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Brick chain filtrations. A report.

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Abstract. We consider the category of finitely generated modules over an artin algebra A. Recall that an object in an abelian category is said to be a *brick* provided its endomorphism ring is a division ring. Simple modules are, of course, bricks, but in case A is connected and not local, there always do exist bricks which are not simple. The aim of this survey is to focus the attention to filtrations of modules where all factors are bricks, with bricks being ordered in some definite way, namely according to a so-called brick chain.

In general, a module category will have many cyclic paths. Recently, Demonet has proposed to look at brick chains in order to deal with a very interesting directedness feature of an arbitrary module category.

The following survey relies on investigations by a quite large group of mathematicians. We have singled out some important observations and have reordered them in order to obtain a self-contained (and elementary) treatment of the relevance of bricks in module categories. (Most of the papers we rely on are devoted to what is called τ-tilting theory, but for nearly all results we are looking at, there is no need to deal with τ-tilting, not even to invoke the Auslander-Reiten translation τ itself).

Outline. This is a report on a very important development in the last 15 years: it focuses the attention to the use of bricks in order to describe the structure of arbitrary modules over artin algebras. The report relies on the work of a quite large number of mathematicians, see section 18. We have singled out decisive observations and have reordered them in order to obtain a self-contained and elementary (however still incomplete) treatment of the relevance of bricks in module categories.

The first three sections describe the main results presented in this survey, they deal with brick chain filtrations and their background. Theorem 1.2 (with its strengthening 3.2) concerns the existence of brick chain filtrations, and there is a corresponding finiteness result, namely Theorem 3.3. The proofs of these results are given in Sections 4 to 11; the main tool is the study of finitely generated torsion classes and their lower neighbors. Theorem 2.2 asserts that finitely generated torsion classes are always generated by finite semibricks. Theorem 2.7 describes the lower neighbors of the torsion class generated by a module M in terms of the so-called top bricks of M.

Given a brick B, we denote by $\mathcal{E}(B)$ the class of all modules which have a filtration with all factors isomorphic to B; these modules will be said to be *homogeneous* of brick type B. The brick type of a non-zero homogeneous module is uniquely determined, see 8.1; thus, for non-isomorphic bricks B, B', the categories $\mathcal{E}(B)$ and $\mathcal{E}(B')$ intersect in zero. The brick chain filtrations studied in this report concern filtrations of modules with factors in suitable subcategories $\mathcal{E}(B)$, namely using bricks which occur in a brick chain. The existence of brick chain filtration is derived from a result for neighbor torsion classes. Neighbor torsion classes $\mathcal{T}' \subset \mathcal{T}$ come with a label: this is a brick B with the following property: any module M in $\mathcal{T}$ has a submodule M' in $\mathcal{T}'$ such that M/M' belongs to $\mathcal{E}(B)$ (see 6.1).

We are going to look quite carefully at torsion classes and their lower neighbors. It seems advisable not to restrict to finitely generated torsion classes: after all, a lower neighbor of a finitely generated torsion class may not be finitely generated. What about upper neighbors? Whereas finitely generated torsion classes always have only finitely many lower neighbors, they may have infinitely many upper neighbors. To get information about the upper neighbors of a torsion class $\mathcal{T}$, one should look at the corresponding torsionfree class $\mathcal{F}$. If $\mathcal{F}$ turns out to be generated by a single module N, then the upper neighbors of $\mathcal{T}$ correspond bijectively to the socle bricks of N, see section 12.

The final sections are devoted to brick finiteness: An algebra A is called *brick finite* iff there are only finitely many bricks. And, A is called *torsion class finite* provided there are only finitely many torsion classes. The bijection between finitely generated torsion classes and finite semibricks yields the important result that an algebra is brick finite iff it is torsion class finite, see section 15. Whereas nearly all our proofs can avoid the use of τ-tilting theory, an interesting characterization of brick finiteness which has to be mentioned, seems to require it: In an appendix (section 17) we provide the corresponding details. Note that this part is not self-contained, in contrast to the previous sections.

1. All modules have brick chain filtrations.

1.1. We deal with an artin algebra A; the modules to be considered are the left A-modules of finite length.

Given a set $\mathcal{X}$ of modules, we denote by $\mathcal{E}(\mathcal{X})$ the class of modules which have a filtration with all factors in $\mathcal{X}$. If $M_1, \dots, M_m$ are modules, let $\mathcal{E}(M_1, \dots, M_m) = \mathcal{E}(\{M_1, \dots, M_m\})$ (such a convention will be used throughout the paper in similar situations).

We recall that a *brick* is a module whose endomorphism ring is a division ring. If B is a brick, the modules in $\mathcal{E}(B)$ will be said to be *homogeneous* of brick type B. A finite sequence $(B_1, \dots, B_m)$ is called a *brick chain,* if all B_i are bricks and $\operatorname{Hom}(B_i, B_j) = 0$ for $i < j$. A filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ will be called a *brick chain filtration,* provided there is a brick chain $(B_1, \dots, B_m)$ (its *type*) such that M_i/M_{i-1} is homogeneous of brick type B_i, for all $1 \le i \le m$.

1.2. Theorem. *Any module has brick chain filtrations.*

The result will be strengthened in 3.2; the proof of 3.2 is given in 7.2.

1.3. Some examples of brick chain filtrations.

(1) Let $S_1, \dots, S_n$ be the simple A-modules. Obviously, $(S_1, \dots, S_n)$ is a brick chain. Let us assume that $\operatorname{Ext}^1(S_i, S_j) = 0$ for all $i > j$. If M is an arbitrary A-module, let M_i be

the submodule of M which is maximal with the property that all its composition factors are of the form $S_1, \dots, S_i$. Then $(M_i)_i$ is a brick chain filtration $(M_i)_i$. If M is sincere, then we obtain a brick chain filtration of type $(S_1, \dots, S_n)$.

In particular, recall that A is said to be *directed,* provided the simple modules $S_1, \dots, S_n$ can be ordered in such a way that $\operatorname{Ext}^1(S_i, S_j) = 0$ for all $i \geq j$. For such a directed algebra A, all sincere A-modules M have a brick chain filtration of type $(S_1, \dots, S_n)$ with the additional property that the factors of the filtration are semisimple.

(2) Let A be a cyclic Nakayama algebra. Its simple modules can be arranged in the form $S_0, S_1, \dots, S_n = S_0$ such that $\operatorname{Ext}^1(S_i, S_{i-1}) \neq 0$ for all $1 \leq i \leq n$. Any indecomposable module M has a brick chain filtration with at most two factors: Without loss of generality, we may assume that the top of M is S_n. If the length of M is at most n, then M itself is a brick. Now assume that the length of M is $an + r$ with $a \geq 1$ and $0 \leq r < n$. There is a unique module H of length n with top S_n; it is obvious that H is a brick. If $r = 0$, then M has a brick chain filtration of type (H). If $r \neq 0$, then M has a brick chain filtration of type (B, H), where B is the factor module of H of length r.

(3) In contrast to many questions in representation theory, looking for brick chain filtrations of modules, it does not seem to be helpful to consider first indecomposable modules. In particular, given brick chain filtrations of modules M and M' may not be useful to obtain a brick chain filtration of $M \oplus M'$.

(4) (Duality) Let us denote by D the usual duality functor. Given a brick chain filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ of type $(B_1, \dots, B_m)$, then clearly D yields a corresponding brick chain filtration for $\mathrm{D}\, M$, namely

$$0 = \mathrm{D}\, M/\mathrm{D}\, M_m \subset \mathrm{D}\, M/\mathrm{D}\, M_{m-1} \subset \cdots \subset \mathrm{D}\, M/\mathrm{D}\, M_1 \subset \mathrm{D}\, M/\mathrm{D}\, M_0 = \mathrm{D}\, M,$$

and its type is $(\mathrm{D}\, B_m, \dots, \mathrm{D}\, B_1)$. In section 15 we will focus the attention to general duality features.

(5) Our construction of a brick chain filtration of a module M will yield quite special filtrations, namely what we call "torsional" ones, see section 3. Let us note already here: if a filtration $(M_i)_i$ of a module M is torsional, then the top of any module M_i is generated by the top of M. Thus, even in the case of a directed algebra, the brick chain filtrations which we will construct are usually different from the obvious filtrations mentioned in example (1) above.

(6) For further examples and remarks, see section 11.

1.4. The proof of Theorem 1.2 and its strengthening 3.2 will be based on the use of torsion classes. Definition and properties of torsion classes will be recalled in the next Section 2 (they are essential for all considerations).

2. Torsion classes, in particular the finitely generated ones.

When we deal with sets of (pairwise non-isomorphic) modules, for example when we consider semibricks (see 2.1), these sets are not necessarily finite (so that we cannot or better do not want to deal with the corresponding direct sum).

2.1. A class $\mathcal{T}$ of modules is said to be a *torsion class* provided $\mathcal{T}$ is closed under factor modules and extensions. The set of all torsion classes is a complete lattice; the meet of a set of torsion classes is just the set-theoretical intersection. Given a class $\mathcal{X}$ of modules, we denote by $T(\mathcal{X})$ the smallest torsion class which contains $\mathcal{X}$ (thus, the closure of $\mathcal{X}$ under factor modules and extensions, or, equivalently, the set-theoretical intersection of all torsion classes containing $\mathcal{X}$). According to the Noether theorems, it is easy to see that $T(\mathcal{X})$ is just the class of modules which have a filtration whose factors are factor modules of modules in $\mathcal{X}$.

A torsion class $\mathcal{T}$ is said to be *finitely generated* provided there is a module M with $\mathcal{T} = T(M)$. Of course, any torsion class $\mathcal{T}$ is the set-theoretical union of the finitely generated torsion classes contained in $\mathcal{T}$.

Bricks B, B' are said to be Hom-*orthogonal* provided $\mathrm{Hom}(B, B') = 0 = \mathrm{Hom}(B', B)$. A *semibrick* is a set of Hom-orthogonal bricks. A torsion class which is generated by a semibrick is said to be *widely generated.*

2.2. Theorem. *For any artin algebra A, the map $M \mapsto T(M)$ provides a bijection between finite semibricks and the finitely generated torsion classes.*

The surjectivity of the map asserts that *any finitely generated torsion class is widely generated.* The injectivity assertion can be extended as follows: the map $\mathcal{B} \mapsto T(\mathcal{B})$ is a bijection between arbitrary semibricks and the widely generated torsion classes, see 9.1.

It follows from Theorem 2.2 that *any torsion class is generated by bricks.* The proof of Theorem 2.2 is given in 5.6 (the surjectivity of the map) and 9.1 (the injectivity of the map). Actually, in Section 5, we construct explicitly an inverse of the map $M \mapsto T(M)$, the main feature should be mentioned already here:

Addendum to Theorem 2.2. Given a module M, there is an effective algorithm for determining the semibrick $\mathcal{B}$ with $T(M) = T(\mathcal{B})$. Namely, for any module M we are going to define its "iterated endotop" $X = \mathrm{et}^\infty M$, see 5.4; this is a factor module of M and the elements of $\mathcal{B}$ are the (isomorphism classes of the) indecomposable direct summands of X; these bricks will be called the *top bricks* of M, see 5.7. To repeat: any finitely generated torsion class $\mathcal{T} = T(M)$ is generated by the set $\mathcal{B}(M)$ of top bricks of M (in particular, by factor modules of M). We also mention: if M is a module, and $T(M) = T(\mathcal{B})$ with $\mathcal{B}$ a semibrick, then $\mathcal{B}$ has to be a finite set, see 9.6.

2.3. Remark. The bijection provided by Theorem 2.2 is of great interest, since it allows to consider the set of finite semibricks as a partially ordered set, using the natural partial ordering of the set of torsion classes given by set-theoretical inclusion.

This poset structure of the set of finite semibricks (thus also of the set of bricks) provides the foundation for the notion of a brick chain as used in Theorem 1.2.

2.4. The algebra A is said to be *brick finite* provided there is only a finite number of isomorphism classes of bricks, and *torsion class finite* provided there is only a finite number of torsion classes.

Corollary. *An algebra is brick finite iff it is torsion class finite, and in this case any torsion class is finitely generated.*

Proof. Clearly, an algebra which is brick finite has also only finitely many semibricks. Assume that A is brick finite, let $\mathcal{T}$ be any torsion class. We start to construct an inclusion chain of torsion classes $\mathcal{T} = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_t$ where $\mathcal{T}_i = \mathcal{T}(B_0, \dots, B_t)$ with bricks $B_0, B_1, \dots, B_t$. If $\mathcal{T}_t$ is a proper subset of $\mathcal{T}$, there is a brick $B_{t+1} \in \mathcal{T} \setminus \mathcal{T}_t$, thus we let $\mathcal{T}_{t+1} = \mathcal{T}(B_0, \dots, B_{t+1})$. Since A is brick finite, the procedure stops, thus $\mathcal{T}$ is generated by a finite number of bricks.

The bijection mentioned in Theorem 2.2 asserts that for any algebra, the number of finite semibricks is equal to the number of finitely generated torsion classes. $\square$

Further characterizations of the brick finite algebras are given in sections 15 and in the Appendix 17.

2.5. Neighbors. The torsion classes $\mathcal{T}' \subset \mathcal{T}$ will be said to be *neighbors* provided there is no torsion class $\mathcal{N}$ with $\mathcal{T}' \subset \mathcal{N} \subset \mathcal{T}$. If $\mathcal{T}' \subset \mathcal{T}$ are neighbor torsion classes, $\mathcal{T}'$ is called a *lower neighbor* of $\mathcal{T}$ and $\mathcal{T}$ is called an *upper neighbor* of $\mathcal{T}'$.

Let $\mathcal{T}$ be a torsion class. We will say that $\mathcal{T}$ has *sufficiently many lower neighbors* provided any torsion class $\mathcal{N}$ with $\mathcal{N} \subset \mathcal{T}$ is contained in a lower neighbor of $\mathcal{T}$. Similarly, we say that $\mathcal{T}$ has *sufficiently many upper neighbors* provided any torsion class $\mathcal{N}$ with $\mathcal{T} \subset \mathcal{N}$ contains an upper neighbor of $\mathcal{T}$.

Given a module class $\mathcal{Y}$, let ${}^\perp\mathcal{Y}$ be the class of all modules X with $\operatorname{Hom}(X,Y) = 0$ for all $Y \in \mathcal{Y}$. It is easy to see that ${}^\perp\mathcal{Y}$ is closed under extensions and under factor modules, thus it is a torsion class. A *torsionfree class* $\mathcal{F}$ is, by definition, a class of modules which is closed under submodules and extensions. Given a module class $\mathcal{X}$, let $\mathcal{X}^\perp$ be the class of all modules Y with $\operatorname{Hom}(X,Y) = 0$ for all $X \in \mathcal{X}$. Then $\mathcal{X}^\perp$ is a torsionfree class. Given a torsion class $\mathcal{T}$, the pair $(\mathcal{T}, \mathcal{T}^\perp)$ is said to be a *torsion pair.* (If $\mathcal{X}, \mathcal{Y}$ are module classes with $\operatorname{Hom}(X,Y) = 0$ for all $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, so that we deal with the situation $\mathcal{X} \subseteq {}^\perp\mathcal{Y}$, or, equivalently, with $\mathcal{Y} \subseteq \mathcal{X}^\perp$, we sometimes prefer to write $\operatorname{Hom}(\mathcal{X}, \mathcal{Y}) = 0$ in order to stress the symmetry and to avoid to focus the attention to the torsion class ${}^\perp\mathcal{Y}$ or to the torsionfree class $\mathcal{X}^\perp$.)

2.6. Theorem. *Let $\mathcal{T}' \subset \mathcal{T}$ be torsion classes. Then $\mathcal{T}' \subset \mathcal{T}$ are neighbors iff $\mathcal{T} \cap (\mathcal{T}')^\perp$ is a homogeneous subcategory.*

In this case, there is precisely one brick B in $\mathcal{T}$ with $\mathcal{T}' = \mathcal{T} \cap {}^\perp B$, namely the type of the homogeneous category $\mathcal{T} \cap (\mathcal{T}')^\perp$.

If $\mathcal{T}' \subset \mathcal{T}$ are neighbors, the brick B in the homogeneous category $\mathcal{T} \cap (\mathcal{T}')^\perp$ is called the *label* of the neighbor torsion classes.

Theorem 2.6 asserts that for neighbor torsion classes $\mathcal{T}' \subset \mathcal{T}$, the label B is the only brick in $\mathcal{T} \cap (\mathcal{T}')^\perp$, and also the only brick B with $\mathcal{T}' = \mathcal{T} \cap {}^\perp B$. Of course, the label B belongs to $\mathcal{T} \setminus \mathcal{T}'$, but it is usually not the only brick in $\mathcal{T} \setminus \mathcal{T}'$. Here is the first example: Take the $\mathbb{A}_2$-quiver $1 \leftarrow 2$ and consider the torsion classes $\mathcal{T}' = T(1) = \operatorname{add} 1$ and $\mathcal{T} = \operatorname{mod} A$. These are neighbors and both 2 and its projective cover $P2$ belong to $\mathcal{T} \setminus \mathcal{T}'$.

The proof of Theorem 2.6 will be given in 8.4. Section 8, as well as sections 10, 12, 13 and 14 provide a quite detailed study of neighbor torsion classes.

Next, let us consider the lower neighbors of a finitely generated torsion class.

2.7. Theorem. *Let M be a module. The map $B \mapsto T(M)_B = T(M) \cap {}^{\perp}B$ provides a bijection between the isomorphism classes of the top bricks B of M and the lower neighbors of $T(M)$. The label of the neighbor classes $T(M)_B \subset T(M)$ is just B. Also, $T(M)$ has sufficiently many lower neighbors.*

The proof of 2.7 will be given in 9.5.

2.8. A torsion class $\mathcal{T}$ is said to be *completely join irreducible* provided the join $\mathcal{T}_*$ of the torsion classes properly contained in $\mathcal{T}$ is still properly contained in $\mathcal{T}$ (and thus $\mathcal{T}_*$ is a lower neighbor of $\mathcal{T}$). Note that $\mathcal{T}$ is completely join irreducible iff $\mathcal{T}$ has a unique lower neighbor and has sufficiently many lower neighbors.

Corollary. *The map $B \mapsto \mathcal{T}(B)$ provides a bijection between the isomorphism classes of the bricks and the completely join irreducible torsion classes.*

Proof. Theorem 2.2 sends a brick to the torsion class $\mathcal{T}(B)$; according to 2.7, $\mathcal{T}(B)$ has a unique lower neighbor, namely $\mathcal{T}_* = \mathcal{T}(B) \cap {}^{\perp}B$ and any torsion class properly contained in $\mathcal{T}$ is contained in $\mathcal{T}_*$. This shows that $\mathcal{T}(B)$ is completely join irreducible.

Conversely, assume that $\mathcal{T}$ is a completely join irreducible torsion class. Clearly, $\mathcal{T}$ is finitely generated: Let M be any module in $\mathcal{T} \setminus \mathcal{T}_*$, where T_* is the join of the torsion classes properly contained in $\mathcal{T}$, then $\mathcal{T} = \mathcal{T}(M)$. Let $B_1, \ldots, B_t$ be the top bricks of M, thus $\mathcal{T} = \mathcal{T}(M) = \mathcal{T}(B_1, \ldots, B_t)$. According to 2.7, $\mathcal{T}$ has t lower neighbors. Since $\mathcal{T}$ is completely join irreducible, we have $t = 1$, thus $\mathcal{T}$ is generated by a brick. $\square$

2.9. Warnings. If $\mathcal{T}(M)$ is a finitely generated torsion class and B a top brick of M, the lower neighbor torsion class $T(M) \cap {}^{\perp}B$ is not necessarily finitely generated! A typical example will be mentioned in 2.10.

Also, we have seen in 2.7 that $T(M)$ has only finitely many lower neighbors. What about upper neighbors? If $T(M)$ is finitely generated and $\mathcal{T}''$ is an upper neighbor of $T(M)$, then trivially $\mathcal{T}''$ is again finitely generated, namely equal to $T(M \oplus N)$, where N is any module in $\mathcal{T}'' \setminus T(M)$. However, whereas a finitely generated torsion class has only finitely many lower neighbors, it may have infinitely many upper neighbors. For a typical example, we again refer to 2.9.

Let us add that there is a recipe for obtaining torsion classes with only finitely many upper neighbors: we have to take into account the corresponding torsionfree class. Namely, assume we deal with a torsion pair $(\mathcal{T}, \mathcal{F})$. We say that $\mathcal{F}$ is finitely generated, if there is a module N such that $\mathcal{F}$ is the smallest torsionfree class containing N (or, equivalently, the closure of $\{N\}$ under submodules and extensions). Similar to 2.7, one observes: given a torsion pair $(\mathcal{T}, \mathcal{F})$ with $\mathcal{F}$ a finitely generated torsionfree class, the torsion class $\mathcal{T}$ has only finitely many upper neighbors, see Section 12.

2.10. An example: The Kronecker algebra. For the benefit of the reader, we want to consider one example in detail, the Kronecker algebra; this is the path algebra of the quiver with two vertices $1, 2$ and two arrows $1 \leftleftarrows 2$. Let us depict the lattice of the torsion classes of the Kronecker algebra A. (Actually, it is the usual example which everyone interested in torsion classes of artin algebras has in mind).

There is the well-known trisection of the indecomposable A-modules: there are the preprojective modules $\mathcal{P}$, the regular modules $\mathcal{R}$ and the preinjective modules $\mathcal{I}$. In terms of torsion classes, this trisection gives rise to two important torsion classes: the torsion class $T(\mathcal{I})$ given by the direct sums of preinjective modules, and the torsion class $T(\mathcal{R})$ given by the direct sums of preinjective and regular modules. Both torsion classes $T(\mathcal{I})$ and $T(\mathcal{R})$ are **not** finitely generated.

The torsion class $T(\mathcal{I})$ of the preinjective modules is the union of a properly ascending chain of completely join irreducible torsion classes, thus it is not finitely generated (but generated by any infinite set of indecomposable preinjective modules). Note that $T(\mathcal{I})$ has no lower neighbor, but it has infinitely many upper neighbors.

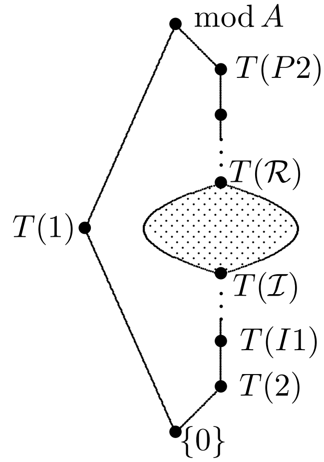
On the other hand, the torsion class $T(\mathcal{R})$ is widely generated, namely by the (infinite!) set of the simple regular modules. Thus, $T(\mathcal{R})$ is not finitely generated. Also, $T(\mathcal{R})$ has no upper neighbor (but it has infinitely many lower neighbors).

If R is a simple regular module, then $T(R)$ is (of course) finitely generated, even completely join irreducible, however its unique lower neighbor $T(R)\cap{}^{\perp}R$ is the torsion class $T(\mathcal{I})$ (and we repeat that this torsion class is not finitely generated). On the other hand, $T(R)$ has infinitely many upper neighbors. Namely, if R' is a simple regular Kronecker module which is not isomorphic to R, then $T(R \oplus R')$ is an upper neighbor of $T(R)$, and there are infinitely many such isomorphism classes of modules R'. The number of upper neighbors of $T(R)$ is $\max(|k|, \aleph_0)$, where k is the base field of A.

Let $\mathcal{X}$ be a non-empty set of pairwise non-isomorphic simple regular Kronecker modules (thus $\mathcal{X}$ is a non-empty semibrick). The torsion classes $T(\mathcal{X})$ are the torsion classes $\mathcal{T}$ with $T(\mathcal{I}) \subset \mathcal{T} \subseteq T(\mathcal{R})$, thus the torsion classes $\mathcal{T}$ with $T(\mathcal{I}) \subseteq \mathcal{T} \subseteq T(\mathcal{R})$ correspond bijectively to the subsets of the set of isomorphism classes of simple regular modules. Note that any torsion class $\mathcal{T}(\mathcal{X})$ has infinitely many neighbors: maybe only finitely many lower neighbors (this happens iff $\mathcal{X}$ is finite, thus iff $\mathcal{T}(\mathcal{X})$ is finitely generated) or only finitely many upper neighbors (this happens iff $\mathcal{X}$ contains representatives from almost all isomorphism classes of simple regular modules). Altogether, *if $\mathcal{T}$ is a torsion class with $T(\mathcal{I}) \subseteq \mathcal{T} \subseteq T(\mathcal{R})$, the number of neighbors of $\mathcal{T}$ is always equal to* $\max(|k|, \aleph_0)$, in particular, infinite (see also 12.7).

For the Kronecker algebra, all torsion classes $\mathcal{T}$ but one are of the form $T(\mathcal{B})$, where $\mathcal{B}$ is a semibrick, the only exception is $\mathcal{T} = T(\mathcal{I})$. Similarly, all torsionfree classes $\mathcal{F}$ but one are of the form $F(\mathcal{B})$ for some semibrick $\mathcal{B}$, the only exception is the torsionfree class $F(\mathcal{P})$. (Here, $F(\mathcal{X})$ denotes the smallest torsionfree class which contains the module class $\mathcal{X}$, see 12.1; as for torsion classes, a torsionfree class of the form $F(\mathcal{B})$ with $\mathcal{B}$ a semibrick is said to be widely generated.).

Here is a sketch of the lattice of all torsion classes of $\operatorname{mod} A$.



The dotted part is the lattice of all torsion classes $\mathcal{T}$ with $T(\mathcal{I}) \subseteq \mathcal{T} \subseteq T(\mathcal{R})$. This dotted lattice looks like the lattice of all subsets of $\mathbb{P}^1(k)$ (where $\mathbb{P}^1(k)$ is just the union of the one element set $\{\infty\}$ and the set of monic irreducible polynomials with coefficients in k). One has to apologize that the dotted part has to be drawn very small, much to small! After all, here we deal with a quite large set: the set of subsets of $\mathbb{P}^1(k)$ is always uncountable (even if k is the field with just 2 elements)!

The remaining vertices correspond to the torsion classes which are functorially finite; there are only countably many. (The usual interest focuses the attention mainly to the functorially finite torsion classes.)

As we have mentioned, for nearly all torsion pairs $(\mathcal{T}, \mathcal{F})$, both $\mathcal{T}$ and $\mathcal{F}$ are widely generated. The only exceptions are the torsion pairs with torsion class equal to $T(\mathcal{I})$ and $T(\mathcal{R})$ (the two bullets seen at the dotted part).

3. Torsional brick chain filtrations.

We have mentioned above that Theorem 1.2 can be strengthened. In order to do so, we need an additional concept.

3.1. A submodule U of a module M is said to be *torsional* provided U belongs to $T(M)$. A filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m$ will be said to be *torsional* provided M_{i-1} is a torsional submodule of M_i, for all $1 \le i \le m$.

If $(M_i)_i$ is a torsional filtration of M, then M_{i-1} belongs to $T(M_i)$, for all $1 \le i \le t$, thus we have the inclusion chain $0 = T(M_0) \subseteq T(M_1) \subseteq \cdots \subseteq T(M_m) = T(M)$, and therefore all the submodules M_i are torsional submodules of M. If a brick chain filtration $(M_i)_i$, say of type $(B_1, \dots, B_m)$ is torsional, then all the bricks B_i belong to $T(M)$, since B_i is a factor module of M_i and M_i belongs to $T(M)$.

Warning. As we have mentioned, given a torsional brick chain filtration $(M_i)_i$ of M, all the modules M_i belong to $T(M)$. Note that the converse is not true. Here is an example: Let M be a serial module with composition factors $1, 2, 2, 1, 2$ going upwards, with an endomorphism with image of length 2; and take the filtration $(M_i)_{0 \le i \le 3}$ with M_i

of length $0, 2, 3, 5$ for $i = 0, 1, 2, 3$; then, all the submodules M_i are torsional submodules of M, but M_1 is not a torsional submodule of M_2.

3.2. Theorem. *Any module has torsional brick chain filtrations.*

The proof will be given in 7.2. As we will see in section 11, the torsional brick chain filtrations of a module M can be constructed easily by induction: Let B be a top brick of M and M' minimal with M/M' in $\mathcal{E}(B)$. Since M' is a proper submodule of M, by induction there is a torsional brick chain filtration of M', say $0 = M_0 \subset M_1 \subset \cdots \subset M_{m-1} = M'$. Let $M_m = M$. Then $(M_i)_{0 \le i \le m}$ is a torsional brick chain filtration of M. As a consequence of this procedure, we have:

3.3. Theorem. *The number of torsional brick chain filtrations of a module is finite.*

Note that this means that any module M determines a finite set of bricks which may be considered as the building blocks for the construction of M, namely the bricks which occur in the types of the finitely many torsional brick chain filtrations of M. The proof will be given in 10.3.

Theorems 3.2 and 3.3 assert that any module M has torsional brick chain filtrations, but only finitely many. In addition, usually there are plenty brick chain filtrations of M which are not torsional. We do not know whether there are modules which have infinitely many brick chain filtrations.

3.4. *A brick B has precisely one torsional brick chain filtration, namely the trivial filtration* $(0 \subset B)$.

Proof. Let $(M_i)_i$ be a torsional brick chain filtration of the brick B. Then M_1 is a non-zero module in $T(B)$. As we will see in 4.1, this implies that there is a non-zero homomorphism $f \colon B \to M_1$. We compose f with the inclusion map $M_1 \subseteq B$ and obtain a non-zero endomorphism $B \to M_1 \subseteq B$ of B. Since B is a brick, it follows that this composition is an isomorphism, thus the inclusion map $M_1 \subseteq B$ is surjective. Therefore $M_1 = B$. □

3.5. Remark. If $(M_i)_i$ is a torsional brick chain filtration of type $(B_1, \ldots, B_m)$, then by definition all the bricks B_i belong to $T(M)$. Now, the brick B_m is obviously a factor module of M, but the remaining bricks B_i do not have to be factor modules of M. Here is a typical example: Let M be serial with composition factors going up: $1, 2, 2, 1, 2$, with torsional brick chain filtration $0 \subset M_1 \subset M$, where $M_1 = B_1$ is the submodule of length three: here, M_1 is not generated by M (but is, of course, in $T(M)$).

4. Some preliminaries.

4.1. Lemma. *Let M' be a non-zero module in $T(M)$. Then* $\operatorname{Hom}(M, M') \neq 0$.

Proof: M' has a filtration $0 = M'_0 \subset M'_1 \subset \cdots \subset M_m = M$, where all the factors M_i/M_{i-1} are non-zero factor modules of M. Since M'_1 it is a factor module of M, we get a non-zero homomorphism $M \to M'_1 \to M'$. □

4.2. Examples of non-isomorphic bricks B', B with $B' \in T(B)$. According to Lemma 4.1, $\operatorname{Hom}(B, B') \neq 0$. (On the other hand, we will see in 6.3 that $\operatorname{Hom}(B', B) = 0$.)

Example 1: $B = \begin{smallmatrix}2\\1\end{smallmatrix}$ and $B' = 2$. Here, we have an epimorphism $B \to B'$. (Or, if we want to have the same support: Let $B = \begin{smallmatrix}2\\1\\1\end{smallmatrix}$, and $B' = \begin{smallmatrix}2\\1\end{smallmatrix}$.)

Example 2: $B = \begin{smallmatrix}2\\1\end{smallmatrix}$, and $B' = \begin{smallmatrix}2\\2\\1\end{smallmatrix}$. Here, we have a monomorphism $B \to B'$ and B, B' have the same support.

Example 3: $B = \begin{smallmatrix}2\\1\end{smallmatrix}$, and $B' = \begin{smallmatrix}3\\2\end{smallmatrix}$. Here, we have a non-zero map $B \to B'$ which is neither epi nor mono.

4.3. Lemma. *A non-zero module is a brick iff it has no non-zero proper torsional submodule.*

Proof. Let M be a module. If M is not a brick, there is an endomorphism f of M such that $f(M)$ is non-zero and a proper submodule. Since $f(M)$ belongs to $T(M)$, we see that $f(U)$ is a torsional submodule of M.

Conversely, assume that U is a non-zero proper submodule which is torsional. Since U belongs to $T(M)$, there is a non-zero submodule U' of U which is a factor module of M. We get a non-zero and not invertible endomorphism $M \to U' \subseteq U \subset M$, thus M is not a brick. $\square$

5. The endotop and the iterated endotop of a module.

We are going to show the surjectivity assertion of Theorem 2. We need the notion of the endotop $\operatorname{et} M$ of a module M.

5.1. Endotop. Denote by $E = \operatorname{End}(M)$ the endomorphism ring of M (operating on the left of M), and $\operatorname{rad} E$ its radical. Then $(\operatorname{rad} E)M$ is a submodule of M and we define $\operatorname{et} M = M/(\operatorname{rad} E)M$, and call it the *endotop* of M; by definition, the endotop of M is a factor module of M.

5.2. Examples: Let A be the local algebra $k\langle x, y\rangle$ with $\operatorname{rad}^3 = 0$. *If M is indecomposable, $\operatorname{et} M$ may be decomposable:* If M is the 3-dimensional indecomposable module with simple socle and top of length 2, then $\operatorname{et} M$ is the direct sum of two copies of the simple module.

Let A be given by the quiver Q with one vertex and two loops and with relations all paths of length 3 (thus A is a local algebra of dimension 7. There is a serial module M of length 3 with $\operatorname{rad} M$ not isomorphic to $M/\operatorname{soc} M$. Then $\operatorname{et} M = M/\operatorname{soc} M$, thus $\operatorname{et} M$ *is indecomposable of length* 2, *and not a brick,* in particular, $\operatorname{et}(\operatorname{et} M)$ is a proper factor module of $\operatorname{et} M$. This leads us below to consider not only et, but the iterations et^i, see 5.4. (Instead of A and M we may consider the following factor algebra A' of A: the subring $A' = k + J$ of the ring of all 3×3-matrices with coefficients in k, where J is the set of nilpotent upper triangular matrices; and $M = k^3$ the A'-module of column vectors.)

If A is the Kronecker algebra, and M a regular Kronecker module, then $\operatorname{et} M$ is just the regular top of M.

5.3. Proposition. *Let M be a module. Then M belongs to $T(\operatorname{et} M)$, therefore $T(M) = T(\operatorname{et} M)$. The kernel of the canonical map $M \to \operatorname{et} M$ is torsional.*

Proof. Let $f_1, \dots, f_t$ be a basis of $E = \operatorname{rad} \operatorname{End} M$. Let $(\operatorname{rad} \operatorname{End} M)^m = 0$. The image of $g = (f_i) \colon \bigoplus_i M \to M$ is $(\operatorname{rad} E)M = \operatorname{rad}_E M = M_1$ and $\operatorname{et} M = M/M_1$. Let $M_{j+1} = g(M_j)$ for all $j \geq 0$ with $M_0 = M$. Then $M_m = 0$. By induction, all modules M_j/M_{j+1} are generated by $\operatorname{et} M$. This shows that $T(M) \subseteq T(\operatorname{et} M)$. On the other hand, we also have $T(\operatorname{et} M) \subseteq T(M)$, since $\operatorname{et} M$ is a factor module of M. Thus M and $\operatorname{et} M$ generate the same torsion-class.

The kernel M' of the canonical map $M \to \operatorname{et} M$ is by definition the image of the map g, thus generated by M. Therefore M' belongs to $T(M)$. $\square$

5.4. We iterate the construction et and get epimorphisms

$$M \to \operatorname{et} M \to (\operatorname{et})^2 M \to \cdots .$$

Since M is of finite length, the sequence stabilizes eventually; in this way we get the *iterated endotop* $\operatorname{et}^\infty M = \operatorname{et}^a M$ for $a \gg 0$.

Example. Let A be a suitable artin algebra with two simple modules 1 and 2. For $n \geq 0$, let $M[n]$ be a serial module of length $n+2$, with composition factors going up: $(1, \dots, 1, 2, 1)$ (thus starting with n factors of the form 1). Then, for $0 \leq i \leq n$, we have $\operatorname{et}^i M[n] = M[n-i]$. For $0 \leq i < n$, the module $M[i]$ is not a brick, but $\operatorname{et}^n M[n] = M[0]$ is a brick (with composition factors $(2, 1)$).

5.5. Proposition. *Let M be a module. The iterated endotop $X = \operatorname{et}^\infty M$ is the direct sum of modules which belong to a semibrick $\mathcal{B}$ and $T(M) = T(X) = T(\mathcal{B})$; the kernel of the canonical map $M \to \operatorname{et}^\infty M$ is a torsional submodule of M.*

Proof. It is obvious that the iterated endotop of a module is always the direct sum of modules which belong to a semibrick, since the sequence $M \to \operatorname{et} M \to (\operatorname{et})^2 M \to \cdots$ stabilizes precisely when $\operatorname{End}(\operatorname{et}^a M)$ is semisimple. Proposition 5.3 yields that the torsion classes $T(\operatorname{et}^i M)$ are equal, for all $i \geq 0$.

The kernel K of the canonical map $M \to \operatorname{et}^\infty M$ has a filtration whose factors are the kernels K_i of the canonical maps $\operatorname{et}^i M \to \operatorname{et}^{i+1} M$, for all $i \geq 0$. According to 5.3, all modules K_i belong to $T(M)$, thus K belongs to $T(M)$. $\square$

5.6. Corollary. *A torsion class $\mathcal{T}$ is finitely generated iff there is a finite semibrick $\mathcal{B}$ with $\mathcal{T} = T(\mathcal{B})$.* $\square$

Corollary 5.6 shows that the map $\mathcal{B} \mapsto T(\mathcal{B})$ from the set of finite semibricks $\mathcal{B}$ to the set of finitely generated torsion classes is surjective. This is part of the assertion of Theorem 2.2.

5.7. Since the iterated endotop of a module M is given by a semibrick, the indecomposable direct summands of the iterated endotop are bricks and will be called the *top bricks* of M.

Examples. (1) *Let M be an indecomposable module. A top brick of M may occur in $\operatorname{et}^\infty M$ with multiplicity greater than* 1. (In particular, $\operatorname{et}^\infty M$ may not be indecomposable.)

For example, consider $A = k[x,y]/\langle x^2, y^2, xy\rangle$ and M the indecomposable injective module. The endotop (and the iterated endotop) of M is just the top of M, thus the direct sum of 2 copies of the simple module k.

(2) *The number of top bricks of an indecomposable module M may be arbitrarily large.* We start with the $(t-1)$-subspace quiver, say with sink 1 and sources $2, 3, \dots, t$, and add a loop at the sink 1. We may consider the corresponding radical-square-zero algebra. There is an indecomposable module M of Loewy length 2 with socle 1 and top $1 \oplus 2 \oplus \cdots \oplus t$. For this module, $\operatorname{et}^\infty M = \operatorname{et} M = 1 \oplus 2 \oplus \cdots \oplus t$, thus all the simple modules are top bricks of M.

6. The essential feature: If B is a brick, $({}^\perp B)\rceil\!\lfloor\mathcal{E}(B)$ is a torsion class.

Given module classes $\mathcal{X}$ and $\mathcal{Y}$, we write $\mathcal{X}\rceil\!\lfloor\mathcal{Y}$ for the class of all modules M which have a submodule M' in $\mathcal{X}$ such that M/M' belongs to $\mathcal{Y}$.

We are going to show: If B is a brick, then

$$T({}^\perp B, B) = ({}^\perp B)\rceil\!\lfloor\mathcal{E}(B).$$

This describes very nicely the torsion class $T({}^\perp B, B)$. Actually, there is the corresponding description for all the torsion classes $\mathcal{T} \subseteq T({}^\perp B, B)$, see the following general Proposition.

6.1. Proposition. *Let B be a brick. Let $\mathcal{T}$ be a torsion class which is contained in $T({}^\perp B, B)$. Then either $\mathcal{T}$ is contained in ${}^\perp B$, or else*

$$\mathcal{T} = (\mathcal{T} \cap {}^\perp B)\rceil\!\lfloor\mathcal{E}(B).$$

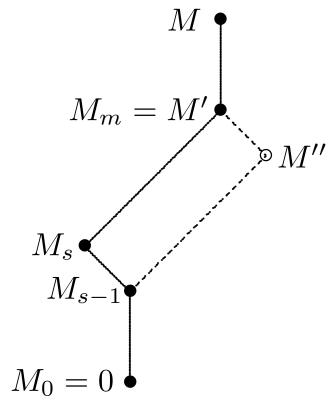
In particular, if $\mathcal{T}$ is not contained in ${}^\perp B$, then B belongs to $\mathcal{T}$ (since the displayed equality asserts that $\mathcal{E}(B) \subseteq \mathcal{T}$).

Let us add: if $0 \to M' \to M \to M/M' \to 0$ is an exact sequence with M' in $\mathcal{T}' = \mathcal{T} \cap {}^\perp B$ and $M/M' \in \mathcal{E}(B)$, then M' is just the torsion submodule of M with respect to the torsion class $\mathcal{T}'$, since obviously B, and therefore $\mathcal{E}(B)$ are contained in $(\mathcal{T}')^\perp$.

Proof of Proposition. Let M' be a submodule of M which belongs to $\mathcal{T}$ with $M/M' \in \mathcal{E}(B)$, and minimal with these two properties. We claim that M' belongs to ${}^\perp B$, thus to $\mathcal{T} \cap {}^\perp B$. (Note that at the moment we do not yet know that B belongs to $\mathcal{T}$, but this does not matter.)

Thus, assume for the contrary that there is a non-zero map $f\colon M' \to B$. Since M' belongs to $\mathcal{T}$, there is a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M'$ such that all factors $F_i = M_i/M_{i-1}$ are factor modules of B or belong to ${}^\perp B$. Let s be minimal such that $f|M_s$ is non-zero. Thus, f vanishes on M_{s-1} and induces a map $\overline{f}\colon M'/M_{s-1}$ with non-zero restriction to $F_s = M_s/M_{s-1}$. Let us denote by $u\colon F_s \to M'/M_{s-1}$ the inclusion map. Thus, the composition $\overline{f} \cdot u\colon F_s \to B$ is a non-zero map.

Now F_s is a factor module of some B or belongs to ${}^\perp B$. Since there is the non-zero map $\overline{f} \cdot u\colon F_s \to B$, we see that F_s is a factor module of B. Also, since B is a brick, there is no non-zero map from a proper factor module of B to B, thus we see that $F_s = B$ and that the composition $\overline{f} \cdot u \colon B = M_s/M_{s-1} \subseteq M'/M_{s-1} \to B$ is an isomorphism. This shows that u is a split monomorphism. It follows that there is a submodule M'' of M' with $M_{s-1} \subseteq M''$, such that $M_s \cap M'' = M_{s-1}$ and $M_s + M'' = M'$.



It follows that $M'/M'' \simeq M_s/M_{s-1} = B$, and that $M''/M_{s-1} \simeq M'/M_s$. Since M/M' and M'/M'' belong to $\mathcal{E}(B)$, also M/M'' belongs to $\mathcal{E}(B)$. On the other hand, $M''/M_{s-1} \simeq M'/M_s$ has a filtration by factors isomorphic to F_i with $s+1 \le i \le t$ and M_{s-1} has the filtration with factors F_i where $1 \le i \le s-1$. Since all the factors F_i belong to $\mathcal{T}$, also M'' belongs to $\mathcal{T}$.

Altogether we see that M'' is a submodule of M which belongs to $\mathcal{T}$ and such that $M/M' \in \mathcal{E}(B)$, Since M'' is a proper submodule of M', this contradicts the minimality of M'. It follows that M' belongs to ${}^{\perp}B$. Since M/M' is a non-zero module in $\mathcal{E}(B)$, is has a factor module of the form B, thus B is a factor module of M, therefore $B \in \mathcal{T}$.

Since M' belongs to $\mathcal{T} \cap {}^{\perp}B$, and M/M' to $\mathcal{E}(B)$, we see that M' is the torsion submodule of M with respect to the torsion class $\mathcal{T} \cap {}^{\perp}B$.

The exact sequence $0 \to M' \to M \to M/M' \to 0$ for an arbitrary module M in $\mathcal{T}$ shows that $\mathcal{T} \subseteq (\mathcal{T} \cap {}^{\perp}B) \rceil \mathcal{E}(B)$. On the other hand, we have $\mathcal{T} \cap {}^{\perp}B \subseteq \mathcal{T}$, and, since $B \in \mathcal{T}$, also $\mathcal{E}(B) \subseteq \mathcal{T}$: This shows the reverse inclusion $(\mathcal{T} \cap {}^{\perp}B) \rceil \mathcal{E}(B) \subseteq \mathcal{T}$, therefore $\mathcal{T} = (\mathcal{T} \cap {}^{\perp}B) \rceil \mathcal{E}(B)$.

This concludes the proof. $\square$

6.2. Corollary. *Let B be a brick. Let M be a module in $T({}^{\perp}B, B)$. Then any non-zero map $M \to B$ is surjective.*

Proof. Let M be a module in $T(B, {}^{\perp}B)$ and $f \colon M \to B$ a non-zero map. The existence of f shows that M does not belong to ${}^{p}erpB$. According to 6.1, there is a submodule M' of M which belongs to ${}^{\perp}B$ such that M/M' belongs to $\mathcal{E}(B)$. Since f vanishes on M', we get an induced map $\overline{f} \colon M/M' \to B$, and $\overline{f}$ is non-zero. However, any non-zero map in $\mathcal{E}(B)$ with target B is an epimorphism. Since $\overline{f}$ is surjective, also f is surjective. $\square$

6.3. Corollary. *Let B, B' be non-isomorphic bricks, and assume that B' is in $T(B)$. Then* $\operatorname{Hom}(B', B) = 0$*, thus $B' \in T(B) \cap {}^{\perp}B$.*

Proof. Assume there is a non-zero map $f \colon B' \to B$. According to 6.2, the map f is surjective. Since B' belongs to $T(B)$, we know from 4.1 that there is a non-zero map $g \colon B \to B'$. Since f is surjective, the composition $gf \colon B' \to B \to B'$ is non-zero. Since

B' is a brick, this means that gf is an isomorphism. Thus f is a (split) monomorphism. Altogether we see that f is bijective, thus B and B' are isomorphic. $\square$

Addition to 6.1. We assume again that B is a brick and $\mathcal{T} \subseteq T({}^{\perp}B, B)$ a torsion class. Proposition 6.1 provides for any module M in $\mathcal{T}$ a submodule $M' \in \mathcal{T} \cap {}^{\perp}$ such that M/M' belongs to $\mathcal{E}(B)$. Claim: *The submodule M' is the largest submodule of M which belongs to ${}^{\perp}B$.*

Proof. Assume that U is a submodule of M and belongs to ${}^{\perp}B$. Let $M \to M/M'$ be the canonical projection. Then $f(U)$ is a submodule of a module in $\mathcal{E}(B)$. If $f(U) \neq 0$, then $f(U)$ has a non-zero factor module which is a submodule of B. Since $U \in {}^{\perp}B$, it follows that $f(U) = 0$, thus U is a submodule of M'. $\square$.

6.5. Remark. If B is a brick and Y a module in ${}^{\perp}B$, then $T(Y)$ is usually properly contained in $T(B, Y) \cap {}^{\perp}B$. For example, consider the quiver with vertices $1, 2$, one arrow $1 \leftarrow 2$ and a loop at 2. Let $Y = 0$ and M a serial module with composition factors going up $1, 2, 2$. Let B be the submodule of M of length two. Then M belongs to $T(B) \cap {}^{\perp}B$ (thus, $T(Y) = 0$, whereas $T(B, Y) \cap {}^{\perp}B \neq 0$).

7. The existence of torsional brick chain filtrations.

7.1. Proposition. *Let B be a top brick of the module M. Then M has a proper submodule M' which belongs to $T(M) \cap {}^{\perp}B$, such that M/M' belongs to $\mathcal{E}(B)$.*

Proof. Let $X = \operatorname{et}^{\infty} M$. Then B is a direct summand of X. Then $T(M) = T(X)$ by Proposition 5.5, thus M belongs to $T(X)$. Now use Proposition 6.1. $\square$

7.2. Proof of Theorem 3.2. According to 7.1, M has a proper submodule M' in $T(M) \cap {}^{\perp}B$ such that M/M' belongs to $\mathcal{E}(B)$ for some brick B.

By induction, M' has a brick chain filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_{m-1} = M'$ of type $(B_1, \dots, B_{m-1})$ such that any M_{i-1} is in $T(M_i)$ for all $1 \leq i \leq m-1$. Note that we have $T(M_0) \subseteq T(M_1) \subseteq \cdots T(M_{m-1}) = T(M')$.

Let $M_m = M$ and $B_m = B$. Now, for $1 \leq i \leq m-1$, the module M_i maps onto B_i. But $M_i \in T(M') \subseteq {}^{\perp}B$. As a consequence, $\operatorname{Hom}(B_i, B) = 0$. This shows that $(B_1, \dots, B_m)$ is a brick chain. Of course, the filtration M_i is of type $(B_1, \dots, B_m)$. Also, M_{i-1} is in $T(M_i)$ for all $1 \leq i \leq m-1$, by induction, and for $i = m$ by 7.1. $\square$

7.4. Some examples of torsional brick chain filtrations.

(1) If M is a brick, the filtration presented by Theorem 3.2 is the trivial one $0 \subset M$. Namely, assume M is a brick and $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ is a torsional brick chain filtration. Then M_1 belongs to $T(M)$, thus $\operatorname{Hom}(M, M_1) \neq 0$, (see Lemma 3.1). A non-zero map $M \to M_1$ gives rise a non-zero composition $M \to M_1 \subseteq M,$, thus the inclusion $M_1 \subseteq M$ is a split epimorphism and therefore the identity map.

(2) Not every brick chain filtration $(M_i)_i$ of a module M is torsional. For example, let A be the quiver $1 \leftarrow 2$ and M the sincere indecomposable module. There is the brick

chain filtration $0 \subset M_1 \subset M_2 = M$, where M_1 is the socle of M. Of course, M_1 is not contained in $T(M)$.

(3) Here is a module with two torsional brick chain filtrations. We start with the quiver with vertices $1, 2, 3$, two arrows $1 \leftleftarrows 2$, one arrow $2 \leftarrow 3$ and one zero relation, and form a node $1 = 3$. The injective module $I(1)$ is of length four, with socle 1, second layer $2 \oplus 2$, and third layer 1. There is the filtration with the following factors going up: first the socle 1, then an indecomposable module of length two, finally a copy of 2. Another filtration has only two factors going up: first the injective envelope of 1 in the category of modules of Loewy length at most two, then the simple module 1.

(4) For a Nakayama algebra, any indecomposable module M has only one torsional brick chain filtration $(M_i)_i$, and this filtration has length at most 2. Namely, let S be the top of M. Then all bricks in $T(M)$ have top S. Assume that M has precisely m composition factors of the form S, and U is the unique submodule of M with top S which is a brick. Then either M is in $\mathcal{E}(U)$, then $0 \subset M$ is the only torsional brick chain filtration of M. Else $0 \subset U \subsetneq M$ is the only torsional brick chain filtration of M.

(5) Duality. We have mention in Section 1 that using the duality functor D, we obtain from a brick chain filtration $(M_i)_i$ of M a corresponding brick chain filtration for $\mathrm{D}\,M$.

Here we want to stress: If the filtration $(M_i)_i$ is torsional, the dual filtration does not have to be torsional. As a typical example, let A be a connected Nakayama algebra with two simple modules and an indecomposable module M of length three, let U be its socle. Then M has the brick chain filtration $(0 \subset U \subset M)$. This filtration is torsional, whereas the dual filtration is not torsional.

There are brick chain filtrations $(M_i)_i$ of modules such that neither the filtration $(M_i)_i$ nor the dual filtration $(\mathrm{D}\,M/\mathrm{D}\,M_i)$ is torsional. Here is an example: Let A be a connected Nakayama algebra with three simple modules and M indecomposable of length four. Let U be the submodule of M of length two. Then $(0 \subset U \subset M)$ is a brick chain filtration, however neither this filtration nor its dual is torsional.

8. Homogeneous subcategories and the labels for neighbor torsion classes.

We will provide a characterization of neighbor torsion classes. This allows to attach to any neighbor torsion classes a uniquely determined brick B called the label of the neighbor torsion classes. First, we show that for a brick B, any non-zero module in $\mathcal{E}(B)$ determines uniquely B.

8.1. *Let B be a brick and M a non-zero module in $\mathcal{E}(B)$. Then M has an endomorphism with image a brick, and B is the only brick which occurs in this way.*

Proof. First, we show that B occurs as the image of an endomorphism of M. Since M belongs to $\mathcal{E}(B)$, there is a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ of M such that all factors are isomorphic to B. A corresponding map $M \to M/M_{m-1} \simeq B \simeq M_1 \subseteq M$ is an endomorphism of M which image isomorphic to B.

Conversely, let f be an endomorphism of M whose image is a brick. Since $\mathcal{E}(B)$ is an exact abelian subcategory, the image M' of f belongs to $\mathcal{E}(B)$. Now M' is a non-zero module in $\mathcal{E}(B)$. As we have seen in the first part of the proof, M' has an endomorphism with image $f(M')$ being isomorphic to B. But we assume that M' is a brick, thus the image of an endomorphism of M' is either zero or M' itself. This shows that $M' = f(M')$, thus M' is isomorphic to B. $\square$

8.2. Lemma. *Let $\mathcal{T}' \subset \mathcal{T}$ be torsion classes. Any module M in $\mathcal{T} \setminus \mathcal{T}'$ of minimal length is a brick and belongs to $(\mathcal{T}')^{\perp}$, thus to $\mathcal{T} \cap (\mathcal{T}')^{\perp}$.*

Proof of 8.2. Let M be a module in $\mathcal{T} \setminus \mathcal{T}'$ of minimal length. We form $X = \operatorname{et}^{\infty} M$. According to 5.5, we have $T(X) = T(M)$, thus also X belongs to $\mathcal{T} \setminus \mathcal{T}'$. There is an indecomposable direct summand X' of X which belongs to $\mathcal{T} \setminus \mathcal{T}'$ and, as we know, X' is a brick (one of the top bricks of M). On the other hand, there are epimorphisms $M \to X \to X'$, thus $|X'| \le |X|$. Since we assume that M is of minimal length, we see that $M = X'$ is a brick.

In order to see that $\mathcal{T}' \subseteq {}^{\perp}M$, consider any homomorphism $f\colon M' \to M$, with $M' \in \mathcal{T}'$. Now $f(M)$ belongs to $\mathcal{T}'$, thus M/M' does not belong to $\mathcal{T}'$. Since M/M' is a module in $\mathcal{T} \setminus \mathcal{T}'$, the minimality of M shows that $M' = 0$. $\square$

8.3. Corollary. *Let $\mathcal{T}' \subset \mathcal{T}$ be neighbor torsion classes. Then there exists a brick $B \in \mathcal{T}$ such that $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B$* (and then $B \in \mathcal{T} \setminus \mathcal{T}'$ and $\mathcal{T} = T(\mathcal{T}', B)$).

We will see in 8.4 that B is uniquely determined. We have mentioned already in section 2 (after Theorem 2.6): If $\mathcal{T}' \subset \mathcal{T}$ are neighbors, there usually will exist many bricks B in $\mathcal{T} \setminus \mathcal{T}'$, all satisfy $\mathcal{T} = T(\mathcal{T}', B)$, but only one B will satisfy the additional condition that $\mathcal{T}' \subseteq {}^{\perp}B$.

Proof. Take as B a module of minimal length in $\mathcal{T} \setminus \mathcal{T}'$. According to 8.2, B is a brick and belongs to $(\mathcal{T}')^{\perp}$. Now $B \in (\mathcal{T}')^{\perp}$ means that $\mathcal{T}' \subseteq {}^{\perp}B$. Therefore $\mathcal{T}' \subseteq \mathcal{T}' \cap {}^{\perp}B \subset \mathcal{T}$ (the last inclusion is proper, since B belongs to $\mathcal{T}$, but not to ${}^{\perp}B$). Since $\mathcal{T}' \subset \mathcal{T}$ are neighbors, we have $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B$.

Of course, if $B \in \mathcal{T}$ and $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B$, then $B \in \mathcal{T} \setminus \mathcal{T}'$ and $\mathcal{T}' \subset T(\mathcal{T}', B) \subseteq \mathcal{T}$ implies that $T(\mathcal{T}', B) = \mathcal{T}$, since we assume that $\mathcal{T}' \subset \mathcal{T}$ are neighbors. $\square$

8.4. Proof of Theorem 2.6. Let $\mathcal{F}' = (\mathcal{T}')^{\perp}$.

First we assume that $\mathcal{T} \cap (\mathcal{T}')^{\perp}$ is a homogeneous category, say of type B (thus B is a brick in $\mathcal{T} \cap \mathcal{F}'$). We want to show that $\mathcal{T}' \subset \mathcal{T}$ are neighbors. Let $\mathcal{N}$ be a torsion class with $\mathcal{T}' \subset \mathcal{N} \subseteq \mathcal{T}$. We claim that B belongs to $\mathcal{N}$. Let M be a module in $\mathcal{N} \setminus \mathcal{T}'$ and let M' be its torsion module with respect to the torsion class $\mathcal{T}'$. Since M does not belong to $\mathcal{T}'$, we see that M' is a proper submodule of M, thus $M/M' \neq 0$. By definition, M/M' belongs to $\mathcal{F}'$. Since M belongs to $\mathcal{T}$, also its factor module M/M' belongs to $\mathcal{T}$, therefore M/M' belongs to $\mathcal{T} \cap \mathcal{F}' = \mathcal{E}(B)$. As a non-zero module in $\mathcal{E}(B)$, the module M/M' has B as a factor module, therefore B belongs to $\mathcal{N}$.

Finally, we show that any module X in $\mathcal{T}$ belongs to $\mathcal{N}$ (thus $\mathcal{N} = \mathcal{T}$). Let X' be the torsion submodule of X with respect to $\mathcal{T}'$, thus X/X' belongs to $\mathcal{F}'$. Also, X/X' is a factor module of $X \in \mathcal{T}$, thus X/X' is in $\mathcal{T}$, therefore in $\mathcal{T} \cap \mathcal{F}' = \mathcal{E}(B)$. This shows that X has a filtration with modules in $\mathcal{T}'$ and in $\mathcal{E}(B)$. Both subcategories are contained in $\mathcal{N}$, thus X belongs to $\mathcal{N}$.

Next we show the reverse direction. Thus, let $\mathcal{T}' \subset \mathcal{T}$ be neighbors. According to 8.3, there is a brick B in $\mathcal{T}$ such that $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B$ (and then also $\mathcal{T} = T(\mathcal{T}', B)$). Since

$\mathcal{T}' \subseteq {}^{\perp}B$, Proposition 6.1 asserts: If M belongs to $\mathcal{T}$, there is a submodule $M' \subseteq M$ which belongs to $\mathcal{T} \cap {}^{\perp}B = \mathcal{T}'$ such that M/M' belongs to $\mathcal{E}(B)$. Assume that M belongs to $\mathcal{T} \cap \mathcal{F}'$, then M' belongs to $\mathcal{T}'$, but, as a submodule of $M \in \mathcal{F}'$ also to $\mathcal{F}'$. Therefore $M' = 0$. It follows that $M = M/M'$ belongs to $\mathcal{E}(B)$. Altogether we see that $\mathcal{T} \cap \mathcal{F}' \subseteq \mathcal{E}(B)$. Since B belongs to $\mathcal{T} \cap \mathcal{F}'$, and $\mathcal{T} \cap \mathcal{F}'$ is closed under extensions, we have $\mathcal{T} \cap \mathcal{F}' = \mathcal{E}(B)$.

Note that we have shown also the last assertion of Theorem: If B is a brick with $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B$, then $\mathcal{T} \cap \mathcal{F}' = \mathcal{E}(B)$. □

Next, 8.5 provides a construction of neighbor torsion classes (we obtain in this way all neighbor torsion classes).

8.5. Proposition. *Let B be a brick. Let $\mathcal{M}$ be a module class contained in ${}^{\perp}B$. Let $\mathcal{T} = T(\mathcal{M}, B)$. Then*

$$\mathcal{T} \cap {}^{\perp}B \ \subset \ \mathcal{T}$$

are neighbor torsion classes (with label B). Conversely, any inclusion of neighbor torsion classes labeled by B is obtained in this way.

Proof. Let B be a brick, and let $\mathcal{M} \subseteq {}^{\perp}B$ be a module class. Then $T(\mathcal{M}, B) \cap {}^{\perp}B \subseteq T(\mathcal{M}, B)$ are torsion classes. The inclusion is strict, since B belongs to $T(\mathcal{M}, B)$, but not to $T(\mathcal{M}, B) \cap {}^{\perp}B$.

In order to show that $T(\mathcal{M}, B) \cap {}^{\perp}B \subset T(\mathcal{M}, B)$ are neighbors, let $\mathcal{N}$ be a torsion class with $T(\mathcal{M}, B) \cap {}^{\perp}B \subset \mathcal{N} \subseteq T(\mathcal{M}, B)$. We want to apply 6.1 to B and $\mathcal{N}$ in order to see that we also have $T(\mathcal{M}, B) \subseteq \mathcal{N}$. Since $\mathcal{M} \subseteq {}^{\perp}B$, we have $\mathcal{N} \subseteq T(\mathcal{M}, B) \subseteq T({}^{\perp}B, B)$. Also, $\mathcal{N} \subseteq T(\mathcal{M}, B)$, but $\mathcal{N} \not\subseteq T(\mathcal{M}, B) \cap {}^{\perp}B$, thus $\mathcal{N} \not\subseteq {}^{\perp}B$. Therefore 6.1 asserts that $T(\mathcal{N} \cap {}^{\perp}B, B) = \mathcal{N}$. It follows that $T(\mathcal{M}, B) \subseteq T(\mathcal{N} \cap {}^{\perp}B, B) = \mathcal{N}$.

Conversely, let $\mathcal{T}' \subset \mathcal{T}$ be neighbor torsion classes labeled by B. By definition, B is a brick, and we have $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B$. Let $\mathcal{M} = \mathcal{T}'$. Then $\mathcal{M} \subseteq {}^{\perp}B$. Of course, we have $\mathcal{T} = T(\mathcal{M}, B)$. On the other hand, $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B = T(\mathcal{M}, B) \cap {}^{\perp}B$. Thus, we see that we recover the given torsion pair. □

8.6. Corollary. *Let $\mathcal{T}' \subset \mathcal{T}$ be torsion classes. If B is a brick in $\mathcal{T} \cap (\mathcal{T}')^{\perp}$, let $\mathcal{N} = T(\mathcal{T}', B)$ and $\mathcal{N}' = \mathcal{N} \cap {}^{\perp}B$, then we have*

$$\mathcal{T}' \subseteq \mathcal{N}' \subset \mathcal{N} \subseteq \mathcal{T}$$

and the torsion classes $\mathcal{N}' \subset \mathcal{N}$ are neighbors labeled by B.

Proof: It is trivial to see that $\mathcal{T}' \subseteq \mathcal{N}' \subseteq \mathcal{N} \subseteq T$. Since B belongs to $\mathcal{N}$, but not to $\mathcal{N}'$, we have $\mathcal{N}' \subset \mathcal{N}$. Finally, 8,5 asserts that $\mathcal{N}' \subset \mathcal{N}$ are neighbor torsion classes. □

8.7. Remark. *Any brick occurs as the label of neighbor torsion classes.* Usually, given a brick B, Corollary 8.6 can be used in order to provide many different neighbor torsion classes with label N. Of course, there are the neighbor classes $\mathcal{T}(B)_B \subset \mathcal{T}(B)$ (with label B) which may be of special interest.

Let us add several equivalent assertions which show that a brick in a torsion class $\mathcal{T}$ is the label for the inclusion of a lower neighbor.

8.8. Proposition. *Let $\mathcal{T}$ be a torsion class, let B be a brick in $\mathcal{T}$. The following conditions are equivalent:*
(i) $\mathcal{T} = T(B, \mathcal{T} \cap {}^{\perp}B \cap B^{\perp})$.
(ii) $\mathcal{T} = T(B, \mathcal{T} \cap {}^{\perp}B)$.
(iii) $\mathcal{T} \subseteq T(B, {}^{\perp}B)$.
(iv) $\mathcal{T} = (\mathcal{T} \cap {}^{\perp}B) \rfloor \mathcal{E}(B)$.
(v) $\mathcal{T} \cap {}^{\perp}B$ *is a lower neighbor of* $\mathcal{T}$.
(vi) *There is a lower neighbor of* $\mathcal{T}$ *with label* B.

Proof. First, we show the equivalence of (i), (ii), (iii), (iv). There is the inclusion chain

$$T(B, \mathcal{T} \cap {}^{\perp}B \cap B^{\perp}) \subseteq T(B, \mathcal{T} \cap {}^{\perp}B) \subseteq \mathcal{T},$$

thus (i) implies (ii). Of course, (ii) implies (iii).

(iii) implies (iv): We assume that $\mathcal{T} \subseteq T({}^{\perp}B, B)$. Since B is in $\mathcal{T}$, we know that $\mathcal{T}$ is not contained in ${}^{\perp}B$. According to 6.1, we have $\mathcal{T} = (\mathcal{T} \cap {}^{\perp}B) \rfloor \mathcal{E}(B)$.

(iv) implies (i): We have to show that $\mathcal{T} \subseteq T(B, \mathcal{T} \cap {}^{\perp}B \cap B^{\perp})$. Thus, let M be a module in $\mathcal{T}$. According to (iv), there is a submodule M' of M which belongs to $\mathcal{T} \cap {}^{\perp}B$ such that M/M' belongs to $\mathcal{E}(B)$. Inductively, we define a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_t \subseteq M'$ such that the factors M_i/M_{i-1} for $1 \leq i \leq t$, are factor modules of B, whereas $N = M'/M_t \in B^{\perp}$. Since N is a factor module of M', and M' belongs to the torsion class $\mathcal{T} \cap {}^{\perp}B$, also N belongs to $\mathcal{T} \cap {}^{\perp}B$. In this way, we see that N belongs to $\mathcal{T} \cap {}^{\perp}B \cap B^{\perp}$. Altogether, we have obtained a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_t \subseteq M' \subseteq M$ such that the first t factors belong to $T(B)$, the next one to $\mathcal{T} \cap {}^{\perp}B \cap B^{\perp}$ and the last one again to $T(B)$. Therefore M belongs to $T(B, \mathcal{T} \cap {}^{\perp}B \cap B^{\perp})$.

(iv) implies (v). We assume that $\mathcal{T} = (\mathcal{T} \cap {}^{\perp}B) \rfloor \mathcal{E}(B)$. Thus, we have $(\mathcal{T} \cap {}^{\perp}B) \subseteq \mathcal{T}$ and this is a proper inclusion, since B belongs to $\mathcal{T}$, but not to ${}^{\perp}B$. It is enough to show that for any module $M \in \mathcal{T} \setminus (\mathcal{T} \cap {}^{\perp}B)$, we have $\mathcal{T} \subseteq T(\mathcal{T} \cap {}^{\perp}B), M)$.

Thus, let M be a module in $\mathcal{T} \setminus (\mathcal{T} \cap {}^{\perp}B)$, According to (iv), M has a submodule M' in $\mathcal{T} \cap {}^{\perp}B$ with $M/M' \in \mathcal{E}(B)$. Since M' has to be a proper submodule of M, we have $M/M' \neq 0$. Now, M/M' belongs to $\mathcal{E}(B)$. Any non-zero object of $\mathcal{E}(B)$ has a factor module of the form B, thus B is a factor module of M/M', and therefore of M. It follows that $\mathcal{T} = T(\mathcal{T} \cap {}^{\perp}B, B) \subseteq T(\mathcal{T} \cap {}^{\perp}B, M)$.

(v) implies (iii): We assume that $\mathcal{T} \cap {}^{\perp}B \subset \mathcal{T}$ are neighbors. We have $\mathcal{T} \cap {}^{\perp}B \subseteq T(\mathcal{T} \cap {}^{\perp}B, B) \subseteq \mathcal{T}$. Since B is not contained in ${}^{\perp}B$, the first inclusion is a proper, thus we have $T(\mathcal{T} \cap {}^{\perp}B, B) = \mathcal{T}$. But this implies that $\mathcal{T} \subseteq T({}^{\perp}B, B)$.

It remains to show that (v) and (vi) are equivalent. It is trivial that (v) implies (vi). In order to show that (vi) implies (v), let $\mathcal{T}'$ be a lower neighbor of $\mathcal{T}$ with label B. Then $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B$, according to 2.6. $\square$

8.9. The brick chains explained in terms of neighbor torsion classes. Let $\mathcal{T}' \subset \mathcal{T}$ be neighbors with label B. Then we have on the one hand: B belongs to $\mathcal{T}$ and not to $\mathcal{T}'$. On the other hand, for every module M in $\mathcal{T}'$, in particular for the bricks in $\mathcal{T}'$, we have $\operatorname{Hom}(M, B) = 0$.

Thus we obtain in this way the Hom-condition which is used in the definition of a brick-chain: If $\mathcal{T}_1 \subset \mathcal{T}_2 \subseteq \mathcal{T}_3 \subset \mathcal{T}_4$ is a chain of torsion classes with $\mathcal{T}_1 \subset \mathcal{T}_2$ as well as

$\mathcal{T}_3 \subset \mathcal{T}_4$ being neighbors, and B is the label for $\mathcal{T}_1 \subset \mathcal{T}_2$, whereas B' is the label for $\mathcal{T}_3 \subset \mathcal{T}_4$, then $\operatorname{Hom}(B, B') = 0$.

9. The lower neighbors of a widely generated torsion class.

We recall that a torsion class $\mathcal{T}$ is said to be widely generated provided there is a semibrick $\mathcal{B}$ with $\mathcal{T} = T(\mathcal{B})$. For $B \in \mathcal{B}$, let $T(\mathcal{B})_B = T(\mathcal{B}) \cap {}^{\perp}B$.

9.1. Theorem. *A torsion class is widely generated iff it has sufficiently many lower neighbors. If $\mathcal{B}$ is a semibrick, and $\mathcal{T} = T(\mathcal{B})$, then $B \mapsto \mathcal{T}_B$ provides a bijection between $\mathcal{B}$ and the set of lower neighbors of $\mathcal{T}$.*

We will need the following Lemma and its Corollary.

9.2. Lemma. *Let $\mathcal{B}$ be a semibrick and $B \in \mathcal{B}$. Let $M \in T(\mathcal{B})$. Any non-zero map $M \to B$ is surjective.*

Proof. Let $B \in \mathcal{B}$ and $\mathcal{B}' = \mathcal{B} \setminus \{B\}$. Since $\mathcal{B}$ is a semibrick, we have $\mathcal{B}' \subseteq {}^{\perp}B$. Therefore $T(\mathcal{B}) \subseteq T({}^{\perp}B, B)$. The assertion of the Lemma follows directly from 6.2. $\square$

9.3. Corollary. *Let $\mathcal{B}$ be a semibrick and $B \in \mathcal{B}$. If M is a module in $T(\mathcal{B})$, but not in $T(\mathcal{B})_B$, then B is a factor module of M.*

Proof. If M is a module in $T(\mathcal{B}) \setminus T(\mathcal{B})_B$, then there is a non-zero map $f\colon M \to B$. According to 9.2, the map f is surjective. $\square$

9.4. Proof of 9.1. We will use proposition 8.4 for the brick B and the semibrick $\mathcal{B}' = \mathcal{B} \setminus \{B\}$. This is possible, since $\mathcal{B}' \subseteq {}^{\perp}B$. According to 8.4, we see that $T(\mathcal{B})_B$ is a lower neighbor of $T(\mathcal{B})$ and that the inclusion $T(\mathcal{B})_B \subset T(\mathcal{B})$ is labeled by B. Now B is the only brick in $\mathcal{B}$ which is not contained in $T(\mathcal{B})_B$, thus the map $B \mapsto T(\mathcal{B})_B$ is **injective.**

Next, we show: $(*)$ *if $\mathcal{T}'$ is a torsion class properly contained in $T(\mathcal{B})$, then $\mathcal{T}' \subseteq T(\mathcal{B})_B$ for some B.* Namely, if $\mathcal{T}'$ is a torsion class properly contained in $T(\mathcal{B})$, there exists some brick B in $\mathcal{B}$ which is not in $\mathcal{T}'$, since otherwise we would have $T(\mathcal{B}) \subseteq \mathcal{T}'$. We claim that $\mathcal{T}' \subseteq T(\mathcal{B})_B$. If not, then 9.3 asserts that B belongs to $\mathcal{T}'$, a contradiction.

A direct consequence of the assertion $(*)$ is that $T(\mathcal{B})$ has sufficiently many lower neighbors. But we also see that the map $B \mapsto T(\mathcal{B})_B$ is **surjective:** namely, if $\mathcal{T}'$ is a lower neighbor of $T(\mathcal{B})$, then $(*)$ yields a brick $B \in \mathcal{B}$ with $\mathcal{T}' \subseteq T(\mathcal{B})_B$. Since by assumption, $\mathcal{T}'$ is a lower neighbor of $T(\mathcal{B})$, we even have $\mathcal{T}' = T(\mathcal{B})_B$.

We have shown already that widely generated torsion classes have sufficiently many lower neighbors. Conversely, let us now assume that $\mathcal{T}$ is an arbitrary torsion class with sufficiently many lower neighbors. Let $\mathcal{B}$ be the set of bricks which are labels of the lower neighbors of $\mathcal{T}$. By definition, any B in $\mathcal{B}$ belongs to $\mathcal{T}$, thus $T(\mathcal{B}) \subseteq \mathcal{T}$. We claim that $\mathcal{T} = T(\mathcal{B})$. Assume, for the contrary, that $T(\mathcal{B}) \subset \mathcal{T}$. Since $\mathcal{T}$ has sufficiently many neighbors, we have $T(\mathcal{B}) \subseteq \mathcal{T}_B = \mathcal{T} \cap {}^{\perp}B$ for some $B \in \mathcal{B}$. But this implies that B is not contained in $T(\mathcal{B})$, a contradiction. $\square$

9.5. Proof of 2.7. Let M be a module. Let $\mathcal{B}$ be the set of top bricks of M. According to 5.5, we have $T(M) = T(\mathcal{B})$. Theorem 9.1 shows that $B \mapsto T(M)_B = T(M) \cap {}^{\perp}B$ provides a bijection between the elements of $\mathcal{B}$ and the lower neighbors of $T(M)$ and that the inclusion $T(M)_B \subset T(M)$ has label B. Also, 9.1 asserts that $T(M)$ has sufficiently many lower neighbors. $\square$

9.6. Corollary. *If $\mathcal{B}$ is a semibrick. Then $T(\mathcal{B})$ is finitely generated iff $\mathcal{B}$ is finite.*

Proof. If $\mathcal{B}$ is finite, then, of course, $T(\mathcal{B})$ is finitely generated. Conversely, assume that $T(\mathcal{B})$ is finitely generated. By definition, there is a module M with $T(\mathcal{B}) = T(M)$. According to Theorem 2.2, there is a finite semibrick $\mathcal{B}'$ with $T(M) = T(\mathcal{B}')$. According to 9.1, we have $\mathcal{B} = \mathcal{B}'$, thus $\mathcal{B}$ is finite. $\square$

10. The torsional brick chain filtrations.

We start with the following consequence of Theorem 2.7.

10.1. Lemma. *Let M be a module, B a brick. Assume that M has a proper torsional submodule Y in $T(M) \cap {}^{\perp}B$ such that M/Y belongs to $\mathcal{E}(B)$. Then B is a top brick of M (and Y is the torsion submodule of M with respect to the torsion class $T(M) \cap {}^{\perp}B$).*

Proof. First, we show that $T(M) = T(Y, B)$. Since Y is a proper submodule of M, we see that M/Y is a non-zero module in $\mathcal{E}(B)$, thus it has a factor module isomorphic to B. Since B is a factor module of M, we know that B belongs to $T(M)$. Also, by assumption, Y belongs to $T(M)$. Thus $T(Y, B) \subseteq T(M)$. On the other hand, M has a filtration with factors of the from Y and B, thus $T(M) \subseteq T(Y, B)$.

Next, we calculate the iterated endotop of $Y \oplus B$. We calculate inductively $\mathrm{et}^a(Y \oplus B)$ for all $a \geq 0$. We claim that $\mathrm{et}^a(Y \oplus B) = Y_a \oplus B$, where Y_a is a factor module of Y with $\mathrm{Hom}(Y_a, B) = 0$. For $a = 0$, we put $Y_a = Y$. Assume we have $\mathrm{et}^a(Y \oplus B) = Y_a \oplus B$, where Y_a is a factor module of Y with $\mathrm{Hom}(Y_a, B) = 0$. Since $\mathrm{Hom}(Y_a, B) = 0$, the radical maps in the endomorphism ring of $Y_a \oplus B$ map into Y_a. If U_a is the sum of these images, then $\mathrm{et}^a(Y \oplus B) = Y_{a+1} \oplus B$ with $Y_{a+1} = Y_a/U_a$. Also, we have $\mathrm{Hom}(Y_{a+1}, B)$, since any non-zero homomorphism $Y_{a+1} \to B$ would yield a non-zero homomorphism $Y_a \to Y_{a+1} \to B$. Since we deal with modules of finite length, there is some a such that $U_a = 0$, and therefore $\mathrm{et}^{\infty}(Y \oplus B) = Y_a \oplus B$. This shows that B is a top brick of $Y \oplus B$.

Since $T(M) = T(Y, B)$, we know from 2.7 that the top bricks of M are just the top bricks of $Y \oplus B$. Thus B is a top brick of M. $\square$

10.2. Corollary. *Let $(M_i)_i$ be a torsional brick chain filtration of M of brick type $(B_1, \dots, B_m)$. Then B_m is a top brick of M and M_{m-1} is the torsion submodule of M for the torsion class $T(M) \cap {}^{\perp}B_m$* (thus uniquely determined by M and B_m).

Proof. We apply Lemma 10.1 to $Y = M_{m-1}$ and $B = B_m$. $\square$

10.3. Proof of Theorem 3.3. Let M be a non-zero module. Let $T_1, \dots T_t$ be its top bricks. For $1 \leq i \leq t$, let $M^{(i)}$ be the torsion submodule of M which respect to the torsion class $T(M) \cap {}^{\perp}T_i$.

For any module M, let $\phi(M) \in \mathbb{N} \cup \{\infty\}$ be the number of torsional brick chain filtrations of M. Of course, we have $\phi(0) = 1$. For $M \neq 0$, let $M^{(i)}$ be the maximal submodule of M which belongs to $T(M) \cap {}^{\perp}T_i$. we claim that

$$\phi(M) = \sum_i \phi(M^{(i)}).$$

This follows from 10.2, since the torsional brick chain filtrations of M are the filtrations obtained from a torsional brick chain filtration of $M^{(i)}$ by adding the inclusion $M^{(i)} \subset M$.

We use induction on the length of M in order to see that $\phi(M)$ is finite for all M. $\square$

11. Further remarks about brick chain filtrations.

11.1. Remark. We have seen in 8.1: If M is a homogeneous module of brick type B, then the endomorphism ring of M shows whether $M = B$ or $M \neq B$. But $\mathrm{End}(M)$ may give only limited information about M. In particular, $\mathrm{End}(M)$ may be a k-algebra of dimension 2, whereas M has a filtration with arbitrarily many factors of the form B.

Here is an example. We consider the subring $A = k + J$ of the ring of $(t \times t)$-matrices with $t \geq 2$, where J is the set of nilpotent upper triangular matrices; and look at the set $M = k^t$ of column vectors. Since A is local, there is the unique brick $B = k$. The module M is a serial module of length t. The image of any non-invertible endomorphism of M has length at most 1, thus $\dim \mathrm{End}(M) = 2$.

11.2. A filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M$ will be said to be *directed* provided $\mathrm{Hom}(M_i/M_{i-1}, M_j/M_{j-1}) = 0$ for all $1 \leq i < j \leq m$.

Proposition. *Let M be a module with a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$. Then $(M_i)_i$ is a brick chain filtration iff $(M_i)_i$ is a directed filtration and all the factors are homogeneous.*

Proof. First, assume that $(M_i)_i$ is a brick chain filtration, say of type $(B_1, \ldots, B_m)$. Since M_i/M_{i-1} belongs to $\mathcal{E}(B_i)$, all the factors of the filtration are homogeneous. Also, for $i < j$, we have $\mathrm{Hom}(B_i, B_j) = 0$. Therefore $\mathrm{Hom}(M_i/M_{i-1}, M_j/M_{j-1}) = 0$.

Conversely, assume that $(M_i)_i$ is a directed filtration (with proper inclusions) and all factors are homogeneous. Since $F_i = M_i/M_{i-1}$ is a homogeneous module, there is a brick B_i with $F_i \in \mathcal{E}(B_i)$. Since F_i is non-zero, B_i occurs both as a submodule and as a factor module of F_i. Thus, any non-zero homomorphism $f\colon B_i \to B_j$ yields a non-zero homomorphism $F_i \to F_j$. Since the given filtration is directed, we see that $\mathrm{Hom}(B_i, B_j) = 0$ for $i < j$. Thus, $(B_1, \ldots, B_m)$ is a brick chain. $\square$

11.3. The composition factors which occur in the top of a module M give rise to interesting brick chain filtrations of M:

Proposition. *Let M be a module. If S is a simple module which occurs in the top of M, then M has a brick chain filtration of type $(B_1, \ldots, B_m)$ with $B_m = S$.*

Proof. Let M' be the minimal submodule of M such that M/M' has only S as composition factor, thus M/M' belongs to $\mathcal{E}(S)$ and S does not occur in the top of M'.

Now take a torsional brick chain filtration $(M_i)_{1\le i\le m-1}$ of M', say of type $(B_1,\dots,B_{m-1})$ and let $M_m = M$. Since we deal with a torsional filtration of M', the modules M_i, thus also the bricks B_i are in $\mathcal{T}(M')$, thus the top of B_i is generated by M'. As a consequence, $\operatorname{Hom}(B_i, S) = 0$. This shows that $(B_1,\dots,B_m)$ with $B_m = S$ is a brick chain, and that the filtration $(M_i)_{1\le i\le m}$ is a brick chain filtration of type $(B_1,\dots,B_m)$. $\square$

11.4. A module M has usually several brick chain filtrations, and the length of these filtrations seem to be quite unrelated. As a typical example, let A be the path algebra of the directed quiver of type $\mathbb{A}_n$ and M the indecomposable sincere A-module. It is easy to see that M has brick chain filtrations of length m, for any $1 \le m \le n$.

By definition, a module M is homogeneous iff $(0 \subseteq M)$ is a brick chain filtration. A homogeneous module which is not a brick has only one brick chain filtration, namely $(0 \subseteq M)$. But bricks usually have several brick chain filtrations:

Proposition. *A brick which is not simple, has at least two brick chain filtrations.*

Proof. Let M be a brick. Then $(0 \subset M)$ is a brick chain filtration of length 1.
Let S be a simple module which occurs in the top of M. According to 11.3, there is a brick chain filtration $(M_i)_{1\le i\le m}$ with M_m/M_{m-1} in $\mathcal{E}(S)$. We claim that $m \ge 2$. Namely, if $m = 1$, then M itself belongs to $\mathcal{E}(S)$. But since M is a brick, we must have $M = S$, thus M is simple. $\square$

We have stressed already in 3.4 that a brick has only one torsional brick chain filtration.

11.5. Question. Let us recall the question asked already in section 3, after the announcement of Theorem 3.3: *Are there modules with infinitely many brick chain filtrations?*

11.6. Brick chain complexity. We say that a module M has *brick chain complexity* at most t provided there is a brick chain filtration with t non-zero factors. The *brick chain complexity* of an algebra A is the maximum of the brick chain complexity of the indecomposable A-modules.

Of course, bricks are modules with brick chain complexity 1. Thus, any representation-directed algebra has complexity 1. Since any indecomposable Kronecker module is homogeneous, the Kronecker algebra has also complexity 1. Of course, if A is a local algebra, then again all modules are homogeneous, thus local algebras also have complexity 1.

Next, Nakayama algebras, tame concealed algebras and tubular algebras have complexity at most 2. For example, if A is a tame concealed, the only indecomposable modules which are not homogeneous are the indecomposable modules M which belong to a tube say of rank r, with regular length not divisible by r and these modules have complexity 2.

Note that any module which belongs to a standard tube has complexity at most 3. The case 3 is possible: see the non-stable tube for the algebra

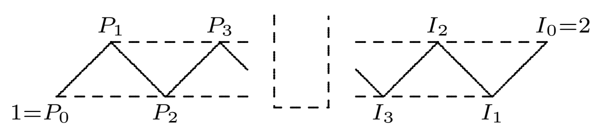
$$\circ \longleftarrow \circ \leftleftarrows \circ \longleftarrow \circ$$

Here, an indecomposable module M with dimension vector $(1,t,t,1)$ has brick chain complexity $\min(t,3)$.

Here is an algebra with brick chain complexity ∞. We take the path algebra of the quiver with 2 vertices $1,2$, with two arrows $1 \leftleftarrows 2$ and a loop γ at the vertex 2, with relation $\gamma^2 = 0$ (thus the algebra has dimension 7). For any $m \in \mathbb{N}_1$, we construct an indecomposable module M_m with brick chain complexits m as follows: The restriction of M_m to the Kronecker quiver $1 \leftleftarrows 2$ shall be $\bigoplus_{i=1}^m I_i$, where I_i is the indecomposable preinjective module of dimension $2i+1$. In order to specify the action of γ, we choose a basis $w_1^{(i)},\dots,w_i^{(i)}$ of the socle of I_i and define $\gamma w_1^{(i)} = w_{i-1}^{(i-1)}$ provided $2 \le i \le m$, and $\gamma w_j^{(i)}$ otherwise (it is clear that γ maps I_i into I_{i-1} and satisfies $\gamma^2 = 0$). We obtain a filtration $0 \subset M_1 \subset \cdots \subset M_m$, and this is a brick chain filtration of M_m of type $(I_1,\dots,I_m)$.

11.7. Looking at the brick chain complexity, we only deal with brick chain filtrations of indecomposable modules, and then only few brick chains may play a role.

But in general, one should be interested in all possible brick chains. Let us exhibit here all brick chains for the Kronecker algebra. As in 2.10, we denote the simple modules as $1,2$ with 1 being projective, 2 being injective. We denote by P_i the indecomposable preprojective modules, by I_i, the indecomposable preinjective modules, so that the Auslander-Reiten quiver of A looks as follows:



Any (finite) brick chain different from $(1,2)$ is of the form

$$(I_{i_1}, I_{i_2}, \cdots, I_{i_q}; R_1,\dots,R_r; P_{j_1}, P_{j_2}, \cdots, P_{j_p}),$$

where $0 \le i_1 < i_2 < \cdots < i_q$, as well as $j_1 > j_2 > \cdots > j_p \ge 0$, with pairwise non-isomorphic simple regular modules $R_1,\dots,R_r$ (in any order); here, p,q,r are non-negative numbers.

12. Opposition: Torsion pairs with a finitely generated torsionfree class.

This section is devoted to formulate opposite (or dual) definitions and results corresponding to the previous presentation. This concerns properties of torsionfree classes, similar to those for torsion classes as discussed in sections 2 to 10. Finally, we show some consequences for the corresponding torsion classes. This concerns, in particular, the upper neighbors of torsion classes.

12.1. Torsionfree classes. Given a set $\mathcal{X}$ of modules, we denote by $F(\mathcal{X})$ the smallest torsionfree class which contains $\mathcal{X}$, it is the class of all modules with a filtration whose factors are submodules of modules in $\mathcal{X}$. We call $F(\mathcal{X})$ the *torsionfree class generated by* $\mathcal{X}$. (We should mention here some hesitation which concerns the use of the words generation and cogeneration; after all, $F(\mathcal{X})$ is the closure under cogeneration and extension, but to say that $F(\mathcal{X})$ is "generated" by $\mathcal{X}$ should not provide any problems. Also note that we avoid to speak of "cotorsional" factor modules, since otherwise there could be some confusion in view of the well-established cotorsion theory.)

Since in this section we focus the attention to arbitrary torsion pairs $(\mathcal{T}, \mathcal{F})$ (mostly with $\mathcal{F}$ finitely generated), the torsion classes $\mathcal{T}$ considered here usually will not be finitely generated!

12.2. The endosocle and the iterated endosocle of a module. Given a module M, the set $\operatorname{es} M$ of elements $m \in M$ with $em = 0$ for all endomorphisms e of M which belong to the radical of $\operatorname{End}(M)$ is a submodule of M and called its *endosocle.* Again, we iterate this construction and obtain the *iterated endosocle* $\operatorname{es}_\infty M$ as the intersection of the submodules $\operatorname{es}^t M$ of M. The indecomposable direct summands of $\operatorname{es}_\infty M$ are called the *socle bricks* of M.

Here is the opposite of Theorem 2.2, and Proposition 5.6.

Theorem. *For any artin algebra A, the map $\mathcal{B} \mapsto F(\mathcal{B})$ provides a bijection between finite semibricks and the finitely generated torsionfree classes. If X is any module, we have $F(X) = F(\mathcal{B})$, where $\mathcal{B}$ is the set of socle bricks of X.*

12.3. The upper neighbors of a torsion class $\mathcal{T}$ with $\mathcal{T}^\perp$ finitely generated. The socle bricks of a module M yield torsionfree classes $\mathcal{F}' \subset F(B)$ such that there is no torsionfree class in between! This is the assertion opposite to 2.7, it concerns the upper neighbors of a torsion class $\mathcal{T}$, where $(\mathcal{T}, \mathcal{F})$ is a torsion pair with $\mathcal{F}$ being finitely generated, say $\mathcal{F} = F(N)$ for some module N.

Theorem. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair with $\mathcal{F}$ being finitely generated, say $\mathcal{F} = F(N)$ for some module N. If B is a socle brick of N, the torsion class $\mathcal{T}^B = T(\mathcal{T}, B)$ is an upper neighbor, with label B, and all upper neighbors are obtained in this way* (in this way, the upper neighbors of $\mathcal{T}$ correspond bijectively to the socle bricks of N). *Also, $\mathcal{T}$ has sufficiently many upper neighbors.*

12.4. Warning and remark: A basic feature. We have mentioned in 2.4 that one of the important consequences of Theorem 2.2 is the fact that we obtain a partial ordering on the set of finite semibricks, using the inclusion order on the set of torsion classes. Of course, similarly, we could use Theorem 2.2$*$ (dual to 2.2) and look at the inclusion order of the torsionfree classes.

The two orderings which we obtain on the set of finite semibricks look already at first sight quite different, as the case of the quiver $1 \leftarrow 2$ of type $\mathbb{A}_2$ seems to indicate: There is the inclusion of the torsion classes $T(2) \subset T(\begin{smallmatrix}2\\1\end{smallmatrix})$, whereas the torsionfree classes $F(2)$ and $F(\begin{smallmatrix}2\\1\end{smallmatrix})$ are incomparable.

However, it is easy to see that the lattice of all torsion classes can be derived from the knowledge of all brick chains. As we know, the brick chains can be defined in terms of the neighbors in the set of torsion classes. If we would use the torsionfree classes instead of the torsion classes, we would have to deal with the opposite of our brick chains (just reversing the total order of the chain)!

There is the following basic feature which is relevant in this context: on the one hand, the lattice of the torsionfree classes is the opposite of the lattice of the torsion classes, on the other hand, if $(\mathcal{T}', \mathcal{F}')$ and $(\mathcal{T}, \mathcal{F})$ are torsion lasses such that $\mathcal{T}' \subset \mathcal{T}$ are neighbors say with label B, then $\mathcal{T} \cap \mathcal{F}' = \mathcal{E}(B)$, and $\mathcal{T} = T(\mathcal{T}', B)$ as well as $\mathcal{F}' = F(\mathcal{F}, B)$.

12.5.. Next, let us mention the opposite of 6.1 and its corollaries 6.2 and 6.3. Whereas 6.1 provided homogeneous factor modules, we now obtain homogeneous submodules.

Proposition. *Let B be a brick. Let $\mathcal{F}$ be a torsionfree class which is contained in $F(B, B^{\perp})$. Then either $\mathcal{F}$ is contained in $B^{\perp}$ or else*

$$\mathcal{F} = \mathcal{E}(B) \rceil\!\!\lfloor (\mathcal{F} \cap B^{\perp}).$$

In particular, if $\mathcal{F}$ is not contained in $B^{\perp}$, then B belongs to $\mathcal{T}$.

Also, we add: if $0 \to M'' \to M \to M/M'' \to 0$ is an exact sequence with M'' in $\mathcal{E}(B)$ and $M/M'' \in \mathcal{F} \cap B^{\perp}$, then M'' is the torsion submodule for the torsion class ${}^{\perp}(\mathcal{F} \cap B^{\perp})$.

Corollary 1. *Let B be a brick. Let M be a module in $F(B, B^{\perp})$. Then any non-zero map $B \to B$ is injective.*

Corollary 2. *Let B, B' be non-isomorphic bricks, and assume that B' is in $F(B)$. Then* $\operatorname{Hom}(B, B') = 0$*, thus $B' \in F(B) \cap B^{\perp}$.*

12.6. Sufficiently many upper neighbors. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair. Then $\mathcal{F}$ is generated by a semibrick iff $\mathcal{T}$ has sufficiently many upper neighbors.*

If $\mathcal{B}$ is a semibrick and $\mathcal{F} = F(\mathcal{B})$, the upper neighbors of the torsion class $\mathcal{T}$ are the torsion classes $T(\mathcal{T}, B)$ with $B \in \mathcal{B}$. □

12.7. Example: The Kronecker algebra. We consider the Kronecker algebra A as in 2.10. For any torsion class $\mathcal{T}$ with $T(\mathcal{I}) \subseteq \mathcal{T} \subseteq T(\mathcal{R})$, the set of labels for the neighbors of $\mathcal{T}$ is the set $\mathcal{B}$ of simple regular modules. Namely, if $B \in \mathcal{B}$ belongs to $\mathcal{T}$, then $\mathcal{T}_B = \mathcal{T} \cap {}^{\perp}B$ is a lower neighbor, and the inclusion $\mathcal{T}_B \subset \mathcal{T}$ has label B. Or else B is not contained in $\mathcal{T}$, then $T(\mathcal{T}, B)$ is an upper neighbor of $\mathcal{T}$ and the inclusion $\mathcal{T} \subset T(\mathcal{T}, B)$ has label B. And there are no other neighbors.

If A is the Kronecker k-algebra, where k is an algebraically closed field, then $\mathcal{B}$ corresponds bijectively to the projective line $\mathbb{P}^1(k)$. Thus we see that for any torsion class $\mathcal{T}$ with $T(\mathcal{I}) \subseteq \mathcal{T} \subseteq T(\mathcal{R})$, the set of neighbors of $\mathcal{T}$ can be indexed in a natural way by the set $\mathbb{P}^1(k)$.

On the other hand, the remaining torsion classes $\mathcal{T}$ have precisely two neighbors (this is a general fact about functorially finite torsion classes as discussed in Section 16: the number of neighbors of a functorially finite torsion class of an algebra A is equal to the number of simple A-modules).

13. More about the brick labeling.

We provide characterizations of the bricks which are labels for given neighbor torsion classes $\mathcal{T}' \subset \mathcal{T}$, first in terms of $\mathcal{T}$, then in terms of $\mathcal{T}'$.

13.1. Proposition. *Let $\mathcal{T}$ be a torsion class, let B be a module. The following assertions are equivalent:*

(i) *The torsion class $\mathcal{T} \cap {}^{\perp}B$ is a lower neighbor of $\mathcal{T}$ with label B.*

(ii) *The module B belongs to $\mathcal{T}$, all proper submodules U of B satisfy* $\mathrm{Hom}(\mathcal{T}, U) = 0$. *and for any exact sequence $0 \to X \to Y \to B \to 0$ with $Y \in \mathcal{T}$ also $X \in \mathcal{T}$.*

Proof. First, let us assume that (i) holds. Let $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B$ and $\mathcal{F}' = (\mathcal{T}')^{\perp}$. By assumption, B is a brick and $\mathcal{T} \cap \mathcal{F}' = \mathcal{E}(B)$. In particular, B belongs to $\mathcal{T}$.

Let U be a proper submodule of B. We want to show that $\mathrm{Hom}(\mathcal{T}, U) = 0$. Let $f\colon M \to U$ be a homomorphism with $M \in \mathcal{T}$. Then $f(M)$ belongs to $\mathcal{T}$ and is a proper submodule of B. Now $f(M)$ cannot have a factor module isomorphic to B. According to 8.8 (iv), it follows that $f(M)$ belongs to ${}^{\perp}B$. Since $f(M)$ is a submodule of B, we have $f(M) = 0$.

Let $0 \to X \to Y \to B \to 0$ be exact and assume that Y belongs to $\mathcal{T}$. Using again 8.8 (iv), Y has a submodule Y' in $\mathcal{T}' = \mathcal{T} \cap B^{\perp}$ with $Y/Y' \in \mathcal{E}(B)$. Since $Y' \in B^{\perp}$, we can assume that Y' is a submodule of X. Now $B = Y/X$ is a factor module of $Y/Y' \in \mathcal{E}(B)$, thus X/Y' is in $\mathcal{E}(B)$. Since both X/Y' and Y' are in $\mathcal{T}$, we see that X is in $\mathcal{T}$. Altogether, we have shown (ii).

Now assume that (ii) holds: thus B belongs to $\mathcal{T}$, all proper submodules U of B satisfy $\mathrm{Hom}(\mathcal{T}, U) = 0$ and given an exact sequence $0 \to X \to Y \to B \to 0$ with $Y \in \mathcal{T}$ also $X \in \mathcal{T}$. First of all, B is a brick, since otherwise B has a non-zero proper submodule U which is an image of B and thus in $\mathcal{T}$. But then $\mathrm{Hom}(\mathcal{T}, U) \neq 0$.

We show by induction that any module M in $\mathcal{T}$ has a submodule M' in $\mathcal{T} \cap B^{\perp}$ such that M/M' belongs to $\mathcal{E}(B)$. This is clear if M belongs to $B^{\perp}$. Thus, we can assume that there is a non-zero map $f\colon M \to B$. Now $f(M)$ is a non-zero submodule of B. Since $f(M)$ is a factor module of M, it belongs to $\mathcal{T}$. Since $\mathrm{Hom}(\mathcal{T}, U) = 0$ for all proper submodules U of B, we see that $f(M) = B$. Let X be the kernel of f, thus we have an exact sequence $0 \to X \to M \to B \to 0$. Since $M \in \mathcal{T}$, (ii) asserts that $X \in \mathcal{T}$. By induction, X has a submodule $X' \in \mathcal{T} \cap B^{\perp}$ with $X/X' \in \mathcal{E}(B)$. Let $M' = X'$. Then M' belongs to $\mathcal{T} \cap B^{\perp}$ and M/M' belongs to $\mathcal{E}(B)$. $\square$

13.2. Proposition. *Let $\mathcal{T}'$ be a torsion class, let B be a module. The following assertions are equivalent:*

(i) *The torsion class $\mathcal{T} = T(\mathcal{T}', B)$ is an upper neighbor of $\mathcal{T}'$ and the neighbor torsion classes $\mathcal{T}' \subset \mathcal{T}$ are labeled by B.*

(ii) *The module B satisfies* $\mathrm{Hom}(\mathcal{T}', B) = 0$, *all proper factor modules of B belong to $\mathcal{T}'$, and if $0 \to B \to Y \to Z \to 0$ is a non-spilt exact sequence with $Z \in \mathcal{T}'$ also $Y \in \mathcal{T}'$.*

Proof. Let $\mathcal{T} = T(\mathcal{T}', B)$.

First, let us assume (i), thus $\mathcal{T}' \subseteq \mathcal{T}$ are neighbors with label B. Then, $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B$, thus $\mathrm{Hom}(\mathcal{T}', B) = 0$. Let U be a non-zero submodule of B and $M = B/U$ belongs to $\mathcal{T}$. Since B is a brick, there is non non-zero homomorphism $M = B/U \to B$, thus M also belongs to ${}^{\perp}B$. Thus B belongs to $\mathcal{T} \cap {}^{\perp}B = \mathcal{T}$.

Finally, consider a non-split exact sequence $0 \to B \to Y \to Z \to 0$ with $Z \in \mathcal{T}'$ and denote the inclusion map $B \to Y$ by u. Let $f\colon Y \to B$ be a homomorphism. Now $fu\colon B \to B$ is not an isomorphism, since u is not split mono. Since B is a brick, $fu = 0$, thus f factors through Z. But Z belongs to ${}^{\perp}B$, therefore $f = 0$. This shows that $Y \in \mathcal{T} \cap {}^{\perp}B = \mathcal{T}'$. Therefore, (ii) holds.

Conversely, let us assume (ii) and let $\mathcal{T} = T(\mathcal{T}', B)$. Clearly, B has to be a brick, since the image of a non-invertible endomorphism is a proper factor module of B.

We have $\mathcal{T}' \subseteq \mathcal{T} \cap {}^{\perp}B$, and we claim that $\mathcal{T} \cap {}^{\perp}B = \mathcal{T}'$. Let us show that $\mathcal{T} \subseteq \mathcal{T}' \rceil\!\lfloor \mathcal{E}(B)$. Let M be a module in $\mathcal{T}$, thus M has a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ where any factor $F_i = M_i/M_{i-1}$ is in $\mathcal{T}'$ or is isomorphic to B. Take such a filtration with length t minimal. We claim that we can assume that the first factors are in $\mathcal{T}'$, the last one are isomorphic to B. Otherwise, there is an exact sequence

$$0 \to F_i \to M_{i+1}/M_{i-1} \to F_{i+1} \to 0.$$

with $F_i = B$ and $F_{i+1} \in \mathcal{T}'$. If it splits, we can reorder the factors. If it does not split, then by assumption M_{i+1}/M_{i-1} belongs to $\mathcal{T}'$, thus we delete M_i from the filtration. Thus, either all factors F_i belong to $\mathcal{T}'$ (and then M belongs to $\mathcal{T}'$), or else M has a factor module isomorphic to B. It follows that $\mathcal{T} \subseteq \mathcal{T}' \rceil\!\lfloor \mathcal{E}(B)$ and therefore $\mathcal{T} \cap {}^{\perp}B = \mathcal{T}'$. According to 8.8, the torsion class $\mathcal{T} \cap {}^{\perp}B$ is a lower neighbor of $\mathcal{T}$ with label B. $\square$

14. Change of support.

We want to exhibit in 14.2 an important special case of 13.2. We use the following definition. We say that a simple module S belongs to the *support* of a module class $\mathcal{M}$ provided there is a module M in $\mathcal{M}$ which has S as a composition factor.

Lemma 14.1. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair. Let S be a simple module. Then S does not belong to the support of $\mathcal{T}$ iff $I(S)$ belongs to $\mathcal{F}$. Similarly, S does not belong to the support of $\mathcal{F}$ iff $P(S)$ belongs to $\mathcal{T}$.*

Proof. Let M be a module. Clearly, S is in the support of M iff $\operatorname{Hom}(M, I(S)) \neq 0$, iff $\operatorname{Hom}(P(S), M) = 0$. $\square$

14.2. Proposition. *Let $\mathcal{T}'$ be a torsion class. Let S be a simple module which does not belong to the support of $\mathcal{T}'$. Assume that $S \subseteq B \subseteq I(S)$ such that B/S is the $\mathcal{T}'$-torsion submodule of $I(S)/S$. Then $T(\mathcal{T}', B)$ is an upper neighbor of $\mathcal{T}'$ with label B.*

Proof. We have to show that B satisfies the conditions (ii) of 13.2. Let $\mathcal{F}' = (\mathcal{T}')^{\perp}$. According to Lemma 14.1, we know that $I(S)$ belongs to $\mathcal{F}'$. Since B is a submodule of $I(S)$, also B belongs to $\mathcal{F}'$. A proper factor module of B is a factor module of B/S. Since B/S belongs to $\mathcal{T}'$, also any factor module of B/S belongs to $\mathcal{T}'$. Finally, assume that there is given the exact sequence $\epsilon\colon 0 \to B \to Y \to Z \to 0$ with $Z \in \mathcal{T}'$. We claim that such a sequence always splits (thus the last condition is trivially satisfied). Namely, we look at the sequence ϵ and embedding $u\colon B \to I(S)$. The injectivity of $I(B)$ yields a map $f\colon Y \to I(S)$ and there is a corresponding map $f'\colon Z \to I(S)/B$ such that the following diagram commutes:

$$\begin{array}{ccccccccc}
\epsilon\colon & 0 & \longrightarrow & B & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0 \\
& & & \| & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle f'} & & \\
& 0 & \longrightarrow & B & \xrightarrow{\ u\ } & I(S) & \longrightarrow & I(S)/B & \longrightarrow & 0
\end{array}$$

Now Z belongs to $\mathcal{T}'$, whereas $I(S)/B$ belongs to $\mathcal{F}'$. Thus $f' = 0$, therefore ϵ splits. $\square$

14.2∗. Proposition. *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair. Let S be a simple module which does not belong to the support of $\mathcal{F}$. Let U be the $\mathcal{T}$-torsion submodule of* $\operatorname{rad} PS$. *Then PS/U is a top brick of $\mathcal{T}$ and S belongs to the support of the torsionfree class $(\mathcal{T}_B)^{\perp}$.* Of course, the neighbor pair $\mathcal{T}_B \subset \mathcal{T}$ has label B.

14.3. Corollary. *Let $(\mathcal{T}, \mathcal{F})$ and $(\mathcal{T}', \mathcal{F}')$ be torsion pairs such that $\mathcal{T}' \subset \mathcal{T}$ are neighbors labeled by the brick B. If the torsion classes $\mathcal{T}'$ and $\mathcal{T}$ have different support, then B is colocal. If the torsionfree classes $\mathcal{F}'$ and $\mathcal{F}$ have different support, then B is local.*

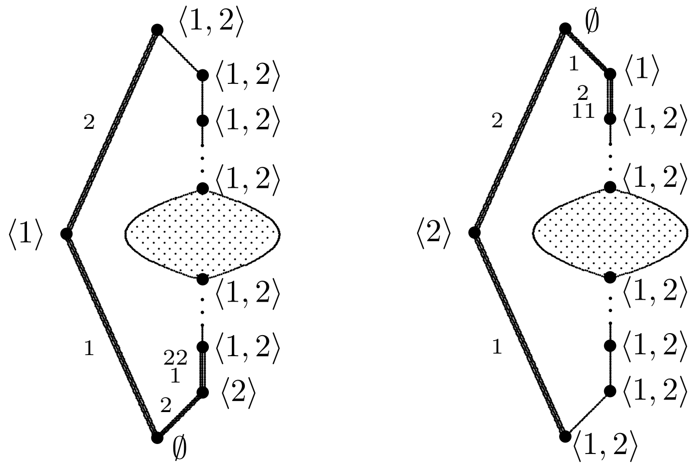
Proof. The first assertion follows from Proposition 14.2, the second by duality. $\square$

Remark. Let $\mathcal{T}' \subset \mathcal{T}$ be neighbor torsion classes with label B. Note that B may be colocal also in case the torsion classes $\mathcal{T}'$ and $\mathcal{F}$ have the same support; and B may be local also in case the torsionfree classes $(\mathcal{T}')^{\perp}$ and $\mathcal{F}^{\perp}$ have the same support. For example, consider for the Kronecker quiver the torsion class $\mathcal{T}$ generated by any 2-dimensional indecomposable module B and let $\mathcal{T}'$ be its unique lower neighbor. Then B is the label of the inclusion $\mathcal{T}' \subset \mathcal{T}$ and B is both local and colocal. The two torsion classes as well as the two corresponding torsionfree classes all have full support.

14.4. Small torsion classes. Let A be an algebra with $n(A) \geq 2$. Let $0 = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \mathcal{T}_2$ be a sequence of neighbor torsion classes with labels B_1, B_2, respectively. Then both neighbor classes are provided by change of support of torsion classes. The brick B_1 and the socle of B_2 are different simple modules, and $B_2/\operatorname{soc} B_2$ belongs to $\mathcal{E}(B_1)$ (and is the submodule of $I(\operatorname{soc} B_2)/\operatorname{soc} B_2$ which is maximal with this property).

14.5. Example. As an example, we look again at the Kronecker quiver. The following two pictures show both the lattice of torsion classes as seen in 2.10. Whereas, on the left, we endow any vertex $\mathcal{T}$ with the support of $\mathcal{T}$; any vertex $\mathcal{T}$ on the right is endowed with the support of the corresponding torsion**free** class $\mathcal{F} = \mathcal{T}^{\perp}$. Such a support is a subset of the set of vertices and is written in angular brackets. The bold edges $\mathcal{T}' \subset \mathcal{T}$ are those with change of support (left for the torsion classes, right for the corresponding torsionfree classes); these edges are endowed with the corresponding brick label B (thus

$\mathcal{T} \cap {}^{\perp}(\mathcal{T}') = \mathcal{E}(B)$).



support of the torsion classes

support of the torsionfree classes

15. Brick finiteness.

15.1. Theorem. *The following conditions are equivalent.*

(i) *There are only finitely many torsion classes.*
(ii) *There are only finitely many finitely generated torsion classes.*
(iii) *There are only finitely many bricks.*
(iv) *There are only finitely many finite semibricks.*
(v) *There are only finitely many semibricks.*
(vi) *All torsion classes are fg, all torsionfree classes are fg.*

Remark. Let us stress that the last condition is not optimal: the two assertions which are combined are actually equivalent: *All torsion classes are fg iff all torsionfree classes are fg,* see the Appendix, section 17. We do not know a direct proof of this fact avoiding the use of functorial finiteness of torsion classes. Section 17 is an appendix which refers to functorial finiteness of subcategories. Using the functorial finiteness of certain torsion classes (the reachable ones) we will be able to add to the equivalent assertions in Theorem 15.1 also the condition that all torsion classes are finitely generated.

Proof of Theorem 15.1. The equivalence of (ii) and (iv) is given by Theorem 2.2. The equivalence of (iii) and (iv) is clear. The equivalence of (iv) and (v) is easy to see, since subsets of semibricks are semibricks.

Of course, (i) implies (ii). Now assume that (ii) holds. Then any increasing chain of torsion classes becomes stationary. Thus any torsion class is finitely generated. As a consequence, there are only finitely many torsion classes, thus condition (i) is satisfied.

Let us show that (i) implies (vi). Now (i) implies (ii) and we have mentioned already that (ii) implies that all torsion classes are fg. Now (i) means also that there are only finitely many torsionfree classes, and therefore that any increasing chain of torsionfree classes becomes stationary, thus that any torsionfree class is finitely generated.

Finally, let us assume (vi) and show that it implies (i). (a) Since all torsion classes are fg, any increasing sequence of torsion classes becomes stationary. (b) Since all torsionfree classes are fg, any increasing sequence of torsionfree classes becomes stationary, thus any decreasing sequence of torsion classes becomes stationary. According to (a) and (b), there are no infinite chains of torsion classes. In particular, for any torsion class $\mathcal{T}$, there is a finite chain of neighbors starting at 0, passing through $\mathcal{T}$ and ending in $\operatorname{mod} A$.

Let us denote by Γ the graph of torsion classes with arrows $\mathcal{T} \to \mathcal{T}'$ provided $\mathcal{T}'$ is a lower neighbor of $\mathcal{T}$. As we have seen, this graph is connected, with unique source $\operatorname{mod} A$.

Since any torsion class $\mathcal{T}$ is finitely generated, we know that $\mathcal{T}$ has only finitely many lower neighbors, thus only finitely many arrows of Γ start at $\mathcal{T}$.

Altogether, we have shown that Γ is a connected graph with a unique source, such that any vertex is the start of only finitely many arrows, and without infinite paths. König's infinite path lemma asserts that Γ has to be finite. $\square$

16. A final observation.

We want to draw again the attention to the bricks and the homogeneous categories generated by bricks. The following quite innocent result may be seen as a key for all the considerations in the report.

Proposition. *A non-zero module M is a brick iff $T(M) \cap F(M) = \mathcal{E}(M)$.*

Proof. First, assume that M is non-zero and not a brick, thus there is a non-zero and non-invertible endomorphism f, say with image N. Then N is a factor module of M, thus in $T(M)$, and also a submodule of M, thus in $F(M)$. However N does not belong to $\mathcal{E}(M)$, since its length is not a multiple of the length of M, whereas the length of any module in $\mathcal{E}(M)$ is a multiple of the length of M.

On the other hand, for any module M we have $\mathcal{E}(M) \subseteq T(M) \cap F(M)$. Now let $M = B$ be a brick. We just have mentioned, that we have $\mathcal{E}(B) \subseteq T(B) \cap F(B)$. We want to show that we also have $T(B) \cap F(B) \subseteq \mathcal{E}(B)$. According to Theorem 2.7, $\mathcal{T}' = T(B)_B = T(B) \cap {}^{\perp} B$ is a lower neighbor of $\mathcal{T} = T(B)$, with label B. Proposition 8.2 asserts that $\mathcal{T} \cap (\mathcal{T}')^{\perp} = \mathcal{E}(B)$.

By definition, $\mathcal{T}' \subseteq {}^{\perp}B$, thus $\operatorname{Hom}(\mathcal{T}', B) = 0$. Therefore B belongs to $(\mathcal{T}')^{\perp}$. Since $(\mathcal{T}')^{\perp}$ is a torsionfree class, we even have $F(B) \subseteq (\mathcal{T}')^{\perp}$. Therefore

$$T(B) \cap F(B) \subseteq \mathcal{T} \cap (\mathcal{T}')^{\perp} = \mathcal{E}(B).$$

$\square$

17. Appendix: Some features of functorially finite torsion classes.

As we have mentioned, Theorem 15.1 is not optimal. The condition (vi) can be split into the following two conditions: *all torsion classes are fg,* and: *all torsionfree classes are fg.* We claim that *these conditions are equivalent to each other.* The proof of the equivalence which we will outline requires to look at functorially finite torsion classes (until now, we have avoided this concept). Also, using functorially finite subcategories, we will provide an improvement to the main theorem 3.2 in case the algebra A is brick finite, or has at least one complete brick chain, see 17.6 and 17.7.

17.1. The definition. A torsion class $\mathcal{T}$ is *functorially finite* (ff) provided any module X has a left $\mathcal{T}$-approximation (that is a map $f_X\colon X \to X'$ with $X' \in \mathcal{T}$ such that any map $g\colon X \to Y$ with $Y \in \mathcal{T}$ has a factorization $g = g' f_X$ with $g'\colon X' \to Y$).

Here are the properties of a functorially finite torsion class which we will need:

A torsion class $\mathcal{T}$ is ff iff $\mathcal{T}$ is generated by a τ-rigid module. (A module M is said to be *τ-rigid* provided $\operatorname{Hom}(M, \tau M) = 0$, where τ is the Auslander-Reiten translation.)

Let $\mathcal{T}$ be a functorially finite torsion class. Then $\mathcal{T}$ has only finitely many upper neighbors, all are functorially finite again, and there are sufficiently many upper neighbors.

Actually, there is a fixed bound on the number of upper neighbors, namely the number n of simple modules: For a functorially finite torsion class, the number of neighbors (the sum of the number of upper neighbors and lower neighbors) is equal to n, a very remarkable result.

17.2. Theorem. *If all torsion classes are finitely generated, then they are even functorially finite and there are only finitely many.*

The proof will be given in 17.4. We need a further definition.

17.3. A torsion class $\mathcal{T}$ will be called *reachable* provided there is a finite chain of neighbors $\{0\} = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_m = \mathcal{T}$. Theorem 17.2 has the following consequence:

Proposition. *Any reachable torsion class is ff.*

Proof. Let $\mathcal{T}$ be a reachable torsion class, thus there is a sequence $\{0\} = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_m = \mathcal{T}$ of neighbors of torsion classes. Now $\{0\}$ is trivially ff. By induction, all torsion classes $\mathcal{T}_i$ are ff, thus $\mathcal{T}$ is ff. $\square$

17.4. Proof of Theorem 17.2. We assume that all torsion classes are finitely generated.

(a) *Any ascending chain of torsion classes gets stationary.* This follows directly from the assumption that all torsion classes are finitely generated.

(b) *All torsion classes are reachable.*

Proof. Assume, for the contrary, that there is a torsion class $\mathcal{T}$ which is not reachable. We construct inductively an infinite chain of neighbors $\{0\} = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_m \subset \cdots$ with $\mathcal{T}_m \subset \mathcal{T}$ for all m. Since $\mathcal{T}$ is not reachable, we have $\{0\} = \mathcal{T}_0 \subset \mathcal{T}$. If $\mathcal{T}_m$ is already constructed, it follows from $\mathcal{T}_m \subset \mathcal{T}$ that there is an upper neighbor $\mathcal{T}_{m+1}$ of $\mathcal{T}_m$ with $\mathcal{T}_{m+1} \subseteq \mathcal{T}$. Since $\mathcal{T}$ is not reachable, we actually have $\mathcal{T}_{m+1} \subset \mathcal{T}$. We get in this way a chain of torsion classes which does not become stationary, a contradiction. $\square$

(c) *There are only finitely many torsion classes.*

Proof: According to (b), the set of torsion classes is connected with $\{0\}$ as the unique smallest element. According to (b) there are no infinite increasing paths. According to 17.2, any vertex has only finitely many upper neighbors. Thus König's infinite path lemma asserts that the set of torsion classes is finite. $\square$

Finally, we note that (b) together with 17.3 asserts that all torsion classes are ff. $\square$

Remark. We have shown: *All torsion classes are fg iff all torsionfree classes are fg.* This can be seen also differently, namely as follows:

A torsionfree class $\mathcal{F}$ is said to be *functorially finite* provided any module X has a right $\mathcal{F}$-approximation (that is a map $h_X : X' \to X$ with $X' \in \mathcal{F}$ such that any map $g: Y \to X$ with $Y \in \mathcal{F}$ has a factorization $g = h_X g'$ with $g' : Y \to X'$).

Already Auslander-Smalø [AS] have shown: *Let $(\mathcal{T}, \mathcal{F})$ be a torsion pair. Then $\mathcal{T}$ is ff, iff $\mathcal{F}$ is ff.* Thus, if all torsion classes are fg, then, as we have seen, they are ff, and then all torsionfree classes are ff, thus fg. And similarly, we have the converse.

We use this appendix to provide an improvement of theorem 3.2 for certain algebras.

17.5. A sequence $(B_1, \dots, B_m)$ of bricks will be said to be a *weak brick chain* provided $\mathrm{Hom}(B_i, B_j) \neq 0$ for $i < j$ implies that $B_i = B_{i+1} = \cdots = B_j$. Of course, the weak brick chains are obtained from the brick chains just by repetitions of some of the entries of the sequence. A filtration $(M_i)_{0 \le i \le t}$ of a module M will be called a *refined brick chain filtration* provided there is a weak brick chain $(B_1, \dots, B_t)$ such that M_i/M_{i-1} belongs to $\mathrm{add}\, B_i$, for $1 \le i \le m$.

17.6. Let us say that a (finite) brick chain $(C_1, \dots, C_m)$ is *complete* provided it is not a proper subsequence of a another brick chain, or, equivalently, provided the torsion classes $\mathcal{T}_i = T(C_1, \dots, C_m)$ with $0 \le i \le m$ provide a sequence of neighbors

$$0 = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_{m-1} \subset \mathcal{T}_m = \mathrm{mod}\, A.$$

Not every brick chain is a subsequence of a (finite) complete brick chain! For example, if A is the Kronecker algebra, then there is just one finite complete brick chain, namely $(1, 2)$. In general, one has to be aware that not every algebra has a finite complete brick chain. The first example of an algebra without a finite complete brick chain is the radical square zero algebra with quiver Q with two vertices $1, 2$, two arrows $1 \rightrightarrows 2$ and two arrows $1 \leftleftarrows 2$.

Here are some examples of finite complete brick chains:

(1) If $A_1, \dots, S_n$ are the simple A-modules and $\mathrm{Ext}^1(S_i, S_j) = 9$ for all $i > j$, then $(S_1, \dots, S_n)$ is a complete brick chain. In particular, all directed algebras have a finite complete brick chain.

(2) Of course, any brick finite algebra has finite complete brick chains! Actually, in this case (and only in this case) any brick chain is a subsequence of a finite complete brick chain.

17.7. Theorem. *Let A be an algebra with a finite complete brick chain. Then there is a natural number c such that any module M has a refined brick chain filtration of length at most c.*

Addendum. One can take $c = \sum_i \lambda(C_i)$, where $(C_1, \dots, C_m)$ is a complete brick chain and $\lambda(C_i)$ is the Loewy number of C_i as defined in 17.8.

The proof of 17.7 and its Addendum will be given in 17.10.

17.8. The Loewy number of a brick. The *Loewy number* $\lambda(B)$ of a brick B is by definition the Loewy length of the category $\mathcal{E}(B)$; it is either a positive integer or ∞.

Examples. (1) Let A be the Kronecker algebra. Then the Loewy number of a preprojective or preinjective brick is 1, whereas the Loewy number of any regular brick is ∞. (In general, if A is hereditary, then the Loewy number of any brick is 1 or ∞; and there are bricks with Loewy number ∞ iff A is representation infinite.)

(2) Let A be a connected Nakayama algebra with n simple modules. Let P be indecomposable projective of length $an+r$ with $a \geq 1$ and $0 \leq r < n$, and let B be the factor module of P of length n. Then B is a brick and its Loewy number is equal to a. Thus, any positive integer can arise as the Loewy number of some brick.

17.9. Proposition. *The Loewy number of a top brick of a τ-rigid module is finite.*

The Nakayama algebras show that any positive number does occur in this way.

Proof. Let B be a top brick of the τ-rigid module M. Since B is a top brick of M, there is an exact sequence $0 \to N \to M \to C \to 0$ with $C \in \mathcal{E}(B)$ and $N \in T(M) \cap {}^{\perp}B$. Let X be a module in $\mathcal{E}(B)$. Now X belongs to $T(M)$. Since M is τ-rigid, any module in $T(M)$ is generated by M, thus X is generated by M: There is an epimorphism $f\colon M^t \to X$ for some natural number t. Since N belongs to ${}^{\perp}B$, we have $f(N^t) = 0$, thus there is an epimorphism $\overline{f}\colon C^t \to X$. But this means that the Loewy length of X is bounded by the Loewy length of C. □

Remark, Let B be a top brink of a τ-rigid module M. Let $0 \to U \to M \to C \to 0$ be exact with U in $T(M) \cap {}^{\perp}B$ and C in $\mathcal{E}(B)$. Then neither B nor C has to be τ-rigid. This is seen by looking at the Asai example of the quiver $1 \leftarrow 2 \leftleftarrows 3$ with one zero relation, with $M = P3 \oplus 2$ and $B = C = I1$.

17.10. Proof of 17.7. Let $(C_1, \dots, C_m)$ be a complete brick chain. We claim that $\lambda(C_i)$ is finite for all i. Namely, given a complete brick chain $(C_1, \dots, C_m)$, then all the torsion classes $\mathcal{T}_i = T(C_1, \dots, C_i)$ are reachable, thus functorially finite according to 17.3. In addition, C_i is a top brick of $\mathcal{T}_i$, thus Loewy-finite according to 17.8 (3).

Let M be any module. We start with a brick chain filtration $(M_i)_i$ of M whose brick type is a subsequence of $(C_1, \dots, C_m)$. Thus, M_i/M_{i-1} belongs to $\mathcal{E}(C_i)$. It follows that M_i/M_{i-1} has a filtration of length at most $\lambda(C_i)$ with factors in $\operatorname{add} C_i$. We refine the filtration $(M_i)_i$ accordingly and obtain a filtration $(N_j)_j$ of M whose length is bounded by $c = \sum \lambda(C_i)$. We can assume that N_j/N_{j-1} is a non-zero module in $\operatorname{add} B_j$, where B_j is one of the bricks $C_1, \dots, C_m$. Then the sequence of the bricks B_j is a weak brick chain (obtained from a subsequence of the brick chain $(C_1, \dots, C_m)$ by suitable repetitions). □

17.10. Question. We have seen: If A is brick finite, then all bricks are Loewy finite. There is the following question: Is the converse true?

18. History and Relevance.

18.1. The results presented here are usually considered as part of the so-called τ-tilting theory (see 18.15). There is a strange reluctance to deal with bricks. For example, many papers prefer to speak about τ-tilting finiteness instead of brick finiteness, but these properties are equivalent (see [DIJ]; here, τ-tilting finiteness means that there are only finitely many τ-tilting modules: Whereas brick finiteness is very easy to grasp, τ-tilting finiteness is much less intuitive). For the main parts of our report, there is no need to mention τ-tilting notions, nor even the Auslander-Reiten translation τ itself, thus we have avoided it. In this way, we stress the completely elementary nature of the corresponding results. I admit that these results may get some special flavor when formulated in terms of τ-tilting theory, but this should be an afterthought. Of course, to shift the τ-tilting interpretation to an afterthought is historically incorrect: the results mentioned where obtained when looking at τ-tilting modules. But mathematics, in contrast to history of mathematics, should put the emphasis on the relevance of the ideas, not on their development. Some remarks on τ-tilting theory will be given in 18.15.

Let me stress that it is astonishing that the relevance of bricks when dealing with tilting modules, with torsion classes, with module categories was observed only so late!

18.2. Bricks and semibricks. The terminology "semibrick" seems to be due to Asai [A]. I used to call a semibrick an "antichain" of bricks, but this is in conflict with Demonet's important notion of a brick chain (and to say that "an antichain of bricks is a brick chain", would sound rather odd).

18.3. Torsion pairs. Torsion pairs $(\mathcal{T}, \mathcal{F})$ were introduced by Dickson [Di] as a generalization of the use of torsion and p-torsion subgroups in abelian group theory, thus generalizing a feature of the category of $\mathbb{Z}$-modules to R-modules, were R is an arbitrary ring. In this paper, torsion classes play a decisive role, but we never mention the corresponding torsionfree class. Of course, since the dual of a torsion class is a torsionfree class, any result about torsion classes provides a corresponding result about torsionfree classes. In this way, the paper yields many assertions about torsionfree classes. But we should mention an intriguing feature of our topic: if we dualize the bijection between chains of torsion classes and brick chains, we obtain a corresponding bijection between chains of torsionfree classes and again brick chains, since the dual of a brick chain is a brick chain.

Throughout the paper, we have used the notation ${}^{\perp}\mathcal{N}$ for the torsion class of modules M with $\operatorname{Hom}(M, N) = 0$ for all $N \in \mathcal{N}$, where $\mathcal{N}$ is an arbitrary class of modules. Correspondingly, given a class $\mathcal{M}$ of modules, one writes $\mathcal{M}^{\perp}$ for the (torsionfree) class of all modules N with $\operatorname{Hom}(M, N) = 0$ for all $M \in \mathcal{M}$. In this way, one obtains all torsion pairs as $({}^{\perp}\mathcal{N}, ({}^{\perp}\mathcal{N})^{\perp})$, or also as $({}^{\perp}(\mathcal{M}^{\perp}), \mathcal{M}^{\perp})$, starting with arbitrary module classes $\mathcal{N}, \mathcal{M}$. Of course, ${}^{\perp}(\mathcal{M}^{\perp})$ is nothing else than the torsion class $T(\mathcal{M})$ generated by $\mathcal{M}$ (and $({}^{\perp}\mathcal{N})^{\perp}$ is the torsionfree class generated by $\mathcal{N}$). The torsion pairs $(\mathcal{T}, \mathcal{F})$ were introduced to focus the attention, for any module M, to the largest submodule U of M which belongs to $\mathcal{T}$, its $\mathcal{T}$-torsion submodule (then M/U is the largest factor module of M which belongs to $\mathcal{F}$) In this light, Proposition 6.1 deals with the torsion pair $({}^{\perp}B, ({}^{\perp}B)^{\perp})$, namely with the ${}^{\perp}B$-torsion submodule M' of M, and asserts that the (torsion-free) factor module M/M' belongs to $\mathcal{E}(B)$.

Note that the main results 1.2 and 3.2 are shown by an iterative use of 6.1: In this way, we deal with a chain of torsion classes in order to obtain a filtration $(M_i)_i$ of the given module M with factors M_i/M_{i-1} in module classes of the form $\mathcal{E}(B_i)$.

18.4. Hereditary torsion pairs, torsional submodules. In contrast to the classical example, torsion classes in general are not hereditary (where *hereditary* means that the torsion class $\mathcal{T}$ is closed under submodules). Of course, the torsion classes $T(M)$ considered in our paper are usually not hereditary. On the other hand, it turns out to be important that looking at a module M and the torsion class $T(M)$ generated by M, to draw attention to the submodules of M which do belong to $T(M)$, namely its torsional submodules. Thus, our focus on torsional submodules and torsional filtrations of modules is an attempt to stress hereditary properties for non-hereditary torsion classes.

The brick-chain theorems 1.2 and 3.2 should be seen in the light of the original example of abelian group theory: any finitely generated module M has a filtration $(M_i)_{0\le i\le m}$ where the factors M_i/M_{i-1} with $0 \le i < m$ are in $\mathcal{E}(\mathbb{Z}/p_i\mathbb{Z})$, for pairwise different prime numbers p_i, whereas M_m/M_{m-1} is in $\mathcal{E}(\mathbb{Z})$. In abelian group theory, this filtration always splits. In our case, we cannot expect that the filtrations provided in 1.2 and 3.2 for a given module M split, just look at indecomposable modules M which are not homogeneous. (It comes as a surprise that actually in first examples one looks at, for example dealing with Kronecker modules, many brick chain filtrations do split.)

18.5. Auslander and Smalø (and Demonet). The relevance of torsion classes when dealing with finite length categories was seen already by Auslander and Smalø [AS].

When dealing with a module category, the existence of cyclic paths, say in the module category or just in the Auslander-Reiten quiver, provides a lot of difficulties. After all, only the representation-directed algebras are easy to visualize, but representation-directedness is a very special feature. There have been many attempts to overcome the difficulties which arise from the presence of cyclic paths. Let us mention the covering theory introduced by Gabriel and his school. Also, the book [ARS] is full of devices: to avoid short chains, to avoid short cycles. However, these methods are designed just for special, well-behaved situations. If one wants to deal with an arbitrary module category, the use of torsion classes always works. As we have mentioned in 2.3, the reference to torsion classes allows to consider the set of semibricks (thus also the set of bricks) as a partially ordered set. In this way, Demonet's proposal to look at brick chains stresses a very interesting directedness feature of an arbitrary module category.

18.6. Wide subcategories and torsion classes. Given an abelian category, the exact abelian subcategories which are closed under extensions are now often called *wide* subcategories. The rather obvious relationship between semibricks and wide subcategories was mentioned in [R1] under the name "simplification". The search for semibricks (or wide subcategories) which generate a given torsion class was initiated by Ingalls and Thomas [IT]. Theorem 2.2 generalizes some of their considerations. The injectivity of the map in 2.2 has been shown by Marks and Stovíček in [MS].

The relevance of the endotop of a module is well-known and was stressed by Asai when looking at τ-rigid modules (our proof of 5.5 follows closely Asai [A]). For a general study of widely generated torsion classes, see Asai and Pfeifer [AP] as well as Marks and Stovicek [MS].

18.7. Homogeneous subcategories. The homogeneous subcategories are equivalent to the module category of a local algebra (not necessarily an artin algebra) and one often uses the representation theory of local algebras just as a black box. But, actually, not much is known about the representation theory of a local algebra A which is not commutative! The commutative local rings are studied very well in commutative algebra, but there never was much interest in the non-commutative ones. But note that often they behave rather differently and really deserve attention.

Let us mention at least one phenomenon which is of relevance for our discussion. If A is a commutative local ring, and M is a serial module, say of length t, then there is an endomorphism of M with image $\operatorname{rad} M$, thus $\operatorname{et} M$ is just the simple module. On the other hand, consider the subring $A = k + J$ of the ring of $(t \times t)$-matrices where J is the set of nilpotent upper triangular matrices, as mentioned already in 5.2 and 11.1. This is a rather nice local ring; it is non-commutative provided $t \geq 3$. The set $M = k^t$ of column vectors is a serial A-module. Since the image of any non-invertible endomorphism of M has length at most 1, we see that $\operatorname{et} M$ has dimension $t-1$; in particular, it is not a brick provided $t \geq 3$ (for $t \geq 3$, we have $\operatorname{et}^\infty M = \operatorname{et}^{t-2} M = k \neq \operatorname{et} M$).

18.8. Neighbors of torsion classes. Neighbor pairs $\mathcal{T}' \subset \mathcal{T}''$ of torsion classes have attracted a lot of interest and several different denominations are used in the literature: that $\mathcal{T}''$ covers $\mathcal{T}'$, that there is an arrow $\mathcal{T}'' \to \mathcal{T}'$ in the Hasse quiver of the lattice of torsion classes, or one speaks about minimal inclusions of torsion classes.

As we have seen, it is easy to determine the lower neighbors of a finitely generated torsion class, and there are only finitely many, but unfortunately, it is difficult to deal with the upper neighbors: usually, there may be infinitely many. In the special case of the functorially finite torsion classes, there is no problem at all, since in this case the number of the neighbors (both lower neighbors and the upper neighbors) is just the number n of simple modules! For any torsion class $\mathcal{T}$, the best way to find its upper neighbors seems to be to look at the corresponding torsion free class $\mathcal{F}$ and to try to determine its lower neighbors, since the lower neighbors of $\mathcal{F}$ correspond to the upper neighbors of $\mathcal{T}$, see section 12. Of course, starting with a finitely generated torsion class $\mathcal{T}$, the corresponding torsionfree class usually will not be finitely generated!

Altogether, it seems that we are in the realm of the second Brauer-Thrall conjecture when we try to find a description of the set of neighbors of a torsion class which is not functorially finite. Looking at the special case of the Kronecker algebra, one has the feeling that an answer may have to take into account both data from the Auslander-Reiten theory as well as data from algebraic geometry (see also 18.16).

18.9. Brick labeling. The brick labeling as presented in sections 8 and 11 to 14 was started for ff torsion classes in [AIR] and Asai [A] identified the labels as bricks. The general case is due to Barnard, Carroll and Zhu [BCZ].

The brick B used as label for the neighbor torsion classes $\mathcal{T}' \subset \mathcal{T}$ is called a *minimal extending module* for $\mathcal{T}'$ in [BCZ], with the characterization given above in 13.2 (2). In [AHL], the modules B considered in 13.1 (2) are said to be *torsion, nearly torsionfree* for the torsion pair $(\mathcal{T}, \mathcal{T}^\perp)$.

The bijection 2.8 between bricks and completely join irreducible torsion classes has been exhibited in Theorem 1.0.5 in [BCZ].

18.10. Brick chains. As we have seen, given a chain of torsion classes, the brick labeling of the neighbor torsion classes yields a brick chain. This observation was used by Demonet [De] to consider not only the finite brick chains as considered in the present paper, but to deal with arbitrarily large totally ordered sets of bricks with the corresponding Hom-condition, called again brick chains. Of special interest are those brick chains which cannot be further refined, since they correspond bijectively to the chains of torsion classes which cannot be refined. This bijection is essential for an understanding of the lattice of all torsion classes.

The relevance of this bijection has been stressed already several times: Since the set of all torsion classes is, in a natural way, a partially ordered set, using the set-theoretical inclusion, the bijection transfers this ordering to bricks and semibricks.

18.11. Complete brick chains. In this report, we restrict the attention to finite brick chains. The reader should be aware: whereas (as an extreme example) the Kronecker algebra A over the field with 2 elements has cardinality 16, thus should be easy to envision even for a child, the lattice of torsion classes in $\operatorname{mod} A$ is uncountable (and there are uncountably many brick chains: one is finite, all others are uncountable)!

There has been some interest in finite complete brick chains, they often are called maximal green sequences, as a reference to cluster theory.

18.12. Special brick chain filtrations. Special brick chain filtrations have been used already a long time ago. We have shown in [R2] that for a hereditary k-algebra, where k is an algebraically closed field, any exceptional module is a tree module, The basis of the proof is Schofield induction, dealing with certain brick chain filtrations of length 2. We stress that the brick chain filtrations used in [R2] are never torsional, since they are filtrations of length 2 of bricks. We see in this way the relevance of brick chain filtrations which are not torsional. The brick chain filtrations used in [R2] have type (B_1, B_2), where B_1, B_2 both are again exceptional modules, thus bricks without self-extensions, and not only the brick chain condition $\operatorname{Hom}(B_1, B_2) = 0$ is satisfied, but B_1, B_2 are even Hom-orthogonal. Note that in this case the modules with a brick chain filtration of type (B_1, B_2) correspond to the representations of the bimodule $\operatorname{Ext}^1(B_2, B_1)$.

18.13. Infinite dimensional modules. All the modules we have been dealing with are finite-dimensional ones. However, this is sort of cheating: when dealing with an infinite semi-brick $\mathcal{B}$ and the torsion class $T(\mathcal{B})$ generated by $\mathcal{B}$, we definitely have in mind the direct sum $\bigoplus_{B\in\mathcal{B}} B$ as a single generator of $T(\mathcal{B})$. It is clear that the use of infinite dimensional modules can be very illuminating. For example, dealing with the Kronecker quiver, the torsion class of all preinjective modules is generated by a single brick, however an infinite dimensional one, namely the so-called generic module.

18.14. Artinian rings. In this report, we have assumed to be in the context of artin algebras. Actually, nearly all the results presented here are valid more generally in arbitrary length categories, thus for example for finitely generated modules over left artinian rings.

18.15. Functorially finite torsion classes. The study of the functorially finite torsion classes was started by Auslander and Smalø [AS1, AS2]. In 1984, Smalø [S] formulated the tie between functorially finite torsion classes and tilting modules for factor algebras: Given a torsion class $\mathcal{T}$ with annihilator I, the maximal basic Ext-projective module of $\mathcal{T}$ is an A/I.tilting modules. The paper [AIR] by Idachi-Iyama-Reiten characterized these modules as the support-τ-tilting modules. It may be considered as a rather innocent addition, but actually, it was the decisive breakthrough! Derksen and Fei outline in [DF2] how the main results in [AIR] follow from their previous paper [DF1], and [DF1] which was available in arXiv already in 2009. The basic concept of τ-tilting theory are the

τ-rigid modules.

An important feature of τ-tilting theory are the mutations. In the setting of tilting theory, the mutation of torsion classes was studied in detail by Happel and Unger. It took a long time that the relevance for dealing with arbitrary module categories was realized.

We refrain to give here even a concise summary of the development of τ-tilting theory. Actually, there are further striking results on bricks which are provided by τ-tilting theory (they deserve a separate report)!

18.16. Number of neighbors. Looking at the Kronecker algebra, one observes that for any torsion class, the number of neighbors (both lower and upper neighbors) is either 2 or infinite. Also, it is an important assertion of τ-tilting theory that any functorially finite torsion class has precisely n neighbors, where n is the number of simple modules. Thus, for any brick finite algebra, any torsion class has precisely n neighbors. I learned from Asai [A2] that in general there do exist torsion classes without any neighbor. He suggested the following example.

Let A be the t-Kronecker algebra A (say over an algebraically closed field) with $t \geq 3$. Let $T(\mathcal{I})$ be the torsion class of all preinjective modules. Of course, $T(\mathcal{I})$ has no lower neighbor. Let us show that $T(\mathcal{I})$ *has also no upper neighbor.* Assume, for the contrary, that $\mathcal{T}$ is an upper neighbor of $T(\mathcal{I})$. Thus there exists a (necessarily regular) brick B with $\mathcal{T} = T(\mathcal{I}, B)$, and, of course, we have $T(\mathcal{I}, B) = T(B)$. Now $T(B)$ has a unique lower neighbor, namely $T(B)_B = T(B) \cap {}^\perp B$. We claim that $T(\mathcal{I})$ is a proper subclass of $T(B)_B$ (thus not a lower neighbor of $T(B)$).

Here is the proof. Let us look at $\mathcal{E}(B)$. There are arbitrarily large modules C in $\mathcal{E}(B)$ with simple relative top. Given C, let $f\colon C \to B$ be an epimorphism. Let U be a simple A-submodule of C which is not contained in the kernel of f. Then C/U has no non-zero factor module which belongs to $\mathcal{E}(B)$, since, up to scalar multiples, f is the only non-zero map $C \to B$. As a consequence, C/U belongs to ${}^\perp B$, thus to $T(B) \cap {}^\perp B = T(B)_B$. The dimension vector of C/U in the Grothendieck group $K_0(A)$ is $\mathbf{dim}\, C/U = m\, \mathbf{dim}\, B - (1,0)$, where m is the relative length of C in $\mathcal{E}(B)$. Looking at the quadratic form $q(x,y) = x^2 + y^2 - txy$, it follows from $q(\mathbf{dim}\, B) < 0$ that also $q(\mathbf{dim}\, C/U) < 0$, provided m is large. Thus, for m large, C/U cannot belong to $T(\mathcal{I})$.

A similar proof shows that for any irrational slope γ inside the cone of the dimension vectors of the regular modules, the torsion class given by the indecomposable modules with slope at least γ has no neighbors. Thus there are uncountably many torsion classes without neighbors. It seems that for a t-Kronecker algebra, any torsion class has either 0, 2, or infinitely many neighbors.

In general, for an arbitrary artin algebra, one may ask in which way the number of neighbors of torsion classes are restricted. For example, if A is a connected artin algebra with uncountable center: has any torsion class either finitely many or uncountably many neighbors?

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