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Brick chain filtrations. A report.

Claus Michael Ringel

Abstract. We consider the category of finitely generated modules over an artin algebra A. Recall that an object in an abelian category is said to be a *brick* provided its endomorphism ring is a division ring. Simple modules are, of course, bricks, but in case A is connected and not local, there always do exist bricks which are not simple. The aim of this survey is to focus the attention to filtrations of modules where all factors are bricks, with bricks being ordered in some definite way, namely according to a so-called brick chain.

In general, a module category will have many cyclic paths. Recently, Demonet has proposed to look at brick chains in order to deal with a very interesting directedness feature of an arbitrary module category.

The following survey is based on investigations by a large group of mathematicians. We have singled out some important observations and have reordered them in order to provide a self-contained (and elementary) treatment of the role of bricks in module categories. (Most of the papers we rely on are devoted to what is called τ-tilting theory, but for the results we are looking at, there is no need to deal with τ-tilting, not even to invoke the Auslander-Reiten translation τ itself).

Outline. This is a report on a very important development in the last 15 years: it focuses the attention to the use of bricks in order to describe the structure of arbitrary modules over artin algebras. The report is based on the work of a quite large number of mathematicians, see section 12. We have singled out decisive observations and have reordered them in order to obtain a self-contained and elementary (however still incomplete) treatment of the role of bricks in module categories.

The first three sections describe the main results presented in the survey, they deal with brick chain filtrations and their background. Theorem 1.2 and its strengthening 3.2 concern the existence of brick chain filtrations (and 3.2 includes a corresponding finiteness assertion). The main tool is the study of torsion classes and their lower neighbors. Theorem 2.3 asserts that finitely generated torsion classes are always generated by finite semibricks. Theorem 2.8 describes the lower neighbors of the torsion class generated by a module M in terms of the so-called top bricks of M.

Given a brick B, we denote by $\mathcal{E}(B)$ the class of all modules which have a filtration with all factors isomorphic to B; these modules will be said to be *homogeneous* of brick type B. The brick type of a non-zero homogeneous module is uniquely determined (see 9.7). The brick chain filtrations studied in this report concern filtrations of modules with factors in suitable subcategories $\mathcal{E}(B)$, namely using bricks B which occur in a brick chain. The existence of brick chain filtrations is derived from a result for neighbor torsion classes. Neighbor torsion classes $\mathcal{T}' \subset \mathcal{T}$ come with a label: this is a brick B with the following property: any module M in $\mathcal{T}$ has a submodule M' in $\mathcal{T}'$ such that M/M' belongs to $\mathcal{E}(B)$, see Theorem 2.7.

1. All modules have brick chain filtrations.

1.1. We deal with an artin algebra A; the modules to be considered are usually 9. left A-modules of finite length. Given a set $\mathcal{X}$ of modules, let $\mathcal{E}(\mathcal{X})$ be the class of modules which have a filtration with all factors in $\mathcal{X}$. If $M_1, \dots, M_m$ are modules, let $\mathcal{E}(M_1, \dots, M_m) = \mathcal{E}(\{M_1, \dots, M_m\})$ (such a convention is used throughout the paper in similar situations).

We recall that a *brick* is a module whose endomorphism ring is a division ring. If B is a brick, the modules in $\mathcal{E}(B)$ will be said to be *homogeneous* of brick type B. A finite sequence $(B_1, \dots, B_m)$ is called a *brick chain,* if all B_i are bricks and $\operatorname{Hom}(B_i, B_j) = 0$ for $i < j$ (in Appendix 11, we also will deal with infinite sequences). A filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ will be called a *brick chain filtration,* provided there is a brick chain $(B_1, \dots, B_m)$ (its *type*) such that M_i/M_{i-1} is homogeneous of brick type B_i, for all $1 \le i \le m$ (the number m is called the *length* of the filtration).

1.2. Theorem. *Any module has brick chain filtrations.*

The result will be strengthened in 3.2. The proof of Theorem 3.2 is given in section 9. The assertion 1.2 is also an immediate consequence of Theorem 11.2.

If all composition factors of a module M are isomorphic, then $(0 \subseteq M)$ is a brick chain filtration of M, and obviously the only one; *otherwise, M has at least two different brick chain filtrations,* as we will show in 10.3.

1.3. Some examples of brick chain filtrations.

(1) Let $S_1, \dots, S_n$ be the simple A-modules. Obviously, $(S_1, \dots, S_n)$ (in any order!) is a brick chain. Let us now assume that $\operatorname{Ext}^1(S_i, S_j) = 0$ for all $i > j$. If M is an A-module M, let M_i be the maximal submodule of M with all composition factors of the form $S_1, \dots, S_i$. If M is sincere, then $(M_i)_i$ is a brick chain filtration of type $(S_1, \dots, S_n)$ (if M is not sincere, $(M_i)_i$ is a brick chain filtration with repetitions, as considered in 10.4).

In particular, recall that A is said to be *directed,* provided the simple modules $S_1, \dots, S_n$ can be ordered in such a way that $\operatorname{Ext}^1(S_i, S_j) = 0$ for all $i \ge j$. *For a directed algebra A, any sincere A-module M has a brick chain filtration of type $(S_1, \dots, S_n)$ such that all factors of the filtration are semisimple.*

(2) Let us consider the special case where $M = B$ itself is a brick. In this case, there is, of course, the trivial brick chain filtration $(0 \subset B)$, but if B is not simple, then there are additional brick chain filtrations (see 10.3).

Let A be the path algebra of the linearly directed quiver of type $\mathbb{A}_n$ and B the indecomposable sincere A-module. Then any proper filtration of B is a brick chain filtration, thus B has 2^{n-1} brick chain filtrations. In particular, B has brick chain filtrations of length m, for any $1 \le m \le n$.

For the quiver with vertices $1, 2$, an arrow $1 \leftarrow 2$ and a loop γ at 1, with relation γ^t for some $t \ge 2$, the projective cover M of the simple module 2 is a brick. It has the brick chain filtration $(0 \subset M_1 \subset M)$ of type (B_1, B_2), where $M_1 = \operatorname{rad} M$, and $B_1 = 1, B_2 = 2$. The relative Loewy length of M_1 in $\mathcal{E}(B_1)$ (of course, also its absolute Loewy length) is t.

(3) In contrast to many questions in representation theory, looking for brick chain filtrations of modules, it does not seem to be helpful to consider first indecomposable

modules. Namely, brick chain filtrations of modules M and M' usually do not provide a brick chain filtration of $M \oplus M'$, see 10.5.

(4) (Duality) Let us denote by D the usual duality functor. Given a brick chain filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ of type $(B_1, \dots, B_m)$, then clearly D yields a corresponding brick chain filtration $(N_i)_i$ of $N = \mathrm{D}\, M$, namely $N_i = \mathrm{D}\, M/\mathrm{D}\, M_{m-i}$, for $0 \le i \le m$. The type of the filtration $(N_i)_i$ is $(\mathrm{D}\, B_m, \dots, \mathrm{D}\, B_1)$.

(5) Our proof of 1.2 will yield quite special brick chain filtrations, namely "torsional" ones, see section 3. Let us note already here: if a filtration $(M_i)_i$ of a module M is torsional, then the top of any M_i is generated by the top of M. Thus, even in the case of a directed algebra, the brick chain filtrations which we will construct are usually different from the obvious filtrations mentioned in (1).

2. Torsion classes, in particular the finitely generated ones.

The proof of Theorem 3.2 will be based on the use of torsion classes, they are essential for all considerations. Here, wee recall the definition and some properties of torsion classes.

2.1. A class $\mathcal{T}$ of modules is said to be a *torsion class* provided $\mathcal{T}$ is closed under factor modules and extensions. The set of all torsion classes is a complete lattice; the meet of a set of torsion classes is just the set-theoretical intersection. Given a class $\mathcal{X}$ of modules, we denote by $T(\mathcal{X})$ the smallest torsion class which contains $\mathcal{X}$ (thus, the closure of $\mathcal{X}$ under factor modules and extensions, or, equivalently, the set-theoretical intersection of all torsion classes containing $\mathcal{X}$). The Noether theorems show that $T(\mathcal{X})$ is the class of modules which have a filtration whose factors are factor modules of modules in $\mathcal{X}$. The *torsion submodule* of a module M with respect to the torsion class $\mathcal{T}$ is by definition the largest submodule of M which belongs to $\mathcal{T}$. Given a module class $\mathcal{Y}$, we denote by ${}^{\perp}\mathcal{Y}$ the class of all modules X such that $\mathrm{Hom}(X, Y) = 0$ for all modules Y in $\mathcal{Y}$. It is clear that ${}^{\perp}\mathcal{Y}$ is closed under factor modules and extensions, thus it is a torsion class.

A torsion class $\mathcal{T}$ is said to be *finitely generated* provided there is a module M with $\mathcal{T} = T(M)$. Of course, any torsion class $\mathcal{T}$ is the set-theoretical union of the finitely generated torsion classes contained in $\mathcal{T}$.

2.2. Let $(M_i)_i$ be a brick chain filtration of the module M, say of type $(B_1, \dots, B_t)$. Then M_i is the torsion submodule of M with respect to the torsion class $T(B_1, \dots, B_i)$ and also the torsion submodule of M with respect to the torsion class ${}^{\perp}\{B_{i+1}, \dots, B_t\}$. Thus, we see: *Given a module M with a brick chain filtration $(M_i)_i$, the submodules M_i are uniquely determined by the type of the filtration.*

2.3. Modules M, M' are defined to be Hom-*orthogonal* provided $\mathrm{Hom}(M, M') = 0 = \mathrm{Hom}(M', M)$. A *semibrick* is a set of pairwise Hom-orthogonal bricks. A torsion class which is generated by a semibrick is said to be *widely generated.* When we deal with sets of (pairwise non-isomorphic) modules, for example when we consider semibricks, these sets will not necessarily be finite (so that we cannot deal or better do not want to deal with the corresponding direct sum).

Theorem. *For any artin algebra A, the map $\mathcal{B} \mapsto T(\mathcal{B})$ provides a bijection between finite semibricks and the finitely generated torsion classes.*

The surjectivity of the map asserts that *any finitely generated torsion class is widely generated.* The injectivity assertion can be extended as follows: the map $\mathcal{B} \mapsto T(\mathcal{B})$ is a bijection between arbitrary semibricks and the widely generated torsion classes, see 2.8.

The proof of Theorem 2.3 is given in 5.6 (the surjectivity of the map), and in section 8 (the injectivity of the map). In Section 5, we construct explicitly an inverse of the map $\mathcal{B} \mapsto T(\mathcal{B})$, for $T(\mathcal{B})$ being finitely generated. Let us outline the construction already here.

Addendum to Theorem. Given a module M, we define in 5.4 its "iterated endotop" $X = \mathrm{et}^{\infty} M$; this is a factor module of M. The indecomposable direct summands of X are called the *top bricks* of M and we denote by $\mathcal{B}(M)$ the set of top bricks of M. The surjectivity assertion in Theorem 2.3 can be strengthened as follows (see sections 5 and 8): *Given any module M, then $T(M) = T(\mathcal{B})$ for a uniquely determined semibrick $\mathcal{B}$, namely the finite semibrick $\mathcal{B} = \mathcal{B}(M)$ of the top bricks of M.* In particular: *$T(M)$ is generated by a finite semibrick $\mathcal{B}$ whose members are factor modules of M.*

2.4. Remark. The bijection provided by Theorem 2.3 is of great importance, since it allows to consider the set of finite semibricks as a partially ordered set, using the natural partial ordering of the set of torsion classes, given by set-theoretical inclusion. This poset structure on the set of finite semibricks (thus also on the set of bricks) provides the foundation for the notion of a brick chain as used in Theorem 1.2 and in section 11.

2.5. Brick finite algebras. An algebra A is said to be *brick finite* provided there are only finitely many isomorphism classes of bricks. The algebra A is said to be *torsion class finite* provided there is only a finite number of torsion classes.

Proposition. *For any algebra, the number of finite semibricks is equal to the number of finitely generated torsion classes.*

An algebra is brick finite iff it is torsion class finite, and in this case any torsion class is finitely generated.

Actually, also the converse of the last assertion is true: *If all torsion classes are finitely generated, then the algebra is brick finite,* see 12.13.

Proof of Proposition. For the first assertion, see 2.3. If A is torsion class finite, then A has only finitely many finitely generated torsion classes, thus only finitely many semibricks, thus only finitely many bricks. Conversely, assume that A is brick finite, thus A has only finitely many finitely generated torsion classes. Given a torsion class $\mathcal{T}$, one can start to construct an inclusion chain of finitely generated torsion classes $\mathcal{T} = \mathcal{T}_0 \subset \mathcal{T}_1 \subset \cdots \subset \mathcal{T}_t \subseteq \mathcal{T}$. This process will stop after finitely many steps, thus $\mathcal{T}$ is finitely generated. We see in this way that all torsion classes are finitely generated. Thus there are only finitely many torsion classes. □

2.6. Neighbors. The torsion classes $\mathcal{T}' \subset \mathcal{T}$ will be said to be *neighbors* provided there is no torsion class $\mathcal{N}$ with $\mathcal{T}' \subset \mathcal{N} \subset \mathcal{T}$. If $\mathcal{T}' \subset \mathcal{T}$ are neighbor torsion classes, $\mathcal{T}'$ is called a *lower neighbor* of $\mathcal{T}$ and $\mathcal{T}$ is called an *upper neighbor* of $\mathcal{T}'$.

2.7. Theorem. *If $\mathcal{T}' \subset \mathcal{T}$ are neighbor torsion classes, then there is a unique brick B in $\mathcal{T}$ such that $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B$.* This brick B is called the *label* of the inclusion $\mathcal{T}' \subset \mathcal{T}$.

If $\mathcal{T}' \subset \mathcal{T}$ are neighbor torsion classes with label B, then we have:

(1) *$\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B$ and $\mathcal{T} = T(\mathcal{T}', B)$.*
(2) *Any module M in $\mathcal{T}$ has a submodule M' in $\mathcal{T}'$ such that M/M' belongs to $\mathcal{E}(B)$.*

Conversely, assume that $\mathcal{T}' \subset \mathcal{T}$ are torsion classes and that B is a brick in $\mathcal{T}$. If one of the conditions (1) *or* (2) *is satisfied, then $\mathcal{T}' \subset \mathcal{T}$ are neighbors with label B.*

For the proof, see 7.2. For a further discussion of neighbor torsion classes, see 7.5.

Next, we consider the lower neighbors of some torsion classes. If $\mathcal{T}$ is a torsion class, we say that $\mathcal{T}$ has *sufficiently many lower neighbors* provided any torsion class $\mathcal{N}$ with $\mathcal{N} \subset \mathcal{T}$ is contained in a lower neighbor of $\mathcal{T}$. Similarly, $\mathcal{T}$ has *sufficiently many upper neighbors* provided any torsion class $\mathcal{N}$ with $\mathcal{T} \subset \mathcal{N}$ contains an upper neighbor of $\mathcal{T}$.

2.8. Theorem. *A torsion class $\mathcal{T}$ is widely generated iff $\mathcal{T}$ has sufficiently many lower neighbors. If $\mathcal{T}$ is widely generated, say $\mathcal{T} = T(\mathcal{B})$ with $\mathcal{B}$ a semibrick, then $B \mapsto \mathcal{T} \cap {}^{\perp}B$ is a bijection between the elements of $\mathcal{B}$ and the lower neighbors of $\mathcal{T}$.*

Theorem 2.8 implies that *the map $\mathcal{B} \mapsto T(\mathcal{B})$ is a bijection between the set of semibricks and the set of widely generated torsion classes.* For the proof of theorem 2.8, see section 8.

2.9. A torsion class $\mathcal{T}$ is said to be *completely join irreducible* provided the join $\mathcal{T}_*$ of the torsion classes properly contained in $\mathcal{T}$ is still properly contained in $\mathcal{T}$ (and thus $\mathcal{T}_*$ is a lower neighbor of $\mathcal{T}$). Note that $\mathcal{T}$ is completely join irreducible iff $\mathcal{T}$ has a unique lower neighbor and has sufficiently many lower neighbors.

Corollary. *The map $B \mapsto T(B)$ provides a bijection between the isomorphism classes of the bricks and the completely join irreducible torsion classes.*

Proof. Theorem 2.3 sends a brick to the torsion class $T(B)$. According to 2.8, $\mathcal{T}(B)$ has a unique lower neighbor, namely $\mathcal{T}_* = T(B) \cap {}^{\perp}B$ and any torsion class properly contained in $\mathcal{T}$ is contained in $\mathcal{T}_*$. This shows that $T(B)$ is completely join irreducible.

Conversely, assume that $\mathcal{T}$ is a completely join irreducible torsion class. Clearly, $\mathcal{T}$ is finitely generated: Let M be any module in $\mathcal{T} \setminus \mathcal{T}_*$, where $\mathcal{T}_*$ is the join of the torsion classes properly contained in $\mathcal{T}$, then $\mathcal{T} = T(M)$. Let $B_1, \dots, B_t$ be the top bricks of M, thus $\mathcal{T} = T(M) = T(B_1, \dots, B_t)$. According to 2.8, $\mathcal{T}$ has t lower neighbors. Since $\mathcal{T}$ is completely join irreducible, we have $t = 1$, thus $\mathcal{T}$ is generated by a brick. □

2.10. Warnings. Let M be a module. If $T(M)$ is a finitely generated torsion class and B a top brick of M, the lower neighbor torsion class $T(M) \cap {}^{\perp}B$ is not necessarily finitely generated! A typical example will be presented in 2.11.

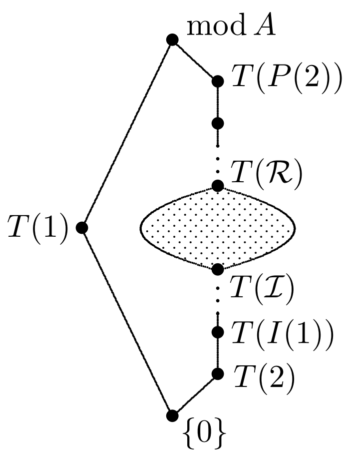
Also, we have seen in 2.8 that $T(M)$ has only finitely many lower neighbors. What about upper neighbors? If $T(M)$ is finitely generated and $\mathcal{T}''$ is an upper neighbor of $T(M)$, then trivially $\mathcal{T}''$ is again finitely generated, namely equal to $T(M \oplus N)$, where N is any module in $\mathcal{T}'' \setminus T(M)$. However, whereas a finitely generated torsion class has only finitely many lower neighbors, it may have infinitely many upper neighbors. For a typical example, we again refer to 2.11.

2.11. An example: The Kronecker algebra. For the benefit of the reader, we want to consider one example in detail, the Kronecker algebra A; this is the path algebra of the quiver with two vertices $1, 2$ and two arrows $1 \leftleftarrows 2$. (It is the usual example which everyone interested in torsion classes of artin algebras has in mind).

If x is a vertex of a quiver, the simple representation corresponding to x will also be denoted by x; and $P(x)$ and $I(x)$ will denote the projective cover or the injective envelope of x, respectively (provided they exist).

For the Kronecker algebra A over the field k, there is the well-known trisection of the indecomposable A-modules: there are the preprojective modules $\mathcal{P}$, the regular modules $\mathcal{R}$ and the preinjective modules $\mathcal{I}$. This trisection gives rise to two important torsion classes: the class $T(\mathcal{I})$ of the direct sums of preinjective modules, and the class $T(\mathcal{R})$ of the direct sums of preinjective and regular modules. *Both torsion classes $T(\mathcal{I})$ and $T(\mathcal{R})$ (as many others) are* **not** *finitely generated.* The class $T(\mathcal{I})$ is the union of a properly ascending chain of torsion classes, thus it is not finitely generated. Note that $T(\mathcal{I})$ has no lower neighbor, but infinitely many upper neighbors. The class $T(\mathcal{R})$ is widely generated, namely by the (infinite!) semibrick of the simple regular modules. Thus, $T(\mathcal{R})$ is not finitely generated. Also, $T(\mathcal{R})$ has no upper neighbor (but infinitely many lower neighbors).

In the following picture of the lattice of all torsion classes of $\operatorname{mod} A$, the sublattice of all torsion classes $\mathcal{T}$ with $T(\mathcal{I}) \subseteq \mathcal{T} \subseteq T(\mathcal{R})$ has been dotted. (As we will outline below: The dotted part is uncountable, even if k is a finite field! In contrast, outside of the dotted part, there are always just countably many torsion classes.)



If $\mathcal{X}$ is a **non-empty** set of pairwise non-isomorphic simple regular Kronecker modules (thus $\mathcal{X}$ is a non-empty semibrick), then $\mathcal{T} = T(\mathcal{X})$ is a torsion class with $T(\mathcal{I}) \subset \mathcal{T} \subseteq T(\mathcal{R})$. Taking also $T(\mathcal{I})$ itself into account, the torsion classes $\mathcal{T}$ with $T(\mathcal{I}) \subseteq \mathcal{T} \subseteq T(\mathcal{R})$ correspond bijectively to all the subsets of $\mathbb{P}^1(k)$ (by definition, $\mathbb{P}^1(k)$ is the union of the one element set $\{\infty\}$ and the set of monic irreducible polynomials with coefficients in k). Of course, the set of subsets of $\mathbb{P}^1(k)$ is uncountable.

For any torsion class $\mathcal{T}$ with $T(\mathcal{I}) \subseteq \mathcal{T} \subseteq T(\mathcal{R})$, the number of neighbors of $\mathcal{T}$ is always equal to $\max(|k|, \aleph_0)$, in particular, infinite. If R is a simple regular module, then $T(R)$ is (of course) finitely generated, however its unique lower neighbor $T(R) \cap {}^{\perp}R$ is the torsion class $T(\mathcal{I})$ which is not finitely generated. On the other hand, $T(R)$ has infinitely many upper neighbors. Namely, if R' is a simple regular Kronecker module, not isomorphic to R, then $T(R \oplus R')$ is an upper neighbor of $T(R)$, and there are infinitely many such modules R'.

We see that for $\operatorname{mod} A$, all torsion classes **but one** (namely $T(\mathcal{I})$) have sufficiently many lower neighbors. Dually, all torsion classes **but one** (namely $T(\mathcal{R})$) have sufficiently many upper neighbors. Also, every torsion class has either two or else infinitely many neighbors (those with two neighbors are the functorially finite ones, as mentioned in 12.13).

For an arbitrary artin algebra, the possible numbers of neighbors of torsion classes are not known. Note that for some algebras (for example for connected wild hereditary algebras and for tubular algebras) there do exists torsion classes without any neighbor.

3. Torsional brick chain filtrations.

In order to strengthen Theorem 1.2, we need an additional notion.

3.1. A submodule U of a module M is said to be *torsional* provided U belongs to $T(M)$. A filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m$ will be said to be *torsional* provided M_{i-1} is a torsional submodule of M_i, for all $1 \le i \le m$.

If $(M_i)_i$ is a torsional filtration of M, then M_{i-1} belongs to $T(M_i)$, for all $1 \le i \le t$, thus we have the inclusion chain $0 = T(M_0) \subseteq T(M_1) \subseteq \cdots \subseteq T(M_m) = T(M)$, and therefore all the submodules M_i are torsional submodules of M. If a brick chain filtration $(M_i)_i$, say of type $(B_1, \dots, B_m)$ is torsional, then all the bricks B_i belong to $T(M)$, since B_i is a factor module of M_i and M_i belongs to $T(M)$.

Warning. *A brick chain filtration $(M_i)_i$ of a module M, with all M_i being torsional submodules of M, is not necessarily a torsional filtration!* It is easy to exhibit modules M with a filtration $(M_i)_i$ which is not torsional, whereas all M_i are torsional submodules of M: Consider for example the radical square zero algebra A with two simple modules $1, 2$, an arrow $1 \leftarrow 2$ and a loop at 1, and take the module $M = I(1)$ of length three. Since $T(M) = \operatorname{mod} A$, all submodules of M are torsional. Let $0 \subset M_1 \subset M_2 \subset M$ be the composition series with M_2/M_1 isomorphic to 2. The submodule $M_1 = 1$ is not a torsional submodule of M_2. (It is slightly more difficult to construct such a filtration $(M_i)_i$ which is in addition a brick chain filtration; an example will be given in 9.6 (4).)

3.2. Theorem. *Any module has at least one, but only finitely many torsional brick chain filtrations. If M has length m, the number of torsional brick chain filtrations of M is bounded by $m!$.*

The proof will be given in section 9. As we will see, the torsional brick chain filtrations of a module M can be constructed easily by induction: Let B be a top brick of M. Then M has a proper submodule M' which belongs to $T(M) \cap {}^{\perp}B$, such that M/M' belongs to $\mathcal{E}(B)$. Since M' is a proper submodule of M, by induction there is a torsional brick chain filtration of M', say $0 = M_0 \subset M_1 \subset \cdots \subset M_{m-1} = M'$. Let $M_m = M$. Then $(M_i)_{0 \le i \le m}$ is a torsional brick chain filtration of M.

Questions. Theorem 3.2 asserts that any module M has only finitely many torsional brick chain filtrations. Usually, M has plenty additional brick chain filtrations (see, for example, 10.2). *Are there modules with infinitely many brick chain filtrations?* And: *Is there a module M of length m with more than $m!$ brick chain filtrations?*

3.3. If $(M_i)_i$ is a torsional brick chain filtration of type $(B_1, \dots, B_m)$, then by definition all the bricks B_i belong to $T(M)$. The last brick B_m is a factor module of M, but *the remaining bricks B_i do not have to be factor modules of M.* Here is a typical example: Let M be serial with composition factors going up: $1, 2, 2, 1, 2$, with torsional brick chain filtration $0 \subset M_1 \subset M$, where M_1 is of length three; here, M_1 is not generated by M.

3.4. Given a direct sum $X = M \oplus N$, any brick chain filtration $(X_i)_i$ of X gives rise to a filtration of M which is, after deleting repetitions, a brick chain filtration of M, we call it the *induced* one (see 10.4 (a)) We will show: *Any brick chain filtration is induced from a torsional brick chain filtration,* see 10.4 (b).

4. Some preliminaries.

4.1. Lemma. *Let M' be a non-zero module in $T(M)$. Then* $\operatorname{Hom}(M, M') \neq 0$.

Proof: M' has a filtration $0 = M'_0 \subset M'_1 \subset \cdots \subset M_m = M$, where all the factors M_i/M_{i-1} are non-zero factor modules of M. Since M'_1 it is a factor module of M, we get a non-zero homomorphism $M \to M'_1 \to M'$. $\square$

4.2. Examples of non-isomorphic bricks B', B with $B' \in T(B)$. According to Lemma 4.1, $\operatorname{Hom}(B, B') \neq 0$. (On the other hand, we will see in 6.3 that $\operatorname{Hom}(B', B) = 0$.) We sometimes will specify modules by a display of the composition factors. For the following examples, we may deal with a quiver with two vertices, labeled 1 and 2, with an arrow $1 \leftarrow 2$, and a loop at 1. The display $\begin{smallmatrix}2\\1\end{smallmatrix}$ stands for a serial module of length two with socle 1 and top 2, and so on . . . .

(1) Let $B = \begin{smallmatrix}2\\1\end{smallmatrix}$ and $B' = 2$. There is an epimorphism $B \to B'$. (Or, if we want that B, B' have the same support: Let $B = \begin{smallmatrix}2\\1\\1\end{smallmatrix}$, and $B' = \begin{smallmatrix}2\\1\end{smallmatrix}$.)

(2) Let $B = \begin{smallmatrix}2\\1\end{smallmatrix}$, and $B' = \begin{smallmatrix}2\\2\\1\end{smallmatrix}$. There is a monomorphism $B \to B'$.

(3) Let $B = \begin{smallmatrix}2\\1\\1\end{smallmatrix}$, and $B' = \begin{smallmatrix}2\\2\\1\end{smallmatrix}$. There is a non-zero map $B \to B'$, neither epi nor mono.

The bricks mentioned here are part of a brick chain of the form $\big(2,\ \begin{smallmatrix}2\\2\\1\end{smallmatrix},\ \begin{smallmatrix}2\\1\end{smallmatrix},\ \begin{smallmatrix}2\\1\\1\end{smallmatrix},\ 1\big)$.

4.3. Lemma. *A non-zero module is a brick iff it has no non-zero proper torsional submodules.*

Proof. Let M be a module. If M is not a brick, there is an endomorphism f of M such that $f(M)$ is a non-zero proper submodule. Since $f(M)$ belongs to $T(M)$, we see that $f(U)$ is a torsional submodule of M.

Conversely, let U be a non-zero proper submodule which is torsional. Since U belongs to $T(M)$, there is a non-zero submodule U' of U which is a factor module of M. We get a non-zero and not invertible endomorphism $M \to U' \subseteq U \subset M$, thus M is not a brick. $\square$

5. The endotop and the iterated endotop of a module.

We are going to show the surjectivity assertion of Theorem 2.3. We need the notion of the endotop $\operatorname{et} M$ of a module M.

5.1. Endotop. Denote by $E = \operatorname{End}(M)$ the endomorphism ring of M (operating on the left of M), and $\operatorname{rad} E$ its radical. Then $(\operatorname{rad} E)M$ is a submodule of M and we define $\operatorname{et} M = M/(\operatorname{rad} E)M$, and call it the *endotop* of M; by definition, the endotop of M is a factor module of M.

One may define the endotop $\operatorname{et} M$ also as follows: $\operatorname{et} M = M/M'$, where M' is the sum of the images of the nilpotent endomorphisms of M (in this way, one avoids the question whether one has to look at the ring $\operatorname{End}(M)$ or its opposite, as well as the related one, whether functions are written left or write of the argument).

5.2. Examples. (1) *If M is an indecomposable module,* $\operatorname{et} M$ *may be decomposable.* For example, let A be a local algebra with radical-square-zero and ${}_AA$ of length $t \geq 2$. If M is the indecomposable injective module, then $\operatorname{et} M$ is the direct sum of $t-1$ copies of the simple module.

(2) Let A be given by the quiver Q with one vertex and two loops and with relations all paths of length 3 (thus A is a local algebra of dimension 7). There is a serial module M of length 3 with $\operatorname{rad} M$ not isomorphic to $M/\operatorname{soc} M$. Then $\operatorname{et} M = M/\operatorname{soc} M$, thus $\operatorname{et} M$ *is indecomposable of length two, and not a brick,* in particular, $\operatorname{et}(\operatorname{et} M)$ is a proper factor module of $\operatorname{et} M$. This leads us below to consider not only et, but the iterations et^i, see 5.4. (Instead of A, we may consider a proper factor algebra A' of A, namely the subring $A' = k + J$ of the ring of all 3×3-matrices with coefficients in k, where J is the set of nilpotent upper triangular matrices; let $M = k^3$ be the A'-module of column vectors.)

(3) If A is the Kronecker algebra, and M is a regular Kronecker module, then $\operatorname{et} M$ is just the regular top of M.

5.3. Proposition. *Let M be a module. Then M belongs to $T(\operatorname{et} M)$, therefore $T(M) = T(\operatorname{et} M)$. The kernel of the canonical map $M \to \operatorname{et} M$ is torsional.*

Proof. Let $E = \operatorname{End} M$ and let $f_1, \dots, f_t$ be a basis of $\operatorname{rad} E$. Let $(\operatorname{rad} E)^m = 0$. The image of $g = (f_i)\colon \bigoplus_i M \to M$ is $(\operatorname{rad} E)M = \operatorname{rad}_E M = M_1$ and $\operatorname{et} M = M/M_1$. Let $M_{j+1} = g(M_j)$ for all $j \geq 0$ with $M_0 = M$. Then $M_m = 0$. By induction, all modules M_j/M_{j+1} are generated by $\operatorname{et} M$. This shows that $T(M) \subseteq T(\operatorname{et} M)$. On the other hand, we also have $T(\operatorname{et} M) \subseteq T(M)$, since $\operatorname{et} M$ is a factor module of M. Thus M and $\operatorname{et} M$ generate the same torsion-class. The kernel M' of the canonical map $M \to \operatorname{et} M$ is by definition the image of the map g, thus generated by M, thus M' belongs to $T(M)$. $\square$

5.4. We iterate the construction et and get epimorphisms

$$M \to \operatorname{et} M \to \operatorname{et}^2 M \to \cdots .$$

Since M is of finite length, the sequence stabilizes eventually. there is a non-negative integer a with $\operatorname{et}^a M = \operatorname{et}^{a+1} M$. In this way, we get the *iterated endotop* $\operatorname{et}^\infty M = \operatorname{et}^a M$ (and we have $\operatorname{et}(\operatorname{et}^\infty M) = \operatorname{et}^\infty M$).

Example. Let A be a suitable artin algebra with two simple modules 1 and 2. For $n \geq 0$, let $M[n]$ be a serial module of length $n+2$, with composition factors going up: $(1, \dots, 1, 2, 1)$ (starting with n factors of the form 1). Then, for $0 \leq i \leq n$, we have $\operatorname{et}^i M[n] = M[n-i]$. For $0 \leq i < n$, the module $M[i]$ is not a brick, but $\operatorname{et}^n M[n] = M[0]$ is a brick of the form $\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}$.

5.5. Proposition. *Let M be a module. The iterated endotop $X = \operatorname{et}^\infty M$ is the direct sum of modules which belong to a semibrick $\mathcal{B}(M)$ and $T(M) = T(X) = T(\mathcal{B}(M))$.* The elements of $\mathcal{B}(M)$ are called the *top bricks* of M. *The kernel of the canonical map $M \to \operatorname{et}^\infty M$ is a torsional submodule of M.*

Proof. It is obvious that the iterated endotop of a module is always the direct sum of modules which belong to a semibrick, since the sequence $M \to \operatorname{et} M \to (\operatorname{et})^2 M \to \cdots$ stabilizes precisely when $\operatorname{End}(\operatorname{et}^a M)$ is semisimple. Proposition 5.3 yields that the torsion classes $T(\operatorname{et}^i M)$ are equal, for all $i \geq 0$.

The kernel K of the canonical map $M \to \operatorname{et}^\infty M$ has a filtration whose factors are the kernels K_i of the canonical maps $\operatorname{et}^i M \to \operatorname{et}^{i+1} M$, for all $i \geq 0$. According to 5.3, all modules K_i belong to $T(M)$, thus K belongs to $T(M)$. □

5.6. Corollary. *A torsion class $\mathcal{T}$ is finitely generated iff there is a finite semibrick $\mathcal{B}$ with $\mathcal{T} = T(\mathcal{B})$.* □

Corollary 5.6 shows that the map $\mathcal{B} \mapsto T(\mathcal{B})$ from the set of finite semibricks $\mathcal{B}$ to the set of finitely generated torsion classes is surjective. This is part of Theorem 2.3.

5.7. Examples. (1) *Let M be an indecomposable module. A top brick of M may occur in* $\operatorname{et}^\infty M$ *with arbitrarily large multiplicity,* see 5.2 (1). In particular, $\operatorname{et}^\infty M$ may not be indecomposable!

(2) *The number of top bricks of an indecomposable module M may be arbitrarily large:* Consider the $(t-1)$-subspace quiver, with sink 1 and sources $2, 3, \dots, t$, and add a loop at the sink 1. For the corresponding radical-square-zero algebra, the indecomposable injective module $M = I(1)$ satisfies $\operatorname{et}^\infty M = \operatorname{et} M = 1 \oplus 2 \oplus \cdots \oplus t$, thus M has t top bricks.

5.8. Warning. Let M be a module and $(M_i)_i$ a brick chain filtration of M of brick type $(B_1, \dots, B_m)$. Then, of course, *B_m is a factor module of M, but B_m is not necessarily a top brick of M.* To have an example in mind, just take any non-simple brick M. Then M has just one top brick, namely M, whereas given any simple module S in the top of M, there is a brick chain filtration of M of type $(B_1, \dots, B_m)$ such that $B_m = S$, see 10.2.

6. The essential feature: If B is a brick, $({}^\perp B) \rceil\!\!\lfloor \mathcal{E}(B)$ is a torsion class.

Given module classes $\mathcal{X}$ and $\mathcal{Y}$, we write $\mathcal{X} \rceil\!\!\lfloor \mathcal{Y}$ for the class of all modules M which have a submodule M' in $\mathcal{X}$ such that M/M' belongs to $\mathcal{Y}$.

We are going to show: If B is a brick, then

$$(*) \qquad T({}^\perp B, B) = ({}^\perp B) \rceil\!\!\lfloor \mathcal{E}(B).$$

This describes very nicely the torsion class $T({}^\perp B, B)$. Actually, there is a corresponding description for some other torsion classes $\mathcal{T} \subseteq T({}^\perp B, B)$, namely the torsion classes $\mathcal{T}$ with $\mathcal{T} = T(\mathcal{T} \cap {}^\perp B, B)$, see the following Proposition. Note that the assumption $\mathcal{T} = T(\mathcal{T} \cap {}^\perp B, B)$ means, in particular, that B belongs to $\mathcal{T}$.

6.1. Proposition. *If B be a brick and $\mathcal{T}$ a torsion class with $\mathcal{T} = T(\mathcal{T} \cap {}^\perp B, B)$, then*

$$\mathcal{T} = (\mathcal{T} \cap {}^\perp B) \rceil\!\!\lfloor \mathcal{E}(B).$$

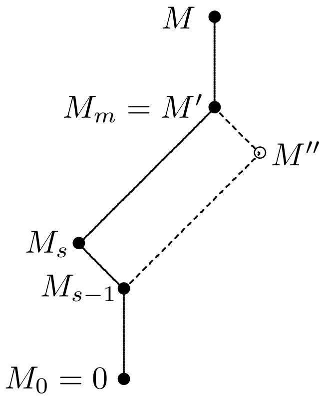
Let us add: If $0 \to M' \to M \to M/M' \to 0$ is an exact sequence with M' in $\mathcal{T}' = \mathcal{T} \cap {}^\perp B$ and $M/M' \in \mathcal{E}(B)$, then M' is just the torsion submodule of M with respect to the torsion class $\mathcal{T}'$, since $\operatorname{Hom}(\mathcal{T}', \mathcal{E}(B)) = 0$.

The description $(*)$ for $\mathcal{T} = T({}^{\perp}B, B)$, where B is an arbitrary brick, is a special case of 6.1. Namely, for this torsion class $\mathcal{T}$, we have ${}^{\perp}B \subseteq \mathcal{T}$, thus $\mathcal{T} \cap {}^{\perp}B = {}^{\perp}B$, so we have $T(\mathcal{T} \cap {}^{\perp}B, B) = T({}^{\perp}B, B) = \mathcal{T}$, thus the assumption is satisfied. This means that we can conclude that $\mathcal{T} = (\mathcal{T} \cap {}^{\perp}B) \rceil \mathcal{E}(B) = ({}^{\perp}B) \rceil \mathcal{E}(B)$, as mentioned in $(*)$.

Proof of Proposition. Let M be a module in $\mathcal{T}$. Let M' be a submodule of M which also belongs to $\mathcal{T}$ with $M/M' \in \mathcal{E}(B)$, and minimal with these two properties. We claim that M' belongs to ${}^{\perp}B$, thus to $\mathcal{T} \cap {}^{\perp}B$.

Thus, assume for the contrary that there is a non-zero map $f \colon M' \to B$. Since M' belongs to $\mathcal{T}$, there is a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M'$ such that all factors $F_i = M_i/M_{i-1}$ are factor modules of B or belong to $\mathcal{T} \cap {}^{\perp}B$ (in particular, all F_i belong to $\mathcal{T}$). Let s be minimal such that $f|M_s$ is non-zero. Thus, f vanishes on M_{s-1} and induces a map $\overline{f} \colon M'/M_{s-1}$ with non-zero restriction to $F_s = M_s/M_{s-1}$. Let us denote by $u \colon F_s \to M'/M_{s-1}$ the inclusion map. Thus, the composition $\overline{f} \cdot u \colon F_s \to B$ is a non-zero map.

Now F_s is a factor module of some B or belongs to ${}^{\perp}B$. Since there is the non-zero map $\overline{f} \cdot u \colon F_s \to B$, we see that F_s is a factor module of B. Also, since B is a brick, there is no non-zero map from a proper factor module of B to B, thus we see that $F_s = B$ and that the composition $\overline{f} \cdot u \colon B = M_s/M_{s-1} \subseteq M'/M_{s-1} \to B$ is an isomorphism. This shows that u is a split monomorphism. It follows that there is a submodule M'' of M' with $M_{s-1} \subseteq M''$, such that $M_s \cap M'' = M_{s-1}$ and $M_s + M'' = M'$.



It follows that $M'/M'' \simeq M_s/M_{s-1} = B$, and that $M''/M_{s-1} \simeq M'/M_s$. Since M/M' and M'/M'' belong to $\mathcal{E}(B)$, also M/M'' belongs to $\mathcal{E}(B)$. On the other hand, $M''/M_{s-1} \simeq M'/M_s$ has a filtration by factors isomorphic to F_i with $s+1 \leq i \leq t$ and M_{s-1} has the filtration with factors F_i where $1 \leq i \leq s-1$. Since all the factors F_i belong to $\mathcal{T}$, also M'' belongs to $\mathcal{T}$.

Altogether we see that M'' is a submodule of M which belongs to $\mathcal{T}$ and such that $M/M' \in \mathcal{E}(B)$. Since M'' is a proper submodule of M', this contradicts the minimality of M'. It follows that M' belongs to ${}^{\perp}B$.

Since M' belongs to $\mathcal{T} \cap {}^{\perp}B$, and M/M' to $\mathcal{E}(B)$, we see that M' is the torsion submodule of M with respect to the torsion class $\mathcal{T} \cap {}^{\perp}B$. The exact sequence $0 \to M' \to M \to M/M' \to 0$ for an arbitrary module M in $\mathcal{T}$ shows that $\mathcal{T} \subseteq (\mathcal{T} \cap {}^{\perp}B) \rceil \mathcal{E}(B)$. On the other hand, we have $\mathcal{T} \cap {}^{\perp}B \subseteq \mathcal{T}$, and, since $B \in \mathcal{T}$, also $\mathcal{E}(B) \subseteq \mathcal{T}$: This shows the reverse inclusion $(\mathcal{T} \cap {}^{\perp}B) \rceil \mathcal{E}(B) \subseteq \mathcal{T}$, therefore $\mathcal{T} = (\mathcal{T} \cap {}^{\perp}B) \rceil \mathcal{E}(B)$. □

Remarks. If B is a brick which belongs to a torsion class $\mathcal{T}$, then it is obvious that $T(\mathcal{T}\cap {}^{\perp}B, B) \subseteq \mathcal{T}$, but usually, this inclusion will be proper. see the remark 7.7.

Let us stress that 6.1 can be rephrased as follows: The pairs $(\mathcal{T}, B)$, where B is a brick, $\mathcal{T}$ a torsion class, and $\mathcal{T} = T(\mathcal{T}\cap {}^{\perp}B, B)$, are just the pairs $(\mathcal{T}, B)$, given by a torsion class $\mathcal{T}$ with a lower neighbor $\mathcal{T}'$ such that B is the label of the inclusion $\mathcal{T}' \subset \mathcal{T}$.

6.2. Corollary. *Let B be a brick. Let M be a module in $T({}^{\perp}B, B)$. Then any non-zero map $M \to B$ is surjective.*

Proof. Let M be a module in $T(B, {}^{\perp}B)$ and $f\colon M \to B$ a non-zero map. The existence of f shows that M does not belong to ${}^{\perp}B$. We have mentioned above that we can use Proposition 6.1 for the torsion class $\mathcal{T} = T({}^{\perp}B, B)$. Thus, there is a submodule M' of M which belongs to ${}^{\perp}B$ such that M/M' belongs to $\mathcal{E}(B)$. Since f vanishes on M', we get an induced map $\overline{f}\colon M/M' \to B$, and $\overline{f}$ is non-zero. However, any non-zero map in $\mathcal{E}(B)$ with target B is an epimorphism. Since $\overline{f}$ is surjective, also f is surjective. $\square$

6.3. Corollary. *Let B, B' be non-isomorphic bricks, and assume that B' is in $T(B)$. Then* $\operatorname{Hom}(B', B) = 0$*, thus $B' \in T(B) \cap {}^{\perp}B$.*

Proof. Assume there is a non-zero map $f\colon B' \to B$. According to 6.2, the map f is surjective. Since B' belongs to $T(B)$, we know from 4.1 that there is a non-zero map $g\colon B \to B'$. Since f is surjective, the composition $gf\colon B' \to B \to B'$ is non-zero. Since B' is a brick, this means that gf is an isomorphism. Thus f is a (split) monomorphism. Altogether we see that f is bijective, thus B and B' are isomorphic. $\square$

7. Neighbors.

7.1. Lemma. *Let $\mathcal{T}' \subset \mathcal{T}$ be torsion classes. Any module M in $\mathcal{T}\setminus\mathcal{T}'$ of minimal length is a brick and satisfies $\mathcal{T}' \subseteq {}^{\perp}M$.*

Proof. Let M be a module in $\mathcal{T}\setminus\mathcal{T}'$ of minimal length. We form $X = \mathrm{et}^{\infty} M$. According to 5.5, we have $T(X) = T(M)$, thus also X belongs to $\mathcal{T}\setminus\mathcal{T}'$. There is an indecomposable direct summand X' of X which belongs to $\mathcal{T}\setminus\mathcal{T}'$ and, as we know, X' is a brick (one of the top bricks of M). On the other hand, there are epimorphisms $M \to X \to X'$, thus $|X'| \le |X|$. Since we assume that M is of minimal length, we see that $M = X'$ is a brick.

In order to see that $\mathcal{T}' \subseteq {}^{\perp}M$, consider any homomorphism $f\colon M' \to M$, with $M' \in \mathcal{T}'$. Now $f(M')$ belongs to $\mathcal{T}'$, thus $M/f(M')$ does not belong to $\mathcal{T}'$. Since $M/f(M')$ is a module in $\mathcal{T}\setminus\mathcal{T}'$, the minimality of M shows that $f(M') = 0$. $\square$

7.2. Proof of 2.7. First, we assume that $\mathcal{T}' \subset \mathcal{T}$ are neighbors. According to 7.1, there is a brick B in $\mathcal{T}$ such that $\mathcal{T}' \subseteq {}^{\perp}B$. Thus, let B be any brick in $\mathcal{T}$ with $\mathcal{T}' \subseteq {}^{\perp}B$. Then we have

$$\mathcal{T}' \subseteq \mathcal{T} \cap {}^{\perp}B \subset \mathcal{T}$$

(the proper inclusion is due to the fact that B belongs to $\mathcal{T}$, but not to ${}^{\perp}B$). Since $\mathcal{T}' \subset \mathcal{T}$ are neighbors, we see that $\mathcal{T}' = \mathcal{T}\cap {}^{\perp}B$. Since B belongs to $\mathcal{T}$ and not to $\mathcal{T}'$, we also have $\mathcal{T} = T(\mathcal{T}', B)$. This shows that (1) is satisfied.

Next, we show (2): any module M in $\mathcal{T}$ has a submodule M' in $\mathcal{T}'$ such that M/M' belongs to $\mathcal{E}(B)$. It follows from $\mathcal{T}' = \mathcal{T}\cap {}^{\perp}B$ and $\mathcal{T} = T(\mathcal{T}', B)$ that $\mathcal{T} = T(\mathcal{T}\cap {}^{\perp}B, B)$, therefore we can use 6.1: For any module $M \in \mathcal{T}$, there is a submodule M' of M which belongs to $\mathcal{T} \cap {}^{\perp}B$ with M/M' in $\mathcal{E}(B)$.

Let us show that B is the unique brick in $\mathcal{T}$ with $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B$. Thus, let C be any brick in $\mathcal{T}$ with $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}C$. As we have seen, this implies that any module M in $\mathcal{T}$ has a submodule M' in $\mathcal{T}'$ with $M/M' \in \mathcal{E}(C)$. Now C cannot belong to $\mathcal{T}'$, since otherwise we would have $\mathcal{T} \subseteq \mathcal{T}'$. The module C has a submodule C' in $\mathcal{T}'$ with C/C' in $\mathcal{E}(B)$. Since C does not belong to $\mathcal{T}'$, we have $C/C' \neq 0$, therefore C maps onto B. On the other hand, B has a submodule B' in $\mathcal{T}'$ with $B/B' \in \mathcal{E}(C)$. Since B is not in $\mathcal{T}'$, we see that B/B' is non-zero, thus B maps onto C. This shows that $C = B$.

Now let $\mathcal{T}' \subset \mathcal{T}$ be any inclusion of torsion classes and let B be a brick.

First, let us assume that condition (2) is satisfied, thus any module M in $\mathcal{T}$ has a submodule M' in $\mathcal{T}'$ such that M/M' belongs to $\mathcal{E}(B)$. Then clearly $\mathcal{T} \subseteq T(\mathcal{T}', B)$. In order to show that $\mathcal{T}' \subset \mathcal{T}$ are neighbors, let $\mathcal{T}''$ be a torsion class with $\mathcal{T}' \subset \mathcal{T}'' \subseteq \mathcal{T}$. We claim that B belongs to $\mathcal{T}''$. Let M be a module in $\mathcal{T}'' \setminus \mathcal{T}'$. According to (2), there is a submodule M' of M in $\mathcal{T}'$ such that M/M' belongs to $\mathcal{E}(B)$. Since M does not belong to $\mathcal{T}$, we see that $M/M' \neq 0$. By definition, M/M' belongs to $\mathcal{E}(B)$, thus it has a factor module isomorphic to B. Since M belongs to $\mathcal{T}''$, also its factor module B belongs to $\mathcal{T}''$. Thus we see that $T(\mathcal{T}', B) \subseteq \mathcal{T}''$. Altogether, we have $T(\mathcal{T}', B) \subseteq \mathcal{T}'' \subseteq \mathcal{T} \subseteq T(\mathcal{T}', B)$, therefore $\mathcal{T}'' = \mathcal{T}$.

We also claim that $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B$. The inclusion $\mathcal{T}' \subseteq {}^{\perp}B$ follows from Lemma 7.1, since B is a module of minimal length in $\mathcal{T} \setminus \mathcal{T}'$. (Namely, if X is any module in $\mathcal{T} \setminus \mathcal{T}'$, then it has a proper submodule $X' \in \mathcal{T}'$ such that X/X' belongs to $\mathcal{E}(B)$. But then B is a factor module of X/X', thus of X.) Since $\mathcal{T}' \subseteq {}^{\perp}B$, we have $\mathcal{T}' \subseteq \mathcal{T} \cap {}^{\perp}B \subset \mathcal{T}$. But $\mathcal{T}' \subset \mathcal{T}$ are neighbors, thus $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B$. This shows that the label of $\mathcal{T}' \subset \mathcal{T}$ is B.

Second, let us assume that (1) is satisfied: $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B$ and $\mathcal{T} = T(\mathcal{T}', B)$. Then we have $\mathcal{T} = T(\mathcal{T} \cap {}^{\perp}B, B)$, thus we can apply 6.1 and see that also condition (2) is satisfied. As we have shown, this implies that $\mathcal{T}' \subset \mathcal{T}$ are neighbors with label B. □

7.3. Proposition. *Let B be a brick and $\mathcal{X} \subseteq {}^{\perp}B$. Then $T(\mathcal{X}, B) \cap {}^{\perp}B \subset T(\mathcal{X}, B)$, and these are neighbors with label B.*

Proof. We write $T(\mathcal{X}, B)_B = T(\mathcal{X}, B) \cap {}^{\perp}B$. Now $T(\mathcal{X}, B)_B \subseteq T(\mathcal{X}, B)$, and this inclusion is proper since B does not belong to ${}^{\perp}B$. Assume that there is a torsion class $\mathcal{T}$ such that $T(\mathcal{X}, B)_B \subset \mathcal{T} \subseteq T(\mathcal{X}, B)$. Since $T(\mathcal{X}, B)_B \subset \mathcal{T}$, there is a module $M \in \mathcal{T}$ which does not belong to ${}^{\perp}B$. Thus, there is a non-zero map $f\colon M \to B$. Since M belongs to $T({}^{\perp}B, B)$, we can apply Corollary 6.2. We see that f is surjective, thus B belongs to $\mathcal{T}$. Of course, also $\mathcal{X} \subseteq \mathcal{T}$. Therefore $T(\mathcal{X}, B) \subseteq \mathcal{T}$. This shows that $T(\mathcal{X}, B) = \mathcal{T}$. Thus, $T(\mathcal{X}, B)_B \subset T(\mathcal{X}, B)$ are neighbors. By definition, the label is B. □

7.4. Corollary. *Let $\mathcal{T}' \subset \mathcal{T}$ be torsion classes. Then there are bricks B in $\mathcal{T}$ such that $\mathcal{T}' \subseteq {}^{\perp}B$. If B is a brick with $\mathcal{T}' \subseteq {}^{\perp}B$, let $\mathcal{N} = T(\mathcal{T}', B)$ and $\mathcal{N}' = \mathcal{N} \cap {}^{\perp}B$, then we have*

$$\mathcal{T}' \subseteq \mathcal{N}' \subset \mathcal{N} \subseteq \mathcal{T}$$

and the torsion classes $\mathcal{N}' \subset \mathcal{N}$ are neighbors with label B.

Proof: It is trivial that $\mathcal{T}' \subseteq \mathcal{N}' \subseteq \mathcal{N} \subseteq \mathcal{T}$. We use 7.3 with $\mathcal{X} = \mathcal{T}'$ in order to see that $\mathcal{N}' \subset \mathcal{N}$ are neighbor torsion classes with label B. $\square$

7.5. Remark. For a torsion class $\mathcal{T}$, there is the corresponding *torsionfree class* $\mathcal{T}^\perp$ (by definition, for any class $\mathcal{X}$ of modules, $\mathcal{X}^\perp$ is defined as the class of all modules Y with $\mathrm{Hom}(X, Y) = 0$ for all $X \in \mathcal{X}$); the pair $(\mathcal{T}, \mathcal{T}^\perp)$ is called a *torsion pair*. Subcategory of the form $\mathcal{E}(B)$ with B a brick will be said to be *homogeneous categories* (of brick type B).

Proposition. *Torsion classes $\mathcal{T}' \subseteq \mathcal{T}$ are neighbors iff $\mathcal{T} \cap (\mathcal{T}')^\perp$ is a homogeneous category. And in this case, the brick type of $\mathcal{T} \cap (\mathcal{T}')^\perp$ is the label of the inclusion $\mathcal{T}' \subset \mathcal{T}$.*

The proposition shows the symmetry between torsion classes and torsionfree classes when dealing with labels: In particular, the labeling of neighbor torsion classes yields a corresponding labeling of neighbor torsionfree classes.

Proof of proposition. Let $\mathcal{T}' \subseteq \mathcal{T}$ be torsion classes, and let $\mathcal{F}' = (\mathcal{T}')^\perp$. First, let us assume that $\mathcal{T} \cap \mathcal{F}'$ is homogeneous, say of type B, with B a brick. Since B belongs to $\mathcal{F}'$, we see that B cannot belong to $\mathcal{T}'$, thus we have $\mathcal{T}' \subset \mathcal{T}$.

In order to show that $\mathcal{T}' \subset \mathcal{T}$ are neighbors, let $\mathcal{T}''$ be a torsion class with $\mathcal{T}' \subset \mathcal{T}'' \subseteq \mathcal{T}$. We claim that B belongs to $\mathcal{T}''$. Let M be a module in $\mathcal{T}'' \setminus \mathcal{T}'$ and let M' be its torsion module with respect to the torsion class $\mathcal{T}'$. Since M does not belong to $\mathcal{T}'$, we see that $M/M' \neq 0$. By definition, M/M' belongs to $\mathcal{F}'$. Since M belongs to $\mathcal{T}$, also its factor module M/M' belongs to $\mathcal{T}$, therefore M/M' belongs to $\mathcal{T} \cap \mathcal{F}' = \mathcal{E}(B)$. As a non-zero module in $\mathcal{E}(B)$, the module M/M' has B as a factor module, therefore B belongs to $\mathcal{T}''$.

Now, let X be any module in $\mathcal{T}$. Let X' be the torsion submodule of X with respect to $\mathcal{T}'$, thus X/X' belongs to $\mathcal{F}'$. Also, X/X' is a factor module of $X \in \mathcal{T}$, thus X/X' is in $\mathcal{T}$, therefore in $\mathcal{T} \cap \mathcal{F}' = \mathcal{E}(B)$. This shows that X has a filtration with modules in $\mathcal{T}'$ and in $\mathcal{E}(B)$, thus X belongs to $\mathcal{T}''$. This shows that $\mathcal{T}'' = \mathcal{T}$.

Conversely, let $\mathcal{T}' \subset \mathcal{T}$ be neighbors, say with label B. Then B belongs to $\mathcal{T}$. Also, $\mathcal{T}' = \mathcal{T} \cap {}^\perp B \subseteq {}^\perp B$ shows that $B \in (\mathcal{T}')^\perp = \mathcal{F}'$. Since $\mathcal{T} \cap \mathcal{F}'$ is closed under extensions, $\mathcal{E}(B) \subseteq \mathcal{T} \cap \mathcal{F}'$. It remains to show that $\mathcal{T} \cap \mathcal{F}' \subseteq \mathcal{E}(B)$.

First, we claim that $\mathcal{T}' = \mathcal{T} \cap {}^\perp B$ and that $T(\mathcal{T}', B) = \mathcal{T}$. Namely, since $\mathcal{T}' \subset \mathcal{T}$ are neighbors and $\mathcal{T}' \subseteq \mathcal{T} \cap {}^\perp B \subset \mathcal{T}$, we have $\mathcal{T}' = \mathcal{T} \cap {}^\perp B$. On the other hand, since $\mathcal{T}' \subset \mathcal{T}$ and $B \in \mathcal{T}$, we have $T(\mathcal{T}', B) \subseteq \mathcal{T}$. Since B does not belong to $\mathcal{T}'$, we have $\mathcal{T}' \subset T(\mathcal{T}', B)$. Altogether, we have $\mathcal{T}' \subset T(\mathcal{T}', B) \subseteq \mathcal{T}$. Thus $T(\mathcal{T}', B) = \mathcal{T}$, since $\mathcal{T}' \subset \mathcal{T}$ are neighbors.

Since $\mathcal{T} = T(\mathcal{T} \cap {}^\perp, B)$ we can use proposition 6.1. It asserts: If M belongs to $\mathcal{T}$ and M' is the torsion submodule of M with respect to $\mathcal{T} \cap {}^\perp B$, then M/M' belongs to $\mathcal{E}(B)$. Now assume that M belongs to $\mathcal{T} \cap \mathcal{F}'$. Since $M \in \mathcal{F}'$, the torsion submodule M' of M with respect to $\mathcal{T} \cap {}^\perp B = \mathcal{T}'$ is zero. Thus, $M = M/M'$ belongs to $\mathcal{E}(B)$. $\square$

7.6. The brick chains explained in terms of neighbor torsion classes. Let $\mathcal{T}' \subset \mathcal{T}$ be neighbors with label B. Then we have on the one hand: B belongs to $\mathcal{T}$ and not to $\mathcal{T}'$. On the other hand, for every module M in $\mathcal{T}'$, in particular for the bricks in $\mathcal{T}'$, we have $\mathrm{Hom}(M, B) = 0$. Thus we obtain in this way the Hom-condition which is used in the definition of a brick-chain: If $\mathcal{T}_1 \subset \mathcal{T}_2 \subseteq \mathcal{T}_3 \subset \mathcal{T}_4$ is a chain of torsion classes with $\mathcal{T}_1 \subset \mathcal{T}_2$ as well as $\mathcal{T}_3 \subset \mathcal{T}_4$ being neighbors, and B is the label for $\mathcal{T}_1 \subset \mathcal{T}_2$, whereas B' is the label for $\mathcal{T}_3 \subset \mathcal{T}_4$, then $\mathrm{Hom}(B, B') = 0$.

7.7. Remark. If $\mathcal{T}' \subset \mathcal{T}$ are neighbors with label B, we have both $T(\mathcal{T} \cap {}^{\perp}B, B) = \mathcal{T}$ and $T(\mathcal{T}' \cap {}^{\perp}B, B) = \mathcal{T}'$. In general, for arbitrary torsion classes $\mathcal{T}$ and $\mathcal{T}'$, and B a brick, there are the obvious inclusions $T(\mathcal{T} \cap {}^{\perp}B, B) \subseteq \mathcal{T}$ provided B belongs to $\mathcal{T}$, as well as $\mathcal{T}' \subseteq T(\mathcal{T}', B) \cap {}^{\perp}B$ provided $\mathcal{T}' \subseteq {}^{\perp}B$. But both inclusions are usually proper inclusions. We are obliged to the referee for pointing out the following example.

Let A be the path algebra of the $\mathbb{A}_2$-quiver $1 \leftarrow 2$ and B the indecomposable module of length two. Here, we have ${}^{\perp}B = T(2) \subset T(B)$. For $\mathcal{T} = \operatorname{mod} A$, we have $T(B) = T(\mathcal{T} \cap {}^{\perp}B, B) \subset \mathcal{T} = \operatorname{mod} A$. For $\mathcal{T}' = \{0\}$, we have $\{0\} = \mathcal{T}' \subset T(\mathcal{T}', B) \cap {}^{\perp}B = T(2)$.

In particular, given an inclusion $\mathcal{T}' \subset \mathcal{T}$ and a brick B, the conditions $\mathcal{T}' \subseteq {}^{\perp}B$ and $\mathcal{T} = T(\mathcal{T}', B)$ do not imply that $\mathcal{T}' \subset \mathcal{T}$ are neighbors.

8. Widely generated torsion classes.

We are going to prove 2.8. If $\mathcal{B}$ is a semibrick and $B \in \mathcal{B}$, we write $T(\mathcal{B})_B = T(\mathcal{B}) \cap {}^{\perp}B$.

8.1. Lemma. *Let $\mathcal{B}$ be a semibrick, and $\mathcal{T} = T(\mathcal{B})$. If the torsion class $\mathcal{T}'$ is properly contained in $\mathcal{T}$, then there is $B \in \mathcal{B}$ with $\mathcal{T}' \subseteq \mathcal{T}_B$ and such that $\mathcal{T}_B \subset \mathcal{T}$ are neighbors.*

Proof. Since $\mathcal{T}'$ is properly contained in $\mathcal{T}$, there is a brick $B \in \mathcal{B}$ which is not contained in $\mathcal{T}'$. Let $\mathcal{B}' = \mathcal{B} \setminus \{B\}$. Since $\mathcal{B}$ is a semibrick, we have $\mathcal{B}' \subseteq {}^{\perp}B$. According to 7.3, $\mathcal{T}_B \subset \mathcal{T}$ are neighbors.

Also, we claim that $\mathcal{T}' \subseteq {}^{\perp}B$. Namely, if $f\colon M \to B$ is a non-zero homomorphism with $M \in \mathcal{T}'$, then 6.2 asserts that f is surjective, thus $B \in \mathcal{T}'$, a contradiction. It follows that $\mathcal{T}' \subseteq {}^{\perp}B$, thus $\mathcal{T}' \subseteq \mathcal{T} \cap {}^{\perp}B = \mathcal{T}_B$. $\square$

8.2. Proof of Theorem 2.8. First, let $\mathcal{B}$ be a semibrick and $\mathcal{T} = T(\mathcal{B})$. According to 8.1, $\mathcal{T}$ has sufficiently many lower neighbors, namely the torsion classes $\mathcal{T}_B$ with $B \in \mathcal{B}$. Also, the map $B \mapsto \mathcal{T}_B$ from $\mathcal{B}$ to the set of lower neighbors of $\mathcal{T}$ is surjective. On the other hand, this map is injective by the unicity of the label.

Conversely, let $\mathcal{T}$ be a torsion class with sufficiently many lower neighbors. Let $\mathcal{B}$ be the set of labels of the lower neighbors. Then $\mathcal{B}$ is a subset of $\mathcal{T}$, thus $T(\mathcal{B}) \subseteq \mathcal{T}$. Let us assume that $T(\mathcal{B}) \subset \mathcal{T}$. Since $\mathcal{T}$ has sufficiently many lower neighbors, there is a lower neighbor $\mathcal{T}'$ of $\mathcal{T}$ such that $T(\mathcal{B}) \subseteq \mathcal{T}'$. Let B be the label of the inclusion $\mathcal{T}' \subset \mathcal{T}$. Then $B \in \mathcal{B}$. Now $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B \subseteq {}^{\perp}B$. Thus we have $B \in T(\mathcal{B}) \subseteq \mathcal{T}' \subseteq {}^{\perp}B$, a contradiction. $\square$

8.3. Corollary. *If $\mathcal{B}$ is a semibrick. Then $T(\mathcal{B})$ is finitely generated iff $\mathcal{B}$ is finite.*

Proof. If $\mathcal{B}$ is finite, then, of course, $T(\mathcal{B})$ is finitely generated. Conversely, assume that $T(\mathcal{B})$ is finitely generated. By definition, there is a module M with $T(\mathcal{B}) = T(M)$. According to Theorem 2.3, there is a finite semibrick $\mathcal{B}'$ with $T(M) = T(\mathcal{B}')$. According to 2.6 and 2.8, we have $\mathcal{B} = \mathcal{B}'$, thus $\mathcal{B}$ is finite. $\square$

9. Torsional brick chain filtrations.

We are going to prove Theorem 3.2.

9.1. Proposition. *Let B be a top brick of the module M. Then M has a proper submodule M' which belongs to $T(M) \cap {}^{\perp}B$, such that M/M' belongs to $\mathcal{E}(B)$.*

Proof. We want to use 6.1 for B and the torsion class $T(M)$. We have to show that $T(M) = T(T(M) \cap {}^{\perp}B, B)$. Of course, $T(T(M) \cap {}^{\perp}B, B) \subseteq T(M)$, since B is a factor module of M. For the reverse inclusion, let $\mathcal{B}'$ be the set of top bricks different from B. According to 5.5, we have $\mathcal{B}' \subseteq {}^{\perp}B$ and $T(M) = T(\mathcal{B}', B)$. As a consequence, we have $\mathcal{B}' \subseteq T(M) \cap {}^{\perp}B$. Therefore $T(M) = T(\mathcal{B}', B) \subseteq T(T(M) \cap {}^{\perp}B, B)$. $\square$

9.2. Corollary. *Let B be a top brick of the module M. Then there is a torsional brick chain filtration $(M_i)_i$ of M of some brick type $(B_1, \dots, B_m)$ with $B_m = B$.*

Proof by induction. According to 9.1, there is a proper submodule M' of M which belongs to $T(M) \cap {}^{\perp}B$, such that M/M' belongs to $\mathcal{E}(B)$. If $M' = 0$, then $(0 \subset M)$ is a brick chain filtration of M of type (B), and this filtration is of course torsional. Otherwise, by induction, M' has a torsional brick chain filtration $(M_i)_i$ say of type $(B_1, \dots, B_{m-1})$. We put $M_{m-1} = M'$ and $M_m = M$. Then the filtration $(M_i)_{0 \le i \le m}$ is the required torsional brick chain filtration. $\square$

Thus any module has at least one torsional brick chain filtration.

9.3. Lemma. *Let M be a module, B a brick. Assume that M has a proper torsional submodule Y in $T(M) \cap {}^{\perp}B$ such that M/Y belongs to $\mathcal{E}(B)$. Then B is a top brick of M (and Y is the torsion submodule of M with respect to the torsion class $T(M) \cap {}^{\perp}B$).*

Proof. First, we show that $T(M) = T(Y, B)$. Since Y is a proper submodule of M, we see that M/Y is a non-zero module in $\mathcal{E}(B)$, thus it has a factor module isomorphic to B. Since B is a factor module of M, we know that B belongs to $T(M)$. Also, by assumption, Y belongs to $T(M)$. Thus $T(Y, B) \subseteq T(M)$. On the other hand, M has a filtration with factors of the from Y and B, thus $T(M) \subseteq T(Y, B)$.

Next, we calculate the iterated endotop of $Y \oplus B$. We calculate inductively $\mathrm{et}^a(Y \oplus B)$ for all $a \ge 0$. We claim that $\mathrm{et}^a(Y \oplus B) = Y_a \oplus B$, where Y_a is a factor module of Y with $\mathrm{Hom}(Y_a, B) = 0$. For $a = 0$, we put $Y_a = Y$. Assume we have $\mathrm{et}^a(Y \oplus B) = Y_a \oplus B$, where Y_a is a factor module of Y with $\mathrm{Hom}(Y_a, B) = 0$. Since $\mathrm{Hom}(Y_a, B) = 0$, the radical maps in the endomorphism ring of $Y_a \oplus B$ map into Y_a. If U_a is the sum of these images, then $\mathrm{et}^a(Y \oplus B) = Y_{a+1} \oplus B$ with $Y_{a+1} = Y_a/U_a$. Also, we have $\mathrm{Hom}(Y_{a+1}, B) = 0$, since any non-zero homomorphism $Y_{a+1} \to B$ would yield a non-zero homomorphism $Y_a \to Y_{a+1} \to B$. Since we deal with modules of finite length, there is some a such that $U_a = 0$, and therefore $\mathrm{et}^{\infty}(Y \oplus B) = Y_a \oplus B$. This shows that B is a top brick of $Y \oplus B$.

Since $T(M) = T(Y, B)$, we know from 2.8 that the top bricks of M are just the top bricks of $Y \oplus B$. Thus B is a top brick of M. $\square$

9.4. Corollary. *Let $(M_i)_i$ be a torsional brick chain filtration of M of brick type $(B_1, \dots, B_m)$. Then B_m is a top brick of M and M_{m-1} is the torsion submodule of M for the torsion class $T(M) \cap {}^{\perp}B_m$.*

Proof. We apply Lemma 9.3 to $Y = M_{m-1}$ and $B = B_m$. $\square$

9.5. Finiteness of the number of torsional brick chain filtrations. For any module M, let $\phi(M)$ be the number of torsional brick chain filtrations of M. We show by induction that $\phi(M) \le m!$, where m is the length of M. Of course, $\phi(0) = 1$. Now, let M be a non-zero module. There are at most m top bricks $B^{(1)}, \dots, B^{(t)}$. For any top brick $B^{(i)}$, let $M^{(i)}$ be the torsion submodule of M with respect to the torsion class $T(M) \cap {}^{\perp}B^{(i)}$. Then $M^{(i)}$ has length at most $m-1$, thus, by induction, $\phi(M^{(i)}) \le (m-1)!$. Therefore $\phi(M) = \sum_i \phi(M^{(i)}) \le m \cdot (m-1)! = m!$. Namely, according to 9.4 any torsional brick chain filtrations of M is obtained from a torsional brick chain filtration of $M^{(i)}$ by adding the inclusion $M^{(i)} \subset M$. □

Remark. The basic semisimple modules show that the bound $\phi(M) \leq m!$ is optimal.

9.6. Some examples.

(1) If M is homogeneous of brick type B, then the only torsional brick chain filtration of M is $(0 \subseteq M)$.) If the brick B is not simple, and M is non-zero, then M has brick chain filtrations which are not torsional, see 10.3.

(2) Let A be a Nakayama algebra and M an indecomposable module. Then *M has precisely one torsional brick chain filtration, and this filtration has length at most two.* Proof. Let B be the (unique) top brick of M. Let U be the smallest submodule of M such that M/U belongs to $\mathcal{E}(B)$. If $U = 0$, then $(0 \subset M)$ is the unique torsional brick chain filtration. If $U \neq 0$, then $(0 \subset U \subset M)$ is the unique torsional brick chain filtration. □

It follows: *If the support of M has cardinality n, then M has at least 2^{n-1} brick chain filtrations.* We may assume that M is faithful. If A is a directed algebra, then see 1.6 (2). We assume that the quiver of A has arrows $(i-1) \leftarrow i$ for $2 \leq i < n$ and $n \leftarrow 1$, and that the top of M is n. Let $1 \leq b_1 < b_2 < \cdots < b_t = n$ be a sequence of integers (the number of such sequences $(b_i)_i$ is 2^{n-1}). For $1 \leq i \leq t$, let M_i be the maximal submodule of M with top b_i. Thus $0 \subset M_1 \subset \cdots \subset M_t = M$ is a filtration of M, and we write $B_i = M_i/M_{i-1}$ for $2 \leq i \leq t$. We now use the torsional brick chain filtration of M_1: Let B_1 be the top brick of M_1. Let U be the smallest submodule of M_1 such that M_1/U belongs to $\mathcal{E}(B_1)$. If $U = 0$, then $0 \subset M_1 \subset \cdots \subset M_t$ is a brick chain filtration of M of type $(B_1, B_2, \ldots, B_t)$. If $U \neq 0$, then $0 \subset U \subset M_1 \subset \cdots \subset M_t$ is a brick chain filtration of M of type $(U, B_1, \ldots, B_t)$. □

(The bound 2^{n-1} is not optimal: If A is a cyclic Nakayama algebra with $n = 3$, any indecomposable module of length 4 has five brick chain filtrations.)

(3) Duality. We obtain from a brick chain filtration $(M_i)_i$ of M a brick chain filtration for $\mathrm{D}\, M$, see 1.3 (4). The dual filtration of a torsional brick chain filtration may not be torsional. As an example, let A be a connected Nakayama algebra with $n = 2$ and let M be an indecomposable module of length three, let U be its socle. Then M has the brick chain filtration $(0 \subset U \subset M)$. This filtration is torsional, but the dual filtration is not.

There are brick chain filtrations $(M_i)_i$ such that neither $(M_i)_i$ nor the dual filtration is torsional. As an example, take a connected Nakayama algebra with $n = 3$ and let M be indecomposable of length 4. Let U be the submodule of M of length 2. Then $(0 \subset U \subset M)$ is a brick chain filtration, neither this filtration nor its dual is torsional.

(4) We have mentioned in 3.1 that a brick chain filtration $(M_i)_i$ of a module M, with all M_i being torsional submodules of M, may not be a torsional filtration. Here is an example: Let Q be the quiver with vertices $1, 2, 3$ and with arrows $1 \to 2,\ 2 \to 1,\ 2 \to 3$, such that the path $1 \to 2 \to 1 \to 2 \to 1$ is a zero relation. The projective module $P(2)$ has a submodule U isomorphic to 3 such that $M = (\operatorname{rad} P(2))/U$ is the direct sum of a serial module V with factors $3, 2, 1, 2, 1$ (going up) and a copy of 3. Take as M_1 and M_2 the submodules of V of length two and four, respectively. Then $0 = M_0 \subset M_1 \subset M_2 \subset M$ is a brick chain filtration of type $(M_1, M_2/M_1, M/M_2)$. Both bricks M_1 and M_2/M_1 are factor modules of M. It follows that M_1 and M_2 are torsional submodules of M. However, M_1 does not belong to $T(M_2)$ (note that M_2 is even a brick, thus it has no non-trivial torsional submodules, see 4.3).

9.7. Lemma. *Let B be a brick and M a non-zero module in $\mathcal{E}(B)$. Then M has an endomorphism with image a brick, and B is the only brick which occurs in this way.* Thus, the type of a non-zero homogeneous module is uniquely determined.

Proof. First, we show that B occurs as the image of an endomorphism of M. Since M belongs to $\mathcal{E}(B)$, there is a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$ of M such that all factors are isomorphic to B. A corresponding map $M \to M/M_{m-1} \simeq B \simeq M_1 \subseteq M$ is an endomorphism of M which image isomorphic to B.

Conversely, let f be an endomorphism of M whose image is a brick. Since $\mathcal{E}(B)$ is an exact abelian subcategory, the image M' of f belongs to $\mathcal{E}(B)$. Now M' is a non-zero module in $\mathcal{E}(B)$. As we have seen in the first part of the proof, M' has an endomorphism with image $f(M')$ being isomorphic to B. But we assume that M' is a brick, thus the image of an endomorphism of M' is either zero or M' itself. This shows that $M' = f(M')$, thus M' is isomorphic to B. □

Remark. Let M be a homogeneous module. According to 9.7, the endomorphism ring of M shows whether M is a brick or not. But $\mathrm{End}(M)$ gives only limited information about M. In particular, $\mathrm{End}(M)$ may be a k-algebra of dimension 2, whereas M has a filtration with arbitrarily many factors of the form B. Here is an example. Consider the subring $A = k + J$ of the ring of $(t \times t)$-matrices with $t \geq 2$, where J is the set of nilpotent upper triangular matrices; and look at the set $M = k^t$ of column vectors. The A-module M is a serial module of length t, but $\dim_k \mathrm{End}(M) = 2$.

10. Further remarks about brick chains and brick chain filtrations.

10.1. A filtration $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_m = M$ will be said to be *solid* provided $\mathrm{Hom}(M_i/M_{i-1}, M_j/M_{j-1}) = 0$ for all $1 \leq i < j \leq m$.

Proposition. *Let M be a module with a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$. Then $(M_i)_i$ is a brick chain filtration iff $(M_i)_i$ is a solid filtration and all the factors are homogeneous.*

Proof. First, assume that $(M_i)_i$ is a brick chain filtration, say of type $(B_1, \ldots, B_m)$. Since M_i/M_{i-1} belongs to $\mathcal{E}(B_i)$, all the factors of the filtration are homogeneous. Also, for $i < j$, we have $\mathrm{Hom}(B_i, B_j) = 0$. Therefore $\mathrm{Hom}(M_i/M_{i-1}, M_j/M_{j-1}) = 0$.

Conversely, assume that $(M_i)_i$ is a solid filtration (with proper inclusions) and all factors are homogeneous. Now $F_i = M_i/M_{i-1}$ belongs to $\mathcal{E}(B_i)$ for some brick B_i. Since F_i is non-zero, B_i occurs both as a submodule and as a factor module of F_i. Thus, any non-zero map $f\colon B_i \to B_j$ yields a non-zero map $F_i \to F_j$. Since the given filtration is solid, we see that $\mathrm{Hom}(B_i, B_j) = 0$ for $i < j$. Thus, $(B_1, \ldots, B_m)$ is a brick chain. □

10.2. The composition factors in the top of a module give rise to brick chain filtrations:

Proposition. *Let M be a module. Let S be a simple module which occurs in the top of M. Let M' be the smallest submodule of M such that M/M' belongs to $\mathcal{E}(S)$. Then, any torsional brick chain filtration $(M_i)_{0 \leq i \leq m'}$ of M' is part of a brick chain filtration $(M_i)_{0 \leq i \leq m'+1}$ of M* (with $M_{m'} = M'$).

This shows: for any simple module S in the top of M, there is a brick chain filtration whose type ends in S. Using duality, we also have: for any simple module S in the socle of M, there is a brick chain filtration whose type starts with S.

Proof. Let $(B_1, \dots, B_{m'})$ be the type of the torsional brick chain filtration of M'. The bricks B_i are in $\mathcal{T}(M')$, thus the top of B_i is generated by M'. As a consequence, $\operatorname{Hom}(B_i, S) = 0$ for $1 \le i \le m'$. Thus $(B_1, \dots, B_{m'}, S)$ is a brick chain, and $(M_i)_{1 \le i \le m'+1}$ is a brick chain filtration of type $(B_1, \dots, B_{m'}, S)$. □

10.3. Proposition. *A module M has only one brick chain filtration iff all composition factors of M are isomorphic.*

Proof. If all composition factors of M are isomorphic, then clearly $(0 \subseteq M)$ is the only brick chain filtration. Conversely, assume that M has only one brick chain filtration. According to 9.2, M has only one top brick, say B. On the other hand, 10.2 shows that M has a brick chain filtration of type $(B_1, \dots, B_m)$ with $B_m = S$ simple. This shows that $\mathcal{B}(M) = \{S\}$. Now 5.5 asserts that $T(M) = T(\mathcal{B}(M))$. Since M belongs to $T(M) = T(\mathcal{B}(M)) = T(S)$, all composition factors of M are isomorphic to S. □

10.4. Induced brick chain filtrations. We say that a filtration $(M_i)_i$ of a module M is *proper* provided all the inclusions $M_{i-1} \subseteq M_i$ are proper. Of course, any filtration $(M_i)_i$ yields a proper filtration by deleting all the submodules M_i with $M_{i-1} = M_i$. Until now, all the filtrations considered in the paper were proper. Let us call an arbitrary filtration of a module M a *brick chain filtration with repetitions* provided the corresponding proper filtration is a brick chain filtration.

Proposition. (a) *Let M, N be modules. Let $(X_i)_i$ be a brick chain filtration of $X = M \oplus N$. Then $(X_i \cap M)_i$ is a brick chain filtration of M with repetitions,* we say that it is *induced from $(X_i)_i$.*

(b) *Let M be a module with a brick chain filtration $(M_i)_i$ say of type $(B_1, \dots, B_t)$. Let $N = \bigoplus B_i$. Then, also the module $M \oplus N$ has a brick chain filtration of type $(B_1, \dots, B_t)$, this filtration is torsional, and it induces the given filtration $(M_i)_i$.* This shows that *any brick chain filtration is induced from a torsional brick chain filtration.*

Proof. (a) Since X_i is the torsion submodule of X for the torsion class $T(B_1, \dots, B_i)$, we have $(X_i \cap M) \oplus (X_i \cap N) = X_i$. Thus $(X_i \cap M)/(X_{i-1} \cap M) \oplus (X_i \cap N)/(X_{i-1} \cap N) = X_i/X_{i-1}$ belongs to $\mathcal{E}(B_i)$, thus $(X_i \cap M)/(X_{i-1} \cap M)$ belongs to $\mathcal{E}(B_i)$. Of course, $(X_i \cap M)/(X_{i-1} \cap M)$ may be zero (thus, the filtration may have repetitions).

(b) Let $N = \bigoplus_{j=1}^{t-1} B_j$ and $X = M \oplus N = X_t$. For $0 \le i < t$, let $X_i = M_i \oplus \bigoplus_{j=1}^{i} B_j$. Then $X_t/X_{t-1} = M_t/M_{t-1} \in \mathcal{E}(B_t)$. Also, for $1 \le i < t$, we have $X_i/X_{i-1} = M_i/M_{i-1} \oplus B_i \in \mathcal{E}(B_i)$. Thus, we see that $(X_i)_i$ is a brick chain filtration of type $(B_1, \dots, B_t)$.

All the bricks B_i with $1 \le i < t$ are factor modules of X, thus $T(B_1, \dots, B_{t-1}) \subseteq T(X)$. Since X_{t-1} belongs to $\mathcal{E}(B_1, \dots, B_{t-1})$, we see that X_{t-1} belongs to $T(X)$. Similarly, for $1 \le i < t$, we have $T(B_1, \dots, B_{i-1}) \subseteq T(X_i)$. Again, X_{i-1} belongs to $\mathcal{E}(B_1, \dots, B_{i-1})$, thus X_{i-1} belongs to $T(X_i)$. This shows that $(X_i)_i$ is a torsional filtration. □

10.5. Remark. According to 10.4, brick chain filtrations of direct sums yield brick chain filtrations of the summands. The converse is not true. For example, let A be a cyclic Nakayama algebra with two simple modules and M, N the two indecomposable modules of length two. Then M, N are bricks, thus they have brick chain filtrations of length one. But any brick chain filtration of the module $M \oplus N$ has length three.

11. Appendix. Complete brick chains.

11.1. Definition. Following Demonet [De], one should also deal with arbitrary (not necessarily finite) brick chains: these are arbitrarily large totally ordered sets $\mathcal{B} = \{B_i \mid i \in I\}$ of bricks with the Hom-condition $\mathrm{Hom}(B_i, B_j) = 0$ for all $i < j$. Given brick chains $\mathcal{B}, \mathcal{B}'$, one calls $\mathcal{B}$ a *refinement* of $\mathcal{B}'$, provided $\mathcal{B}'$ is obtained from $\mathcal{B}$ by deleting some elements. And, a brick chain is said to be *complete* provided it has no proper refinement. Similarly, dealing with arbitrary chains of torsion classes, there is the corresponding concept of refinements and of completeness of such chains.

Demonet has shown the following assertions.

(1) *Any brick chain has a complete refinement.*

(2) *The complete chains of torsion classes correspond bijectively to the complete brick chains, sending a complete chain $(\mathcal{T}_i)_i$ of torsion classes to the brick chain given by the labels of the neighbor torsion classes in the chain $(\mathcal{T}_i)_i$.*

(3) *Any torsion class is generated by the bricks of a (usually infinite) brick chain.*

(4) *Any simple module occurs in any complete brick chain.*

The proofs require to deal with sets of arbitrary cardinality, but otherwise they are easy. For (2), Corollary 7.4 is essential. Assertion (3) is a direct consequence of (2). $\square$

11.2. Theorem. *Let $\mathcal{B}$ be a complete brick chain. Any module M has a unique brick chain filtration $(M_i)_i$ such that $\mathcal{B}$ is a refinement of the type of $(M_i)_i$.*

Proof, by induction on the length of M. The assertion is clear for the zero module. Thus, assume that $M \neq 0$. Let $(\mathcal{T}_i)_{i\in I}$ be a complete chain of torsion classes. Let $\mathcal{T} = \bigcap_{M\in\mathcal{T}_i} \mathcal{T}_i$ and $\mathcal{T}' = \bigcup_{M\notin\mathcal{T}_i} \mathcal{T}_i$. Since we deal with a complete chain of torsion classes, both $\mathcal{T}, \mathcal{T}'$ belong to the chain. Of course, we have $\mathcal{T}' \subseteq \mathcal{T}$, but even $\mathcal{T}' \subset \mathcal{T}$, since M belongs to $\mathcal{T}$ and does not belong to $\mathcal{T}'$. It follows, that $\mathcal{T}' \subset \mathcal{T}$ are neighbors, say with label $B \in \mathcal{B}$, thus $\mathcal{T}' = \mathcal{T} \cap {}^{\perp}B$.

First, we show that M has at most one brick chain such that the bricks of the type belong to $\mathcal{B}$. Let $(M_i)_i$ be a brick chain of M, say of type $(B_1, \dots, B_t)$ and assume that all the bricks B_i belong to $\mathcal{B}$. We want to show that $B_t = B$. Not all the modules B_i can belong to $\mathcal{T}'$, since otherwise M is in $\mathcal{T}'$. Thus, B_t is not in $\mathcal{T}'$. Now M belongs to $\mathcal{T}$ and B_t is a factor module of M, thus B_t belongs to $\mathcal{T}$. But B is the only brick in $\mathcal{B}$ which belongs to $\mathcal{T} \setminus \mathcal{T}'$, therefore $B_t = B$. By induction, also the bricks $B_1, \dots, B_{t-1}$ are determined by M_{t-1} and $\mathcal{B}$.

Conversely, we show that M has a brick chain filtration say with type $\mathcal{B}'$, such that $\mathcal{B}$ is a refinement of $\mathcal{B}'$. Since $\mathcal{T} = T(\mathcal{T}', B) = T(\mathcal{T} \cap {}^{\perp}B, B)$, we can apply 6.1 and obtain a submodule M' of M belonging to $\mathcal{T}'$ such that M/M' is a non-zero module in $\mathcal{E}(B)$. By induction, M' has a brick chain filtration say of type $\mathcal{B}''$, where $\mathcal{B}$ is a refinement of $\mathcal{B}''$. $\square$

12. History and relevance (as well as additions).

12.1. The results presented here are usually considered as part of the so-called τ-tilting theory (see 12.13). There is a strange reluctance to deal with bricks. For example, many papers prefer to speak about τ-tilting finiteness instead of brick finiteness, but these properties are equivalent (see [DIJ]; here, τ-tilting finiteness means that there are only finitely many basic support-τ-tilting modules: In my opinion, brick finiteness is very easy to grasp, whereas τ-tilting finiteness is much less intuitive). For our report, there was no need to mention τ-tilting notions, nor even the Auslander-Reiten translation τ itself, thus we have avoided it. In this way, we stress the completely elementary nature of the corresponding results. Some remarks on τ-tilting theory will be given in 12.13.

It is astonishing that the relevance of bricks when dealing with tilting modules and torsion classes, was observed only so late!

12.2. Bricks and semibricks. The terminology "semibrick" is due to Asai [A1]. I used to call a semibrick an "antichain" of bricks, but this is in conflict with Demonet's important notion of a brick chain (to say that "an antichain of bricks is a brick chain", would sound rather odd).

12.3. Torsion pairs $(\mathcal{T}, \mathcal{F})$. Torsion pairs were introduced by Dickson [Di] as a generalization of the use of torsion and p-torsion subgroups of abelian groups, for dealing with arbitrary R-modules, were R is any ring.

12.4. Hereditary torsion pairs, torsional submodules. In contrast to the classical example, torsion classes in general are not hereditary (where *hereditary* means that the torsion class $\mathcal{T}$ is closed under submodules). The torsion classes $T(M)$ considered in our paper are usually not hereditary. But it turns out that dealing with a module M, it is important to look at submodules of M which do belong to $T(M)$ (the torsional submodules). Our focus on torsional submodules is an attempt to stress heredity properties for non-hereditary torsion classes.

Theorems 1.2 and 3.2 should be seen in the light of the original example of abelian group theory: any finitely generated abelian group M has a filtration $(M_i)_{0\le i\le m}$ where the first factors M_i/M_{i-1} are in $\mathcal{E}(\mathbb{Z}/p_i\mathbb{Z})$, for pairwise different prime numbers p_i, whereas M_m/M_{m-1} is in $\mathcal{E}(\mathbb{Z})$, and this filtration always splits! In our case, we cannot expect that the filtrations in 1.2 and 3.2 split. (It comes as a surprise that actually in first examples one looks at, say dealing with Kronecker modules, many brick chain filtrations do split.)

12.5. Auslander and Smalø (and Demonet). The relevance of torsion classes when dealing with finite length categories was seen already by Auslander and Smalø [AS].

Looking at a module category, the existence of cyclic paths in the category or even in the Auslander-Reiten quiver, provides a lot of difficulties. Only the representation-directed algebras are easy to visualize, but representation-directedness is a very special property. There have been many attempts to overcome the difficulties which arise from the presence of cyclic paths. There is the covering theory by Gabriel and his school; also, the book of Auslander, Reiten, Smalø is full of helpful devices: to avoid short chains, to avoid short cycles. However, all these methods are designed just for special, well-behaved situations. If one wants to deal with an arbitrary module category, the use of torsion classes always works. The reference to torsion classes allows to consider the set of semibricks as a partially ordered set. In this way, Demonet's proposal to look at brick chains stresses a very interesting directedness feature of an arbitrary module category.

12.6. Wide subcategories and torsion classes. Given an abelian category, the exact abelian subcategories which are closed under extensions are now often called *wide* subcategories. The rather obvious relationship between semibricks and wide subcategories was mentioned in [R1] under the name "simplification". The search for semibricks (or wide subcategories) which generate a given torsion class was initiated by Ingalls and Thomas [IT]. Theorem 2.3 generalizes some of their considerations.

The relevance of the endotop of a module is well-known and was stressed by Asai when looking at τ-rigid modules (our proof of 5.5 follows closely Asai [A]). For a general study of widely generated torsion classes, see Asai and Pfeifer [AP] and Marks and Stovičcek [MS].

12.7. Homogeneous subcategories. The homogeneous subcategories are equivalent to the module category of a local algebra (not necessarily an artin algebra). Actually, not much is known about the representation theory of local algebras which are not commutative! The commutative rings are studied very well in commutative algebra, but who cares about the non-commutative ones? Note that they can behave rather differently and really deserve attention.

For example, if A is a commutative local ring, and M is a serial module, then there is an endomorphism of M with image $\operatorname{rad} M$, thus $\operatorname{et} M$ is just the simple module. On the other hand, consider the subring $A = k + J$ of the ring of $(t \times t)$-matrices where J is the set of nilpotent upper triangular matrices, as mentioned already in 5.2 (2). This is a rather nice local ring; it is non-commutative provided $t \geq 3$. The set $M = k^t$ of column vectors is a serial A-module. Here, the image of any non-invertible endomorphism of M has length at most one, thus we see that $\operatorname{et} M$ has dimension $t-1$; in particular, it is not a brick provided $t \geq 3$ (for $t \geq 3$, we have $\operatorname{et}^\infty M = \operatorname{et}^{t-2} M = k \neq \operatorname{et} M$).

12.8. Neighbors of torsion classes. Neighbor torsion classes $\mathcal{T}' \subset \mathcal{T}''$ have attracted a lot of interest and several different denominations are used in the literature: that $\mathcal{T}''$ covers $\mathcal{T}'$, that there is an arrow $\mathcal{T}'' \to \mathcal{T}'$ in the Hasse quiver of the lattice of torsion classes, or one speaks about minimal inclusions of torsion classes.

As we have seen, it is easy to determine the lower neighbors of a finitely generated torsion class (and there are only finitely many), but unfortunately, it is difficult to deal with the upper neighbors: usually, there may be infinitely many. For any torsion class $\mathcal{T}$, the best way to find its upper neighbors seems to be to look at the corresponding torsion free class $\mathcal{F}$ and to try to determine its lower neighbors, since the lower neighbors of $\mathcal{F}$ correspond to the upper neighbors of $\mathcal{T}$.

12.9. Brick labeling. The brick labeling as presented in section 7 was started for functorially finite torsion classes in [AIR] and Asai [A] identified the labels as bricks. The general case is due to Barnard, Carroll and Zhu [BCZ]. The brick B used as label for the neighbor torsion classes $\mathcal{T}' \subset \mathcal{T}$ is called a *minimal extending module* for $\mathcal{T}'$ in [BCZ]. In [AHL], the labels are said to be *torsion, nearly torsionfree* for the torsion pair $(\mathcal{T}, \mathcal{T}^\perp)$. The bijection 2.9 between bricks and completely join irreducible torsion classes has been exhibited in Theorem 1.0.5 in [BCZ].

12.10. Brick chains. Given chains of torsion classes, the brick labeling of the neighbor torsion classes yields the Hom-condition which defines the brick chains. This observation was used by Demonet [De] to deal with arbitrarily large totally ordered sets of bricks with this Hom-condition, see 10.8. But note that already for the Kronecker algebra, the sets which occur explode: The Kronecker algebra A over the field with 2 elements has cardinality 16, thus it is very easy to envision, but there are uncountably many complete brick chains (one is finite, all others are countable).

Demonet's bijection 10.8 (2) is very charming: whereas torsion classes are complicated categories (usually they are quite difficult to exhibit), a brick is just a brick, a well-behaved single module! Of course, there is the next level: to look at brick chains: they are not so easy to grasp, since usually these chains may be very large sets. But for getting a feeling for large brick chains, one may first restrict the attention to finite brick chains (as we do in this report). Actually, in some cases, there is no problem to overview well the complete brick: for example, an infinite complete brick chain for the Kronecker algebra has a preprojective and a preinjective part (both are easy to visualize); and in-between, there is the regular part: it is $\mathbb{P}^1(k)$ with an arbitrary (and really irrelevant) total ordering.

12.11. Special brick chain filtrations have been used already in [R2]: In this paper we have shown that for a hereditary k-algebra, with k an algebraically closed field, any brick without self-extensions is a tree module. The basis of the proof is Schofield induction: it deals with certain brick chain filtrations of length two. The brick chain filtrations used have type (B_1, B_2), where both B_1, B_2 are again bricks without self-extensions. These brick chain filtrations are never torsional (since they are filtrations of bricks). This shows the relevance of brick chain filtrations which are not torsional.

12.12. Artinian rings. In this report, we have assumed to be in the context of artin algebras. Actually, nearly all the results presented here are valid more generally in arbitrary length categories, thus for finitely generated modules over left artinian rings.

12.13. Functorially finite torsion classes. Functorially finite torsion classes were first considered by Auslander and Smalø. In 1984, Smalø formulated the tie between functorially finite torsion classes in $\operatorname{mod} A$ and tilting modules for suitable factor algebras of A. The basic objects of τ-tilting theory are the τ-rigid modules (a module M is *τ-rigid* provided $\operatorname{Hom}(M, \tau M) = 0$, where τ is the Auslander-Reiten translation): The Adachi-Iyama-Reiten paper [AIR] (published in 2014) showed that the functorially finite torsion classes are just the torsion classes generated by τ-rigid modules, a very important observation! (Actually, several of the main results of [AIR] also follow from investigations by Derksen and Fei which were available in arXiv already in 2009, but it seems that this paper is not really appreciated by the tau-tilting community).

We cannot give here even a concise summary of τ-tilting theory, but there do exist already many surveys which can be consulted. Only few details should be mentioned: For an algebra with n simple modules, any functorially finite torsion class has precisely n, and sufficiently many, neighbors, and all neighbors are again functorially finite. Asai [A2] calls a torsion class *bicompact* provided it has finitely many, and sufficiently many, neighbors, and he conjectures that a bicompact torsion class has to be functorially finite (as we have mentioned, the converse is true). An important feature of τ-tilting theory is the use of mutations in order to describe for a given functorially finite torsion class its neighbors. In the setting of tilting theory, mutations were studied by Happel and Unger (and others), but it took a long time that the relevance for arbitrary module categories was realized.

We have seen in 2.5 that an algebra is brick finite iff it is torsion class finite. It turns out that in this case, all torsion classes are not only finitely generated, but even functorially finite. And conversely, if any torsion class is functorially finite, then the algebra has to be brick finite. One can use this fact in order to show: if all torsion classes are finitely generated, then the algebra is brick finite, as mentioned in 2.5.

12.14. Further developments. We say that a module M has *brick chain complexity* at most t provided there is a brick chain filtration of M with t factors. The *brick chain complexity* of an algebra A is the supremum of the brick chain complexity of the indecomposable A-modules (it is a natural number or ∞); it will be discussed in [R3[.

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Claus Michael Ringel
Fakultät für Mathematik, Universität Bielefeld
POBox 100131, D-33501 Bielefeld, Germany

ringel@math.uni-bielefeld.de