Combinatorial Representation Theory
History and Future

Claus Michael Ringel

ABSTRACT. This is an outline of the combinatorial approach to the representation theory of finite-dimensional algebras, as it was developed and successfully used during the last thirty years. We will present an overview of the essential ideas and open problems. The blooming period say between 1970 and 1985 was followed by a sort of stagnation: with still hundreds of papers but no real final results. This slow down seems to be surprising if one keeps in mind the many open, but feasible problems and the wealth of new methods not yet used to their full potentiality. These notes are based on two invited lectures given by the author at the International Conference on Representations of Algebras ICRA 9, held at Beijing Normal University 2000.

0. The Basic Examples: Representation-finite Algebras.

Let k be a field. The algebras to be considered will be associative k-algebras with sufficiently many idempotents, usually they will be finite-dimensional (and then with unit element). Given such a k-algebra A, representation theory considers *representations* of A, these are the algebra homomorphisms $\phi\colon A \to \mathrm{End}_k(V)$ where V is a k-space and $\mathrm{End}(V)$ denotes the algebra of all endomorphisms of V. Equivalently, one may deal with the map $\overline{\phi}\colon A \times V \to V$ adjoint to ϕ; the vector space V together with the bilinear map obtained in this way is called an A-*module.* For a fixed algebra A we denote by $\operatorname{mod} A$ the category of all finite-dimensional A-modules, it is a *length category:* an abelian category such that every object has finite length.

Recall that an algebra A is said to be *representation-finite,* provided there are only finitely many isomorphism classes of indecomposable A-modules, and then all indecomposable A-modules are finite-dimensional and any A-module can be written as a direct sum of indecomposables. In case k is algebraically closed, a basic result due to Bautista, Gabriel, Roiter and Salmeron [BGRS] asserts that a representation finite k-algebra A has a multiplicative Cartan basis $\mathcal{B}$: the multiplicativity means that the product of two elements of $\mathcal{B}$ is either zero or belongs again to $\mathcal{B}$, to say that we deal with a Cartan basis means that $\mathcal{B}$ contains sufficiently many primitive idempotents and that $\mathcal{B}$ contains a generating set for the radical NA of A (and then we find in $\mathcal{B}$ generating sets for all the powers of the radical of A). Of course, algebras with a multiplicative basis are combinatorial objects, but their representation theory usually will depend on the given field k, in particular on the characteristic of k: the modular representation theory of finite groups gives ample evidence (and by the very definition these groups algebras have a multiplicative basis). It is the requirement to deal with a Cartan basis which allows a combinatorial construction of the representations of A and then even of the category of all A-modules. The relevant references are the following: Let us first assume that we deal with what is called a standard algebra A, then we may invoke Gabriel's covering theory [G3]: it yields a G-grading on A, where G is a finitely generated free group, with the following two properties: every A-module is gradable and the category of graded A-modules is directed. As we want to stress, there is a strong relationship between directedness conditions on length categories and the possibility to describe them in combinatorial terms. Starting with a representation-finite k-algebra A, where k is algebraically closed, covering theory reduces the classification problem for the indecomposable A-modules to the corresponding problem for k-algebras which are in addition tilted algebras [HR], and the latter problem is well-known to be a purely combinatorial one: it amounts to knit preprojective translation quivers. The procedure of covering, tilting and knitting is very powerful and provides a clear picture of the individual representations as well as the structure of the category $\operatorname{mod} A$ itself. In particular, Dräxler [Dx1] was able to show along these lines that for any finite dimensional representation $\phi\colon A \to \operatorname{End}_k(V)$, we can choose a basis of V such that the only entries of the matrices $\phi(b)$ with $b \in \mathcal{B}$ are 0 and 1. Similarly[1], we can write any A-module as the cokernel of a map between free modules whose entries are just $0, 1$ and -1. Of course, such assertions stress the combinatorial nature of the objects we are dealing with.

We have assumed that A is a standard algebra. Only in case the characteristic of k is 2, there do exist algebras A which are not standard (see Riedtmann [Rm]) and then we cannot directly use covering theory. However, according to Bongartz [Bn], A will still have sufficiently many standard factor algebras in the following sense: given an indecomposable A-module M, there exists an ideal I which annihilates M such that A/I is standard. This means that all the indecomposable modules are combinatorially given, and the problem of dealing with non-standard algebras is only to patch together a finite number of combinatorially given categories, indeed a feasible and still combinatorial situation.

In order to describe the category $\operatorname{mod} A$ globally, one relies on investigations of Auslander and Reiten which lead to the so called Auslander-Reiten quiver of A, its vertices are the isomorphism classes of the indecomposable A-modules, and the arrows indicate the existence of irreducible maps. Indeed, for a standard algebra the Auslander-Reiten quiver yields a presentation of the category $\operatorname{mod} A$ by generators and relations: Note that an additive category is an algebraic object like a group or a ring, thus one may look for presentations by generators and relations. In fact, an additive category is really just a ring if one extends the only partially defined composition by zero in order to obtain an everywhere defined multiplication; thus, additive categories are often called "rings with several objects".

[1] We are grateful to Brüstle, Simson and Crawley-Boevey for pointing this out after the lecture.

We have outlined here in which way representation-finite algebras over an algebraically closed field may be handled combinatorially; as starting point one has to refer to the treatment of hereditary algebras by Gabriel in 1972 [G1] using quivers (in order to describe the algebras) and positive roots (in order to describe the indecomposables). For any object M in a length category $\mathcal{A}$, the first invariant to be mentioned is its class in the Grothendieck group $K_0(\mathcal{A})$ of objects in $\mathcal{A}$ modulo exact sequences; note that $K_0(\mathcal{A})$ is free abelian with basis $\mathcal{S}$ the set of isomorphism classes of simple objects. We usually will denote the class of M in $K_0(\mathcal{A})$ by $\dim M$ and call it the *dimension vector;* the coefficients of $\dim M$ in terms of the basis $\mathcal{S}$ are the Jordan-Hölder multiplicities of the simple objects in M. Of course, these Jordan-Hölder multiplicities are combinatorial invariants, whereas the study of the class of all objects with a fixed dimension vector tends to be of algebraic-geometrical nature.

But combinatorial methods are by no means restricted to representation-finite cases; indeed, already the 1970 classification of pairs of annihilating operators on a vector space due to Gelfand and Ponomarev [GP] used a combinatorial description of the indecomposables as strings and bands, namely words in some alphabet.

1. The Basic Setting: Algebras, Modules, Categories.

Let us describe the general procedure of the combinatorial approach to representation theory. As we have mentioned in the previous section, we deal successively with

- algebras,
- modules,
- categories,

and aim at a combinatorial description at all three levels. Actually, this is a repetitive scheme[2], since the categories obtained are additive categories, thus they may be considered again as algebras (with several objects) and one may start anew. To put it differently, given an algebra A, the category $\operatorname{mod} A$ may be just considered as a functor category with values in the category of all k-spaces, and we may continue to consider functor categories $\operatorname{mod}\operatorname{mod} A$ and so on. On the other hand, instead of dealing with the whole category $\operatorname{mod} A$ of all finite-dimensional A-modules, it may be appropriate to look for small subcategories and their behavior. In particular, if we are interested in a finite number of indecomposables $M_1, \dots, M_m$, the full subcategory of all their direct sums is encoded in the endomorphism ring E of the direct sum $\bigoplus_{i=1}^m M_i$; and such an E is again a finite dimensional algebra. These successive steps from algebras to modules to algebras form the core of representation theory, and we are going to discuss in which way combinatorial data at one stage yield combinatorial ones at the next step.

[2] The repetitive character of comparing A and $\operatorname{mod} A$ by iterating this process and looking also at $\operatorname{mod}\operatorname{mod} A$ and $\operatorname{mod}\operatorname{mod}\operatorname{mod} A$ and so on, was stressed in particular by Auslander, for an outline of some relevant features we refer to [R8].

What kind of combinatorial data one may use in order to describe algebras, modules and additive categories?

Algebras. The idea of describing finite-dimensional algebras by quivers and relations is due to Gabriel [G1], but the concept of a quiver and its representations is much older: it was used by Grothendieck under the name diagram scheme in order to deal with say commutative diagrams as they appear everywhere in homological algebra. Note that two rings are said to be Morita equivalent provided their module categories are equivalent; finite dimensional algebras A, B are Morita equivalent if and only if the categories $\operatorname{mod} A$ and $\operatorname{mod} B$ are equivalent. Any finite dimensional algebra A is Morita equivalent to a basic algebra B (*basic* means that B/NB is a finite product of division rings) and B is unique up to isomorphism. Thus, we always may assume that we deal with a basic algebra A. In case the base field k is algebraically closed, A is the factor algebra of the path algebra kQ of a finite **quiver** modulo an admissible ideal I (to be admissible means $J^m \subseteq I \subseteq J^2$, for some $m \geq 2$, where J is the ideal of kQ generated by the arrows). Such an ideal is generated by linear combinations $\sum_{w \in W(a,b)} c_w w$, where $W(a,b)$ is the set of paths w starting at a and ending at b, with length $l(w)$ bounded by $2 \leq l(w) \leq m$, coefficients $c_w \in k$ and arbitrary vertices a, b. Note that the quiver Q is uniquely determined by A, and this is a purely combinatorial invariant. There are usually many possible choices for I and its generators and only for very special choices of the coefficients c_w one may interpret A as being combinatorially given. This is the case if one may choose coefficients c_w in $\{0, 1\}$ (or in $\{0, 1, -1\}$). In particular, we should mention the incidence algebra of a finite **poset** P, here one deals with a quiver without oriented cycles and without multiple arrows, and one takes as generators for I all possible differences $w - w'$, where w, w' are paths starting at a vertex and ending at some other vertex.

Modules. As we have mentioned already, the first invariant of a module of finite length is its **dimension vector,** often it turns out that the dimension vectors for the indecomposable A-modules are just the **roots** of a quadratic form q defined on the Grothendieck group $K_0(A)$. The root systems which one encounters in this way are related to those which one knows from Lie theory and here one has a very fruitful connection to be discussed at the end of these lectures.

But there are other ways to use combinatorial concepts for characterizing or specifying indecomposable modules. In the case of a special biserial algebra, one follows Gelfand and Ponomarev [GP] using **words** in some finite alphabet (given by the arrows in the quiver and formal inverses of the arrows); the modules which can be described in this way are called strings and bands and the corresponding word indicates in which way a suitable basis of the algebra operates on a suitable basis of the module in question. For the so called clannish algebras one has to modify this procedure slightly, taking into account the internal symmetry of such words. In general, for any representation of a quiver one may work with matrices and consider the corresponding **coefficient quiver** [R7], a well-known object of interest in linear algebra.

Additive Categories. The main combinatorial data for describing additive categories are **translation quivers.** As long as we deal with a category with almost split sequences such as the category $\operatorname{mod} A$ where A is an artin algebra, there is defined its Auslander-Reiten quiver $\Gamma(A)$. The shape of its components often gives a lot of insight. Since an artin algebra A has only finitely many isomorphism classes of indecomposable modules which are projective or injective, all but finitely components of $\Gamma(A)$ are stable.

Some kinds of components allow to recover the precise structure of the corresponding modules without any further knowledge. In particular, this is true for preprojective and for preinjective components (in case we work over an algebraically closed base field).

The study of tame algebras and their representations tends to be the study of one-parameter families of homogeneous tubes (a homogeneous tube is a component of the form $\mathbb{Z}A_\infty/\langle\tau\rangle$). The modules belonging to a homogeneous tube will be said to be *homogeneous* modules, those on the mouth of the tube will be said to be *primitive homogeneous.* Note that in a homogeneous tube there is a unique primitive homogeneous module M and all the modules in the tube have a filtration with all factors isomorphic to M. Let A be a tame k-algebra, k algebraically closed. According to Crawley-Boevey [C1], for any natural number d almost all indecomposable A-modules of dimension d are homogeneous.

Other translation quiver which have been used in combinatorial representation theory are all kinds of **hammocks,** see in particular [RV] and [Sr].

CREP. The combinatorial approach to representation theory allows an effective use of computer algorithms. Such programs have been developed by several mathematicians and a package of programs has been made available by Dräxler under the name CREP, an abbreviation of what is also title for these lectures: Combinatorial REPresentation theory. We refer to the manuals [DN1] and the survey [DN2] by Dräxler and Nörenberg.

2. The representation type of an algebra.

We have mentioned above that a representation-finite algebra over an algebraically closed field always has a multiplicative Cartan basis. If P is any property an algebra may have or not have, we say that an algebra A is minimal with the property P, provided A has this property, but any proper factor algebra does not have property P. For example, an algebra A is said to be minimal without a multiplicative Cartan-basis, provided A does not have multiplicative Cartan-basis, but any proper factor algebra of A has one.

Problem 1. *Determine all minimal algebras without a multiplicative Cartan-basis.*

Of course, according to [BGRS], all algebras without a multiplicative Cartan-basis have to be representation-infinite. But having the list of all minimal algebras without a multiplicative basis, it should be easy to show directly that these algebras are representation-infinite. Thus, a solution of problem 1 should yield a new (and hopefully shorter) proof for the multiplicative basis theorem.

When looking at the representation type of an algebra, there clearly is a hierarchy of behavior, starting with the representation-finite ones, then the so-called domestic ones, and so on, finally ending with the wild ones. The hierarchy of complication should not be thought as a linear ordering of complications (see [R5]), but it clearly is a partial ordering. Now, given such a property P of complicatedness, one may be interested to know all the algebras not having this property. But in general it turns out that there may be a vast number of such algebras, whereas the number of minimal algebras having the property P may be rather small.

For example, the last 25 years have seen a strong endeavor to find all tame algebras, but it seems that such a list should be really large and thus quite useless. In contrast, one may hope to be able to deal with the minimal wild algebras. It seems that if A is minimal wild, then the number $s(A)$ of isomorphism classes of simple A-modules should be quite small:

Problem 2. *Are there minimal wild algebras A with $s(A) > 10$?*

The answer should be NO. When dealing with algebras with a preprojective component, the Unger list [U] provides typical minimal wild algebras; of course, for all of them we have $s(A) \leq 10$. Note that the minimal wild algebras A with $s(A) = 1$ (this means that A is local) have been classified in [R1] - there are only few and anyone satisfies $(NA)^3 = 0$. The partial results in order to classify the minimal wild algebras A with $s(A) = 2$ due to Hoshino and Miyachi [HM] have now been completed by Brüstle and Han [BH,H2].

A more subtle minimality concept has been introduced by Nagase. Again, let P be an algebra property. We say that A is Nagase-minimal with respect to P provided A is minimal with respect to this property, and, in addition, if B is an algebra with property P and there is a full exact embedding $\operatorname{mod} B \to \operatorname{mod} A$, then $s(B) \geq s(A)$. Of course, the second condition usually will reduce the number of cases: there will be less Nagase-minimal algebras than minimal algebras. For example, there is only one Nagase-minimal strictly wild algebra, namely the path algebra $K(3)$ of the quiver

$$\circ \;\substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow}\; \circ$$

whereas there are many minimal strictly wild algebras. On the other hand, there is the following obvious observation:

Lemma. *Let A be an algebra with property P. Then there is a Nagase-minimal algebra B with respect to P and a full exact embedding* $\operatorname{mod} B \to \operatorname{mod} A$.

Proof, by induction on $s(A)$ and the dimension of A. If A is not minimal with respect to P, then there is a proper factor algebra A' of A with property P, and, of course, there is a full and exact embedding $\operatorname{mod} A' \to \operatorname{mod} A$. Note that the dimension of A' is smaller than that of A. If A is minimal with respect to P and A is not Nagase-minimal with respect to B, then there exists an algebra B with $s(B) < s(A)$ and a full exact embedding $\operatorname{mod} B \to \operatorname{mod} A$.

If we are interested in module theoretical properties of all the algebras with property P, then the Lemma shows that usually it will be sufficient to consider just the Nagase-minimal algebras with respect to B.

Problem 3. *Determine all Nagase-minimal wild algebras.*

What happens if we bound the nilpotency index of the radical NA of A, say if we assume that $(NA)^t = 0$? And how much information about A is encoded into $A/(NA)^t$ for a given t?

Let us start with $t = 1$. The condition $NA = 0$ means that A is semisimple, thus all the A-modules are semisimple: the only indecomposable A-modules are the simple ones. For a general algebra A, the modules annihilated by NA are the semisimple ones, thus we see that the information which we may recover form A/NA is just that concerning all the simple A-modules.

Next, let $t = 2$. If $(NA)^2 = 0$, then one knows that A is stably equivalent to a hereditary algebra, the reduction process was outlined by Auslander [A,AR1] and by Gabriel [G1]. Thus the indecomposable A-modules correspond to the indecomposable B-modules for some hereditary algebra B, and thus at least the corresponding dimension vectors are known by Kac [K]. The regular Auslander-Reiten components of A are just the same as the regular Auslander-Reiten components of B: they are of tree class A_∞. For a general algebra A, the A-modules annihilated by $(NA)^2$ are just those of Loewy length at most 2, thus the factor algebra $A/(NA)^2$ encodes all the information concerning A-modules of Loewy length 2. Note that the algebras A with $(NA)^2 = 0$ are very special. For example, such an algebra can be tame only if it is domestic!

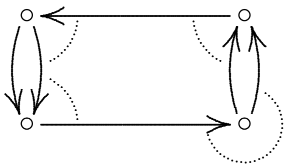
Let us consider $t = 3$. Whereas for $t \le 2$ the algebras A with $(NA)^t = 0$ where very special, it seems to us, that all the possible representation theoretical behavior occur already for $t = 3$.

Problem 4. *Determine the representation type of algebras with* $(NA)^3 = 0$. *Second: which algebras A have the same representation type as $A/(NA)^3$?*

Let k be algebraically closed and A a k-algebra. Recall that an algebra is tame, provided for every dimension d there is a finite number n of A-$k[T]$-bimodules M such that almost all primitive homogeneous modules of dimension d are of the form $M/M(T-\lambda)$ with $\lambda \in k$. The smallest number n will be denoted by $\pi_A(d)$. If $p = \max_d \pi_A(d)$ is finite, then A is said to be *domestic* or better *p-domestic.*

Problem 5. *Assume A is tame. Are all the regular components which are not tubes of the form $\mathbb{Z}A_\infty^\infty$ and $\mathbb{Z}D_\infty$?*

Crawley-Boevey's result implies that a tame algebra has at most countably many components which are not homogeneous tubes. There do exist already 2-domestic algebras which have countably many components which are of the form $\mathbb{Z}A_\infty^\infty$, for example



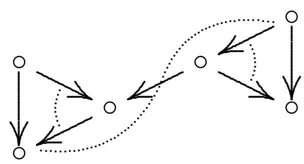
but no 1-domestic algebra with this property is known.

Problem 6. *Assume that A is 1-domestic. Are all but finitely many components homogeneous tubes?*

Actually, this may be the most urgent problem: To develop a structure theory for $\operatorname{mod} A$, where A is domestic. Domestic algebras are the closest relatives of those of finite type, and it seems to be of great interest in which way domestic module categories deviate from those of finite type. Actually, the usually infinite τ-orbits for a representation-infinite algebras should show a more regular behavior than the always finite τ-orbits of a representation-finite algebra. As long as the module categories even for domestic algebras are not yet understood, we are far away from a general theory of tame algebras. The most innocent question to be asked seems to be the following:

Problem 7. *What are the domestic algebras A with a faithful homogeneous tube?*

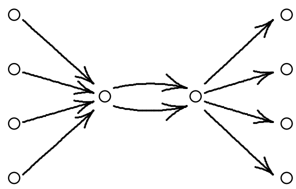
We conjecture that such an algebra has to be 1-domestic. There are several such algebras obtained from a connected tame hereditary algebra by glueing together parts of the preprojective and the preinjective components. Here is a less trivial example:



(with two zero relations of length 2 and one of length 3). It has a family of homogeneous tubes, index by $k \setminus \{0\}$ and one planar component with a hole.

In general, the study of algebras with faithful homogeneous tubes should be very rewarding. Of course, they usually will not be domestic. For example, all the tubular algebras belong to this class, the pg-critical ones, and many non-domestic special biserial ones. There are also wild algebras which have faithful homogeneous tubes. For example, let B be a self-injective local algebra and let $A = B[T]/\langle T^2, NB \cdot T\rangle$. Then $B = A/TA$ (considered as an A-module) belongs to a faithful homogeneous tube in $\Gamma(A)$.

It should be stressed that combinatorial representation theory is mainly concerned with combinatorial invariants of modules, not of algebras. Algebras are less combinatorial objects than modules, even if we work with algebras k, where k is an algebraically closed field. The case of representation-finite algebras is definitely misleading: here, all the modules are given by 0-1-matrices, but also the algebras themselves have a multiplicative basis, thus they also can be exhibited using as structure constants just 0 and 1. But if we consider tame algebras, the situation gets more complicated: it seems that all the indecomposables can be exhibited by matrices using as entries $0,\ 1$ and at most one additional $\lambda \in k$. whereas there are families of connected tame algebras depending on several parameters, for example there is a 5-parameter family of tame algebras with the following quiver



with eight relations of length 2; there is one which starts at any of the four sources, and one which ends at any of the four sinks. Thus, the algebras are obtained from the Kronecker algebra by forming four tubular extensions and four tubular coextensions and we require that the extension modules and the coextension modules belong to pairwise different tubes. Altogether we see that eight tubes are fixed, thus eight points on a projective line. Three of these points may be labeled $0, 1, \infty$, then the remaining 5 points are invariants. Note that all the algebras are tame, they are iterated tubular. Deleting one sink and one source we obtain a still three-parameter family of 1-domestic algebras.

Problem 8. *Assume A is a tame algebra with a faithful homogeneous tube. Does there exist a basis $\mathcal{B}$ of A such that for any indecomposable representation ϕ of A, if we consider the set of matrices $\phi(b)$ with $b \in \mathcal{B}$ with respect to a suitable basis, all but at most one of all the entries are 0 or 1.*

Let us add one question concerning wild hereditary algebras (the tree modules considered here are those as introduced in [R7]: there are appropriate bases so that the coefficient quivers are trees).

Problem 9. *Let d be a positive root. Is there an indecomposable tree module with dimension vector d? If d is imaginary, then there should be more than one isomorphism classes of indecomposable tree modules with dimension vector d.*

3. Controlled Embedding and Fractal Behavior of Algebras.

Recall that additive categories may be considered just as rings. When comparing different rings, one may ask whether one is isomorphic to a subring or a factor ring of the other. Thus, comparing additive categories, we may ask whether one is isomorphic (or at least equivalent) to a subcategory or a factor category of the other. Such questions have to be considered when we deal with the old concept of wildness of algebras. First of all, we note the following: To assert that an additive category $\mathcal{A}$ has many different kinds of exact subcategories (without further specification of these embeddings) is not really of interest, just look at the category $\mathcal{A} = \operatorname{mod} k$ of all finite dimensional k-spaces: for any k-algebra B, the category $\operatorname{mod} B$ embeds as an exact subcategory into $\mathcal{A}$ via the forgetful functor. The strongest assertion seems to be to require that for any k-algebra B, there is a **full** and exact embedding of $\operatorname{mod} B$ into $\operatorname{mod} A$; then A will be said to be *strictly wild.* It is clear that a strictly wild algebra A will have A-modules with prescribed endomorphism ring: if we need a module with endomorphism ring B, just take the image of the regular representation ${}_BB$ under a full embedding $\operatorname{mod} B \to \operatorname{mod} A$. However, there are obvious examples of algebras which have to be called "wild", but which are not strictly wild. For example, no local k-algebra A can be strictly wild, since the endomorphism ring E of any A-module is quite special: for example, the only central idempotents of such a ring E are 0 and 1. In particular $k \times k$ can never occur as endomorphism ring of an A-module. The old concept of wildness was based on the condition that the wild algebras A should be characterized by the property that any k-algebra B occurs as a factor ring of the endomorphism ring of an A-module. Let us call the algebras with this property the algebras *with prescribed endomorphism rings.*

The notion of wildness used nowadays is due to Drozd [Dr]. It is based on the use of tensor functors $F = M \otimes_B -$, where M is an A-B-bimodule, with B a strictly wild k-algebra: often one uses $B = k\langle X, Y\rangle$, the free algebra in two generators, or, if one prefers to work with finite-dimensional algebras, $B = K(3)$. Since one wants that F is an exact embedding, one has to require that M_B is faithful and projective. Without further conditions on M we are in the situation of having just an exact embedding of $\operatorname{mod} B$ into $\operatorname{mod} A$, a quite useless fact. What Drozd requires in addition is that F respects indecomposability and detects isomorphy. Unfortunately, it does not seem to be obvious that the wild algebras in the sense of Drozd are just the algebras with prescribed endomorphism rings. Indeed, if A is wild, how can one use a given embedding functor $F = M \otimes_B -$ in order to produce factor algebras of endomorphism rings? But also the opposite implication is not clear at all: after all, there do exist tame algebras where already small modules have quite large endomorphism rings; what is needed here are results which provide restrictions on the endomorphism rings say of indecomposable modules over tame algebras. Only few results are known in this direction. In particular, Krause [Kr] discussed the endomorphism rings of the string modules for a special biserial algebra and showed that the factor algebras with two generators are very restricted.

Drozd's wildness definition is based on prescribing subrings of endomorphism rings, the classical concept aimed at a realization as factor rings. An optimal solution would be to obtain a semidirect product: a subring which is complemented by an ideal. In order to present the relevant definition, we need the following preparation: Let $\mathcal{C}$ be a set (or class) of objects in the additive category $\mathcal{A}$. Given objects A, A' in $\mathcal{A}$, we denote by $\operatorname{Hom}_{\mathcal{A}}(A, A')_{\mathcal{C}}$ the set of maps $A \to A'$

in $\mathcal{A}$ which factor through a finite direct sum of objects in $\mathcal{C}$. We obtain in this way an ideal in the category $\mathcal{A}$, more precisely we obtain an ideal which is generated by idempotents (namely the identity maps of the objects in $\mathcal{C}$); thus we call it the ideal generated by $\mathcal{C}$. As usual, we denote by $\operatorname{add}\mathcal{C}$ the additive closure of $\mathcal{C}$, it consists of all finite direct sums of direct summands of objects in $\mathcal{C}$. Clearly, $\operatorname{add}\mathcal{C}$ and $\mathcal{C}$ generate the same ideal.

Let $\mathcal{A}$, $\mathcal{B}$ be abelian (or at least exact) categories. We say that an embedding functor

$$F\colon \mathcal{B} \to \mathcal{A}$$

is *controlled* by a class $\mathcal{C}$ of objects in $\mathcal{A}$ provided for all objects B, B' in $\mathcal{B}$, we have

$$\operatorname{Hom}_{\mathcal{A}}(F(B), F(B')) = F \operatorname{Hom}_{\mathcal{B}}(B, B') \oplus \operatorname{Hom}_{\mathcal{A}}(F(B), F(B'))_{\mathcal{C}}.$$

For rings, the parallel situation would be as follows. One has to consider rings with identity, without requiring that ring inclusions preserve the identity. A subring R of a ring S is a controlled subring provided there exists an idempotent e in S such that $1_R S 1_R = R \oplus 1_R SeS 1_R$.

In order to show that a functor $F\colon \mathcal{B} \to \mathcal{A}$ is controlled, one often proceeds as follows: First, select a full subcategory $\mathcal{U}$ of $\mathcal{A}$ which contains the image of $\mathcal{B}$, for example the full subcategory given by all the objects $F(B)$, where B is an object of $\mathcal{B}$. Next, look for a functor $G\colon \mathcal{U} \to \mathcal{B}$ such that $G \circ F$ is equivalent to the identity functor on $\mathcal{B}$. Finally, show that the kernel of G is the intersection of $\mathcal{U}$ with an ideal of $\mathcal{A}$ generated by some class $\mathcal{C}$ of objects.

Lemma. *Let $\mathcal{A}$ be a length category. Let $F\colon \mathcal{B} \to \mathcal{A}$ be an embedding, controlled by $\mathcal{C} = \operatorname{add}\mathcal{C}$. If B is indecomposable in $\mathcal{B}$, then $F(B) = A \oplus U$ where A is indecomposable and does not belong to $\mathcal{C}$, whereas U belongs to $\mathcal{C}$. If B, B' are non-isomorphic objects of $\mathcal{B}$, and $F(B) = A \oplus U$, $F(B') = A' \oplus U'$ with A, A' indecomposable and not in $\mathcal{C}$, whereas $U, U' \in \mathcal{C}$, then A, A' are not isomorphic.*

For the proof, we just note the following: Since $\mathcal{A}$ is a length category, we may write every object A as a direct sum $A = A' \oplus A''$ with $A'' \in \mathcal{C}$ and such that A' has no indecomposable direct summand in $\mathcal{C}$. The controlled embedding functors F used in representation theory often have the additional property that F preserves indecomposability[3]. We say that a k-algebra A is *controlled wild* provided there is a controlled exact embedding $\operatorname{mod} K(3) \to \operatorname{mod} A$. Of course, any strictly wild algebra is controlled wild, since a full embedding is controlled by the zero module. Another class of example: any wild algebra A with $(NA)^2 = 0$ is controlled wild, with control class consisting of semisimple modules. It has been shown by Rosenthal and Han [H1] that all the wild local algebras are controlled wild.

[3] But it should be stressed that also other cases are definitely of interest, see for example wildness results for separable abelian groups [E]!

If A is controlled wild, say with an embedding functor $F\colon \operatorname{mod} K(3) \to \operatorname{mod} A$ controlled by $\mathcal{C}$, and if B is any k-algebra, then we may choose a $K(3)$-module M with endomorphism ring B (this is possible since $K(3)$ is strictly wild) and consider $F(M)$. Under F the endomorphism ring B of M embeds into $\operatorname{End}_A(F(M))$ and there is the complementary ideal $I = \operatorname{End}_A(F(M))_{\mathcal{C}}$, thus $\operatorname{End}_A(F(M))/I$ is isomorphic to B.

Problem 10. *Is every wild algebra controlled wild?*

For partial results see Han [H1] and Dräxler [Dx2]; in particular, note that these investigations show that the notion of controlled embeddings fits well to covering and cleaving functors.

Given two algebras A, B, one may ask whether there is a controlled exact embedding $\operatorname{mod} B \to \operatorname{mod} A$. In particular, one may wonder whether for a given algebra A there is a proper controlled exact embedding of $\operatorname{mod} A$ into itself (where proper means that the image of the functor is a proper subcategory of $\operatorname{mod} A$). If A is controlled wild, then clearly there do exist proper controlled embeddings $\operatorname{mod} A \to \operatorname{mod} A$. This is a kind of fractal behavior: we obtain a nice subquotient of the category which is equivalent to the category itself.

Example [RSr]. Let $k[X,Y]$ be the polynomial ring in two variables. We denote by $\mathcal{A}$ the category of all finite-dimensional $k[X,Y]$-modules such that both X and Y operate as nilpotent endomorphisms. *Let* $A = k[X,Y]/\langle XY, X^3, Y^3\rangle$. *There is a controlled exact embedding of* $\mathcal{A}$ *into* $\operatorname{mod} A$. Its restriction to $\operatorname{mod} A$ is a proper controlled exact embedding of $\operatorname{mod} A$ into itself.

As usual, we consider words $w = l_1 l_2 \dots l_{t-1} l_t$ with letters $l_i \in \{X, Y^{-1}\}$; in particular, let $z = XY^{-1}$. If $w = l_1 l_2 \dots l_{t-1} l_t$ is such a word, let $e(w) = z l_1 z l_2 z \cdots l_{t-1} z l_t z$. Example: the word $e(w)$, for $w = X^2 Y^{-1} X$

Similarly, if $w = l_1 \dots l_t$ is a primitive cyclic word, let $c(w) = z l_1 z l_2 z \cdots l_{n-1} z l_t$. Consider the functor F which sends $M(w)$ to $M(e(w))$ for any word w and which sends $M(w, \lambda, n)$ to $M(c(w), \lambda, n)$, where w is primitive cyclic, $\lambda \in k$ and $n \in \mathbb{N}_1$. We claim that F is a full embedding and controlled by the string modules of length at most 2.

For the proof, let $\mathcal{U}$ be the full subcategory of all modules M in $\mathcal{A}$ such that the kernels of the multiplications by X and by Y are contained in the radical of M. Define a functor

$$G\colon \mathcal{U} \longrightarrow \operatorname{mod} K[X,Y] \quad \text{by} \quad G(M) = (\operatorname{top} M, Y^{-1}X^2, X^{-1}Y^2),$$

where $\operatorname{top} M = M/\operatorname{rad} M$. The assumption that the kernel of Y is contained in $\operatorname{rad} M$ implies that $Y^{-1}X^2$ is an endomorphism of $\operatorname{top} M$, similarly $X^{-1}Y^2$ is an endomorphism of $\operatorname{top} M$. It is easy to see that $G \circ F$ is the identity on $\mathcal{A}$, thus it remains to consider the kernel of G. Given two modules M, M' in $\mathcal{U}$ and a map $f\colon M \to M'$, then $G(f) = 0$ if and only if the image of f is contained in $\operatorname{rad} M'$. However, for every object M'' in $\mathcal{A}$, the radical of $F(M'')$ is a direct sum of copies of the three string modules $M(1), M(X), M(Y)$ of length at most 2. This completes the proof.

Problem 11. *Let A be a tame algebra. For which algebras B does there exist a controlled exact embedding* $\operatorname{mod} B \to \operatorname{mod} A$ *?*

We may call A, B controlled equivalent provided there are controlled exact embeddings $\operatorname{mod} A \to \operatorname{mod} B$ and $\operatorname{mod} B \to \operatorname{mod} A$. The corresponding equivalence classes seem to be of interest, they provide a hierarchy of algebras which should be studied carefully.

Problem 12. Let $\mathcal{L}$ be the exact subcategory of $\operatorname{mod} K(2)$ of all $K(2)$ modules without simple direct summands (thus, a $K(2)$-module M belongs to $\mathcal{L}$ if and only if the radical of M is equal to the socle of M). *Given a tame connected algebra A over an algebraically closed field, is any homogeneous tube in the image of a controlled exact embedding* $\mathcal{L} \to \operatorname{mod} A$ *?*

4. Artin Algebras.

When dealing with problems in representation theory, the usual procedure was to look first for a solution in the case of an algebraically closed base field and only afterwards for the general case. The general case always introduces some additional complications, but one never did encounter insurmountable difficulties to extend the results. For example, when dealing with the representation-finite hereditary algebras, Gabriel discovered that in case the base field is algebraically closed one has to deal with quivers of type A_n, D_n, E_6, E_7, E_8; in the joint work with Dlab we extended this to arbitrary base fields and it turned out that one obtains in addition the remaining Dynkin diagrams B_n, C_n, F_4, G_2. For k being algebraically closed, the algebra itself and all their representations are given purely combinatorially; in contrast, when dealing with the additional cases B_n, C_n, F_2, a field extension of degree 2, in case G_2 a field extension of degree 3 comes into play, a really non-combinatorial ingredient which plays a role both for the algebra as well as its representation. However, as soon as we fix the algebra and thus the necessary field extension, the corresponding module category can be derived in a purely combinatorial way, without any further non-combinatorial construction.

A very interesting result of Crawley-Boevey [C2] has to be mentioned: *Assume that A is a representation finite algebra, and M an indecomposable A-module. Then there exists a simple A-module S such that* $\operatorname{End}(M)/N\operatorname{End}(M)$ *is isomorphic to* $\operatorname{End} S$.

It clearly is desirable to delete the condition of dealing with an algebraically closed base field, but up to this point our discussion still concentrated on k-

algebras, where k is a field. But as we know from the work of Auslander and Reiten, all the basic results of modern representation theory are valid in the broader context of artin algebras, and there is a definite need to do so: one of the basic sources for artinian rings is in number theory, and the rings A considered there are k-algebras where k is a factor ring of the integers, such that A as a k-module has finite length. But already in abelian group theory there are several problems for which the use of methods from the representation theory of artin algebras seems to be very helpful.

Given an artin algebra A, it is a quite natural, but usually hopeless question to ask for the classification of all the indecomposable A-modules. Only for few classes of algebras such a classification is known, one of the first such class were the serial (or generalized uniserial or Nakayama) algebras: here one requires that the indecomposable projective A-modules as well as the indecomposable projective A^{op}-modules are serial (i.e. have a unique composition series). This then implies that all the indecomposable modules are serial, and are uniquely determined by the length and the isomorphism class of the socle. The prominent examples of such algebras are the proper factor rings of $\mathbb{Z}$, thus artin algebras which definitely are not algebras (in case the radical is non-zero). It is the combinatoric of partitions and the corresponding Young diagrams which is used in order to deal with questions concerning representations of serial algebras. We will return to these considerations in the last section.

Let me draw attention to a more complicated classification problem in abelian group theory, this report is based on joint work with Schmidmeier [RSm]. Let Λ be an artin algebra. We denote by $\mathcal{S}(\Lambda)$ the category of pairs (M, U), where M is a finitely generated Λ-module and $U \subseteq M$ is a submodule of M; a map $f\colon (M, U) \to (M', U')$ in $\mathcal{S}(\Lambda)$ is just a Λ-linear map $f\colon M \to M'$ such that $f(U) \subseteq U'$. The case of $\Lambda = \mathbb{Z}/p^n\mathbb{Z}$ with p a prime number and n a positive integer has attracted a lot of interest since the categories $\mathcal{S}(\mathbb{Z}/p^n\mathbb{Z})$ describe the possible subgroups of finite abelian p-groups.

Problem 13. *Assume that A is a representation-finite artin algebra. Is it possible to obtain* $\operatorname{mod} A$ *from A purely combinatorially?*

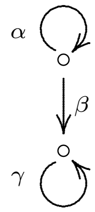
Let A be an artin algebra. Of course, we can assume that A/NA is a product of division rings. One expects that for a sincere representation-finite artin algebra A, at most two different division rings occur as factors of A/NA and in case two different ones do occur, then one should be an extension of the other of degree at most 2. Of course, a structure theory for artin algebras in general never will be purely combinatorial, since division rings will be involved.

Problem 14. *Definition of one-parameter families for tame artin algebras.*

Whereas for k-algebras A with k being separable one may hope to get all one-parameter families by looking for controlled embeddings $\operatorname{mod} B \to \operatorname{mod} A$, where B is a finite-dimensional tame hereditary algebra with two simple modules, it seems to be more difficult to describe the source for the one-parameter families for a general artin algebra. For example, it will be necessary to take into account also categories of a form like $\operatorname{mod} B$, where $B = \mathbb{Z}[T]/\langle p^2, pT, T^2\rangle$.

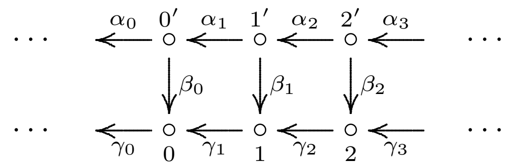
Let me return to the abelian group theory problem mentioned above. In order to handle the problem of finding the representation type of $\mathcal{S}(\Lambda)$, where $\Lambda = \mathbb{Z}/p^n\mathbb{Z}$, it is helpful first to look at the corresponding case where $\Lambda = k[T]/T^n$ with k a field. The objects in $\mathcal{S}(k[T]/T^n)$ are just the representations of the quiver

Q



which satisfy the relations $\alpha\beta = \beta\gamma$, and $\gamma^n = 0$ such that the map β is an inclusion map. Using the universal covering

$\widetilde{Q}$



one can show that $\mathcal{S}(k[T]/T^n)$ has finitely many indecomposables, for $n \le 5$, is tame for $n = 6$ and wild for $n \ge 7$. In fact, one can present a complete classification of the indecomposable objects in $\mathcal{S}(k[T]/T^n)$ for $n \le 6$. Note that $\mathcal{S}(k[T]/T^n)$ is an exact category, and one can show easily that there are sufficiently many relative projective objects as well as sufficiently many relative injective objects and they coincide; also, it is a category with almost split sequences. The stable Auslander-Reiten quivers have tree class A_2, D_4, E_6, E_8 for $n = 2, 3, 4, 5$ respectively; for $n = 6$ one obtains a tubular behavior, of tubular class $\widetilde{E}_8$. For $n \ge 7$, each of the categories $\mathcal{S}(k[T]/T^n)$ has wild representation type, but it is possible to determine the representation type of all the full categories $\mathcal{S}_m(k[T]/T^n)$ consisting of those representations of Q for which the additional relation $\alpha^m = 0$ is satisfied, see [RSm].

Having determined the representation types for the problems with $\Lambda = k[T]/T^n$, where k is a field, it is possible to transfer these results from the case $k = \mathbb{Z}/p\mathbb{Z}$ to the corresponding categories with $\Lambda = \mathbb{Z}/p^n\mathbb{Z}$. One obtains in this way, for all the finite and the tame cases, a natural bijection between the indecomposable objects in the corresponding categories.

5. Changing Algebras.

If we start with a well-behaved algebra and slightly change the structure constants, one may expect to keep the well-behavior. For the property of being representation-finite, this is a well-known result of Gabriel [G2]: Finite representation type is open. For tameness, the corresponding assertion is not yet known.

There is however a partial result in this direction due to Geiß [Ge]: If an algebra B is a degeneration of an algebra A and B is tame, then also A is tame. What is missing is the following: Assume we have a one-parameter family of algebras, and one of the algebras is tame, does this imply that almost all of these algebras have to be tame? Note that Kasjan recently has shown that the class of tame algebras is axiomatizable (in first order language) and that finite axiomatizability would be equivalent to know that tame type is open [Kj].

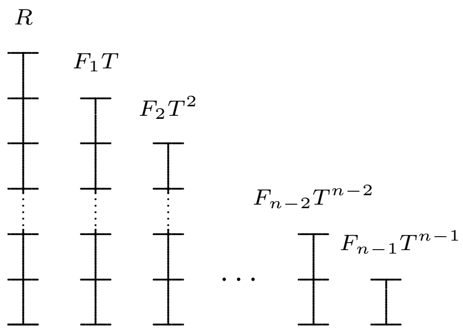
As we have mentioned above, it is the class of domestic algebras which deserves to be studied in detail.

Problem 15. *Fix some $n \in \mathbb{N}_0$. Is the class of algebras which are m-domestic with $m \leq n$ open?*

Filtered rings. Let R be a ring and $R = F_0 \supseteq F_1 \supseteq \cdots \supseteq F_n = 0$ a chain of ideals with $F_iF_j \subseteq F_{i+j}$. Then we call (R, F) a filtered ring. For example, if I is a nilpotent ideal, say $I^n = 0$, we may take $F_i = I^i$. Given a filtered ring $R = (R, F)$, consider the polynomial ring $R[T]$ and note that

$$\widetilde{R} = \bigoplus\nolimits_{i \geq 0} F_iT^i \subseteq R[T]$$

is a subring of $R[T]$.



We also consider the ideal

$$J = \bigoplus\nolimits_{i \geq 0} F_{i+1}T^i \subseteq \widetilde{R}.$$

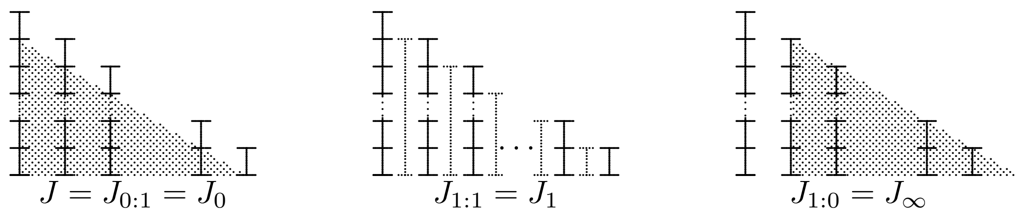
If a, b belong to the center of R, we may consider the maps $u_{a:b} \colon J \to \widetilde{R}$ defined by

$$u_{a:b}(rT^i) = arT^{i+1} + brT^i = rT^i(aT + b).$$

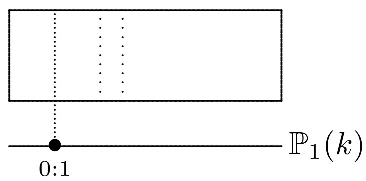
We denote by $J_{a:b}$ the image of $u_{a:b}$, this is an ideal of $\widetilde{R}$ and we define $R_{a:b} = \widetilde{R}/J_{a:b}$. We note the following:

Consider first the special index $0 : 1$. Clearly, $J_{0:1} = J$, thus $R_{0:1}$ is the graded ring $R_{0:1} = \mathrm{gr}(R, F) = \bigoplus_{i \geq 0} F_i/F_{i+1}$ with respect to F.

Next, let $a \neq 0$, thus we may assume $a = 1$. The ideal $J_{1:(-\lambda)}$ is the kernel of the composition ϵ_λ of the inclusion $\widetilde{R} \to R[T]$ and the evaluation map $R[T] \to R$, $f(T) \mapsto f(\lambda)$. Note that ϵ_λ is surjective, thus $\widetilde{R}/J_{1:\lambda} \simeq R$. (Proof: Note that $J_{1:\lambda}$ is contained in the principal ideal of $R[T]$ generated by $(T-\lambda)$, and we have $R \oplus J_{1:\lambda} = \widetilde{R}$.)



What we have constructed in this way is a one-parameter family of rings $R_{a:b} = \widetilde{R}/J_{a:b}$ indexed over the projective line $\mathbb{P}_1(k)$, where $R_{0:1} = \mathrm{gr}(R, F)$ is the graded ring with respect to F, whereas all the remaining rings are isomorphic to R.



If R is a finite-dimensional k-algebra, it follows that $\mathrm{gr}(R, F)$ is a degeneration of the algebra R (if we consider the variety of the k-algebras of dimension $\dim R$, then the orbit of $\mathrm{gr}(R, F)$ is contained in the closure of the orbit of R). According to Geiß [Ge], this has to following consequence: *If $\mathrm{gr}(R, F)$ is tame, then also R itself is tame.*

Application 1. Assume that R is a finite-dimensional k-algebra and take $F_i = (NR)^i$. If $\mathrm{gr}(R, F)$ is tame, then R is tame. In this way, we recover the Geiß result that the quaternions (and similar algebras) are tame.

Let us return to Problem 4 asking for the representation type of algebras A with $(NA)^3 = 0$ and for the relationship between the representation type of A and that of $A/(NA)^3$. Note that in [BH] the following is shown: Let k be an algebraically closed field and A a finite dimensional k-algebra A with at most 2 simple modules and with no loop in its quiver. Then A is tame if and only if $A/(NA)^3$ is tame if and only if A degenerates to a biserial algebra.

Application 2. We also may take other filtrations: Take an arbitrary finite dimensional k-algebra, take its quiver, and add arbitrary positive integers to the arrows. In this way, any path has a degree, let F_i be the ideal generated by all paths of degree i.

For example, consider Erdmann's example of an algebra A with two vertices a, b and arrows $\alpha\colon a \to a$, $\beta\colon a \to b$, $\gamma\colon b \to a$ and $\eta\colon b \to b$, with relations

$$\gamma\beta = \eta^2, \quad \beta\eta = \alpha\beta\gamma\alpha\beta, \quad \eta\gamma = \gamma\alpha\beta\gamma, \quad \alpha^2 = \beta\gamma\alpha\beta\gamma + (\beta\gamma\alpha)^2$$

and all paths of length 7. If we define $d(\alpha) = d(\beta) = d(\gamma) = 1$ and $d(\eta) = 2$, we obtain as corresponding graded algebra a special biserial algebra. This yields a proof of the Geiß result that the algebra A is tame, again avoiding calculations.

There is the following general observation: a commutative square $\alpha\beta = \gamma\delta$ may be destroyed in forming the graded algebra, provided we attach numbers with $d(\alpha) + d(\beta) \neq d(\gamma) + d(\delta)$. This is possible provided $\{\alpha, \beta\} \neq \{\gamma, \delta\}$. Of interest here is the case of the local algebra (c) in [R1].

Problem 16. Which algebras degenerate to a special biserial algebra?

When considering degenerations of algebras and modules one always has to keep in mind that algebras and modules behave quite differently with respect to degenerations. Just recall that for algebras, the semisimple ones are open, whereas the semisimple modules form a closed set.

6. Algebraic Combinatorics.

The prototype of the considerations which we will discuss here is the classical Hall algebra introduced by Ph. Hall as the "algebra of partitions" in 1975, but actually introduced (and in the meanwhile forgotten) by Steinitz in 1900. Let us start with a discrete valuation ring Λ with radical $N\Lambda$ and denote by $\mathcal{A}$ the category of finite length Λ-modules. Since the indecomposable Λ-modules of finite length are uniquely determined by their length, we see that the set of isomorphism classes of indecomposables in $\mathcal{A}$ can be identified with the set $\mathbb{N}_1$ of natural numbers, the number $n \in \mathbb{N}_1$ corresponding to the Λ-module $\Lambda/(N\Lambda)^n$. Therefore, the set of isomorphism classes of objects in $\mathcal{A}$ can be identified with the set of partitions. For example, if $\lambda = (\lambda_1, \dots, \lambda_t)$ is a partition (thus all the λ_i are natural numbers and $\lambda_1 \geq \lambda_2 \geq \cdots$), then this partition stands for the Λ-module $\bigoplus_i \Lambda/(N\Lambda)^{\lambda_i}$. It should be stressed that the identification of the isomorphism classes of objects in $\mathcal{A}$ with partitions serves as the guiding example for the combinatorial representation theory. Of course, we may interpret partitions as functions $\mathbb{N}_1 \to \mathbb{N}_0$ with finite support: here, $\mathbb{N}_1$ is the index set for the indecomposables in $\mathcal{A}$ and $a(n)$ denotes the multiplicity of $\Lambda/(N\Lambda)^n$ when we write the Λ-module corresponding to a as a direct sum of indecomposables.

The ring of symmetric functions is one of the fundamental objects of mathematics. It plays a role in seemingly different parts of mathematics. Its identification as the classical Hall-algebra allows the use of methods from representation theory in order to get a better understanding of this ring. As it turns out, combinatorial considerations concerning partitions can be interpreted well in terms of finite length modules over a discrete valuation ring.

Now consider the parallel situation of dealing with a root system Φ for a finite-dimensional complex semisimple Lie algebra $\mathbf{g}$ of Dynkin type Δ. Choose a Cartan subalgebra $\mathbf{h}$ and a root basis, thus we obtain a triangular decomposition $\mathbf{g} = \mathbf{n}_- \oplus \mathbf{h} \oplus \mathbf{n}_+$ and denote by Φ^+ the set of positive roots. Let U^+ be the universal enveloping algebra of $\mathbf{n}_+$. The dimension of the weight spaces U^+_λ is given by Kostant's partition function π, the value $\pi(\lambda)$ is the number of ways λ can be written as a sum of positive roots, thus of functions $a\colon \Phi^+ \to \mathbb{N}_0$ such that $\sum_\alpha a(\alpha) = \lambda$. For any quiver or species of type Δ, the indecomposable representations correspond bijectively to the positive roots, thus the isomorphism classes of finite length modules correspond bijectively to the functions $a\colon \Phi^+ \to \mathbb{N}_0$. If we deal with algebras defined over finite fields, we may consider the corresponding Hall algebras $\mathcal{H}(\vec{\Delta})$, and it turns out that after a multiplicative twist one obtains a ring which is independent of the chosen orientation. This twisted Hall algebra $\mathcal{H}_*(\vec{\Delta})$ is just the Drinfeld-Jimbo quantization U_q^+ of U^+, see [R3].

These considerations have been extended to arbitrary finite quivers or species. Green [Gr] has shown that in the general case, U_q^+ can be interpreted as a subring of the corresponding twisted Hall algebra $\mathcal{H}_*(\vec{\Delta})$, namely the subring generated by the simple representations. Recent investigations [SV, DX] show that $\mathcal{H}_*(\vec{\Delta})$ itself is a polynomial ring in countably many variables over U_q^+ and corresponds to a quantization of the positive part of a generalized Kac-Moody algebra as introduced by Borcherds [Bo]. On the other hand, it should be noted that it is possible to obtain the Drinfeld-Jimbo quantization of the universal enveloping algebra of all of $\mathbf{g}$ (and not only the positive part) by replacing the module category by the corresponding derived category.

When Green started to work on Hall algebras, he realized that it is possible to introduce a comultiplication similar to that of the classical Hall algebra and that one obtains in this way a twisted bialgebra. These considerations are valid for arbitrary hereditary categories defined over finite fields. Thus it will be worthwhile to consider the derived categories of the canonical algebras, since it is well-known that these equivalent to the derived categories of suitable hereditary abelian categories (introduced by Geigle and Lenzing [GL] as the categories of coherent sheaves over weighted projective lines). Whereas the quivers and species correspond to generalized Cartan matrices or generalized Cartan data, the tubular algebras correspond just to the generalized intersection matrices [Sl,SY] of type $D_4^{(1,1)}, E_6^{(1,1)}, E_7^{(1,1)}, E_8^{(1,1)}$. Thus, it seems that the Hall algebra approach will provide quantizations of the universal enveloping algebras of some of the elliptic Lie algebras first studied by Slodowy and now in great detail by Saito and Yoshii [SY].

Acknowledgment. The author is indebted to Th. Brüstle, Chr. Geiß, J. Schröer and the referee for remarks concerning the presentation of the paper. In particular, Th. Brüstle has outlined examples which helped to prevent some serious misunderstandings.

References.

[A] M. Auslander: Representation Dimension of Artin Algebras. Queen Mary College. Math. Notes. (1971)

[AR1] M. Auslander, I. Reiten: Stable equivalence of artin algebras. Springer LNM 353 (1973), 8-71.

[AR2] M. Auslander, I. Reiten: Representation theory of artin algebras. III. Comm. Algebra 3 (1975), 239–294; IV. Comm. Algebra 5 (1977), 443-518.

[BGRS] R. Bautista, P. Gabriel, A.V. Roiter, L. Salmeron. Representation-finite algebras and multiplicative bases. Invent. Math. 81 (1985), 217-285.

[Bn] K. Bongartz: Indecomposables are standard. Comment. Math. Helv. 60 (1985), 400-410.

[Bo] R. Borcherds: Generalized Kac-Moody algebras. J. Algebra 115 (1988), 501-512.

[BH] Th. Brüstle, Y. Han: Tame two-point algebras without loops. Erscheint in: Comm. Algebra. SFB-preprint 00-040

[C1] W. Crawley-Boevey: On tame algebras and bocses. Proc. London Math. Soc.(3), 56 (1988),451-483.

[C2] W. Crawley-Boevey: Matrix reduction for artinian rings, and an application to rings of finite representation type. Preprint 90-020 SFB 343 Bielefeld.

[DR] V. Dlab, C.M. Ringel: Indecomposable representations of graphs and algebras. Mem. Amer. Math. Soc. 173 (1976).

[DX] B. Deng, J. Xiao: On double Ringel-Hall algebras. Preprint.

[Dr] Y. Drozd: Tame and wild matrix problems. In: Springer LNM 832 (1980), 242-258.

[Dx1] P. Dräxler: Normal forms for representations of representation-finite algebras. Preprint 98-028 of SFB 343.

[Dx2] P. Dräxler: Cleaving functors and controlled wild algebras. Preprint 00-041 of SFB 343.

[DN1] P. Dräxler, R. Nörenberg: Classification problems in the combinatorial representation theory of finite-dimensional algebras. In: Progress in Mathematics 173 (1999)

[DN2] P. Dräxler, R. Nörenberg: CREP Manual, Parts 1 - 4 Volumes E 99-007, 00-005 of SFB 343.

[E] P.C. Eklof: Modules with strange decomposition properties. In: Infinite length modules. Trends in Mathematics. Birkhäuser Verlag. (2000). 75-87.

[G1] P. Gabriel: Unzerlegbare Darstellungen I. Manuscripta Math. 6 (1972), 71–103.

[G2] P. Gabriel: Finite representation type is open. Springer LNM 488 (1975), 132-155.

[G3] P. Gabriel: The universal cover of a representation-finite algebra. Springer LNM 903 (1981) 68-105.

[GL] W. Geigle, H. Lenzing: A class of weighted projective curves arising in the representation theory of finite dimensional algebras. Springer LNM 1273 (1987),

265-297.
[Ge] Chr. Geiß: On degenerations of tame and wild algebras. Archiv Math. 64 (1995), 11-16.
[GP] I.M. Gelfand, V.A. Ponomarev: Indecomposable representations of the Lorentz group. Russian Math. Surveys 23 (1968), 1-58.
[Gr] A.J. Green: Hall algebras, hereditary algebras and quantum groups. Invent. math. 120 (1995), 361-377.
[H1] Y. Han: Controlled wild algebras. Proc. London Math. Soc. (to appear)
[H2] Y. Han: Wild two-point algebras. Preprint.
[HR] D. Happel, C.M. Ringel: Tilted algebras. Trans. Amer. Math. Soc. 274 (1982), 399-443.
[HM] M. Hoshino, J. Miyachi: Tame two–point algebras. Tsukuba J. Math. 12 (1988), 65-96.
[K] V. Kac. Infinite root systems, representations of graphs and invaraint theory. Invent. Math. 56 (1989), 57-92.
[Kj] S. Kasjan: On the problem of axiomatization of tame representation type. Preprint.
[Kr] H. Krause: Endomorphismen von Worten in einem Köcher. Dissertation. Bielefeld 1991.
[Lu] G. Lusztig: Canonical bases arising from quantized enveloping algebras. J. Amer. Math. Soc. 3 (1990), 447-498
[N] H. Nagase: Controlled Wild Algebras and τ-Wild Algebras. SFB-Preprint-Reihe 00-074.
[P] L. Peng: IM-Lie algebras and Kac-Moody Lie algebras. Preprint.
[PX] L. Peng, J. Xiao: Triangulated categories and Kac-Moody algebras. Inventiones Math. 140 (2000), 563-603.
[Rm] Chr. Riedtmann: Many algebras with the same Auslander-Reiten quiver. Bull. London Math. Soc. 15 (1983), 43-47.
[R1] C.M. Ringel: The representation type of local algebras. In: Representations of Algebras. Springer LNM 488 (1975), 282-305.
[R2] C.M. Ringel: *Tame Algebras and Integral Quadratic Forms.* Springer LNM 1099 (1984).
[R3] C.M. Ringel: Hall algebras and quantum groups. Invent. Math. 101 (1990), 583-592
[R5] C.M. Ringel: The development of the representation theory of finite dimensional algebras. 1968 - 1975. In: London Math. Soc. Lecture Note Series 238. Cambridge University Press, Cambridge 1997, p.89-115.
[R6] C.M. Ringel, Recent advances in the representation theory of finite dimensional algebras, in: Progress in Math. 95. Birkhäuser-Verlag, Basel (1991), 141–192.
[R7] C.M. Ringel: Exceptional modules are tree modules. Lin. Alg. Appl. 275-276 (1998) 471-493.
[R8] C.M. Ringel: Infinite length modules. Some examples as introduction. In: Infinite length modules. Trends in Mathematics. Birkhäuser Verlag. (2000),

1-73.
[RSm] C.M. Ringel, M. Schmidmeier. In preparation.
[RSr] C.M. Ringel, J. Schröer. In preparation.
[RV] C.M. Ringel, D. Vossieck: Hammocks. Proc. London Math. Soc. (3) 54 (1987), 216-246.
[SY] K. Saito, D. Yoshii: Extended affine root systems IV (simply laced elliptic Lie algebras). Preprint (2000)
[Se] T. Scheuer: More hammocks. In: Topics in Algebra. Banach Centre Publ. 26. Warszawa (1990), 493-498.
[Sr] J. Schröer: Hammocks for string algebras. Dissertation. Bielefeld 1997.
[SV] B. Sevenhant, M. Van den Bergh: A relation between a conjecture of Kac and the structure of the Hall algebra. Preprint.
[Sl] P. Slodowy: Beyond Kac-Moody algebras and inside. Can. Math. Soc. Proc. 5 (1986), 361-371.
[U] L. Unger: The concealed algebras of the minimal wild hereditary algebras. Bayreuther Math. Schriften 31 (1990), 145-154.

Fakultät für Mathematik, Universität Bielefeld,
POBox 100 131, D-33 501 Bielefeld
E-mail address: `ringel@mathematik.uni-bielefeld.de`