Version 3.

Brick chain filtrations. A report.

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Abstract. We consider the category of finitely generated modules over an artin algebra A. Recall that an object in an abelian category is said to be a *brick* provided its endomorphism ring is a division ring. Simple modules are, of course, bricks, but in case A is connected and not local, there always do exist bricks which are not simple. The aim of this survey is to focus the attention to filtrations of modules where all factors are bricks, with bricks being ordered in some definite way, namely according to a so-called brick chain.

In general, a module category will have many cyclic paths. Recently, Demonet has proposed to look at brick chains in order to deal with a very interesting directedness feature of an arbitrary module category.

The following survey relies on investigations by a large group of mathematicians. We have singled out some important observations and have reordered them in order to provide a self-contained (and elementary) treatment of the role of bricks in module categories. (Most of the papers we rely on are devoted to what is called τ-tilting theory, but for the results we are looking at, there is no need to deal with τ-tilting, not even to invoke the Auslander-Reiten translation τ itself).

Outline. This is a report on a very important development in the last 15 years: it focuses the attention to the use of bricks in order to describe the structure of arbitrary modules over artin algebras. The report relies on the work of a quite large number of mathematicians, see section 11. We have singled out decisive observations and have reordered them in order to obtain a self-contained and elementary (however incomplete) treatment of the role of bricks in module categories.

The first three sections describe the main results presented in this survey, they deal with brick chain filtrations and their background. Theorem 1.2 and its strengthening 3.2 concerns the existence of brick chain filtrations (and 3.2 includes a corresponding finiteness assertion). The main tool is the study of torsion classes and their lower neighbors. Theorem 2.3 asserts that finitely generated torsion classes are always generated by finite semibricks. Theorem 2.8 describes the lower neighbors of the torsion class generated by a module M in terms of the so-called top bricks of M.

Given a brick B, we denote by $\mathcal{E}(B)$ the class of all modules which have a filtration with all factors isomorphic to B; these modules will be said to be *homogeneous* of brick type B. The brick type of a non-zero homogeneous module is uniquely determined (see 10.1). The brick chain filtrations studied in this report concern filtrations of modules with factors in suitable subcategories $\mathcal{E}(B)$, namely using bricks which occur in a brick chain. The existence of brick chain filtrations is derived from a result for neighbor torsion classes. Neighbor torsion classes $\mathcal{T}' \subset \mathcal{T}$ come with a label: this is a brick B with the following property: any module M in $\mathcal{T}$ has a submodule M' in $\mathcal{T}'$ such that M/M' belongs to $\mathcal{E}(B)$, see Theorem 2.7.

1. All modules have brick chain filtrations.

1.1. We deal with an artin algebra A; the modules to be considered are the left A-modules of finite length. Given a set $\mathcal{X}$ of modules, let $\mathcal{E}(\mathcal{X})$ be the class of modules which have a filtration with all factors in $\mathcal{X}$. If $M_1, \dots, M_m$ are modules, let $\mathcal{E}(M_1, \dots, M_m) = \mathcal{E}(\{M_1, \dots, M_m\})$ (such a convention is used throughout the paper in similar situations).

We recall that a *brick* is a module whose endomorphism ring is a division ring. If B is a brick, the modules in $\mathcal{E}(B)$ will be said to be *homogeneous* of brick type B. A finite sequence $(B_1, \dots, B_m)$ is called a *brick chain,* if all B_i are bricks and $\operatorname{Hom}(B_i, B_j) = 0$ for $i < j$. A filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ will be called a *brick chain filtration,* provided there is a brick chain $(B_1, \dots, B_m)$ (its *type*) such that M_i/M_{i-1} is homogeneous of brick type B_i, for all $1 \le i \le m$.

1.2. Theorem. *Any module has brick chain filtrations.*

The result will be strengthened in 3.2; the proof of 3.2 is given in section 9.

1.3. Some examples of brick chain filtrations.

(1) Let $S_1, \dots, S_n$ be the simple A-modules. Obviously, $(S_1, \dots, S_n)$ is a brick chain. Let us now assume that $\operatorname{Ext}^1(S_i, S_j) = 0$ for all $i > j$. If M is any A-module, let M_i be the submodule of M which is maximal with the property that all its composition factors are of the form $S_1, \dots, S_i$. Then $(M_i)_i$ is a brick chain filtration. If M is sincere, then we obtain a brick chain filtration of type $(S_1, \dots, S_n)$.

In particular, recall that A is said to be *directed,* provided the simple modules $S_1, \dots, S_n$ can be ordered in such a way that $\operatorname{Ext}^1(S_i, S_j) = 0$ for all $i \ge j$. *For a directed algebra A, any sincere A-module M has a brick chain filtration of type $(S_1, \dots, S_n)$ with the additional property that the factors of the filtration are semisimple.*

(2) Let A be a connected Nakayama algebra with n simple modules. *Any indecomposable module M has a brick chain filtration of length at most two:* If the length of M is at most n, then M itself is a brick. Now assume that the length of M is equal to $an + r$ with $a \ge 1$ and $0 \le r < n$. Let B be the factor module of M of length n. Then B is a brick. If $r = 0$, then M has a brick chain filtration of type (B). If $r \ne 0$, then B has a brick chain filtration of type (B', B), where B' is the factor module of B of length r.

(3) In contrast to many questions in representation theory, looking for brick chain filtrations of modules, it does not seem to be helpful to consider first indecomposable modules. Namely, brick chain filtrations of modules M and M' usually do not provide a brick chain filtration of $M \oplus M'$, see 10.8. (However, a brick chain filtration of a direct sum yields brick chain filtrations of the direct summands, see 10.7.)

(4) (Duality) Let us denote by D the usual duality functor. Given a brick chain filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ of type $(B_1, \dots, B_m)$, then clearly D yields a corresponding brick chain filtration $(N_i)_i$ of $N = \mathrm{D}\,M$, namely $N_i = \mathrm{D}\,M/\mathrm{D}\,M_{m-i}$, for $0 \le i \le m$. The type of the filtration $(N_i)_i$ is $(\mathrm{D}\,B_m, \dots, \mathrm{D}\,B_1)$.

(5) Our proof of 1.2 will yield quite special brick chain filtrations, namely "torsional" ones, see section 3. Let us note already here: if a filtration $(M_i)_i$ of a module M is torsional, then the top of any M_i is generated by the top of M. Thus, even in the case of

a directed algebra, the brick chain filtrations which we will construct are usually different from the obvious filtrations mentioned in (1).

2. Torsion classes, in particular the finitely generated ones.

The proof of Theorem 1.2 and its strengthening 3.2 will be based on the use of torsion classes. Definition and properties of torsion classes will be recalled in this section. they are essential for all considerations.

2.1. A class $\mathcal{T}$ of modules is said to be a *torsion class* provided $\mathcal{T}$ is closed under factor modules and extensions. The set of all torsion classes is a complete lattice; the meet of a set of torsion classes is just the set-theoretical intersection. Given a class $\mathcal{X}$ of modules, we denote by $T(\mathcal{X})$ the smallest torsion class which contains $\mathcal{X}$ (thus, the closure of $\mathcal{X}$ under factor modules and extensions, or, equivalently, the set-theoretical intersection of all torsion classes containing $\mathcal{X}$). The Noether theorems show that $T(\mathcal{X})$ is the class of modules which have a filtration whose factors are factor modules of modules in $\mathcal{X}$. The *torsion submodule* of a module M with respect to the torsion class $\mathcal{T}$ is by definition the largest submodule of M which belongs to $\mathcal{T}$. Given a module class $\mathcal{Y}$, we denote by ${}^{\perp}\mathcal{Y}$ the class of all modules X such that $\operatorname{Hom}(X, Y) = 0$ for all modules Y in $\mathcal{Y}$. It is clear that ${}^{\perp}\mathcal{Y}$ is closed under factor modules and extensions, thus it is a torsion class.

A torsion class $\mathcal{T}$ is said to be *finitely generated* provided there is a module M with $\mathcal{T} = T(M)$. Of course, any torsion class $\mathcal{T}$ is the set-theoretical union of the finitely generated torsion classes contained in $\mathcal{T}$.

2.2. Let $(M_i)_i$ be a brick chain filtration of the module M, say of type $(B_1, \dots, B_t)$. Then M_i is the torsion submodule of M with respect to the torsion class $T(B_1, \dots, B_i)$ and also the torsion submodule of M with respect to the torsion class ${}^{\perp}\{B_{i+1}, \dots, B_t\}$. Thus, we see: *Given a module M with a brick chain filtration $(M_i)_i$, the submodules M_i are uniquely determined by the type of the filtration.*

2.3. Bricks B, B' are defined to be Hom-*orthogonal* provided $\operatorname{Hom}(B, B') = 0 = \operatorname{Hom}(B', B)$. A *semibrick* is a set of pairwise Hom-orthogonal bricks. A torsion class which is generated by a semibrick is said to be *widely generated.* When we deal with sets of (pairwise non-isomorphic) modules, for example when we consider semibricks, these sets are not necessarily finite (so that we cannot or better do not want to deal with the corresponding direct sum).

Theorem. *For any artin algebra A, the map $M \mapsto T(M)$ provides a bijection between finite semibricks and the finitely generated torsion classes.*

The surjectivity of the map asserts that *any finitely generated torsion class is widely generated.* The injectivity assertion can be extended as follows: the map $\mathcal{B} \mapsto T(\mathcal{B})$ is a bijection between arbitrary semibricks and the widely generated torsion classes, see 2.8.

The proof of Theorem 2.3 is given in 5.6 (the surjectivity of the map), and in section 8 (the injectivity of the map). In Section 5, we construct explicitly an inverse of the map $\mathcal{B} \mapsto T(\mathcal{B})$, for $T(\mathcal{B})$ being finitely generated. Let us outline the construction already here.

Addendum to Theorem. Given a module M, we define in 5.4 its "iterated endotop" $X = \operatorname{et}^{\infty} M$; this is a factor module of M. The indecomposable direct summands of X

are called the *top bricks* of M and we denote by $\mathcal{B}(M)$ the set of top bricks of M. The surjectivity assertion in Theorem 2.3 can be strengthened as follows, see sections 5 and 8. *Given any module M, then $T(M) = T(\mathcal{B})$ for a uniquely determined semibrick $\mathcal{B}$, namely the finite semibrick $\mathcal{B} = \mathcal{B}(M)$ of the top bricks of M.* In particular: *$T(M)$ is generated by a finite semibrick $\mathcal{B}$ whose members are factor modules of M.*

2.4. Remark. The bijection provided by Theorem 2.3 is of great interest, since it allows to consider the set of finite semibricks as a partially ordered set, using the natural partial ordering of the set of torsion classes, given by set-theoretical inclusion. This poset structure on the set of finite semibricks (thus also on the set of bricks) provides the foundation for the notion of a brick chain as used in Theorem 1.2.

2.5. The algebra A is said to be *brick finite* provided there is only a finite number of isomorphism classes of bricks, and *torsion class finite* provided there is only a finite number of torsion classes.

Corollary. *For any algebra, the number of finite semibricks is equal to the number of finitely generated torsion classes. An algebra is brick finite iff it is torsion class finite, and in this case any torsion class is finitely generated.*

Actually, also the converse of the last sentence is true: *If all torsion classes are finitely generated, then the algebra is brick finite.* And there are many more characterizations of the brick finite algebras. For both assertions, see 11.13.

Proof of Corollary. The first assertion follows directly from Theorem 2.3.

If A is torsion class finite, then A has only finitely many finitely generated torsion classes, thus only finitely many semibricks, thus only finitely many bricks.

Conversely, assume that A is brick finite, thus A has only finitely many finitely generated torsion classes. Given a torsion class $\mathcal{T}$, one can start to construct an inclusion chain of finitely generated torsion classes $\mathcal{T} = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_t \subseteq \mathcal{T}$. This process has to stop after finitely many steps, thus $\mathcal{T}$ is finitely generated. But if all torsion classes are finitely generated, there are only finitely many torsion classes. □

2.6. Neighbors. The torsion classes $\mathcal{T}' \subset \mathcal{T}$ will be said to be *neighbors* provided there is no torsion class $\mathcal{N}$ with $\mathcal{T}' \subset \mathcal{N} \subset \mathcal{T}$. If $\mathcal{T}' \subset \mathcal{T}$ are neighbor torsion classes, $\mathcal{T}'$ is called a *lower neighbor* of $\mathcal{T}$ and $\mathcal{T}$ is called an *upper neighbor* of $\mathcal{T}'$.

Let $\mathcal{T}$ be a torsion class. We will say that $\mathcal{T}$ has *sufficiently many lower neighbors* provided any torsion class $\mathcal{N}$ with $\mathcal{N} \subset \mathcal{T}$ is contained in a lower neighbor of $\mathcal{T}$. Similarly, we say that $\mathcal{T}$ has *sufficiently many upper neighbors* provided any torsion class $\mathcal{N}$ with $\mathcal{T} \subset \mathcal{N}$ contains an upper neighbor of $\mathcal{T}$.

2.7. Theorem. *Assume that $\mathcal{T}' \subset \mathcal{T}$ are neighbor torsion classes. Then there is a unique brick B in $\mathcal{T}$ such that $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B$. This module B is also the unique brick in $\mathcal{T}$ with the following property: Any module M in $\mathcal{T}$ has a submodule M' in $\mathcal{T}'$ such that M/M' belongs to $\mathcal{E}(B)$.* This brick B is called the *label* of the inclusion $\mathcal{T}' \subset \mathcal{T}$.

For the proof, see 7.2. Next, we consider the lower neighbors of some torsion classes.

2.8. Theorem. *A torsion class $\mathcal{T}$ is widely generated iff $\mathcal{T}$ has sufficiently many lower neighbors. If $\mathcal{T}$ is widely generated, say $\mathcal{T} = T(\mathcal{B})$ for the semibrick $\mathcal{B}$, then $B \mapsto \mathcal{T} \cap {}^{\perp}B$ is a bijection between the elements of $\mathcal{B}$ and the lower neighbors of $\mathcal{T}$.*

The proof will be given in section 8. Note that Theorem 2.8 implies that *the map $\mathcal{B} \mapsto T(\mathcal{B})$ is a bijection between the set of semibricks and the set of widely generated torsion classes.*

2.9. A torsion class $\mathcal{T}$ is said to be *completely join irreducible* provided the join $\mathcal{T}_*$ of the torsion classes properly contained in $\mathcal{T}$ is still properly contained in $\mathcal{T}$ (and thus $\mathcal{T}_*$ is a lower neighbor of $\mathcal{T}$). Note that $\mathcal{T}$ is completely join irreducible iff $\mathcal{T}$ has a unique lower neighbor and has sufficiently many lower neighbors.

Corollary. *The map $B \mapsto T(B)$ provides a bijection between the isomorphism classes of the bricks and the completely join irreducible torsion classes.*

Proof. Theorem 2.3 sends a brick to the torsion class $T(B)$. According to 2.8, $\mathcal{T}(B)$ has a unique lower neighbor, namely $\mathcal{T}_* = T(B) \cap {}^{\perp}B$ and any torsion class properly contained in $\mathcal{T}$ is contained in $\mathcal{T}_*$. This shows that $T(B)$ is completely join irreducible.

Conversely, assume that $\mathcal{T}$ is a completely join irreducible torsion class. Clearly, $\mathcal{T}$ is finitely generated: Let M be any module in $\mathcal{T} \setminus \mathcal{T}_*$, where T_* is the join of the torsion classes properly contained in $\mathcal{T}$, then $\mathcal{T} = T(M)$. Let $B_1, \dots, B_t$ be the top bricks of M, thus $\mathcal{T} = T(M) = T(B_1, \dots, B_t)$. According to 2.8, $\mathcal{T}$ has t lower neighbors. Since $\mathcal{T}$ is completely join irreducible, we have $t = 1$, thus $\mathcal{T}$ is generated by a brick. □

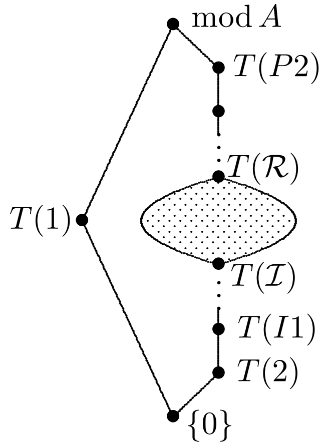
2.10. Warnings. Let M be a module. If $T(M)$ is a finitely generated torsion class and B a top brick of M, the lower neighbor torsion class $T(M) \cap {}^{\perp}B$ is not necessarily finitely generated! A typical example will be mentioned in 2.11.

Also, we have seen in 2.8 that $T(M)$ has only finitely many lower neighbors. What about upper neighbors? If $T(M)$ is finitely generated and $\mathcal{T}''$ is an upper neighbor of $T(M)$, then trivially $\mathcal{T}''$ is again finitely generated, namely equal to $T(M \oplus N)$, where N is any module in $\mathcal{T}'' \setminus T(M)$. However, whereas a finitely generated torsion class has only finitely many lower neighbors, it may have infinitely many upper neighbors. For a typical example, we again refer to 2.11.

2.11. An example: The Kronecker algebra. For the benefit of the reader, we want to consider one example in detail, the Kronecker algebra A, the path algebra of the quiver with two vertices $1, 2$ and two arrows $1 \leftleftarrows 2$. (Actually, it is the usual example which everyone interested in torsion classes of artin algebras has in mind).

If x is a vertex of a quiver, the simple representation corresponding to x will also be denoted by x; and Px and Ix will denote the projective cover or the injective envelop of x, respectively (provided they exist). For the Kronecker algebra A, there is the well-known trisection of the indecomposable A-modules: there are the preprojective modules $\mathcal{P}$, the regular modules $\mathcal{R}$ and the preinjective modules $\mathcal{I}$. In terms of torsion classes, this trisection gives rise to two important torsion classes: the torsion class $T(\mathcal{I})$ given by the direct sums of preinjective modules, and the torsion class $T(\mathcal{R})$ given by the direct sums of preinjective and regular modules. *Both torsion classes $T(\mathcal{I})$ and $T(\mathcal{R})$ (as many others) are* **not** *finitely generated.*

The torsion class $T(\mathcal{I})$ is the union of a properly ascending chain of torsion classes, thus it is not finitely generated. Note that $T(\mathcal{I})$ has no lower neighbor, but it has infinitely many upper neighbors. On the other hand, the torsion class $T(\mathcal{R})$ is widely generated, namely by the (infinite!) semibrick of the simple regular modules. Thus, $T(\mathcal{R})$ is not finitely generated. Also, $T(\mathcal{R})$ has no upper neighbor, but infinitely many lower neighbors.



The dotted part is the lattice of all torsion classes $\mathcal{T}$ with $T(\mathcal{I}) \subseteq \mathcal{T} \subseteq T(\mathcal{R})$. If $\mathcal{X}$ is a non-empty set of pairwise non-isomorphic simple regular Kronecker modules (thus $\mathcal{X}$ is a semibrick), then $T(\mathcal{X})$ is a torsion class with $T(\mathcal{I}) \subset \mathcal{T} \subseteq T(\mathcal{R})$. Taking also $T(\mathcal{I})$ into account, we see that the torsion classes $\mathcal{T}$ with $T(\mathcal{I}) \subseteq \mathcal{T} \subseteq T(\mathcal{R})$ correspond bijectively to the subsets of $\mathbb{P}^1(k)$ (by definition, $\mathbb{P}^1(k)$ is the union of the one element set $\{\infty\}$ and the set of monic irreducible polynomials with coefficients in k). Note that for any torsion class $\mathcal{T}$ with $T(\mathcal{I}) \subseteq \mathcal{T} \subseteq T(\mathcal{R})$, the number of neighbors of $\mathcal{T}$ is always equal to $\max(|k|, \aleph_0)$, in particular, infinite. (Note that the dotted part is very large: the set of subsets of $\mathbb{P}^1(k)$ is always uncountable, even if k is just the field with 2 elements!)

If R is a simple regular module, then $T(R)$ is (of course) finitely generated, however its unique lower neighbor $T(R) \cap {}^{\perp}R$ is the torsion class $T(\mathcal{I})$ (and we repeat: $T(\mathcal{I})$ is not finitely generated). On the other hand, $T(R)$ has infinitely many upper neighbors. Namely, if R' is a simple regular Kronecker module, not isomorphic to R, then $T(R \oplus R')$ is an upper neighbor of $T(R)$, and there are infinitely many such modules R'.

Finally, let us stress that here all torsion classes but one are widely generated, the only exception is $\mathcal{T} = T(\mathcal{I})$.

3. Torsional brick chain filtrations.

In order to strengthen Theorem 1.2, we need an additional notion.

3.1. A submodule U of a module M is said to be *torsional* provided U belongs to $T(M)$. A filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m$ will be said to be *torsional* provided M_{i-1} is a torsional submodule of M_i, for all $1 \le i \le m$.

If $(M_i)_i$ is a torsional filtration of M, then M_{i-1} belongs to $T(M_i)$, for all $1 \le i \le t$, thus we have the inclusion chain $0 = T(M_0) \subseteq T(M_1) \subseteq \cdots \subseteq T(M_m) = T(M)$, and therefore all the submodules M_i are torsional submodules of M. If a brick chain filtration $(M_i)_i$, say of type $(B_1, \ldots, B_m)$ is torsional, then all the bricks B_i belong to $T(M)$, since B_i is a factor module of M_i and M_i belongs to $T(M)$.

Warning. Note that *a brick chain filtration $(M_i)_i$ of a module M with all M_i being torsional submodules of M, may not be a torsional filtration.* Here is an example: Let Q be the quiver with vertices $1,2,3$ and with arrows $1 \to 2,\ 2 \to 1,\ 2 \to 3$, such that the paths $2 \to 1 \to 2 \to 3$ and $2 \to 1 \to 2 \to 1 \to 2 \to 1$ are zero relations. The projective module $M = P(2)$ has the following brick chain filtration: $0 \subset M_1 \subset M_2 \subset M$ with $M_1 = \begin{smallmatrix} 2 \\ 3 \end{smallmatrix}$, $M_2/M_1 = \begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$. Both M_1 and M_2 are isomorphic to factor modules of M, thus torsional submodules, but M_1 is not contained in $T(M_2)$.

3.2. Theorem. *Any module has at least one, but only finitely many torsional brick chain filtrations.*

The proof will be given in section 9. As we will see, the torsional brick chain filtrations of a module M can be constructed easily by induction: Let B be a top brick of M Then M has a proper submodule M' which belongs to $T(M) \cap {}^{\perp}B$, such that M/M' belongs to $\mathcal{E}(B)$. Since M' is a proper submodule of M, by induction there is a torsional brick chain filtration of M', say $0 = M_0 \subset M_1 \subset \cdots \subset M_{m-1} = M'$. Let $M_m = M$. Then $(M_i)_{0 \le i \le m}$ is a torsional brick chain filtration of M.

Note that Theorem 3.2 shows that any module M determines a finite set of bricks which may be considered as the building blocks for the construction of M, namely the bricks which occur in the types of the finitely many torsional brick chain filtrations of M.

Question. Theorem 3.2 assert that any module M has only finitely many torsional brick chain filtrations. Usually, there are plenty additional brick chain filtrations of M which are not torsional. *Are there modules with infinitely many brick chain filtrations?*

3.3. *A brick B has just one torsional brick chain filtration: namely the trivial filtration* $(0 \subset B)$; after all, a brick has no non-zero proper torsional submodules, see 4.3.

3.4. Remark. If $(M_i)_i$ is a torsional brick chain filtration of type $(B_1, \dots, B_m)$, then by definition all the bricks B_i belong to $T(M)$. The brick B_m is a factor module of M, but *the remaining bricks B_i do not have to be factor modules of M.* Here is a typical example: Let M be serial with composition factors going up: $1,2,2,1,2$, with torsional brick chain filtration $0 \subset M_1 \subset M$, where M_1 is of length three; here, M_1 is not generated by M.

4. Some preliminaries.

4.1. Lemma. *Let M' be a non-zero module in $T(M)$. Then* $\operatorname{Hom}(M, M') \neq 0$.

Proof: M' has a filtration $0 = M'_0 \subset M'_1 \subset \cdots \subset M_m = M$, where all the factors M_i/M_{i-1} are non-zero factor modules of M. Since M'_1 it is a factor module of M, we get a non-zero homomorphism $M \to M'_1 \to M'$. □

4.2. Examples of non-isomorphic bricks B', B with $B' \in T(B)$. According to Lemma 4.1, $\operatorname{Hom}(B, B') \neq 0$. (On the other hand, we will see in 6.3 that $\operatorname{Hom}(B', B) = 0$.) We sometimes will specify modules by a display of the composition factors. For example, in the following example 1, we may deal with a quiver with two vertices, labeled 1 and 2, with an arrow $1 \leftarrow 2$, and a loop at 1. The display $\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}$ stands for a serial module of length two with socle 1 and top 2, and so on . . . .

Example 1: $B = \begin{smallmatrix}2\\1\end{smallmatrix}$ and $B' = 2$. Here, we have an epimorphism $B \to B'$. (Or, if we want to have the same support: Let $B = \begin{smallmatrix}2\\1\\1\end{smallmatrix}$, and $B' = \begin{smallmatrix}2\\1\end{smallmatrix}$.)

Example 2: $B = \begin{smallmatrix}2\\1\end{smallmatrix}$, and $B' = \begin{smallmatrix}2\\2\\1\end{smallmatrix}$. Here, we have a monomorphism $B \to B'$ and B, B' have the same support.

Example 3: $B = \begin{smallmatrix}2\\1\end{smallmatrix}$, and $B' = \begin{smallmatrix}3\\2\end{smallmatrix}$. Here, we have a non-zero map $B \to B'$ which is neither epi nor mono.

4.3. Lemma. *A non-zero module is a brick iff it has no non-zero proper torsional submodules.*

Proof. Let M be a module. If M is not a brick, there is an endomorphism f of M such that $f(M)$ is non-zero and a proper submodule. Since $f(M)$ belongs to $\mathcal{T}(M)$, we see that $f(U)$ is a torsional submodule of M.

Conversely, let U be a non-zero proper submodule which is torsional. Since U belongs to $\mathcal{T}(M)$, there is a non-zero submodule U' of U which is a factor module of M. We get a non-zero and not invertible endomorphism $M \to U' \subseteq U \subset M$, thus M is not a brick. $\square$

5. The endotop and the iterated endotop of a module.

We are going to show the surjectivity assertion of Theorem 2.3. We need the notion of the endotop $\operatorname{et} M$ of a module M.

5.1. Endotop. Denote by $E = \operatorname{End}(M)$ the endomorphism ring of M (operating on the left of M), and $\operatorname{rad} E$ its radical. Then $(\operatorname{rad} E)M$ is a submodule of M and we define $\operatorname{et} M = M/(\operatorname{rad} E)M$, and call it the *endotop* of M; by definition, the endotop of M is a factor module of M.

5.2. Examples. (1) *If M is an indecomposable module,* $\operatorname{et} M$ *may be decomposable.* For example, let A be a local algebra with radical-square-zero and ${}_AA$ of length 3. If M is the indecomposable injective module, then $\operatorname{et} M$ is the direct sum of two copies of the simple module.

(2) Let A be given by the quiver Q with one vertex and two loops and with relations all paths of length 3 (thus A is a local algebra of dimension 7). There is a serial module M of length 3 with $\operatorname{rad} M$ not isomorphic to $M/\operatorname{soc} M$. Then $\operatorname{et} M = M/\operatorname{soc} M$, thus $\operatorname{et} M$ *is indecomposable of length two, and not a brick,* in particular, $\operatorname{et}(\operatorname{et} M)$ is a proper factor module of $\operatorname{et} M$. This leads us below to consider not only et, but the iterations et^i, see 5.4. (Instead of A, we may consider a proper factor algebra A' of A, namely the subring $A' = k + J$ of the ring of all 3×3-matrices with coefficients in k, where J is the set of nilpotent upper triangular matrices; let $M = k^3$ be the A'-module of column vectors.)

(3) If A is the Kronecker algebra, and M a regular Kronecker module, then $\operatorname{et} M$ is just the regular top of M.

5.3. Proposition. *Let M be a module. Then M belongs to $\mathcal{T}(\operatorname{et} M)$, therefore $\mathcal{T}(M) = \mathcal{T}(\operatorname{et} M)$. The kernel of the canonical map $M \to \operatorname{et} M$ is torsional.*

Proof. Let $f_1, \dots, f_t$ be a basis of $E = \operatorname{rad}\operatorname{End} M$. Let $(\operatorname{rad}\operatorname{End} M)^m = 0$. The image of $g = (f_i)\colon \bigoplus_i M \to M$ is $(\operatorname{rad} E)M = \operatorname{rad}_E M = M_1$ and $\operatorname{et} M = M/M_1$. Let $M_{j+1} = g(M_j)$ for all $j \geq 0$ with $M_0 = M$. Then $M_m = 0$. By induction, all modules M_j/M_{j+1} are generated by $\operatorname{et} M$. This shows that $T(M) \subseteq T(\operatorname{et} M)$. On the other hand, we also have $T(\operatorname{et} M) \subseteq T(M)$, since $\operatorname{et} M$ is a factor module of M. Thus M and $\operatorname{et} M$ generate the same torsion-class.

The kernel M' of the canonical map $M \to \operatorname{et} M$ is by definition the image of the map g, thus generated by M. Therefore M' belongs to $T(M)$. □

5.4. We iterate the construction et and get epimorphisms

$$M \to \operatorname{et} M \to (\operatorname{et})^2 M \to \cdots .$$

Since M is of finite length, the sequence stabilizes eventually; in this way we get the *iterated endotop* $\operatorname{et}^\infty M = \operatorname{et}^a M$ for $a \gg 0$.

Example. Let A be a suitable artin algebra with two simple modules 1 and 2. For $n \geq 0$, let $M[n]$ be a serial module of length $n+2$, with composition factors going up: $(1, \dots, 1, 2, 1)$ (thus starting with n factors of the form 1). Then, for $0 \leq i \leq n$, we have $\operatorname{et}^i M[n] = M[n-i]$. For $0 \leq i < n$, the module $M[i]$ is not a brick, but $\operatorname{et}^n M[n] = M[0]$ is a brick of the form $\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$.

5.5. Proposition. *Let M be a module. The iterated endotop $X = \operatorname{et}^\infty M$ is the direct sum of modules which belong to a semibrick $\mathcal{B}$ and $T(M) = T(X) = T(\mathcal{B})$; the kernel of the canonical map $M \to \operatorname{et}^\infty M$ is a torsional submodule of M.*

Proof. It is obvious that the iterated endotop of a module is always the direct sum of modules which belong to a semibrick, since the sequence $M \to \operatorname{et} M \to (\operatorname{et})^2 M \to \cdots$ stabilizes precisely when $\operatorname{End}(\operatorname{et}^a M)$ is semisimple. Proposition 5.3 yields that the torsion classes $T(\operatorname{et}^i M)$ are equal, for all $i \geq 0$.

The kernel K of the canonical map $M \to \operatorname{et}^\infty M$ has a filtration whose factors are the kernels K_i of the canonical maps $\operatorname{et}^i M \to \operatorname{et}^{i+1} M$, for all $i \geq 0$. According to 5.3, all modules K_i belong to $T(M)$, thus K belongs to $T(M)$. □

5.6. Corollary. *A torsion class $\mathcal{T}$ is finitely generated iff there is a finite semibrick $\mathcal{B}$ with $\mathcal{T} = T(\mathcal{B})$.* □

Corollary 5.6 shows that the map $\mathcal{B} \mapsto T(\mathcal{B})$ from the set of finite semibricks $\mathcal{B}$ to the set of finitely generated torsion classes is surjective. This is part of Theorem 2.3.

5.7. Since the iterated endotop of a module M is given by a semibrick, the indecomposable direct summands of the iterated endotop are bricks and will be called the *top bricks* of M.

Examples. (1) *Let M be an indecomposable module. A top brick of M may occur in $\operatorname{et}^\infty M$ with multiplicity greater than* 1. (In particular, $\operatorname{et}^\infty M$ may not be indecomposable.) For example, consider $A = k[x,y]/\langle x^2, y^2, xy\rangle$ and M the indecomposable injective module. The endotop (and the iterated endotop) of M is just the top of M, thus the direct sum of 2 copies of the simple module k.

(2) *The number of top bricks of an indecomposable module M may be arbitrarily large.* We start with the $(t-1)$-subspace quiver, with sink 1 and sources $2, 3, \dots, t$, and add a loop at the sink 1. We consider the corresponding radical-square-zero algebra. There is an indecomposable module M of Loewy length two with socle 1 and top $1 \oplus 2 \oplus \cdots \oplus t$. For this module, $\operatorname{et}^\infty M = \operatorname{et} M = 1 \oplus 2 \oplus \cdots \oplus t$, thus all the simple modules are top bricks of M.

6. The essential feature: If B is a brick, $({}^\perp B)\rceil\!\lfloor \mathcal{E}(B)$ is a torsion class.

Given module classes $\mathcal{X}$ and $\mathcal{Y}$, we write $\mathcal{X}\rceil\!\lfloor \mathcal{Y}$ for the class of all modules M which have a submodule M' in $\mathcal{X}$ such that M/M' belongs to $\mathcal{Y}$.

We are going to show: If B is a brick, then

$$T({}^\perp B, B) = ({}^\perp B)\rceil\!\lfloor \mathcal{E}(B).$$

This describes very nicely the torsion class $T({}^\perp B, B)$. Actually, there is the corresponding description for all the torsion classes $\mathcal{T} \subseteq T({}^\perp B, B)$, see the following general Proposition.

6.1. Proposition. *Let B be a brick. Let $\mathcal{T}$ be a torsion class which is contained in $T({}^\perp B, B)$. Then either $\mathcal{T}$ is contained in ${}^\perp B$, or else*

$$\mathcal{T} = (\mathcal{T} \cap {}^\perp B)\rceil\!\lfloor \mathcal{E}(B).$$

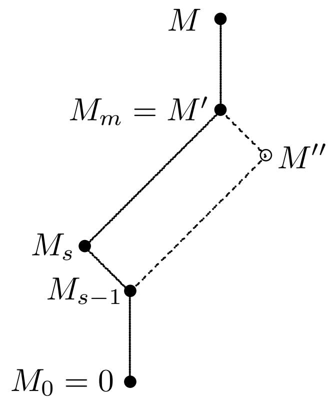
In particular, if $\mathcal{T}$ is not contained in ${}^\perp B$, then B belongs to $\mathcal{T}$ (since the displayed equality asserts that $\mathcal{E}(B) \subseteq \mathcal{T}$).

Let us add: if $0 \to M' \to M \to M/M' \to 0$ is an exact sequence with M' in $\mathcal{T}' = \mathcal{T} \cap {}^\perp B$ and $M/M' \in \mathcal{E}(B)$, then M' is just the torsion submodule of M with respect to the torsion class $\mathcal{T}'$, since $\operatorname{Hom}(\mathcal{T}', \mathcal{E}(B)) = 0$.

Proof of Proposition. Let M' be a submodule of M which belongs to $\mathcal{T}$ with $M/M' \in \mathcal{E}(B)$, and minimal with these two properties. We claim that M' belongs to ${}^\perp B$, thus to $\mathcal{T} \cap {}^\perp B$. (Note that at the moment we do not yet know that B belongs to $\mathcal{T}$, but this does not matter.)

Thus, assume for the contrary that there is a non-zero map $f\colon M' \to B$. Since M' belongs to $\mathcal{T}$, there is a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M'$ such that all factors $F_i = M_i/M_{i-1}$ are factor modules of B or belong to ${}^\perp B$. Let s be minimal such that $f|M_s$ is non-zero. Thus, f vanishes on M_{s-1} and induces a map $\overline{f}\colon M'/M_{s-1}$ with non-zero restriction to $F_s = M_s/M_{s-1}$. Let us denote by $u\colon F_s \to M'/M_{s-1}$ the inclusion map. Thus, the composition $\overline{f} \cdot u\colon F_s \to B$ is a non-zero map.

Now F_s is a factor module of some B or belongs to ${}^\perp B$. Since there is the non-zero map $\overline{f} \cdot u\colon F_s \to B$, we see that F_s is a factor module of B. Also, since B is a brick, there is no non-zero map from a proper factor module of B to B, thus we see that $F_s = B$ and that the composition $\overline{f} \cdot u\colon B = M_s/M_{s-1} \subseteq M'/M_{s-1} \to B$ is an isomorphism. This shows that u is a split monomorphism. It follows that there is a submodule M'' of M' with $M_{s-1} \subseteq M''$, such that $M_s \cap M'' = M_{s-1}$ and $M_s + M'' = M'$.



It follows that $M'/M'' \simeq M_s/M_{s-1} = B$, and that $M''/M_{s-1} \simeq M'/M_s$. Since M/M' and M'/M'' belong to $\mathcal{E}(B)$, also M/M'' belongs to $\mathcal{E}(B)$. On the other hand, $M''/M_{s-1} \simeq M'/M_s$ has a filtration by factors isomorphic to F_i with $s+1 \leq i \leq t$ and M_{s-1} has the filtration with factors F_i where $1 \leq i \leq s-1$. Since all the factors F_i belong to $\mathcal{T}$, also M'' belongs to $\mathcal{T}$.

Altogether we see that M'' is a submodule of M which belongs to $\mathcal{T}$ and such that $M/M' \in \mathcal{E}(B)$. Since M'' is a proper submodule of M', this contradicts the minimality of M'. It follows that M' belongs to ${}^{\perp}B$. Since M/M' is a non-zero module in $\mathcal{E}(B)$, is has a factor module of the form B, thus B is a factor module of M, therefore $B \in \mathcal{T}$.

Since M' belongs to $\mathcal{T} \cap {}^{\perp}B$, and M/M' to $\mathcal{E}(B)$, we see that M' is the torsion submodule of M with respect to the torsion class $\mathcal{T} \cap {}^{\perp}B$.

The exact sequence $0 \to M' \to M \to M/M' \to 0$ for an arbitrary module M in $\mathcal{T}$ shows that $\mathcal{T} \subseteq (\mathcal{T} \cap {}^{\perp}B)\rceil\!\lfloor\mathcal{E}(B)$. On the other hand, we have $\mathcal{T} \cap {}^{\perp}B \subseteq \mathcal{T}$, and, since $B \in \mathcal{T}$, also $\mathcal{E}(B) \subseteq \mathcal{T}$: This shows the reverse inclusion $(\mathcal{T} \cap {}^{\perp}B)\rceil\!\lfloor\mathcal{E}(B) \subseteq \mathcal{T}$, therefore $\mathcal{T} = (\mathcal{T} \cap {}^{\perp}B)\rceil\!\lfloor\mathcal{E}(B)$. □

6.2. Corollary. *Let B be a brick. Let M be a module in $T({}^{\perp}B, B)$. Then any non-zero map $M \to B$ is surjective.*

Proof. Let M be a module in $T(B, {}^{\perp}B)$ and $f\colon M \to B$ a non-zero map. The existence of f shows that M does not belong to ${}^{p}erpB$. According to 6.1, there is a submodule M' of M which belongs to ${}^{\perp}B$ such that M/M' belongs to $\mathcal{E}(B)$. Since f vanishes on M', we get an induced map $\overline{f}\colon M/M' \to B$, and $\overline{f}$ is non-zero. However, any non-zero map in $\mathcal{E}(B)$ with target B is an epimorphism. Since $\overline{f}$ is surjective, also f is surjective. □

6.3. Corollary. *Let B, B' be non-isomorphic bricks, and assume that B' is in $T(B)$. Then* $\operatorname{Hom}(B', B) = 0$*, thus $B' \in T(B) \cap {}^{\perp}B$.*

Proof. Assume there is a non-zero map $f\colon B' \to B$. According to 6.2, the map f is surjective. Since B' belongs to $T(B)$, we know from 4.1 that there is a non-zero map $g\colon B \to B'$. Since f is surjective, the composition $gf\colon B' \to B \to B'$ is non-zero. Since B' is a brick, this means that gf is an isomorphism. Thus f is a (split) monomorphism. Altogether we see that f is bijective, thus B and B' are isomorphic. □

7. Neighbors.

7.1. Lemma. *Let $\mathcal{T}' \subset \mathcal{T}$ be torsion classes. Any module M in $\mathcal{T} \setminus \mathcal{T}'$ of minimal length is a brick and satisfies $\mathcal{T}' \subseteq {}^{\perp}M$.*

Proof. Let M be a module in $\mathcal{T} \setminus \mathcal{T}'$ of minimal length. We form $X = \operatorname{et}^{\infty} M$. According to 5.5, we have $T(X) = T(M)$, thus also X belongs to $\mathcal{T} \setminus \mathcal{T}'$. There is an indecomposable direct summand X' of X which belongs to $\mathcal{T} \setminus \mathcal{T}'$ and, as we know, X' is a brick (one of the top bricks of M). On the other hand, there are epimorphisms $M \to X \to X'$, thus $|X'| \leq |X|$. Since we assume that M is of minimal length, we see that $M = X'$ is a brick.

In order to see that $\mathcal{T}' \subseteq {}^{\perp}M$, consider any homomorphism $f\colon M' \to M$, with $M' \in \mathcal{T}'$. Now $f(M')$ belongs to $\mathcal{T}'$, thus M/M' does not belong to $\mathcal{T}'$. Since $M/f(M')$ is a module in $\mathcal{T} \setminus \mathcal{T}'$, the minimality of M shows that $f(M') = 0$. □

7.2. Proof of 2.7. We assume that $\mathcal{T}' \subset \mathcal{T}$ are neighbors. According to 7.1, there is a brick B in $\mathcal{T}$ such that $\mathcal{T}' \subseteq {}^{\perp}B$. Thus, we have

$$\mathcal{T}' \subseteq \mathcal{T} \cap {}^{\perp}B \subset \mathcal{T}$$

(the proper inclusion is due to the fact that B does not belong to ${}^{\perp}B$). Since $\mathcal{T}' \subset \mathcal{T}$ are neighbors, we see that $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B$.

Next, we show: $(*)$ If B is a brick in $\mathcal{T}$ with $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B$, then for any module $M \in \mathcal{T}$, there is a submodule M' of M which belongs to $\mathcal{T} \cap {}^{\perp}B$ such that M/M' in $\mathcal{E}(B)$. Here is the proof. Since $\mathcal{T}' \subset \mathcal{T}$ are neighbors, and B is in $\mathcal{T}$, but not in $\mathcal{T}'$, we have $T(\mathcal{T}', B) = \mathcal{T}$. Now $\mathcal{T}' \subseteq {}^{\perp}B$, thus $\mathcal{T} = T(\mathcal{T}', B) \subseteq T({}^{\perp}B, B)$. We see that we can apply 6.1: For any module $M \in \mathcal{T}$, there is a submodule M' of M which belongs to $\mathcal{T} \cap {}^{\perp}B$ with M/M' in $\mathcal{E}(B)$. We have shown that B has the two properties mentioned in 2.7.

It remains to show that B is the unique brick in $\mathcal{T}$ with these properties. First, assume that C is a brick in $\mathcal{T}$ such that any module M in $\mathcal{T}$ has a submodule M' in $\mathcal{T}'$ with $M/M' \in \mathcal{E}(C)$. Now C cannot belong to $\mathcal{T}'$)since otherwise we would have $\mathcal{T}' \subsetneq \mathcal{T}$. Now, C has a submodule C' in $\mathcal{T}'$ with C/C' in $\mathcal{E}(B)$. Since C does not belong to $\mathcal{T}'$, we have $C/C' \neq 0$, therefore C maps onto B. On the other hand, B has a submodule B' in $\mathcal{T}'$ with $B/B' \in \mathcal{E}(C)$. Since B is not in $\mathcal{T}'$, we see that B/B' is non-zero, thus B maps onto C. This shows that $C = B$.

Finally, assume that C is a brick with $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}C$. According to $(*)$, we know that for any module $M \in \mathcal{T}$, there is a submodule M' of M which belongs to $\mathcal{T} \cap {}^{\perp}B$ such that M/M' in $\mathcal{E}(C)$. But as we have seen already, this implies that $C = B$. □

7.3. Proposition. *Let B be a brick and $\mathcal{X} \subseteq {}^{\perp}B$. Then $T(\mathcal{X}, B) \cap {}^{\perp}B \subset T(\mathcal{X}, B)$ are neighbors with label B.*

Proof. We write $T(\mathcal{X}, B)_B = T(\mathcal{X}, B) \cap {}^{\perp}B$. Now $T(\mathcal{X}, B)_B \subseteq T(\mathcal{X}, B)$, and this inclusion is proper since B does not belong to ${}^{\perp}B$. Assume that there is a torsion class $\mathcal{T}$ such that $T(\mathcal{X}, B)_B \subset \mathcal{T} \subseteq T(\mathcal{X}, B)$. Since $T(\mathcal{X}, B)_B \subset \mathcal{T}$, there is a module $M \in \mathcal{T}$ which does not belong to ${}^{\perp}B$. Thus, there is a non-zero map $f\colon M \to B$. Since M belongs to $T({}^{\perp}B, B)$, we can apply Corollary 6.2. We see that f is surjective, thus B belongs to $\mathcal{T}$. Of course, also $\mathcal{X} \subseteq \mathcal{T}$. Therefore $T(\mathcal{X}, B) \subseteq \mathcal{T}$. This shows that $T(\mathcal{X}, B) = \mathcal{T}$. Thus, $T(\mathcal{X}, B)_B \subset T(\mathcal{X}, B)$ are neighbors. By definition, the label is B. □

7.4. Corollary. *Let $\mathcal{T}' \subset \mathcal{T}$ be torsion classes. There there are bricks B in $\mathcal{T}$ such that $\mathcal{T}' \subseteq {}^\perp B$. If B is a brick with $\mathcal{T}' \subseteq {}^\perp B$, let $\mathcal{N} = T(\mathcal{T}', B)$ and $\mathcal{N}' = \mathcal{N} \cap {}^\perp B$, then we have*

$$\mathcal{T}' \subseteq \mathcal{N}' \subset \mathcal{N} \subseteq \mathcal{T}$$

and the torsion classes $\mathcal{N}' \subset \mathcal{N}$ are neighbors with label B.

Proof: It is trivial to see that $\mathcal{T}' \subseteq \mathcal{N}' \subseteq \mathcal{N} \subseteq T$. Since B belongs to $\mathcal{N}$, but not to $\mathcal{N}'$, we have $\mathcal{N}' \subset \mathcal{N}$. Finally, we use 7.3 with $\mathcal{X} = \mathcal{T}'$ in order to see that $\mathcal{N}' \subset \mathcal{N}$ are neighbor torsion classes with label B. $\square$

7.5. Remark. Dealing with torsion classes $\mathcal{T}$, there are the corresponding module classes $\mathcal{T}^\perp$ (for any module class $\mathcal{X}$, the module class $\mathcal{X}^\perp$ is the class of all modules Y with $\mathrm{Hom}(X, Y) = 0$ for all $X \in \mathcal{X}$; these module classes are called the *torsionfree classes*); the pair $(\mathcal{T}, \mathcal{T}^\perp)$ is called a *torsion pair.* Using this notation, there is the following characterization of neighbor torsion classes:

Let $\mathcal{T}' \subset \mathcal{T}$ be torsion classes. Then these are neighbors iff $\mathcal{T} \cap (\mathcal{T}')^\perp = \mathcal{E}(B)$ for some brick B, and then B is the label of the inclusion.

7.6. The brick chains explained in terms of neighbor torsion classes. Let $\mathcal{T}' \subset \mathcal{T}$ be neighbors with label B. Then we have on the one hand: B belongs to $\mathcal{T}$ and not to $\mathcal{T}'$. On the other hand, for every module M in $\mathcal{T}'$, in particular for the bricks in $\mathcal{T}'$, we have $\mathrm{Hom}(M, B) = 0$.

Thus we obtain in this way the Hom-condition which is used in the definition of a brick-chain: If $\mathcal{T}_1 \subset \mathcal{T}_2 \subseteq \mathcal{T}_3 \subset \mathcal{T}_4$ is a chain of torsion classes with $\mathcal{T}_1 \subset \mathcal{T}_2$ as well as $\mathcal{T}_3 \subset \mathcal{T}_4$ being neighbors, and B is the label for $\mathcal{T}_1 \subset \mathcal{T}_2$, whereas B' is the label for $\mathcal{T}_3 \subset \mathcal{T}_4$, then $\mathrm{Hom}(B, B') = 0$.

7.7. Remark. Let B be a brick. If $\mathcal{T}' \subset \mathcal{T}$ are neighbors with label B, we have both $T(\mathcal{T} \cap {}^\perp B, B) = \mathcal{T}$ and $T(\mathcal{T}' \cap {}^\perp B, B) = \mathcal{T}'$. In general, for arbitrary torsion classes $\mathcal{T}$ and $\mathcal{T}'$, we have $T(\mathcal{T} \cap {}^\perp B, B) \subseteq \mathcal{T}$, provided B belongs to $\mathcal{T}$, and $T(\mathcal{T}', B) \cap {}^\perp B \supseteq \mathcal{T}'$, provided $\mathcal{T}' \subseteq {}^\perp B$. However, both inclusions are usually proper inclusions: Look at the path algebra A of the $\mathbb{A}_2$-quiver $1 \leftarrow 2$ and B the indecomposable module of length two. For $\mathcal{T} = \mathrm{mod}\, A$, we have $T(\mathcal{T} \cap {}^\perp B, B) = T(B) \subset \mathcal{T}$. For $\mathcal{T}' = \{0\}$, we have $T(\mathcal{T}', B) \cap {}^\perp B = T(2) \supset \mathcal{T}'$.

8. Widely generated torsion classes.

We are going to provide a proof of Theorem 2.8. If $\mathcal{B}$ is a semibrick and $B \in \mathcal{B}$, we write $T(\mathcal{B})_B = T(\mathcal{B}) \cap {}^\perp B$.

8.1. Lemma. *Let $\mathcal{B}$ be a semibrick, and $\mathcal{T} = T(\mathcal{B})$. If the torsion class $\mathcal{T}'$ is properly contained in $\mathcal{T}$, then there is $B \in \mathcal{B}$ with $\mathcal{T}' \subseteq \mathcal{T}_B$ and such that $\mathcal{T}_B \subset \mathcal{T}$ are neighbors.*

Proof. Since $\mathcal{T}'$ is properly contained in $\mathcal{T}$, there is a brick $B \in \mathcal{B}$ which is not contained in $\mathcal{T}'$. Let $\mathcal{B}' = \mathcal{B} \setminus \{B\}$. Since $\mathcal{B}$ is a semibrick, we have $\mathcal{B}' \subseteq {}^\perp B$. According to 7.3, $\mathcal{T}_B \subset \mathcal{T}$ are neighbors.

Also, we claim that $\mathcal{T}' \subseteq {}^{\perp}B$. Namely, if $f\colon M \to B$ is a non-zero homomorphism with $M \in \mathcal{T}'$, then 6.2 asserts that f is surjective, thus $B \in \mathcal{T}'$, a contradiction. It follows that $\mathcal{T}' \subseteq {}^{\perp}B$, thus $\mathcal{T}' \subseteq \mathcal{T} \cap {}^{\perp}B = \mathcal{T}_B$. □

8.2. Proof of Theorem 2.8. First, let $\mathcal{B}$ be a semibrick and $\mathcal{T} = T(\mathcal{B})$. According to 8.1, $\mathcal{T}$ has sufficiently many lower neighbors, namely the torsion classes $\mathcal{T}_B$ with $B \in \mathcal{B}$. Also, the map $B \mapsto \mathcal{T}_B$ from $\mathcal{B}$ to the set of lower neighbors of $\mathcal{T}$ is surjective. On the other hand, this map is injective by the unicity of the label.

Conversely, let $\mathcal{T}$ be a torsion class with sufficiently many lower neighbors. Let $\mathcal{B}$ be the set of labels of the lower neighbors. Then $\mathcal{B}$ is a subset of $\mathcal{T}$, thus $T(\mathcal{B}) \subseteq \mathcal{T}$. Let us assume that $T(\mathcal{B}) \subset \mathcal{T}$. Since $\mathcal{T}$ has sufficiently many lower neighbors, there is a lower neighbor $\mathcal{T}'$ of $\mathcal{T}$ such that $T(\mathcal{B}) \subseteq \mathcal{T}'$. Let B be the label of the inclusion $\mathcal{T}' \subset \mathcal{T}$. Then $B \in \mathcal{B}$. Now $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B \subseteq {}^{\perp}B$. Thus we have $B \in T(\mathcal{B}) \subseteq \mathcal{T}' \subseteq {}^{\perp}B$, a contradiction. □

8.3. Corollary. *If $\mathcal{B}$ is a semibrick. Then $T(\mathcal{B})$ is finitely generated iff $\mathcal{B}$ is finite.*

Proof. If $\mathcal{B}$ is finite, then, of course, $T(\mathcal{B})$ is finitely generated. Conversely, assume that $T(\mathcal{B})$ is finitely generated. By definition, there is a module M with $T(\mathcal{B}) = T(M)$. According to Theorem 2.3, there is a finite semibrick $\mathcal{B}'$ with $T(M) = T(\mathcal{B}')$. According to 2.8, we have $\mathcal{B} = \mathcal{B}'$, thus $\mathcal{B}$ is finite. □

9. Torsional brick chain filtrations.

We are going to prove Theorem 3.2.

9.1. Proposition. *Let B be a top brick of the module M. Then M has a proper submodule M' which belongs to $T(M) \cap {}^{\perp}B$, such that M/M' belongs to $\mathcal{E}(B)$.*

Proof. Let $X = \operatorname{et}^{\infty} M$. Then B is a direct summand of X. Then $T(M) = T(X)$ by Proposition 5.5, thus M belongs to $T(X)$. Now use Proposition 6.1. □

9.2. Existence. According to 9.1, M has a proper submodule M' in $T(M) \cap {}^{\perp}B$ such that M/M' belongs to $\mathcal{E}(B)$ for some brick B.

By induction, M' has a brick chain filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_{m-1} = M'$ of type $(B_1, \dots, B_{m-1})$ such that any M_{i-1} is in $T(M_i)$ for all $1 \le i \le m-1$. Note that we have $T(M_0) \subseteq T(M_1) \subseteq \cdots T(M_{m-1}) = T(M')$.

Let $M_m = M$ and $B_m = B$. Now, for $1 \le i \le m-1$, the module M_i maps onto B_i. But $M_i \in T(M') \subseteq {}^{\perp}B$. As a consequence, $\operatorname{Hom}(B_i, B) = 0$. This shows that $(B_1, \dots, B_m)$ is a brick chain. Of course, the filtration M_i is of type $(B_1, \dots, B_m)$. Also, M_{i-1} is in $T(M_i)$ for all $1 \le i \le m-1$, by induction, and for $i = m$ by 9.1. □

Theorem 2.8 has the following consequence.

9.3. Lemma. *Let M be a module, B a brick. Assume that M has a proper torsional submodule Y in $T(M) \cap {}^{\perp}B$ such that M/Y belongs to $\mathcal{E}(B)$. Then B is a top brick of M (and Y is the torsion submodule of M with respect to the torsion class $T(M) \cap {}^{\perp}B$).*

Proof. First, we show that $T(M) = T(Y, B)$. Since Y is a proper submodule of M, we see that M/Y is a non-zero module in $\mathcal{E}(B)$, thus it has a factor module isomorphic to B.

Since B is a factor module of M, we know that B belongs to $T(M)$. Also, by assumption, Y belongs to $T(M)$. Thus $T(Y,B) \subseteq T(M)$. On the other hand, M has a filtration with factors of the from Y and B, thus $T(M) \subseteq T(Y,B)$.

Next, we calculate the iterated endotop of $Y \oplus B$. We calculate inductively $\mathrm{et}^a(Y \oplus B)$ for all $a \geq 0$. We claim that $\mathrm{et}^a(Y \oplus B) = Y_a \oplus B$, where Y_a is a factor module of Y with $\mathrm{Hom}(Y_a, B) = 0$. For $a = 0$, we put $Y_a = Y$. Assume we have $\mathrm{et}^a(Y \oplus B) = Y_a \oplus B$, where Y_a is a factor module of Y with $\mathrm{Hom}(Y_a, B) = 0$. Since $\mathrm{Hom}(Y_a, B) = 0$, the radical maps in the endomorphism ring of $Y_a \oplus B$ map into Y_a. If U_a is the sum of these images, then $\mathrm{et}^a(Y \oplus B) = Y_{a+1} \oplus B$ with $Y_{a+1} = Y_a/U_a$. Also, we have $\mathrm{Hom}(Y_{a+1}, B)$, since any non-zero homomorphism $Y_{a+1} \to B$ would yield a non-zero homomorphism $Y_a \to Y_{a+1} \to B$. Since we deal with modules of finite length, there is some a such that $U_a = 0$, and therefore $\mathrm{et}^\infty(Y \oplus B) = Y_a \oplus B$. This shows that B is a top brick of $Y \oplus B$.

Since $T(M) = T(Y,B)$, we know from 2.8 that the top bricks of M are just the top bricks of $Y \oplus B$. Thus B is a top brick of M. □

9.4. Corollary. *Let $(M_i)_i$ be a torsional brick chain filtration of M of brick type $(B_1, \dots, B_m)$. Then B_m is a top brick of M and M_{m-1} is the torsion submodule of M for the torsion class $T(M) \cap {}^\perp B_m$.*

Proof. We apply Lemma 9.3 to $Y = M_{m-1}$ and $B = B_m$. □

9.5. Finiteness. Let M be a non-zero module. Let $T_1, \dots T_t$ be its top bricks. For $1 \leq i \leq t$, let $M^{(i)}$ be the torsion submodule of M which respect to the torsion class $T(M) \cap {}^\perp T_i$.

For any module M, let $\phi(M) \in \mathbb{N} \cup \{\infty\}$ be the number of torsional brick chain filtrations of M. Of course, we have $\phi(0) = 1$. For $M \neq 0$, let $M^{(i)}$ be the maximal submodule of M which belongs to $T(M) \cap {}^\perp T_i$. we claim that

$$\phi(M) = \sum_i \ \phi(M^{(i)}).$$

This follows from 9.4, since the torsional brick chain filtrations of M are the filtrations obtained from a torsional brick chain filtration of $M^{(i)}$ by adding the inclusion $M^{(i)} \subset M$.

We use induction on the length of M in order to see that $\phi(M)$ is finite for all M. □

9.6. Some examples.

(1) If M is a brick, the only torsional brick chain filtration is $(0, M)$, since a brick has no non-zero proper torsional submodules, see 4.3. If M is a brick and not simple, then we will see in 10.4 that M has at least two brick chain filtrations. Thus, not every brick chain filtration $(M_i)_i$ of a module M is torsional.

(2) To see an example of a module M with at least two torsional brick chain filtrations, take any module with at least two top bricks, see 5.7.

(3) For a Nakayama algebra, any indecomposable module M has only one torsional brick chain filtration $(M_i)_i$, and this filtration has length at most two. Namely, let S be the top of M. Then all bricks in $T(M)$ have top S. Assume that M has precisely m

composition factors of the form S, and U is the unique submodule of M with top S which is a brick. Then either M is in $\mathcal{E}(U)$, then $0 \subset M$ is the only torsional brick chain filtration of M. Else $0 \subset U \subseteq M$ is the only torsional brick chain filtration of M.

(4) Duality. We have mention in Section 1 that using the duality functor D, we obtain from a brick chain filtration $(M_i)_i$ of M a corresponding brick chain filtration for $\mathrm{D}\,M$. But note: If the filtration $(M_i)_i$ is torsional, the dual filtration may not be torsional. As a typical example, let A be a connected Nakayama algebra with two simple modules and an indecomposable module M of length three, let U be its socle. Then M has the brick chain filtration $(0 \subset U \subset M)$. This filtration is torsional, whereas the dual filtration is not torsional.

There are brick chain filtrations $(M_i)_i$ of modules such that neither the filtration $(M_i)_i$ nor the dual filtration $(\mathrm{D}\,M/\,\mathrm{D}\,M_i)$ is torsional. Here is an example: Let A be a connected Nakayama algebra with three simple modules and M indecomposable of length four. Let U be the submodule of M of length two. Then $(0 \subset U \subset M)$ is a brick chain filtration, however neither this filtration nor its dual is torsional.

10. Further remarks about brick chain filtrations.

10.1. Lemma. *Let B be a brick and M a non-zero module in $\mathcal{E}(B)$. Then M has an endomorphism with image a brick, and B is the only brick which occurs in this way.*

Proof. First, we show that B occurs as the image of an endomorphism of M. Since M belongs to $\mathcal{E}(B)$, there is a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ of M such that all factors are isomorphic to B. A corresponding map $M \to M/M_{m-1} \simeq B \simeq M_1 \subseteq M$ is an endomorphism of M which image isomorphic to B.

Conversely, let f be an endomorphism of M whose image is a brick. Since $\mathcal{E}(B)$ is an exact abelian subcategory, the image M' of f belongs to $\mathcal{E}(B)$. Now M' is a non-zero module in $\mathcal{E}(B)$. As we have seen in the first part of the proof, M' has an endomorphism with image $f(M')$ being isomorphic to B. But we assume that M' is a brick, thus the image of an endomorphism of M' is either zero or M' itself. This shows that $M' = f(M')$, thus M' is isomorphic to B. □

Remark. The Lemma shows: If M is a homogeneous module of brick type B, then the endomorphism ring of M shows whether $M = B$ or $M \neq B$. But even for a homogeneous module M, the ring $\mathrm{End}(M)$ gives only limited information about M. In particular, $\mathrm{End}(M)$ may be a k-algebra of dimension 2, whereas M has a filtration with arbitrarily many factors of the form B. Here is an example. Consider the subring $A = k + J$ of the ring of $(t \times t)$-matrices with $t \geq 2$, where J is the set of nilpotent upper triangular matrices; and look at the set $M = k^t$ of column vectors. Since A is local, there is the unique brick $B = k$. The module M is a serial module of length t. The image of any non-invertible endomorphism of M has length at most one, thus $\dim \mathrm{End}(M) = 2$.

10.2. A filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M$ will be said to be *solid* provided $\mathrm{Hom}(M_i/M_{i-1}, M_j/M_{j-1}) = 0$ for all $1 \leq i < j \leq m$.

Proposition. *Let M be a module with a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$. Then $(M_i)_i$ is a brick chain filtration iff $(M_i)_i$ is a solid filtration and all the factors are homogeneous.*

Proof. First, assume that $(M_i)_i$ is a brick chain filtration, say of type $(B_1, \dots, B_m)$. Since M_i/M_{i-1} belongs to $\mathcal{E}(B_i)$, all the factors of the filtration are homogeneous. Also, for $i < j$, we have $\mathrm{Hom}(B_i, B_j) = 0$. Therefore $\mathrm{Hom}(M_i/M_{i-1}, M_j/M_{j-1}) = 0$.

Conversely, assume that $(M_i)_i$ is a solid filtration (with proper inclusions) and all factors are homogeneous. Now $F_i = M_i/M_{i-1}$ belongs to $\mathcal{E}(B_i)$ for some brick B_i. Since F_i is non-zero, B_i occurs both as a submodule and as a factor module of F_i. Thus, any non-zero map $f\colon B_i \to B_j$ yields a non-zero map $F_i \to F_j$. Since the given filtration is solid, we see that $\mathrm{Hom}(B_i, B_j) = 0$ for $i < j$. Thus, $(B_1, \dots, B_m)$ is a brick chain. □

10.3. The composition factors in the top of a module give rise to brick chain filtrations:

Proposition. *Let M be a module. If S is a simple module which occurs in the top of M, then M has a brick chain filtration of type $(B_1, \dots, B_m)$ with $B_m = S$.*

Proof. Let M' be the minimal submodule of M such that M/M' has only S as composition factor, thus M/M' belongs to $\mathcal{E}(S)$ and S does not occur in the top of M'. Now take a torsional brick chain filtration $(M_i)_{1\le i\le m-1}$ of M', say of type $(B_1, \dots, B_{m-1})$ and let $M_m = M$. Since we deal with a torsional filtration of M', the modules M_i, thus also the bricks B_i are in $\mathcal{T}(M')$, thus the top of B_i is generated by M'. As a consequence, $\mathrm{Hom}(B_i, S) = 0$. This shows that $(B_1, \dots, B_m)$ with $B_m = S$ is a brick chain, and that the filtration $(M_i)_{1\le i\le m}$ is a brick chain filtration of type $(B_1, \dots, B_m)$. □

10.4. A module M has usually several brick chain filtrations, and the length of these filtrations seem to be quite unrelated. As a typical example, let A be the path algebra of the directed quiver of type $\mathbb{A}_n$ and M the indecomposable sincere A-module. It is easy to see that M has brick chain filtrations of length m, for any $1 \le m \le n$.

By definition, a module M is homogeneous iff $(0 \subseteq M)$ is a brick chain filtration. A homogeneous module which is not a brick has only one brick chain filtration, namely $(0 \subseteq M)$. But bricks usually have several brick chain filtrations:

Proposition. *A brick which is not simple, has at least two brick chain filtrations.*

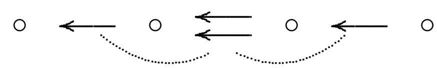
Proof. Let M be a brick. Then $(0 \subset M)$ is a brick chain filtration of length one. Let S be a simple module which occurs in the top of M. According to 10.3, there is a brick chain filtration $(M_i)_{1\le i\le m}$ with M_m/M_{m-1} in $\mathcal{E}(S)$. We claim that $m \ge 2$. Namely, if $m = 1$, then M belongs to $\mathcal{E}(S)$. Since M is a brick, $M = S$, thus M is simple. □

10.5. Brick chain complexity. We say that a module M has *brick chain complexity* at most t provided there is a brick chain filtration with t non-zero factors. The *brick chain complexity* of an algebra A is the maximum of the brick chain complexity of the indecomposable A-modules.

Of course, bricks are modules with brick chain complexity 1. Thus, any representation-directed algebra has complexity 1. Since any indecomposable Kronecker module is homogeneous, the Kronecker algebra has also complexity 1. Next, if A is a local algebra, then again all modules are homogeneous, thus local algebras also have complexity 1.

Nakayama algebras, tame concealed algebras and tubular algebras have complexity at most 2. For example, if A is a tame concealed, the only indecomposable modules which are not homogeneous are the indecomposable modules M which belong to a tube say of rank r, with regular length not divisible by r and these modules have complexity 2.

Note that any module which belongs to a standard tube has complexity at most 3. The case 3 is possible: see the non-stable tube for the algebra



An indecomposable module M with dimension vector $(1, t, t, 1)$ has brick chain complexity $\min(t, 3)$.

Here is an algebra with brick chain complexity ∞. Take the path algebra of the quiver with two vertices $1, 2$, with two arrows $1 \leftleftarrows 2$ and with a loop γ at the vertex 1, with relation $\gamma^2 = 0$ (thus the algebra has dimension 7). For any $m \in \mathbb{N}_1$, we construct an indecomposable module M_m with brick chain complexity m as follows: The restriction of M_m to the Kronecker quiver $1 \leftleftarrows 2$ shall be $\bigoplus_{i=1}^m I_i$, where I_i is the indecomposable preinjective module of dimension $2i + 1$. In order to specify the action of γ, we choose a basis $w_1^{(i)}, \dots, w_i^{(i)}$ of the socle of I_i and define $\gamma w_1^{(i)} = w_{i-1}^{(i-1)}$ provided $2 \le i \le m$, and $\gamma w_j^{(i)}$ otherwise (thus γ maps I_i into I_{i-1} and satisfies $\gamma^2 = 0$). We obtain inclusions $0 \subset M_1 \subset \cdots \subset M_m$. This is a brick chain filtration of M_m of type $(I_1, \dots, I_m)$.

10.6. Looking at the brick chain complexity, we only deal with brick chain filtrations of indecomposable modules, and then only few brick chains may play a role.

But in general, one should be interested in all possible brick chains. Here are the brick chains for the Kronecker algebra. Let us denote by P_i and I_i the indecomposable modules of dimension $2i + 1$, where P_i is preprojective, I_i preinjective. Any (finite) brick chain different from (P_0, I_0) is of the form $(I_{i_1}, I_{i_2}, \cdots, I_{i_q}; R_1, \dots, R_r; P_{j_1}, P_{j_2}, \cdots, P_{j_p})$, where $0 \le i_1 < i_2 < \cdots < i_q$, as well as $j_1 > j_2 > \cdots > j_p \ge 0$, with pairwise non-isomorphic simple regular modules $R_1, \dots, R_r$ (in any order); here, p, q, r are non-negative numbers.

10.7. We say that a filtration $(M_i)_i$ of a module M is *proper* provided all the inclusions $M_{i-1} \subseteq M_i$ are proper. Of course, any filtration $(M_i)_i$ yields a proper filtration by deleting all the submodules M_i with $M_{i-1} = M_i$. Until now, all the filtrations considered in the paper were proper. Let us call an arbitrary filtration of a module M a *brick chain filtrations with repetitions* provided the corresponding proper filtration is a brick chain filtration.

Proposition. (a) *Let M, N be modules. Let $(X_i)_i$ be a brick chain filtration of $X = M \oplus N$. Then $(X_i \cap M)_i$ is a brick chain filtration of M with repetition,* we say that it is *induced from $(X_i)_i$.*

(b) *Given a module M and a brick chain filtration $(M_i)_i$, of type $(B_1, \dots, B_t)$. Then there is a module N and a torsional brick chain filtration $(X_i)_i$ of $X = M \oplus N$ such that the filtration $(M_i)_i$ of M is induced from $(X_i)_i$.*

Proof. (a) Since X_i is the torsion submodule of X for the torsion class $T(B_1, \dots, B_i)$, we have $(X_i \cap M) \oplus (X_i \cap N) = X_i$. Thus $(X_i \cap M)/(X_{i-1} \cap M) \oplus (X_i \cap N)/(X_{i-1} \cap N) = X_i/X_{i-1}$ belongs to $\mathcal{E}(B_i)$, thus $(X_i \cap M)/(X_{i-1} \cap M)$ belongs to $\mathcal{E}(B_i)$. Of course, $(X_i \cap M)/(X_{i-1} \cap M)$ may be zero (thus, the filtration may have repetitions).

(b) Let $N = \bigoplus_{j=1}^{t-1} B_j$ and $X = M \oplus N = X_t$. For $0 \le i < t$, let $X_i = M_i \oplus \bigoplus_{j=1}^{i} B_j$. Then $X_t/X_{t-1} = M_t/M_{t-1} \in \mathcal{E}(B_t)$. Also, for $1 \le i < t$, we have $X_i/X_{i-1} = M_i/M_{i-1} \oplus B_i \in \mathcal{E}(B_i)$. Thus, we see that $(X_i)_i$ is a brick chain filtration of type $(B_1, \dots, B_t)$.

All the bricks B_i with $1 \le i < t$ are factor modules of X, thus $T(B_1, \dots, B_{t-1}) \subseteq T(X)$. Since X_{t-1} belongs to $\mathcal{E}(B_1, \dots, B_{t-1})$, we see that X_{t-1} belongs to $T(X)$. Similarly, for $1 \le i < t$, we have $T(B_1, \dots, B_{i-1}) \subseteq T(X_i)$. Again, X_{i-1} belongs to $\mathcal{E}(B_1, \dots, B_{i-1})$, thus X_{i-1} belongs to $T(X_i)$. This shows that $(X_i)_i$ is a torsional filtration. □

In the proof of (b), we also could have used $N = \bigoplus_{j=1}^{t-1} M_j$.

10.8. Remark. We have seen in 10.7 that brick chain filtrations of direct sums yield brick chain filtrations of the summands. The converse is not true. For example, let A be a cyclic Nakayama algebra with two simple modules and M, N the two indecomposable modules of length two. Then M, N are bricks, thus they have brick chain filtrations of length one. Any brick chain filtration of the module $M \oplus N$ has length three and induces on one of the summands a brick chain filtration of length two.

11. History and Relevance.

11.1. The results presented here are usually considered as part of the so-called τ-tilting theory (see 11.13). There is a strange reluctance to deal with bricks. For example, many papers prefer to speak about τ-tilting finiteness instead of brick finiteness, but these properties are equivalent (see [DIJ]; here, τ-tilting finiteness means that there are only finitely many τ-tilting modules: In my opinion, brick finiteness is very easy to grasp, whereas τ-tilting finiteness is much less intuitive). For our report, there is no need to mention τ-tilting notions, nor even the Auslander-Reiten translation τ itself, thus we have avoided it. In this way, we stress the completely elementary nature of the corresponding results. Some remarks on τ-tilting theory will be given in 11.13.

Let me stress that it is astonishing that the relevance of bricks when dealing with tilting modules, with torsion classes, with module categories was observed only so late!

11.2. Bricks and semibricks. The terminology "semibrick" seems to be due to Asai [A]. I used to call a semibrick an "antichain" of bricks, but this is in conflict with Demonet's important notion of a brick chain (and to say that "an antichain of bricks is a brick chain", would sound rather odd).

11.3. Torsion pairs $(\mathcal{T}, \mathcal{F})$. Torsion pairs were introduced by Dickson [Di] as a generalization of the use of torsion and p-torsion subgroups of abelian groups, for dealing with arbitrary R-modules, were R is any ring.

11.4. Hereditary torsion pairs, torsional submodules. In contrast to the classical example, torsion classes in general are not hereditary (where *hereditary* means that the torsion class $\mathcal{T}$ is closed under submodules). For example, the torsion classes $T(M)$ considered in our paper are usually not hereditary. On the other hand, it turns out that dealing with a module M it is important to look at submodules of M which do belong to $T(M)$, namely its torsional submodules. Thus, our focus on torsional submodules is an attempt to stress hereditary properties for non-hereditary torsion classes.

The brick-chain theorems 1.2 and 3.2 should be seen in the light of the original example of abelian group theory: any finitely generated abelian group M has a filtration $(M_i)_{0 \le i \le m}$

where the factors M_i/M_{i-1} with $0 \leq i < m$ are in $\mathcal{E}(\mathbb{Z}/p_i\mathbb{Z})$, for pairwise different prime numbers p_i, whereas M_m/M_{m-1} is in $\mathcal{E}(\mathbb{Z})$. Note that this filtration always splits. In our case, we cannot expect that the filtrations provided in 1.2 and 3.2 split, just look at indecomposable modules M which are not homogeneous. (It comes as a surprise that actually in first examples one looks at, for example dealing with Kronecker modules, many brick chain filtrations do split.)

11.5. Auslander and Smalø (and Demonet). The relevance of torsion classes when dealing with finite length categories was seen already by Auslander and Smalø [AS].

Looking at a module category, the existence of cyclic paths in the category or even in the Auslander-Reiten quiver, provides a lot of difficulties. Only the representation-directed algebras are easy to visualize, but representation-directedness is a very special property. There have been many attempts to overcome the difficulties which arise from the presence of cyclic paths. There is the covering theory by Gabriel and his school; also, the book of Auslander, Reiten, Smalø is full of helpful devices: to avoid short chains, to avoid short cycles. However, all these methods are designed just for special, well-behaved situations. If one wants to deal with an arbitrary module category, the use of torsion classes always works. As we have mentioned in 2.4, the reference to torsion classes allows to consider the set of semibricks as a partially ordered set. In this way, Demonet's proposal to look at brick chains stresses a very interesting directedness feature of an arbitrary module category.

11.6. Wide subcategories and torsion classes. Given an abelian category, the exact abelian subcategories which are closed under extensions are now often called *wide* subcategories. The rather obvious relationship between semibricks and wide subcategories was mentioned in [R1] under the name "simplification". The search for semibricks (or wide subcategories) which generate a given torsion class was initiated by Ingalls and Thomas [IT]. Theorem 2.3 generalizes some of their considerations.

The relevance of the endotop of a module is well-known and was stressed by Asai when looking at τ-rigid modules (our proof of 5.5 follows closely Asai [A]). For a general study of widely generated torsion classes, see Asai and Pfeifer [AP] and Marks and Stovicek [MS].

11.7. Homogeneous subcategories. The homogeneous subcategories are equivalent to the module category of a local algebra (not necessarily an artin algebra) and one often uses the representation theory of local algebras just as a black box. But, actually, not much is known about the representation theory of a local algebra A which is not commutative! The commutative local rings are studied very well in commutative algebra, whereas there never was much interest in the non-commutative ones. But note that often they behave rather differently and really deserve attention.

Let us mention at least one phenomenon which is of relevance for our discussion. If A is a commutative local ring, and M is a serial module, say of length t, then there is an endomorphism of M with image $\operatorname{rad} M$, thus $\operatorname{et} M$ is just the simple module. On the other hand, consider the subring $A = k + J$ of the ring of $(t \times t)$-matrices where J is the set of nilpotent upper triangular matrices, as mentioned already in 5.2 and 10.1. This is a rather nice local ring; it is non-commutative provided $t \geq 3$. The set $M = k^t$ of column vectors is a serial A-module. Since the image of any non-invertible endomorphism of M has length at most one, we see that $\operatorname{et} M$ has dimension $t-1$; in particular, it is not a brick provided $t \geq 3$ (for $t \geq 3$, we have $\operatorname{et}^\infty M = \operatorname{et}^{t-2} M = k \neq \operatorname{et} M$).

11.8. Neighbors of torsion classes. Neighbor torsion classes $\mathcal{T}' \subset \mathcal{T}''$ have attracted a lot of interest and several different denominations are used in the literature: that $\mathcal{T}''$ covers $\mathcal{T}'$, that there is an arrow $\mathcal{T}'' \to \mathcal{T}'$ in the Hasse quiver of the lattice of torsion classes, or one speaks about minimal inclusions of torsion classes.

As we have seen, it is easy to determine the lower neighbors of a finitely generated torsion class (and there are only finitely many), but unfortunately, it is difficult to deal with the upper neighbors: usually, there may be infinitely many. For any torsion class $\mathcal{T}$, the best way to find its upper neighbors seems to be to look at the corresponding torsion free class $\mathcal{F}$ and to try to determine its lower neighbors, since the lower neighbors of $\mathcal{F}$ correspond to the upper neighbors of $\mathcal{T}$.

11.9. Brick labeling. The brick labeling as presented in section 7 was started for functorially finite torsion classes in [AIR] and Asai [A] identified the labels as bricks. The general case is due to Barnard, Carroll and Zhu [BCZ]. The brick B used as label for the neighbor torsion classes $\mathcal{T}' \subset \mathcal{T}$ is called a *minimal extending module* for $\mathcal{T}'$ in [BCZ]. In [AHL], the labels are said to be *torsion, nearly torsionfree* for the torsion pair $(\mathcal{T}, \mathcal{T}^{\perp})$. The bijection 2.9 between bricks and completely join irreducible torsion classes has been exhibited in Theorem 1.0.5 in [BCZ].

11.10. Brick chains. Given a chain of torsion classes, the brick labeling of the neighbor torsion classes yields a brick chain. This observation was used by Demonet [De] to introduce not only the finite brick chains as considered in the present paper, but to deal with arbitrarily large totally ordered sets of bricks with the corresponding Hom-condition. (But note that already for the Kronecker algebra, the sets which occur explode: The Kronecker algebra A over the field with 2 elements has cardinality 16, thus it is very easy to envision, but the lattice of torsion classes in $\operatorname{mod} A$ is uncountable, and there are uncountably many complete brick chains: one is finite, all others are uncountable!)

11.11. Special brick chain filtrations. Special brick chain filtrations have been used already a long time ago. We have shown in [R2] that for a hereditary k-algebra, where k is an algebraically closed field, any exceptional module is a tree module, The basis of the proof is Schofield induction, dealing with certain brick chain filtrations of length two. We stress that the brick chain filtrations used in [R2] are never torsional, since they are filtrations of length two of bricks. We see in this way the relevance of brick chain filtrations which are not torsional. The brick chain filtrations used in [R2] have type (B_1, B_2), where B_1, B_2 both are again exceptional modules, thus bricks without self-extensions, and B_1, B_2 are even Hom-orthogonal.

11.12. Artinian rings. In this report, we have assumed to be in the context of artin algebras. Actually, nearly all the results presented here are valid more generally in arbitrary length categories, thus for finitely generated modules over left artinian rings.

11.13. Functorially finite torsion classes. Functorially finite torsion classes were first considered by Auslander and Smalø. In 1984, Smalø formulated the tie between functorially finite torsion classes and tilting modules for suitable factor algebras. The basic objects of τ-tilting theory are the τ-rigid modules (a module M is *τ-rigid* provided $\operatorname{Hom}(M, \tau M) = 0$, where τ is the Auslander-Reiten translation): Adachi-Iyama-Reiten [AIR] showed that the functorially finite torsion classes are just the torsion classes generated by τ-rigid modules, a very important observation (but actually, the main results of

[AIR] also follow from investigations by Derksen and Fei which were available in arXiv already in 2009). An important feature of τ-tilting theory are the mutations which describe the functorially finite torsion classes which are neighbors. In the setting of tilting theory, mutations were studied by Happel and Unger, but it took a long time that the relevance for arbitrary module categories was realized.

We have seen in 2.5 that an algebra is brick finite iff it is torsion class finite. In this case, all torsion classes are not only finitely generated but even functorially finite. And conversely, if any torsion class is functorially finite, then the algebra is brick finite.

We cannot give here even a concise summary of the development of τ-tilting theory, but can refer to the many survey papers devoted to τ-tilting theory.

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12. References.

[AIR] T. Adachi, O. Iyama, I. Reiten. τ-tilting theory. Compos. Math., 150(3), 415–452 (2014).

[A] S. Asai. Semibricks. Int. Math. Res. Not., (16), 4993–5054. doi.org/10.1093/imrn/rny150 (2020).

[AP] S. Asai, C. Pfeifer. Wide Subcategories and Lattices of Torsion Classes. Algebras and Representation Theory. Vol. 25, 1611–1629 (2022). arXiv:1905.01148

[AS] M. Auslander, S. O. Smalø. Addendum to: Almost split sequences in subcategories. J. Algebra, 71, 592–594 (1981).

[BCZ] E. Barnard, A. Carroll, S. Zhu. Minimal inclusions of torsion classes. Algebr. Comb., 2(5), 879–901 (2019).

[De] L. Demonet. Maximal chains of torsion classes. Appendix to: B. Keller. A survey on maximal green sequences, In: Representation theory and beyond, Contemp. Math. 758, Amer. Math. Soc., Providence, RI, 267-286 (2020).

[Di] S. E. Dickson. A torsion theory for Abelian categories. Trans. Amer. Math. Soc., 121, 223–235 (1966).

[DIJ] L. Demonet, O. Iyama, G. Jasso. τ–tilting finite algebras, bricks, and g-vectors. Int. Math. Res. Not. IMRN, (3), 852–892 (2019).

[IT] C. Ingalls, H. Thomas. Noncrossing partitions and representations of quivers. Comp. Math. 145, 1533156 (2009).

[MS] F. Marks, J. Stovǐcek. Torsion classes, wide subcategories and localisations. Bulletin of the London Mathematical Society. Bull. Lond. Math. Soc. 49, 405–416, doi:10.1112/blms.12033 (2017).

[R1] C. M. Ringel. Representations of K-species and bimodules. J. Algebra 41, 269–302 (1976).

[R2] C. M. Ringel. Exceptional modules are tree modules. Lin. Alg. Appl. 275-276, 471–493 (1998).

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