Indecomposables live in all smaller lengths.

Claus Michael Ringel

Abstract. Let $\Lambda$ be a finite-dimensional $k$-algebra with $k$ algebraically closed. Bongartz has recently shown that the existence of an indecomposable $\Lambda$-module of length $n > 1$ implies that also indecomposable $\Lambda$-modules of length $n - 1$ exist. Using a slight modification of his arguments, we strengthen the assertion as follows: If there is an indecomposable module of length $n$, then there is also a constructible one. Here, the constructible modules are defined inductively, as follows: First, the simple modules are constructible. Second, a module of length $n \geq 2$ is constructible provided it is indecomposable and there is a submodule or a factor module of length $n - 1$ which is constructible.

Let $k$ be an algebraically closed field. Let $\Lambda$ be a finite-dimensional $k$-algebra, we may (and will) assume that $\Lambda$ is basic. We are interested in (finite-dimensional left) $\Lambda$-modules. A recent preprint [B3] of Bongartz with the same title is devoted to a proof of the following important result:

**Theorem (Bongartz 2009).** Let $\Lambda$ be a finite-dimensional $k$-algebra with $k$ algebraically closed. If there exists an indecomposable $\Lambda$-module of length $n > 1$, there exists an indecomposable $\Lambda$-module of length $n - 1$.

Unfortunately, the statement does not assert any relationship between the modules of length $n$ and those of length $n - 1$. There is the following open problem: **Given an indecomposable $\Lambda$-module $M$ of length $n \geq 2$. Is there an indecomposable submodule or factor module of length $n - 1$?** This is the case for $\Lambda$ being representation-finite or tame concealed, as Bongartz [B1, B2] has shown already in 1984 and 1996, respectively. Two remarks should be added:

1. It is definitely necessary to look both for submodules and factor modules, since for suitable algebras $\Lambda$, there are indecomposable modules $M$ which have no maximal submodules which are indecomposable. Any local module of length at least 3 and Loewy length 2 is such an example. And dually, there are indecomposable modules $M$ of length $n \geq 3$ such that all factor modules of length $n - 1$ are decomposable.

2. In case we weaken the assumption on the base field $k$, then we may find counterexamples. For example, let $k$ be the field with 2 elements, $Q$ the 3-subspace quiver (this is the quiver of type $\mathbb{D}_4$ with one sink and 3 sources) and $M$ the (unique) indecomposable $kQ$-module of length 5. There is also only one indecomposable $kQ$-module of length 4. Now $N$ cannot be a submodule of $M$, since we even have $\text{Hom}(N, M) = 0$. But $N$ is also not a factor module of $M$, since $\text{Hom}(M, N)$ is a 2-dimensional $k$-space and the three non-zero elements in $\text{Hom}(M, N)$ all have images of length 3.

The present note modifies slightly the arguments of Bongartz in [B3] in order to strengthen his assertion. We define inductively constructible modules: First, the simple

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modules are constructible. Second, a module of length \( n \geq 2 \) is constructible provided it is indecomposable and there is a submodule or a factor module of length \( n - 1 \) which is constructible. The problem mentioned above can be reformulated as follows: Are all indecomposable modules constructible?

**Theorem.** Let \( \Lambda \) be a finite-dimensional \( k \)-algebra with \( k \) algebraically closed. If there is an indecomposable module of length \( n \), then there is a constructible one of length \( n \).

As we have mentioned, for a representation-finite algebra all the indecomposable modules are constructible, thus we can assume that \( \Lambda \) is representation-infinite. According to Roiter’s solution of the first Brauer-Thrall conjecture, a representation-infinite algebra has indecomposable modules of arbitrarily large length, thus we have to show that \( \Lambda \) has constructible modules of any length, and we can assume that \( \Lambda \) is minimal representation-infinite (this means that \( \Lambda \) is representation-infinite and that any proper factor algebra is representation-finite).

According to Bongartz [B3, section 3.2] we only have to consider algebras with non-distributive ideal lattice: Namely, if \( \Lambda \) is minimal representation-infinite and the ideal lattice of \( \Lambda \) is distributive, then the universal covering is interval-finite and the fundamental group is free. Using covering theory, the problem is reduced to representation-directed and to tame concealed algebras, but for both classes all the indecomposable modules are constructible.

It seems to be surprising that here we deal with a question not yet settled only for algebras with non-distributive ideal lattice. After all, the class of algebras with non-distributive ideal lattice was the first major class of representation-infinite algebras studied in representation theory, see Jans [J], 1957.

Thus, let \( \Lambda \) be minimal representation-infinite and assume that the ideal lattice of \( \Lambda \) is non-distributive. Let \( J \) be the radical of \( \Lambda \). Then there are (not necessarily different) primitive idempotents \( e, e' \) and linearly independent elements \( \phi, \psi \) in \( eJe' \) such that \( J\phi = J\psi = \phi J = \psi J = 0 \).

Let \( I(e) \) be the injective envelope of the simple module \( \Lambda e/Je \). In \( I(e) \), there are elements \( x = e'x, y = e'y \) such that

\[
\phi x = 0, \quad u := \psi x = \phi y \neq 0, \quad \psi y = 0.
\]

Note that \( u \) is necessarily an element of the socle \( \text{soc} I(e) \). Let \( X = \Lambda x, Y = \Lambda y \) and \( V = X + Y \). Note that \( \phi(X + JY) = 0 \) as well as \( \psi(JX + Y) = 0 \).

We consider direct sums of copies \( V_{(i)} = V \), say \( V^n = \bigoplus_{i=1}^n V_{(i)} \). An element \( v \in V \) will be denoted by \( v_{(i)} \) when considered as an element of \( V_{(i)} \subseteq V^n \); similarly, a submodule \( U \subseteq V \) will be denoted by \( U_{(i)} \) when considered as a submodule of \( V_{(i)} \subseteq V^n \). For \( 1 \leq i < n \) let \( z_i = y_{(i)} + x_{(i+1)} \).

The following three submodules of \( V^n \) (with \( n \geq 1 \)) will be of interest:

\[
M(n-1) = \sum_{i=1}^{n-1} \Lambda z_i, \quad \text{for} \quad n \geq 2, \quad \text{and} \quad M(0) = \Lambda u, \\
R(n) = X_{(1)} + M(n-1), \\
W(n) = R(n) + Y_{(n)}.
\]
We want to refine the inclusion $M(n - 1) \subseteq W(n)$ by a chain of indecomposable submodules $U_i$, say

$$M(n - 1) = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_t = W(n),$$

such that $U_i/U_{i-1}$ is simple for $1 \leq i \leq t$.

We call an inclusion of modules $N \subseteq M$ uniform, provided any submodule $U$ with $N \subseteq U \subseteq M$ is indecomposable (this is related to the well-accepted notion of a uniform module: a module $M$ is uniform provided it is non-zero and any inclusion $N \subseteq M$ with $N \neq 0$ is uniform).

**Lemma.** The inclusions $M(n - 1) \subseteq JX(1) + M(n - 1)$ and $R(n) + JY(n) \subseteq W(n)$ are uniform, also the module $W(n)$ is indecomposable.

**Proof.** Let $M(n - 1) \subseteq U \subseteq JX(1) + M(n - 1)$ or $R(n) \subseteq U \subseteq R(n) + JY(n)$, or $U = W(n)$. We have to show that $U$ is indecomposable.

For $n = 1$ this is clear, since $W(1)$ is uniform and $M(0) \neq 0$. Thus, we can assume that $n \geq 2$. We will show: If $U = U' \oplus U''$, then the socle of $U$ is contained in one of the summands, say in $U'$, and therefore $U = U'$.

We claim that

$$(\ast) \quad \text{soc} U = \phi U + \psi U$$

First of all, since $U \subseteq V^n$, we see that \(\text{soc} U \subseteq \text{soc} V^n = ku(1) + \cdots + ku(n)\). Second, $U$ contains $M(n-1)$, thus the elements $z_1, \ldots, z_{n-1}$. Now $\phi U$ contains the elements $u(i) = \phi z_i$, for $1 \leq i \leq n - 1$, whereas $\psi U$ contains the elements $u(i+1) = \phi z_i$, for $1 \leq i \leq n - 1$. As a consequence, we see that the elements $u(1), \ldots, u(n)$ are contained in $\phi U + \psi U$ (since $n \geq 2$). Thirdly, $\phi U + \psi U \subseteq \text{soc} U$, since $J\phi = 0 = J\psi$. Altogether, we see that

$$\text{soc} U \subseteq ku(1) + \cdots + ku(n) \subseteq \phi U + \psi U \subseteq \text{soc} U,$$

this completes the proof of (\ast).

Recall that the Kronecker quiver is given by two vertices, say $a$ and $b$, and two arrows $a \to b$, the representations of the Kronecker quiver are called Kronecker modules, and the classification of the indecomposable Kronecker modules is known (see for example [R]).

Define a functor $F$ from the category of $\Lambda$-modules to the category of Kronecker modules, by attaching to a $\Lambda$-module $M$ the Kronecker module $F(M) = (M, \text{soc} M; \phi, \psi)$. Here again we use that the multiplication by $\phi$ and $\psi$ maps $M$ into $\text{soc} M$.

The structure of $F(U)$ for our $\Lambda$-modules $U$ is easy to determine:

(a) If $M(n - 1) \subseteq U \subseteq JX(1) + M(n - 1)$, then $F(U)$ is the direct sum of the indecomposable preprojective Kronecker module of dimension $2n - 1$ and several copies of the simple injective Kronecker module $T$.

(b) If $R(n) \subseteq U \subseteq R(n) + JY(n)$, then $F(U)$ is the direct sum of an indecomposable regular module of dimension $2n$ and again several copies of $T$.

(c) Finally, if $U = W(n)$, then $F(U)$ is the direct sum of the indecomposable preinjective Kronecker module of dimension $2n + 1$ and several copies of $T$.  

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In all cases we denote by $G$ the indecomposable direct summand of $F(U)$ which is different from $T$.

Let us calculate the radical of such a Kronecker module $F(U)$. For any $\Lambda$-module $M$, the radical $\text{rad} F(M)$ of $F(M)$ is equal to $(0, \phi M + \psi M; 0, 0)$, thus according to $(\ast)$, $\text{rad} F(U) = (0, \text{soc} M; 0, 0)$. On the other hand, $\text{rad} F(U)$ is just $\text{rad} G$ (since the copies of $T$ do not contribute to the radical of $F(U)$)

Now assume that $U$ is decomposable, say $U = U' \oplus U''$. Since $F$ is a functor, $F(U) = F(U') \oplus F(U'')$. We know that $F(U)$ is the direct sum of an indecomposable Kronecker module $G$ with socle of dimension $n$ and several copies of $T$. According to the Krull-Remak-Schmidt theorem, one of the summands, say $F(U')$ has to contain a direct summand $G'$ isomorphic to $G$, but then $\text{rad} G = \text{rad} G' \subset F(U')$. This shows that $\text{soc} U \subset U'$, and therefore $U = U'$. This concludes the proof of the Lemma.

Note that $R(n)/(M(n-1) + JX_1)$ and $W(n)/(R(n) + JY_n)$ are simple modules. Thus, maximal refinements of the inclusions

$$M(n-1) \subseteq M(n-1) + JX_1 \quad \text{and} \quad R(n) \subseteq R(n) + JY_n$$

yield a submodule chain $M(n-1) = U_0 \subset U_1 \subset \cdots \subset U_t = W(n)$, such that all the factors $U_i/U_{i-1}$ are simple for $1 \leq i \leq t$ and all the modules $U_i$ for $0 \leq i \leq t$ are indecomposable. In particular, we see: if $M(n-1)$ is constructible, then also $W(n)$ is constructible.

Using the dual arguments, we similarly see: if $W(n-1)$ is constructible, then also $M(n)$ is constructible. But $M(0)$ is a simple module, thus constructible. It follows by induction that all the modules $M(n)$ and $W(n)$ are constructible. This completes the proof of the theorem.

Remark. Note that in general the inclusion $M(n-1) \subset W(n)$ is not uniform. Consider for $\Lambda$ the Kronecker algebra $kQ$ itself, and look at the submodules $N, N'$ of $W(2)$ generated by the elements $z = x_{(1)} + y_{(1)} + x_{(2)} + y_{(2)}$ and $z' = x_{(1)} - y_{(1)} - x_{(2)} + y_{(2)}$, respectively. We have $\dim N = \dim N' = 2$. Assume now that the characteristic of $k$ is different from 2. Then $N \neq N'$ and even $N \cap N' = 0$. Thus $N \oplus N'$ is a decomposable submodule of $W(2)$. Also, $M(1)$ is contained in $N \oplus N'$ (as the submodule generated by $\frac{1}{2}(z - z')$).

References.


Fakultät für Mathematik, Universität Bielefeld
POBox 100 131, D-33501 Bielefeld, Germany
e-mail: ringel@math.uni-bielefeld.de