(S^+)$ is representation-finite, S is subspace-finite. However, the converse is not true. For example, the poset:  is subspace-finite, however $I(S^+)$ is representation-infinite.

Here is Kleiner's list of the minimal subspace-infinite posets S:



$\widetilde{D}_4$ $\widetilde{E}_6$ $\widetilde{E}_7$ $\widetilde{E}_8$ $\widetilde{E}_8$

For all these posets S, the incidence algebra $I(S^+)$ is a tame concealed algebra, it is a tilted algebra of the indicated Euclidean type (actually, in all cases but the last, $I(S^+)$ is hereditary).

As we have seen, the incidence algebra of a poset need not to be hereditary, it is given by a quiver with all possible commutativity relations. In order to deal with such algebras (but also with related ones which are of interest when we consider a "species" instead of a quiver) Bautista has introduced the notion of l-hereditary: A finite dimensional algebra Λ is said to be *l-hereditary* provided any local submodule of a projective module is projective again (a local module is by definition a module with a unique maximal submodule). There is the following equivalent condition: if $f\colon P \to Q$ is a non-zero homomorphism, where P, Q are indecomposable projective modules, then f is a monomorphism.

Representation-finite incidence algebras have been investigated already in the early seventies by several mathematicians (in particular Kleiner and Loupias); in all cases it turned out to be quite easy to determine inductively all the indecomposable modules. The reason is explained by the following result of Bautista which he published in 1981 [B5]: Any representation-finite l-hereditary algebra Λ is representation-directed. Thus we are in a setting where we can use the knitting procedure in order to construct all the indecomposable modules!

The investigation of l-hereditary algebras and their representations has been a leading theme of Raymundo's collaboration with several other mathematicians; let us name at least Martinez, Simson, Kleiner and Norieta. This work stretched over a long period. As we have mentioned, the category of S-spaces is a full subcategory of the category $\operatorname{mod} I(S^+)$, thus one may wonder whether suitable full subcategories of a module category have (relative) Auslander-Reiten sequences. Again, this is a question which has attracted a lot of interest by many mathematicians, and which was raised for the first time by the Mexican school. In the Antwerp paper of 1979, Bautista and Martinez gave the positive answer for this question for the category of S-spaces.

We have seen that it is of interest to deal with posets with a unique maximal element. In order to have good duality properties, one may assume that we deal with posets with the additional property of having a unique minimal element. The incidence algebra $I(S)$ of such a poset S (with a unique minimal and a unique maximal element) is an l-hereditary 1-Gorenstein algebra, and such algebras were studied very carefully by Bautista and his collaborators. The importance of the 1-Gorenstein condition is well-known in ring and module theory, one can read it off from the various and quite different characterizations of these algebras (the injective hull of the regular representation is projective; there exists a faithful module which occurs as a direct summand of any faithful module; there exists a faithful module which is both projective and injective,...) as well as from the fact that many different names have been introduced as labels (the 1-Gorenstein algebras are the algebras of dominant dimension at least 1, they are also just the QF-3 algebras in the sense of Thrall).

Final Remark. I have quoted at the beginning Raymundo's comment on the 1976 lectures of Auslander in Mexico: *We were impressed mainly in the part of the lectures related to almost split sequences, then recently discovered by M. Auslander and I. Reiten.* It is clear that this part of the lectures had a very decisive impact on Raymundo's further work, with relations to all the developments which we have covered in this lecture. His pioneering investigations on the structure of Auslander-Reiten quivers have to be highly praised: Bautista has to be considered as one of the architects of the modern representation theory of artin algebras which is based on the combinatorics of the Auslander-Reiten quiver.

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