Indecomposables live in all smaller lengths.

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Abstract. Let Λ be a finite-dimensional k-algebra with k algebraically closed. Bongartz has recently shown that the existence of an indecomposable Λ-module of length $n>1$ implies that also indecomposable Λ-modules of length $n-1$ exist. Using a slight modification of his arguments, we strengthen the assertion as follows: If there is an indecomposable module of length n, then there is also an accessible one. Here, the accessible modules are defined inductively, as follows: First, the simple modules are accessible. Second, a module of length $n\geq 2$ is accessible provided it is indecomposable and there is a submodule or a factor module of length $n-1$ which is accessible.

Let k be an algebraically closed field. Let Λ be a finite-dimensional k-algebra, we may (and will) assume that Λ is basic. We are interested in (usually finite-dimensional left) Λ-modules. A recent preprint [B3] of Bongartz with the same title is devoted to a proof of the following important result:

Theorem (Bongartz 2009). *Let Λ be a finite-dimensional k-algebra with k algebraically closed. If there exists an indecomposable Λ-module of length $n > 1$, then there exists an indecomposable Λ-module of length $n - 1$.*

Unfortunately, the statement does not assert any relationship between the modules of length n and those of length $n - 1$. There is the following open problem: *Given an indecomposable Λ-module M of length $n \geq 2$. Is there an indecomposable submodule or factor module of length $n - 1$?*

Remarks. (1) This is the case for Λ being representation-finite or tame concealed, as Bongartz [B1, B2] has shown already in 1984 and 1996, respectively, but the answer is unknown in general. A positive answer would have to be considered as a strong finiteness condition — after all, if we consider for example any quiver of type $\mathbb{A}_\infty^\infty$, then there is a unique minimal faithful representation M, it is indecomposable, but all its maximal submodules as well as all the factor modules M/S with S simple, are decomposable.

(2) It is definitely necessary to look both for submodules and factor modules, since for suitable algebras Λ, there are indecomposable modules M which are not simple and have no maximal submodules which are indecomposable. Any local module of length at least 3 and Loewy length 2 is an example. And dually, there are indecomposable modules M of length $n \geq 3$ such that all factor modules of length $n - 1$ are decomposable.

(3) In case we weaken the assumption on the base field k, then we may find counterexamples. For instance, let k be the field with 2 elements, Q the 3-subspace quiver (this is the quiver of type $\mathbb{D}_4$ with one sink and 3 sources) and M the (unique) indecomposable kQ-module of length 5. There is also only one indecomposable kQ-module N of length 4. Now N cannot be a submodule of M, since we even have $\mathrm{Hom}(N, M) = 0$. But N is also not a factor module of M, since $\mathrm{Hom}(M, N)$ is a 2-dimensional k-space and the three non-zero elements in $\mathrm{Hom}(M, N)$ all have images of length 3. For dealing with an arbitrary field k, one may ask: *Given an indecomposable Λ-module M of length $n \geq 2$, is there an indecomposable module N of length $n - 1$, generated or cogenerated by M ?*

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The present note modifies slightly the arguments of Bongartz in [B3] in order to strengthen his assertion. We define inductively *accessible* modules: First, the simple modules are accessible. Second, a module of length $n \geq 2$ is accessible provided it is indecomposable and there is a submodule or a factor module of length $n-1$ which is accessible. The open problem mentioned above can be reformulated as follows: Are all indecomposable modules accessible? For a certain class of algebras, we are going to construct a suitable number of accessible modules of arbitrarily large length.

We call an inclusion of modules $M' \subseteq M$ *uniform*, provided any submodule U with $M' \subseteq U \subseteq M$ is indecomposable (this is related to the well-accepted notion of a uniform module: a module M is uniform provided it is non-zero and any inclusion $M' \subset M$ with $M' \neq 0$ is uniform). If $M' \subseteq M$ is a uniform inclusion, then $\operatorname{soc} M' = \operatorname{soc} M$. The converse is not true: the inclusion of a module M' into its injective envelope $E(M')$ is uniform only in case M' itself is uniform, however M' and $E(M')$ always have the same socle. If $M' \subseteq M$ is a uniform inclusion, then the module M is obtained from the indecomposable module M' by successive extensions (from above) using simple modules, with all the intermediate modules being indecomposable. In particular, if $M' \subseteq M$ is a uniform inclusion and M' is accessible, then also M is accessible. There is the dual notion of a couniform projection: If X is a submodule of M, then the canonical map $M \to M/X$ is said to be a *couniform projection* provided all the modules M/X' with X' a submodule of X are indecomposable. Of course, if $M \to M''$ is a couniform projection and M'' is accessible, then also M is accessible.

Our aim is to show that all representation-infinite algebras have accessible modules of arbitrarily large length. As Bongartz has pointed out (see the proof of the Corollary below), it is actually enough to look at non-distributive algebras. We recall that a finite-dimensional algebra is said to be *non-distributive* in case its ideal lattice is not distributive.

Theorem. *Let Λ be a non-distributive algebra. Then there are Λ-modules $M(n)$, $R(n)$, $W(n)$ and non-invertible homomorphisms*

$$\begin{array}{c} W(1) \leftarrow R(2) \leftarrow M(2) \rightarrow R(3) \rightarrow W(3) \leftarrow \dots \\ \dots \rightarrow W(2n-1) \leftarrow R(2n) \leftarrow M(2n) \rightarrow R(2n{+}1) \rightarrow W(2n{+}1) \leftarrow \dots \end{array}$$

where the arrows pointing to the left are couniform projections and those pointing to the right are uniform inclusions, and such that $W(1)$ is a uniform module.

By induction it follows that all these modules $M(n), R(n), W(n)$ are accessible. In particular, we see that *a non-distributive algebra Λ has accessible modules of arbitrarily large length.*

It seems to be surprising that here we deal with a very natural question that had not yet been settled for non-distributive algebras. Note that the class of non-distributive algebras was the first major class of representation-infinite algebras studied in representation theory, see Jans [J], 1957. Before we turn to the proof of the Theorem, let us derive the following consequence.

Corollary. *Let Λ be a finite-dimensional k-algebra with k algebraically closed. If there is an indecomposable module of length n, then there is an accessible one of length n.*

Proof of Corollary. As we have mentioned, for a representation-finite algebra all the indecomposable modules are accessible, thus we can assume that Λ is representation-infinite. According to Roiter's solution [R] of the first Brauer-Thrall conjecture, a representation-infinite algebra has indecomposable modules of arbitrarily large length, thus we have to show that Λ has accessible modules of any length. Clearly, we can assume that Λ is minimal representation-infinite (this means that Λ is representation-infinite and that any proper factor algebra is representation-finite).

According to Bongartz [B3, section 3.2] we only have to consider algebras with non-distributive ideal lattice: Namely, if Λ is minimal representation-infinite and the ideal lattice of Λ is distributive, then the universal cover is interval-finite and the fundamental group is free; using covering theory, the problem is reduced in this way to representation-directed and to tame concealed algebras, but for both classes all the indecomposable modules are accessible. This completes the proof of the Corollary.

From now on, let Λ be a non-distributive algebra and let J be the radical of Λ. Since the ideal lattice of Λ is non-distributive, there are pairwise different ideals $I_0, \dots, I_3$ such that $I_1 \cap I_2 = I_2 \cap I_3 = I_3 \cap I_1 = I_0$ and $I_1 + I_2 = I_2 + I_3 = I_3 + I_1$. We can assume that $I_0 = 0$, since with Λ also Λ/I_0 is non-distributive and the Λ/I_0-modules constructed can be considered as Λ-modules (annihilated by I_0). Note that the existence of I_3 implies that the ideals I_1 and I_2 (considered as Λ-Λ-bimodules) are isomorphic and we can assume that these bimodules are simple bimodules. But since Λ is a basic k-algebra and k is algebraically closed, a simple Λ-Λ-bimodule I is one-dimensional and there are primitive idempotents e, f of Λ (not necessarily different) such that $I = eIf$. Thus, taking generators ϕ of I_1 and ψ of I_2, these elements of Λ are linearly independent, there are primitive idempotents e, f of Λ such that $\phi = e\phi f$, $\psi = e\psi f$ and $J\phi = J\psi = \phi J = \psi J = 0$ (conversely, the existence of such elements $\phi, \psi \in \Lambda$ implies that Λ is non-distributive).

Let $E(e)$ be the injective envelope of the simple module $\Lambda e/Je$. In $E(e)$, there are elements $x = fx,\ y = fy$ such that

$$\phi x = 0, \quad u := \psi x = \phi y \neq 0, \quad \psi y = 0.$$

Note that u is necessarily an element of the socle of $E(e)$. Let $V = \Lambda x + \Lambda y \subseteq E(e)$

We consider direct sums of copies $V_{(i)} = V$, say $V^n = \bigoplus_{i=1}^n V_{(i)}$. An element $v \in V$ will be denoted by $v_{(i)}$ when considered as an element of $V_{(i)} \subseteq V^n$. For $1 \leq i < n$ let $z_i = y_{(i)} + x_{(i+1)}$.

The following three submodules of V^n (with $n \geq 1$) will be used:

$$\begin{aligned} M(n-1) &= \sum\nolimits_{i=1}^{n-1} \Lambda z_i, \quad \text{for} \quad n \geq 2, \quad \text{and} \quad M(0) = \Lambda u \subset V \\ R(n) &= \Lambda x_{(1)} + M(n-1), \\ W(n) &= R(n) + \Lambda y_{(n)}. \end{aligned}$$

Proposition 1. *The inclusions $M(n-1) \subset R(n)$ and $R(n) \subset W(n)$ are uniform.*

The proof will use the following restriction lemma. Here, we denote by B the subalgebra of Λ with basis $1, \phi, \psi$. It is a local algebra with radical square zero. If we consider a Λ-module M as a B-module, then we write ${}_BM$.

Restriction Lemma 1. *Let M be a Λ-module. Assume that ${}_BM = N \oplus N'$ where N is an indecomposable non-simple B-submodule and N' is a semisimple B-module. Also, assume that* $\operatorname{soc}_\Lambda M = \operatorname{soc}_B N$ *(as vector spaces). Then M is an indecomposable Λ-module.*

Proof. Let $M = M_1 \oplus M_2$ be a direct decomposition of M as a Λ-module, thus also ${}_BM = {}_B(M_1) \oplus {}_B(M_2)$. We apply the theorem of Krull-Remak-Schmidt to the direct decompositions $N \oplus N' = {}_BM = {}_B(M_1) \oplus {}_B(M_2)$ and see that one of the summands ${}_B(M_1), {}_B(M_2)$, say ${}_B(M_1)$ can be written in the form $N_1 \oplus N_1'$ with N_1 isomorphic to N and N_1' semisimple and then ${}_B(M_2)$ is also semisimple. Since N is an indecomposable non-simple B-module, we have $\operatorname{soc} N = \operatorname{rad} N$. On the other hand, $\operatorname{rad} N' = 0 = \operatorname{rad} N_1'$ and also $\operatorname{rad}{}_B(M_2) = 0$. Thus,

$$\begin{aligned}\operatorname{soc} M = \operatorname{soc} N = \operatorname{rad} N &= \operatorname{rad} N \oplus \operatorname{rad} N' = \operatorname{rad}{}_BM \\ &= \operatorname{rad} N_1 \oplus \operatorname{rad} N_1' \oplus \operatorname{rad}{}_B(M_2) = \operatorname{rad} N_1 \subseteq M_1.\end{aligned}$$

But this implies that M_2 is zero (if $M_2 \neq 0$, then also $\operatorname{soc} M_2 \neq 0$ and of course $\operatorname{soc} M = \operatorname{soc} M_1 \oplus \operatorname{soc} M_2$).

The indecomposable B-modules are well-known, since B is stably equivalent to the Kronecker algebra kQ (see for example [ARS], exercise X.3, or [Be], chapter 4.3; recall that the Kronecker quiver Q is given by two vertices, say a and b, and two arrows $a \to b$). For any $n > 1$, there are up to isomorphism precisely indecomposable B-modules of length $2n+1$, one is said to be *preprojective* (its socle has length $n+1$, its top length n), the other one *preinjective* (with socle of length n and top of length $n+1$. The remaining non-simple indecomposables are said to be *regular;* they have even length (and the length of the socle coincides with the length of the top). For any $n \geq 1$, there is a up to isomorphism a unique indecomposable regular module of length $2n$ such that the kernel of the multiplication by ϕ has dimension $n+1$.

Proof of proposition 1. We will consider Λ-modules U with $M(n-1) \subseteq U \subseteq V^n$; note that for such a module U, one has $\operatorname{soc} U = \sum_{i=1}^n ku_{(i)}$. Always, we will see that ${}_BU$ is the direct sum of an indecomposable B-module N and a semisimple B-module N'.

(1) *The inclusion $M(n-1) \subseteq Jx_{(1)} + M(n-1)$ is uniform for $n \geq 1$.*

Proof. Consider a Λ-module U with $M(n-1) \subseteq U \subseteq Jx_{(1)} + M(n-1)$. If $n = 1$, then U is a non-zero submodule of the uniform module V, thus indecomposable. Let $n \geq 2$. Let

$$N = \sum_{i=1}^{n-1} Bz_i = \sum_{i=1}^{n-1} kz_i + \sum_{i=1}^{n} ku_{(i)},$$

here we use that $\phi(z_{(i)}) = u_{(i)}$ and $\psi(z_{(i)}) = u_{(i+1)}$, for $1 \leq i < n$. Note that N is the indecomposable preprojective B-module of length $2n-1 > 1$ and its socle is $\operatorname{soc}_B N = \sum_{i=1}^n ku_{(i)}$. Thus, we see that $\operatorname{soc}_B N = \operatorname{soc} U$. On the other hand, $M(n-1) = JM(n-1) + N$, thus $Jx_{(1)} + M(n-1) = Jx_{(1)} + JM(n-1) + N$. Since $\phi J = 0 = \psi J$, it follows that $Jx_{(1)} + JM(n-1)$ is semisimple as a B-module. Thus $Jx_{(1)} + M(n-1)$ is as a B-module the sum of N and a semisimple B-module, and therefore also U is as a B-module the sum of N

and a semisimple B-module N'. Altogether we see that we can apply the restriction lemma to the Λ-module U and the B-modules N, N' and conclude that U is indecomposable.

(2) *The inclusion $R(n) \subseteq Jy_{(n)} + R(n)$ is uniform for $n \geq 1$.*

The proof is similar to that of (1), now we consider a Λ-module U with $R(n) \subseteq U \subseteq Jy_{(n)} + R(n)$ and can again assume that $n \geq 2$. This time, let

$$N = Bx_{(1)} + \sum_{i=1}^{n-1} Bz_i = kx_{(1)} + \sum_{i=1}^{n-1} kz_i + \sum_{i=1}^{n} ku_{(i)}.$$

The B-module N is regular indecomposable of length $2n > 1$ and the kernel of the multiplication by ϕ has dimension $n+1$. The socle of N is $\sum_{i=1}^{n} ku_{(i)} = \operatorname{soc} U$. On the other hand, $R(n) = JR(n) + N$, thus $Jy_{(n)} + R(n) = Jy_{(n)} + JR(n) + N$, and $Jy_{(n)} + JR(n)$ is semisimple as a B-module. Since $Jy_{(n)} + R(n)$ is as a B-module the sum of N and a semisimple B-module, also ${}_BU$ is the sum of N and a semisimple B-module N'. We apply again the restriction lemma to the Λ-module U and the B-modules N, N'.

(3) *The module $W(n)$ is indecomposable for $n \geq 1$.*

The proof is again similar: let $U = W(n)$ and $n \geq 2$. Now let

$$N = Bx_{(1)} + By_{(n)} + \sum_{i=1}^{n-1} Bz_i = kx_{(1)} + ky_{(n)} + \sum_{i=1}^{n-1} kz_i + \sum_{i=1}^{n} ku_{(i)}.$$

The B-module N is the preinjective indecomposable B-module of length $2n+1 > 1$, and its socle is $\sum_{i=1}^{n} ku_{(i)} = \operatorname{soc} U$. On the other hand, $W(n) = JW(n) + N$, and $JW(n)$ is semisimple as a B-module. As before, we see that ${}_BU$ is the sum of N and a semisimple B-module N'. The restriction lemma shows that U is indecomposable.

(4) *Let $M = M' + L$ be an indecomposable Λ-module with submodules M' and L such that L is local. If U is a Λ-module with $M' \subseteq U \subseteq M$, then $U \subseteq M' + JL$ or else $U = M$.*

Proof. Let $U \subseteq M$ be a submodule which is not contained in $M' + JL$. Then, in particular, $M' + JL$ is a proper submodule of M, and actually $M' + JL$ is a maximal submodule of M (namely, the composition of the inclusion map $L \subseteq M = M' + L$ and the projection $M \to M/(M' + JL)$ is surjective and contains JL in its kernel, but L/JL is simple).

It follows that $M = U + (M' + JL) = U + JL$. Let $L = \Lambda m$ for some $m \in L$. Since $M = U + Jm$, we see that $m = u + am$ with $u \in U$ and $a \in J$, thus $(1-a)m = u \in U$. But since $a \in J$, we know that $1 - a$ is invertible in the ring Λ, therefore also $m \in U$. As a consequence, $M = M' + \Lambda m \subseteq U$ and therefore $M = U$.

It follows from (2) that $R(n)$ is indecomposable, thus (1) and (4) show that the inclusion $M(n-1) \subset R(n)$ is uniform. Similarly, (2), (3) and (4) show that the inclusion $R(n) \subset W(n)$ is uniform. This completes the proof of proposition 1.

Proposition 2. *For $n \geq 1$, there are couniform projections $M(n) \to R(n)$ and $R(n+1) \to W(n)$.*

Proof: First, consider the embedding $R(n+1) \subset V^{n+1} = \bigoplus_{i=1}^{n+1} V_i$ and the submodule $X = R(n+1) \cap V_{n+1} \subset R(n+1)$. Note that $R(n+1)/X = W(n)$, since for the canonical projection $R(n+1) \to R(n+1)/X$ we have $z_n \mapsto y_{(n)}$, whereas $x_{(1)} \mapsto x_{(1)}$, $z_i \mapsto z_i$ for $1 \le i \le n-1$.

Similarly, consider the embedding $M(n) \subset \bigoplus_{i=1}^{n+1} V_i$ and the submodule $Y = M(n) \cap V_{(1)} \subset M(n)$. For the canonical projection $M(n) \to M(n)/Y$, we have $z_1 \mapsto x_{(2)}$, and $z_i \mapsto z_{i+1}$ for $1 \le i \le n-1$, thus we can identify $M(n)/Y$ with $R(n)$ (where $R(n)$ is now considered as a submodule of $\bigoplus_{i=2}^{n+1} V_i$).

In order to see that these projections $R(n+1) \to R(n+1)/X$ and $M(n) \to M(n)/Y$ are couniform, we proceed as in the proof of Proposition 1, or better dually. In particular, we have to use the dual of the restriction lemma 1 (here, instead of looking at the socles of ${}_\Lambda M$ and ${}_B N$, we assume that the tops of ${}_\Lambda M$ and ${}_B N$ coincide):

Restriction Lemma 2. *Let M be a Λ-module. Assume that ${}_B M = N \oplus N'$ where N is an indecomposable non-simple B-submodule and N' is a semisimple B-module. Also, assume that there is a vector subspace T of N such that $M = T \oplus \operatorname{rad}_\Lambda M$ and $N = T \oplus \operatorname{rad}_B N$ as vector spaces. Then M is an indecomposable Λ-module.*

This completes the proof of proposition 2 and also that of the theorem.

Remark. Note that in general the inclusion $M(n\text{–}1) \subset W(n)$ is not uniform. Consider for Λ the Kronecker algebra kQ, and look at the submodules U, U' of $W(2)$ generated by the elements $z = x_{(1)} + y_{(1)} + x_{(2)} + y_{(2)}$ and $z' = x_{(1)} - y_{(1)} - x_{(2)} + y_{(2)}$, respectively. We have $\dim U = \dim U' = 2$. Assume now that the characteristic of k is different from 2. Then $U \neq U'$ and even $U \cap U' = 0$. Thus $U \oplus U'$ is a decomposable submodule of $W(2)$. Also, $M(1)$ is contained in $U \oplus U'$ (as the submodule generated by $\frac{1}{2}(z - z')$).

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