

KRONECKER MODULES GENERATED BY MODULES OF LENGTH 2

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ABSTRACT. Let Λ be a ring and \mathcal{N} a class of Λ -modules. A Λ -module is said to be generated by \mathcal{N} provided that it is a factor module of a direct sum of modules in \mathcal{N} . The semi-simple Λ -modules are just the Λ -modules which are generated by the Λ -modules of length 1. It seems that the modules which are generated by the modules of length 2 (we call them bristled modules) have not attracted the interest they deserve.

In this paper we deal with the basic case of the Kronecker modules, these are the (finite-dimensional) representations of an n -Kronecker quiver, where n is a natural number. We show that for $n \geq 3$, there is an abundance of bristled Kronecker modules.

1. INTRODUCTION

1.1. Let Λ be a ring and \mathcal{N} a class of Λ -modules. A Λ -module is said to be generated by \mathcal{N} provided that it is a factor module of a direct sum of modules in \mathcal{N} . A Λ -module M is said to be \mathcal{N} -saturated provided that $\text{Ext}^1(N, M) = 0$ for all modules $N \in \mathcal{N}$. Of course, *the class of modules which are both generated by \mathcal{N} and \mathcal{N} -saturated is closed under extensions. In case Λ is hereditary, the class of modules which are both generated by \mathcal{N} and \mathcal{N} -saturated is also closed under factor modules.* We consider the case where \mathcal{N} is a class of modules of length 2. The modules which are generated by modules of length 2 will be called *bristled* modules; a *bristle* is by definition an indecomposable module of length 2, and we denote by \mathcal{B} the set of isomorphism classes of bristles (we should stress that a bristled module is not necessarily the sum of its bristle submodules — it is the sum of submodules which are either bristles or simple).

Given any artin algebra Λ , we denote by $\text{mod } \Lambda$ the category of all (finitely generated) Λ -modules and by τ the Auslander-Reiten translation in $\text{mod } \Lambda$.

1.2. We focus the attention to the n -Kronecker algebras and their modules, the n -Kronecker modules (for convenience, we sometimes will refer to these algebras and their modules just as the Kronecker algebras and the Kronecker modules). Despite of the importance of the Kronecker algebras, not much is known about Kronecker modules. Our approach to look at the bristled Kronecker modules may be seen as one of the first attempts to get insight into the structure of Kronecker modules in general. On the other hand, the Appendix B will outline in which way the study of bristled Kronecker modules is an essential step for understanding bristled Λ -modules for any finite-dimensional k -algebra, where k is an algebraically closed field.

Let k be a field. The n -Kronecker quiver $K(n)$ has two vertices, denoted by 1 and 2, and n arrows $\alpha_i: 1 \rightarrow 2$, with $1 \leq i \leq n$.

$$\begin{array}{ccc} & \alpha_1 & \\ & \longrightarrow & \\ 1 & & 2 \\ & \vdots & \\ & \longrightarrow & \\ & \alpha_n & \end{array}$$

The n -Kronecker algebra $kK(n)$ is the path algebra of $K(n)$, the n -Kronecker modules are the representations of $K(n)$ (over k). Kronecker modules are written in the form $M = (M_1, M_2; \alpha_1, \dots, \alpha_n)$, where M_1, M_2 are two vector spaces and $\alpha_i: M_1 \rightarrow M_2$ are linear transformations, for $1 \leq i \leq n$.

Note that $kK(n)$ is a hereditary artin algebra. Given any hereditary artin algebra Λ , an indecomposable Λ -module is called *preinjective* provided that it is of the form $\tau^t I$ for some injective Λ -module I and some $t \geq 0$.

An n -Kronecker module which is a bristle is of the form $B(\lambda) = (k, k; \lambda_1, \dots, \lambda_n)$, where $(\lambda_1, \dots, \lambda_n)$ is a non-zero element of k^n ; the bristles $B(\lambda)$ and $B(\lambda')$ are isomorphic if and only if $(\lambda'_1, \dots, \lambda'_n)$ is a scalar multiple of $(\lambda_1, \dots, \lambda_n)$ (thus the set \mathcal{B} of isomorphism classes of bristles may be identified with the projective space \mathbb{P}^{n-1}).

1.3. The essential part of the paper concerns the class of bristled Kronecker modules which are \mathcal{B} -saturated. As we have mentioned, this class of modules is closed under factor modules and extensions. As we will see, there is an abundance of such modules (but note that for $n \geq 2$, the bristles themselves do not belong to this class of modules, since they are not \mathcal{B} -saturated). Actually, it seems to be appropriate to select a suitable finite set \mathcal{B}' of (isomorphism classes of) bristles and to look at the (slightly smaller) class of modules which are generated by \mathcal{B}' and which are \mathcal{B} -saturated (also this class is closed under factor modules and extensions). Here is the main theorem.

Main Theorem. *Let $n \geq 3$. There is a set \mathcal{B}_0 of $n + 2$ bristles in $\text{mod } kK(n)$ with the following properties:*

- (a) *Any preinjective module is generated by \mathcal{B}_0 and is \mathcal{B} -saturated.*
- (b) *If M is any module, then there is a number $t(M)$ such that all the modules $\tau^t M$ with $t \geq t(M)$ are generated by \mathcal{B}_0 and are \mathcal{B} -saturated.*

For the choice of the set \mathcal{B}_0 , see section 4.2. As we will see in the Addendum 4.23, the number $n + 2$ is best possible: *There is no set of $n + 1$ bristles which generates all the preinjective modules.*

As a consequence of Main Theorem, we see that for $n \geq 3$ there exists a regular module which generates all the preinjective modules, for example the direct sum of the modules in \mathcal{B}_0 (and there are also indecomposable regular modules with this property, see 4.21). In 1994, Kerner [3] had asked whether such a module does exist. In 2014, Kerner gave a non-constructive proof for the existence of such a module [4]; here is now an explicit example. Note that for $n = 2$, any regular module generates only finitely many indecomposable preinjective modules, see 4.22 and Appendix C.

Using duality, we see that also the corresponding dual statements hold:

- (a)* *Any preprojective module M is cogenerated by \mathcal{B}_0 and \mathcal{B} -cosaturated.*

(b)* If M is any module, then there is a number $t^*(M)$ such that all the modules $\tau^{-t}M$ with $t \geq t^*(M)$ are cogenerated by \mathcal{B}_0 and are \mathcal{B} -cosaturated.

Of course, a module M is said to be *cogenerated* by a class \mathcal{N} of modules provided that it is a submodule of a direct sum of modules in \mathcal{N} (note that the modules considered here are modules of finite length; otherwise we would have to take direct products instead of direct sums). And M is said to be *\mathcal{N} -cosaturated* provided $\text{Ext}^1(M, N) = 0$ for all modules N in \mathcal{N} .

1.4. Outline of the paper. Section 2 contains some preliminaries: we fix the notation and recall some general properties of the categories $\text{mod } kK(n)$. Of course, the structure of the categories $\text{mod } kK(1)$ and $\text{mod } kK(2)$ is well-known (whereas only little is known about the categories $\text{mod } kK(n)$ with $n \geq 3$).

The proof of theorem 1.3 will be presented in section 4 and some related results can be found in section 5. In particular, we will show in section 5 that an indecomposable \mathcal{B} -saturated module which is not simple is faithful. Section 4 (and most parts of section 5) deal with n -Kronecker modules, where $n \geq 3$. The cases $n = 1$ and $n = 2$ are discussed in section 3.

There are three appendices. The first is a short introduction to the paper [6]. The set of bristle submodules of a Kronecker module M without simple injective submodules is in a natural way a projective variety. In [6] we show that any projective variety occurs in this way. Appendix B deals with arbitrary artin algebras Λ and their bristled modules. We are going to point out in which way the study of bristled Λ -modules can be reduced to the study of bristled Kronecker modules. As we have mentioned already, the existence of a regular module generating all preinjective modules holds true for $K(n)$ with $n \geq 3$, but not for $K(2)$. It does not hold for any tame hereditary artin algebras, see the Appendix C.

1.5. Acknowledgment. A weaker version of the Main Theorem (as well as the result of [6]) were first presented at the conference in honor of Jerzy Weyman's 60th birthday, April 2015, at the University of Connecticut. The weaker version used a set of $2n - 1$ bristles instead of the set \mathcal{B}_0 with the optimal cardinality $n + 2$; it was written up at a stay at the IPM in Isfahan, October 2016 and the author is grateful to the Iranian hospitality. The present version was finished in April 2017. The author is indebted to various mathematicians for questions and comments.

2. PRELIMINARIES

The aim of this section is to recall some general properties of the categories $\text{mod } kK(n)$ of the n -Kronecker modules, see for example [5].

Given any acyclic quiver Q and a vertex x of Q , we write $S(x)$, $P(x)$ and $I(x)$ for the corresponding indecomposable representation of Q which is simple, projective, or injective, respectively. Thus, $S(x)_x = k$ and $S(x)_y = 0$ for any vertex $y \neq x$. The representation $P(x)$ is the projective cover, the representation $I(x)$ the injective envelope of $S(x)$.

2.1. There are two simple n -Kronecker modules, namely the injective module $S(1) = I(1) = (k, 0; 0, \dots, 0)$ and the projective module $S(2) = P(2) = (0, k; 0, \dots, 0)$.

If $M = (M_1, M_2; \alpha_1, \dots, \alpha_n)$ is an n -Kronecker module, we denote by $\mathbf{dim} M = (\dim M_1, \dim M_2)$ (or also by $\begin{pmatrix} \dim M_1 \\ \dim M_2 \end{pmatrix}$) its *dimension vector*, it is an element of the Grothendieck group $K_0(kK(n)) = \mathbb{Z}^2$.

On $K_0(kK(n))$, there is defined the bilinear form $\langle -, - \rangle = \langle -, - \rangle_n$ via

$$\left\langle \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix} \right\rangle = aa' + bb' - nab',$$

and one knows that

$$\langle \mathbf{dim} M, \mathbf{dim} M' \rangle = \dim \text{Hom}(M, M') - \dim \text{Ext}^1(M, M'),$$

for any pair M, M' of n -Kronecker modules.

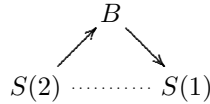
On $K_0(kK(n))$, the Coxeter transformation $\Phi: K_0(kK(n)) \rightarrow K_0(kK(n))$ is defined via $\Phi(a, b) = (n^2a - nb - a, na - b)$. It mimics the Auslander-Reiten translation as follows: Let M be an indecomposable n -Kronecker module. If M is non-projective, then

$$\mathbf{dim} \tau M = \Phi(\mathbf{dim} M).$$

If M is projective, then, of course, $\tau M = 0$ (and we have $\Phi(\mathbf{dim} P(i)) = -\mathbf{dim} I(i)$ for $i = 1, 2$).

For $n \geq 1$, a module is bristled if and only if it is generated by the bristles and $S(2)$; and an indecomposable bristled module is either isomorphic to $S(2)$ or else generated by the set \mathcal{B} of bristles.

2.2. The Auslander-Reiten quiver of $\text{mod } kK(1)$. There are just three isomorphism classes of indecomposable 1-Kronecker modules, namely $S(2) = P(2), B = P(1) = I(2)$ and $I(1) = S(1)$, and the Auslander-Reiten quiver of $\text{mod } kK(1)$ is of the form



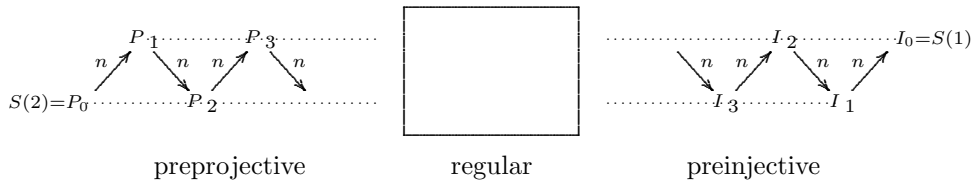
(we write B instead of $P(1) = I(2)$ in order to stress that this module is a bristle; of course, it is the only bristle).

2.3. The Auslander-Reiten quiver of $\text{mod } kK(n)$ with $n \geq 2$. We recall that the indecomposable representations of $K(n)$ fall into three classes: there are the preprojective ones, the regular ones and the preinjective ones.

There are countably many (isomorphism classes of) indecomposable preinjective modules, we denote these modules by I_t with $t \in \mathbb{N}_0$, as follows: $I_0 = S(1) = I(1)$, $I_1 = I(2)$, and $I_t = \tau I_{t-2}$ for $t \geq 2$, thus they are ordered in such a way that $\text{Hom}(I_t, I_{t'}) = 0$ if and only if $t < t'$.

Similarly, there are the indecomposable preprojective modules $P_0 = S(2) = P(2), P_1 = P(1), P_2, P_3, \dots$, but this notation will not be needed in an essential way in the paper.

The Auslander-Reiten quiver looks as follows:



where we have endowed the arrows by the number n in order to indicate that the corresponding bimodule of irreducible maps is n -dimensional. Note that there are no non-zero maps from a regular module to a preprojective module, or from

a preinjective module to a preprojective or a regular module. All non-zero maps in the preprojective component and in the preinjective component go from left to right. The components in the regular part are of type \mathbb{A}_∞ . For $n = 2$, they are regular tubes; for $n \geq 3$ they are of the form $\mathbb{Z}A_\infty$.

It is important to be aware that the bristles are regular. Thus, any indecomposable bristled module different from $S(2)$ is generated by bristles and therefore is regular or preinjective.

3. THE BRISTLED MODULES FOR $n = 1$ AND $n = 2$.

3.1. The case $n = 1$. As we have mentioned, there is (up to isomorphism) a unique bristle, namely $B = P(1) = I(2)$; this bristle is projective and injective. All the 1-Kronecker modules are \mathcal{B} -saturated bristled modules.

3.2. The case $n = 2$. Recall that we have indexed the (isomorphism classes of the) bristles by the elements of the projective space $\mathbb{P}^{n-1} = \mathbb{P}^1$. For $c \in k$, let $B_c = B(1 : c)$, and let $B_\infty = B(0 : 1)$.

Proposition. *Let $n = 2$. Let J be a subset of $k \cup \{\infty\}$. Then I_t is generated by the bristles B_c with $c \in J$ if and only if $|J| \geq t + 1$.*

Proof. Note that the dimension vector of I_t is $(t + 1, t)$. First, assume that I_t is generated by the bristles B_c with $c \in J$. Since $\dim \text{Hom}(B_c, M) = 1$ for all bristles B_c , it follows that I_t is a factor module of the module $\bigoplus_{c \in J} B_c$ with dimension vector $(|J|, |J|)$, therefore $|J| \geq t + 1$.

On the other hand, let $I_t = (M_1, M_2; \alpha_1, \alpha_2)$. We can assume that M_1 has the basis e_0, e_1, \dots, e_t , that M_2 has the basis e'_1, \dots, e'_t , that $\alpha_1(e_i) = e'_{i+1}$ for $0 \leq i < t$ and $\alpha_1(e_t) = 0$ and that $\alpha_2(e_0) = 0$ and $\alpha_2(e_i) = e'_i$ for $1 \leq i \leq t$, thus M may be visualized as follows:

$$\begin{array}{ccccccc}
 e_0 & & e_1 & & e_2 & & \dots & & e_{t-1} & & e_t \\
 & \searrow^{\alpha_1} & & \searrow^{\alpha_2} & & \searrow^{\alpha_1} & & \searrow^{\alpha_2} & & \searrow^{\alpha_1} & & \searrow^{\alpha_2} \\
 & & e'_1 & & e'_2 & & \dots & & e'_t & & \\
 & & & & & & \dots & & & & &
 \end{array}$$

For any $c \in k$, the element $m_c = \sum_{i=0}^t c^i e_i$ in M_1 is sent under α_1 to $\alpha_1(m_c) = \sum_{i=1}^t c^{i-1} e'_i$ and under α_2 to $\alpha_2(m_c) = \sum_{i=1}^t c^i e'_i = c \cdot \alpha_1(m_c)$. This shows that m_c generates a bristle of the form B_c . The Vandermonde determinant shows that for $t + 1$ pairwise different elements c in k , the corresponding elements m_c in M_1 are linearly independent, thus form a basis. But this implies that M is a bristled module.

There is one additional submodule of M which is a bristle, namely the submodule generated by $m_\infty = e_t$; it is isomorphic to B_∞ . Given t pairwise different scalars c_1, \dots, c_t in k , then also the elements $m_{c_1}, \dots, m_{c_t}, m_\infty$ are linearly independent: the determinant of the $(t + 1) \times (t + 1)$ matrix whose rows are the vectors $m_{c_1}, \dots, m_{c_t}, m_\infty$ is the product of the $t \times t$ Vandermonde determinant for the scalars c_1, \dots, c_t and the determinant of the 1×1 -matrix with entry 1, thus non-zero. \square

3.3. Corollary. *Let $n = 2$ and $q = |k|$. An indecomposable module M different from $S(2)$ is bristled if and only if M is one of the bristles B_c with $c \in k \cup \{\infty\}$ or $M = I_t$ is preinjective with $0 \leq t \leq q$.*

The preinjective bristled modules are \mathcal{B} -saturated, the module $S(2)$ and the bristles are not \mathcal{B} -saturated.

Proof. Let M be an indecomposable bristled module. First, assume that M is regular. If the regular socle U of M is a bristle B , then $\dim \operatorname{Hom}(B, M) = 1$ and $\operatorname{Hom}(B', M) = 0$ for any bristle not isomorphic to B , thus $M = U = B$. Or else $\operatorname{Hom}(B, M) = 0$ for any bristle B , thus $M = 0$ (since M is generated by bristles), a contradiction. This shows that the only indecomposable regular bristled modules are the bristles.

It remains to decide whether the bristled modules are \mathcal{B} -saturated or not. For $M = S(2)$, we have $\operatorname{Ext}^1(B, M) \neq 0$ for any bristle B . For $M = B_c$, we have $\operatorname{Ext}^1(B_c, M) \neq 0$. This shows that the bristled modules $S(2)$ and B_c are not \mathcal{B} -saturated. On the other hand, we have $\operatorname{Ext}^1(X, Y) = 0$ for X regular and Y preinjective. This implies that the preinjective modules are \mathcal{B} -saturated modules. It follows that the only \mathcal{B} -saturated bristled modules are the modules $M = I_t$ with $0 \leq t \leq q$. \square

3.4. Remark. There is the following consequence: For $n = 2$, the set of indecomposable bristled modules is characterized by their dimension vectors.

Let us mention already here that for $n \geq 3$, the set of indecomposable bristled modules (even the set of faithful ones) is not characterized by their dimension vectors, as already the dimension vector $(3, 2)$ shows:



4. PROOF OF THE MAIN THEOREM.

In this section, we will assume that $n \geq 3$. We will identify the index set $\{1, 2, \dots, n\}$ of the arrow set of $K(n)$ with $\mathbb{Z}/n\mathbb{Z}$, thus sometimes we will write α_{n+1} instead of α_1 .

4.1. We will use the following notation for some bristles (this convention deviates from the previous notation, but there should be no confusion). If $1 \leq r \leq n$, let $B(r)$ be defined by $B(r)_1 = B(r)_2 = k$ with α_i being the identity map in case $i = r$, and the zero map otherwise. If $r \neq s$ are numbers between 1 and n , let $B(r, s)$ be defined by $B(r, s)_1 = B(r, s)_2 = k$ with α_i being the identity map in case $i \in \{r, s\}$, and the zero map otherwise.

4.2. Let \mathcal{B}_0 be the set of (the isomorphism classes of) the bristles $B(n-1), B(n)$ and the bristles $B(r, r+1)$ with $1 \leq r \leq n$ (recall that $B(n, n+1)$ is just $B(n, 1)$). Note that the cardinality of \mathcal{B}_0 is $n+2$. Let \mathcal{B}'_0 be obtained from \mathcal{B}_0 by deleting $B(n-1, n)$, thus the cardinality of \mathcal{B}'_0 is $n+1$.

4.3. For the proof of the assertion (b) of Theorem 1.3, we may assume that M is indecomposable. In case M is preprojective, nothing has to be shown since $\tau^t M = 0$ for t large. The case that M is preinjective is considered in assertion (a). Finally, there is the case that M is regular. Since the class of modules which are generated by \mathcal{B}_0 and \mathcal{B} -saturated is closed under extensions, it is sufficient to consider just the elementary modules as introduced by Crawley-Boevey (see [3] and

[4], or section 4.14 below). The proof of (a) dealing with the preinjective modules and of the statement (b) for elementary modules proceeds in several steps and will be completed in 4.20.

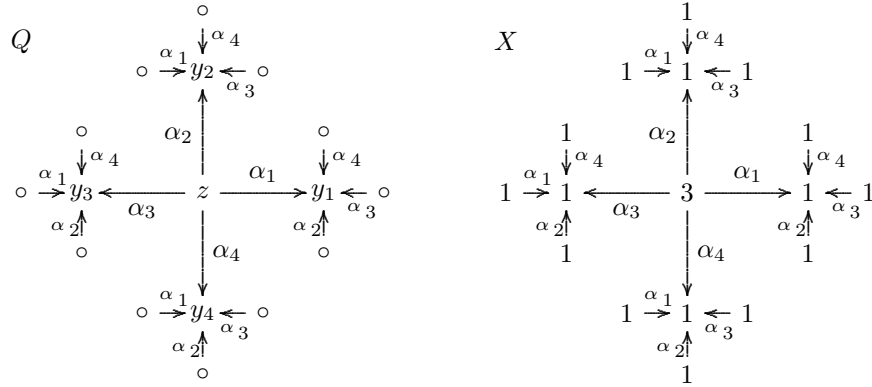
4.4. *The preinjective module I_2 is generated by \mathcal{B}'_0 , thus by \mathcal{B}_0 .*

For the proof of 4.4 (and also for the proof of 4.6) we are going to use the universal cover $\tilde{K}(n)$ of the n -Kronecker quiver $K(n)$ (see for example [1] and [2]) in order to exhibit sufficiently many bristles as submodules of the modules I_2 and $\tau B(1)$. Note that the quiver $\tilde{K}(n)$ is the n -regular tree with bipartite orientation. The representations of $\tilde{K}(n)$ will be called $\tilde{K}(n)$ -modules (we may consider them as modules over the corresponding path algebra; we should note that the path algebra of such a quiver with infinitely many vertices does not have a unit element, but at least sufficiently many primitive idempotents).

The quiver $\tilde{K}(n)$ is a “labeled” quiver: The projection $\pi: \tilde{K}(n) \rightarrow K(n)$ sends the arrow α of $\tilde{K}(n)$ to an arrow $\pi(\alpha)$ of $K(n)$, this is one of $\alpha_1, \dots, \alpha_n$, and we use $\pi(\alpha)$ (or its index i with $1 \leq i \leq n$) as a label for α . By abuse of notation, we sometimes denote α by its label $\pi(\alpha)$. We also denote by π the corresponding push-down functor $\text{mod } \tilde{K}(n) \rightarrow \text{mod } K(n)$.

Proof of 4.4. Let $M = I_2$ and let X be a representation of $\tilde{K}(n)$ with $\pi(X) = M$, thus there is a vertex z of $\tilde{K}(n)$ such that the support Q of X consists of the vertices in $\tilde{K}(n)$ with distance at most 2 from z and $\dim X_z = n - 1$, whereas $\dim X_a = 1$ for the remaining vertices a in Q .

Here is, in the case $n = 4$, the quiver Q as well as the dimension vector of X



Always, we denote by y_j the neighbor of z such that the arrow $z \rightarrow y_j$ has label j . Recall that a *leaf* of a tree is a vertex with at most one neighbor. The leaves of Q are just the vertices which have distance 2 from the center z ; in the picture on the left, they are indicated by small circles \circ . If x is a leaf of Q and the arrow starting in x has label α_i and ends in y_j , we write $x = x(j, i)$ and we say that x is of *type* i .

In order to find bristle submodules in M , we will exhibit some submodules of X ; all these submodules are indecomposable and their support is a path in Q .

For $1 \leq j \leq n$, let $Q(j)$ be the full subquiver of Q with vertices y_j and $x(j, i)$ where $1 \leq i \leq n$ and $i \neq j$. Note that $Q(j)$ is a quiver with one sink and $n - 1$ sources (thus it is a so-called subspace quiver).

For $1 \leq j \leq n$, let $Y(j) = X|Q(j)$. Since the subquiver $Q(j)$ of Q is closed under successors, we see that $Y(j)$ is a submodule of X . Since the subquivers $Q(j)$ are pairwise disjoint, the submodules $Y(j)$ form a direct sum $Y = \bigoplus_j Y(j)$ and the factor module X/Y is nothing else than X_z , considered as a direct sum of copies of the simple representation $S(z)$.

Let $N(j) = \pi(Y(j))$ and $N = \bigoplus_j N(j)$; these are submodules of M . The factor module M/N is a direct sum of copies of $S(1)$ and may be identified with M_z . We are going to look on the one hand at N , and will show that any $N(j)$ is generated by bristles which are denoted by $D(x)$ and $E(x)$, where x is a leaf in $Q(j)$, see (I) below. On the other hand, we have to find bristle submodules $M(i, j)$ of M which are not contained in N , see (II) below.

For any leaf x , let $P(x)$ be the indecomposable projective representation of Q corresponding to the vertex x . Note that $P(x)$ is a bristle, and, of course, also $D(x) = \pi(P(x))$ is a bristle. If x is of type i , the bristle $D(x)$ is isomorphic to $B(i)$.

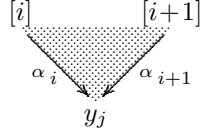
(I) The bristle submodules $D(x)$ and $E(x)$ of M . We are going to show the following:

For any j , the submodule $N(j)$ is generated by bristle submodules which belong to \mathcal{B}'_0 .

Proof. Fix some j and write $[i] = x(j, i)$. In order to reduce the number of brackets, we will write $P[i]$ instead of $P(x(j, i))$ and so on.

Let $G(j)$ be the set of leaves in $Q(j)$ of types $n-1$ and n . If $x \in G(j)$, then $D(x) = \pi(P(x))$ belongs to \mathcal{B}'_0 . If $j \in \{n-1, n\}$, then $G(j)$ is a single leaf, otherwise $G(j)$ consists of two leaves.

We consider in $Q(j)$ subquivers with three vertices, namely those of the following kind:



such that $i \neq n-1$ (note that we also have that $i \neq j$ and $i+1 \neq j$, thus $i \neq j-1$) and we denote by $V[i] = V(x(j, i))$ the indecomposable kQ -module with this support $\{y_j, [i], [i+1]\}$. Let $H(j)$ be the set of all $1 \leq i \leq n$ with $i \notin \{j-1, j, n-1\}$. Note that $i \in H(j)$ if and only if $V[i]$ is defined. We claim that

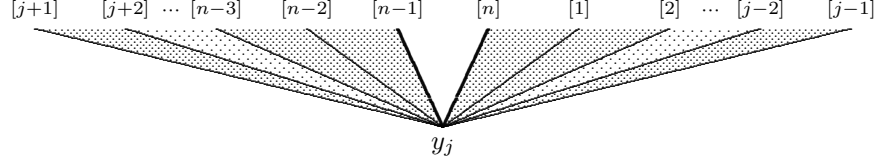
$$(1) \quad Y(j) = \sum_{i \in G(j)} P[i] + \sum_{i \in H(j)} V[i].$$

Any $V[i]$ has two bristle submodules, namely $P[i]$ and $P[i+1]$. The Kronecker module $\pi(V[i])$ has the bristle submodules $D[i] = \pi(P[i])$ and $D[i+1] = \pi(P[i+1])$, but actually many more bristle submodules: any non-zero element in the top of $\pi(V[i])$ generates a bristle. We are interested in the submodule $E[i] \subset \pi(V[i])$ which is isomorphic to $B(i, i+1)$, thus belongs to \mathcal{B}'_0 (since $i \neq n-1$). We claim that

$$(2) \quad N(j) = \pi(Y(j)) = \sum_{i \in G(j)} D[i] + \sum_{i \in H(j)} E[i].$$

For the proof of (1) and (2) we distinguish the cases whether j is one of the numbers $n-1, n$ or not.

Consider first the case $j \notin \{n-1, n\}$. Then $G(j) = \{n-1, n\}$ and $H(j) = \{j+1, j+2, \dots, n-2, n, 1, \dots, j-2\}$. The support of the modules $V[i]$ with $i \in H(j)$ are the various shaded triangles in the following picture:



Let us show (1) and (2) in this case. Consider first (1). Of course, the submodules mentioned on the right side of (1) have support in $Q(j)$, thus they are submodules of $Y(j)$. What we have to show is that for any leaf $x = [i]$ of $Q(j)$, the module $P(x)$ is included in the right side. We only have to deal with the vertices $[i]$, where $1 \leq i \leq n-2$ (and, of course, $i \neq j$). First, let $1 \leq i \leq j-1$. We know that $P[i]$ is contained in $V[i-1]$ and $i-1$ is either equal to n or $1 \leq i-1 \leq j-2$, thus always $i-1$ belongs to $H(j)$. Second, let $j+1 \leq i \leq n-2$. In this case, we use that $P[i]$ is contained in $V[i]$ and that i itself belongs to $H(j)$. This completes the proof of (1).

The equality (1) implies that

$$\pi(Y(j)) = \sum_{i \in G(j)} \pi(P[i]) + \sum_{i \in H(j)} \pi(V[i]),$$

and $\pi(P[i]) = D[i]$. It remains to be seen that for any $i \in H(j)$, the submodule $\pi(V[i])$ is contained in the right hand side R of (2). For $i = n$, we have $\pi(V[n]) = D[n] + E[n]$ and n belongs both to $G(j)$ and to $H(j)$. Next, let $1 \leq i \leq j-2$. We use induction on i in order to see that $\pi(V[i])$ is contained in R . Now

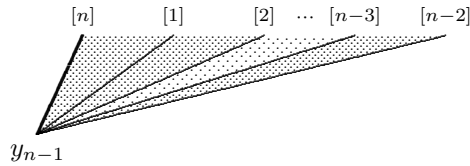
$$\pi(V[i]) = \pi(P[i]) + E[i] \subset \pi(V[i-1]) + E[i].$$

For $i = 1$, we know already that $\pi(V[i-1]) = \pi(V[n])$ belongs to R ; for $i > 1$, we know by induction that $\pi V[i-1] \subset R$. Thus, we see that let all $1 \leq i \leq j-2$, the module $\pi(V[i])$ is contained in R .

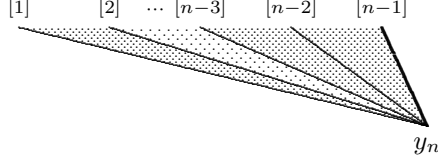
Second, we have to show that $\pi(V[i])$ is contained in R , for $j+1 \leq i \leq n-2$. This time, we use induction going downwards, starting with $i = n-2$, and the fact that $\pi(V[i]) = E[i] + \pi(P[i+1])$. For the start $i = n-2$ of the induction, we have $i+1 = n-1$ and this number belongs to $G(j)$, thus $\pi(P[i+1]) = D[i+1] = D[n-1]$ belongs to R . For $i < n-2$, we have $\pi(P[i+1]) \subset \pi(V[i+1])$ and $\pi(V[i+1]) \subset R$, by induction. This completes the proof of (2) in the case that $j \notin \{n-1, n\}$.

Now, we deal with the two cases where $j \in \{n-1, n\}$. In both cases, the proof will be similar to the previous proof, thus we only describe the setting in detail.

If $j = n-1$, then $G(j) = \{n\}$ and $H(j) = \{n, 1, \dots, n-3\}$. The support of the modules $V[i]$ with $i \in H(j)$ are the various shaded triangles in the following picture:

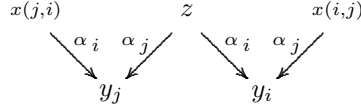


For $j = n$, we have $G(j) = \{n - 1\}$ and $H(j) = \{1, \dots, n - 2\}$. The support of the modules $V[i]$ with $i \in H(j)$ are the various shaded triangles in the following picture:



As we have mentioned, in both case $j = n - 1$ and $j = n$, the proof of the equalities (1) and (2) proceeds as in the case when $j \notin \{n - 1, n\}$, but using just one induction and not two. In the case $j = n - 1$, we have to use the upgoing induction, in the case $j = n$ the downgoing induction.

(II) The bristle submodules $M(i, j)$ of M . For any pair $i \neq j$ with $1 \leq i, j \leq n$, there is a (unique) path $p(i, j) = \alpha_i^{-1} \alpha_j \alpha_i^{-1} \alpha_j$ in Q between $x(j, i)$ and $x(i, j)$:



Let $W(i, j)$ be the indecomposable kQ -module whose support is the path $p(i, j)$.

Observe that $W(i, j)$ is a submodule of X ; it is defined by

$$W(i, j)_z = \bigcap_{s \notin \{i, j\}} \text{Ker}(\alpha_s)$$

(this is a one-dimensional subspace of X_z) and $W(i, j)_a = X_a$ for the remaining vertices a on the path $p(i, j)$.

Let us look at $\pi(W(i, j))$. We claim that the n -Kronecker module $\pi(W(i, j))$ contains a bristle $M(i, j)$ which is isomorphic to $B(i, j)$. The top of $\pi(W(i, j))$ is given by

$$W(i, j)_{x(j, i)} \oplus W(i, j)_z \oplus W(i, j)_{x(i, j)} = k \oplus k \oplus k.$$

The element $(1, 1, 1) \in k^3$ is mapped under both α_i and α_j to $(1, 1) \in W(i, j)_{y_i} \oplus W(i, j)_{y_j}$. This shows that the $kK(n)$ -submodule $M(i, j)$ of $\pi(W(i, j))$ generated by $(1, 1, 1) \in W(i, j)_{x(j, i)} \oplus W(i, j)_z \oplus W(i, j)_{x(i, j)}$ is isomorphic to $B(i, j)$. Altogether, we see that

$$\pi(W(i, j)) = \pi(P(x(j, i))) + M(i, j) + \pi(P(x(i, j))).$$

Since both $\pi(P(x(j, i)))$ and $\pi(P(x(i, j)))$ are contained in N , it follows that

$$(3) \quad \pi(W(i, j)) \subset N + M(i, j).$$

The pairs (i, j) which we will consider are those of the form $(i, i + 1)$. Let I be a subset of $\{1, 2, \dots, n\}$ of cardinality $n - 1$, thus obtained from $\{1, 2, \dots, n\}$ by deleting one of its elements. We will use the submodules $W(i, i + 1)$ with $i \in I$.

For example, for $n = 4$, and $I = \{1, 2, 3\}$, here are the dimension vectors of the submodules $W(i, i + 1)$ of X with $i \in I$ (as well as the corresponding paths

$\alpha_i^{-1}\alpha_{i+1}\alpha_i^{-1}\alpha_{i+1}$):

$$\begin{array}{ccc}
 W(1,2) & W(2,3) & W(3,4) \\
 \begin{array}{c}
 0 \\
 \vdots \\
 1 \xrightarrow{\alpha_1} 1 \cdots 0 \\
 \vdots \\
 \alpha_2 \uparrow \\
 0 \cdots 0 \cdots 1 \xrightarrow{\alpha_1} 1 \cdots 0 \\
 \vdots \\
 \alpha_2 \uparrow \\
 0 \cdots 0 \cdots 0 \\
 \vdots \\
 0
 \end{array} &
 \begin{array}{c}
 0 \\
 \vdots \\
 0 \cdots 1 \xleftarrow{\alpha_3} 1 \\
 \vdots \\
 \alpha_2 \uparrow \\
 0 \cdots 1 \xleftarrow{\alpha_3} 1 \cdots 0 \cdots 0 \\
 \vdots \\
 \alpha_2 \uparrow \\
 1 \\
 \vdots \\
 0 \cdots 0 \cdots 0 \\
 \vdots \\
 0
 \end{array} &
 \begin{array}{c}
 0 \\
 \vdots \\
 0 \cdots 0 \cdots 0 \\
 \vdots \\
 1 \downarrow \alpha_4 \\
 0 \cdots 1 \xleftarrow{\alpha_3} 1 \cdots 0 \cdots 0 \\
 \vdots \\
 \alpha_4 \downarrow \\
 0 \cdots 1 \xleftarrow{\alpha_3} 1 \\
 \vdots \\
 0
 \end{array} \\
 \alpha_1^{-1}\alpha_2\alpha_1^{-1}\alpha_2 & \alpha_2^{-1}\alpha_3\alpha_2^{-1}\alpha_3 & \alpha_3^{-1}\alpha_4\alpha_3^{-1}\alpha_4
 \end{array}$$

Here is the essential assertion: *If $I \subset \{1, 2, \dots, n\}$ has cardinality $n - 1$, then*

$$(4) \quad M = N + \sum_{i \in I} M(i, i + 1).$$

Proof of (4). We can assume that $I = \{1, 2, \dots, n - 1\}$. First, we show that there is a vector space decomposition

$$(5) \quad X_z = \bigoplus_{i \in I} W(i, i + 1)_z.$$

For the proof of (5), we recall that the arrow of type α_i starting in z ends in the vertex y_i . The Auslander-Reiten sequence in $\text{mod } kQ$ ending in $S(z)$ is of the form

$$0 \longrightarrow X \xrightarrow{\iota} \bigoplus_{1 \leq i \leq n} I(y_i) \xrightarrow{[\pi(1), \dots, \pi(n)]} S(z) \longrightarrow 0$$

where the support of $I(y_i)$ consists of y_i and its neighbors, where $\pi(i): I(y_i) \rightarrow S(z)$ is the canonical projection. We can assume that ι is an inclusion map. At the vertex z , there is the corresponding exact sequence of vector spaces

$$0 \longrightarrow X_z \xrightarrow{\iota_z} \bigoplus_{1 \leq i \leq n} I(y_i)_z \xrightarrow{[\pi(1)_z, \dots, \pi(n)_z]} S(z)_z \longrightarrow 0$$

with $I(y_i)_z = k$, $S(z)_z = k$ and $\pi(i)_z$ the identity map. Thus X_z is the kernel of the map

$$[\pi(1)_z, \dots, \pi(n)_z] = [1, \dots, 1]: k^n \rightarrow k$$

and therefore generated by the elements $e(i) - e(i + 1)$, with $1 \leq i < n$; here $e(i)$ is the canonical generator of $k = I(y_i)_z$. Of course, $e(i) - e(i + 1)$ belongs to the kernel of $\alpha_s: X_z \rightarrow X_{a_s}$ for $s \notin \{i, i + 1\}$, thus to $W(i, i + 1)_z$. This shows that

$$X_z = \bigoplus_{1 \leq i < n} \langle e(i) - e(i + 1) \rangle = \bigoplus_{1 \leq i < n} W(i, i + 1)_z.$$

This completes the proof of (5).

As a consequence of (5), we have

$$X = Y + \sum_{i \in I} W(i, i + 1).$$

We apply π and obtain

$$M = \pi(X) = N + \sum_{i \in I} \pi(W(i, i+1)).$$

Now we use (3) in order to see that

$$M = N + \sum_{i \in I} \pi(W(i, i+1)) \subseteq N + \sum_{i \in I} M(i, i+1) \subseteq M,$$

thus we obtain (4). \square

In order to complete the proof of 4.4, we choose $I = \{1, 2, \dots, n-2, n\}$. Then all the submodules $M(i, i+1)$ with $i \in I$ belong to \mathcal{B}'_0 . We know already by (I) that N is generated by \mathcal{B}'_0 . Thus the equality (4) shows that M is generated by \mathcal{B}'_0 . \square

4.5. Remark: Book-keeping. For the sake of book-keeping, let us mention for any bristle type the number of bristles used for generating $M = I_2$. The number of leaves of Q of types $n-1$ and n is $n-1$. Thus, as bristles of the form $D(x)$ we have used $n-1$ bristles isomorphic to $B(n-1)$ as well as $n-1$ bristles isomorphic to $B(n)$. Let $I = \{1, 2, \dots, n-2, n\}$. For any $i \in I$, there are $n-2$ modules of the form $V[i]$, namely with support inside $Q(j)$, where $j \notin \{i, i+1\}$, thus we have used $n-2$ bristles of the form $E[i]$ and these bristles are isomorphic to $B(i, i+1)$. Finally, for $i \in I$, we needed the module $M(i, i+1)$. Now, $M(i, i+1)$ again is isomorphic to $B(i, i+1)$. Altogether, we see that for any bristle $B \in \mathcal{B}'_0$, the number of bristles isomorphic to B used in order to generate I_2 is $n-1$.

Therefore, the total number of bristles used for generating I_2 is $(n+1)(n-1) = n^2 - 1$. Since $\dim I_2 = (n^2 - 1, n)$, the module $\text{top } I_2$ is of length $n^2 - 1$, thus a minimal number of bristle submodules which generate I_2 consists of precisely $n^2 - 1$ bristles. We also should mention that for any bristle B ,

$$\dim \text{Hom}(B, I_2) = \left\langle \binom{1}{1}, \binom{n^2 - 1}{n} \right\rangle = n^2 - 1 + n - n^2 = n - 1$$

(since $\text{Ext}^1(B, I_2) = 0$). Looking at any minimal set of bristle submodules which generate I_2 , this set can contain at most $n-1$ bristles isomorphic to B .

4.6. *The module $\tau B(1)$ is generated by \mathcal{B}_0 and it has a proper submodule isomorphic to $B(1)$.*

Proof. The proof of the first assertion is similar to the considerations in 4.4 dealing with I_2 . Actually, one may consider $\tau B(1)$ as a submodule of I_2 , generated by some of the bristles constructed above.

Here, we consider a representation X' of $\tilde{K}(n)$ with $\pi(X') = \tau B(1)$. We denote by Q' its support: it is obtained from Q defined above as follows: Deleting the arrow $z \rightarrow y_1$, we obtain two connected components and we delete the component which contains y_1 (this is just $Q(1)$ as considered in 4.4). Thus Q' consists of the vertices a of $\tilde{K}(n)$ which have distance at most 2 from z , and such that the path from z to a does not start with the arrow with label α_1 . We have $\dim X'_z = n-2$, and $\dim X'_a = 1$ for the remaining vertices a in Q' .

The module X' can be described as a submodule of X as follows: X'_z is the kernel of the map $X_z \rightarrow X_{y_1}$, and $X'_a = X_a$ for the remaining vertices a in Q' .

As in the proof of 4.4, we first consider the subquivers $Q(j)$ and the restriction of X' to $Q(j)$. Of course, now we deal only with $j \neq 1$. Actually, the restrictions

of X' to $Q(j)$ is just $Y(j)$ as considered in 4.4. We recall that all the modules $Y(j)$ are generated by \mathcal{B}_0 .

Second, we need bristles which are not contained in $\pi(Y')$, where $Y' = \bigoplus_{j=2}^n Y(j)$. We start with the submodules $W(i, i+1) \subset X'$, where $2 \leq i \leq n-1$ and take the corresponding submodules $M(i, i+1)$ of $\pi(W(i, i+1))$. As we know, $M(i, i+1)$ is isomorphic to $B(i, i+1) \in \mathcal{B}_0$.

Since $\tau B(1) = \pi(Y') + \sum_{i=2}^{n-1} M(i, i+1)$, we see that $\tau B(1)$ is generated by \mathcal{B}_0 . This yields the first assertion.

For the proof of the second assertion, just take any submodule $\pi(P(x(j), 1))$ with $j \neq 1$. Of course, one may also refer to the Auslander-Reiten formula which yields non-zero homomorphisms (and this embeddings) $B(1) \rightarrow \tau B(1)$, since obviously $\text{Ext}^1(B(1), B(1)) \neq 0$. \square

Here are, for $n = 4$, on the left the dimension vector of X' , in the middle and on the right those of $W(2, 3)$ and $W(3, 4)$, respectively.

$$\begin{array}{ccc}
 X' & W(2, 3) & W(3, 4) \\
 \begin{array}{c}
 \begin{array}{c}
 1 \\
 \downarrow \alpha_4 \\
 1 \xrightarrow{\alpha_1} 1 \xleftarrow{\alpha_3} 1 \\
 \uparrow \alpha_2 \\
 2 \\
 \downarrow \alpha_4 \\
 1 \xrightarrow{\alpha_1} 1 \xleftarrow{\alpha_3} 1 \\
 \uparrow \alpha_2 \\
 1
 \end{array}
 \end{array}
 &
 \begin{array}{c}
 0 \\
 \vdots \\
 0 \cdots 1 \xleftarrow{\alpha_3} 1 \\
 \uparrow \alpha_2 \\
 0 \cdots 1 \xleftarrow{\alpha_3} 1 \\
 \uparrow \alpha_2 \\
 0 \\
 \vdots \\
 0
 \end{array}
 &
 \begin{array}{c}
 0 \\
 \vdots \\
 0 \cdots 0 \cdots 0 \\
 \downarrow \alpha_4 \\
 0 \cdots 1 \xleftarrow{\alpha_3} 1 \\
 \downarrow \alpha_4 \\
 0 \cdots 1 \xleftarrow{\alpha_3} 1 \\
 \downarrow \alpha_4 \\
 0
 \end{array}
 \end{array}
 \end{array}$$

$$\begin{array}{cc}
 \alpha_2^{-1} \alpha_3 \alpha_2^{-1} \alpha_3 & \alpha_3^{-1} \alpha_4 \alpha_3^{-1} \alpha_4
 \end{array}$$

4.7. Remark: Again the book-keeping. The number of leaves of Q' of type $n-1$ is $n-2$, as is the number of leaves of type n . Thus, as bristles of the form $D(x)$ we have used $n-2$ bristles isomorphic to $B(n-1)$ as well as $n-2$ bristles isomorphic to $B(n)$. The number of bristles $E(x)$ which we have used are as follows: We have used $n-2$ bristles $E(x)$ isomorphic to $B(1, 2)$, we have used $n-3$ bristles $E(x)$ isomorphic to $B(i, i+1)$, for any i with $2 \leq i \leq n-2$, and we have used $n-2$ bristles $E(x)$ isomorphic to $B(n, 1)$. In addition, there are the bristles $M(i, i+1)$ isomorphic to $B(i, i+1)$. Altogether we have used $n-2$ bristles of the form $B(n-1), B(n)$ and $B(i, i+1)$ with $1 \leq i \leq n-2$ and with $i = n$ and a single bristle of the form $B(n-1, n)$.

In 4.6 we have presented a set of cardinality $n+2$ in order to generate $\tau B(1)$. As we will see in section 4.24, there are bristle sets of cardinality $n+1$ which generate $\tau B(1)$, but they are not helpful for the remaining proof of 1.3.

4.8. Corollary to 4.6. *If B is a bristle, then τB is generated by bristles, but is not a bristle, and $\tau^{-1} B$ is cogenerated by bristles and is not a bristle.*

Proof. Using an automorphism of the Kronecker algebra, we may shift B to $B(1)$, apply 4.6 and shift back. This shows that τB is generated by bristles. Using duality, it follows that τ^{-1} is cogenerated by bristles. \square

4.9. *If B, B' are non-isomorphic bristles, then $\text{Ext}^1(B, \tau B') = 0$.*

Proof. This follows from the isomorphism $\text{Ext}^1(B, \tau B') \simeq D \text{Hom}(B', B)$, where D denotes k -duality, since non-isomorphic bristles are Hom-orthogonal. \square

4.10. *Let M be an indecomposable module and U a submodule of M isomorphic to $B(1)$. If $\tau(M/U)$ is generated by \mathcal{B}_0 , then τM is generated by \mathcal{B}_0 .*

Proof. We apply τ to the exact sequence

$$0 \rightarrow U \rightarrow M \rightarrow M/U \rightarrow 0$$

and obtain the exact sequence

$$0 \rightarrow \tau U \rightarrow \tau M \rightarrow \tau(M/U) \rightarrow 0.$$

Consider any homomorphism $f: B \rightarrow \tau(M/U)$ with $B \in \mathcal{B}_0$. Since B is not isomorphic to $B(1)$, 4.9 shows that $\text{Ext}^1(B, \tau B(1)) = 0$, thus f can be lifted to τM . And 4.6 asserts that τU is generated by \mathcal{B}_0 . Since by assumption also $\tau(M/U)$ is generated by \mathcal{B}_0 , it follows that τM is generated by \mathcal{B}_0 . \square

4.11. *All preinjective modules are generated by \mathcal{B}_0 and are \mathcal{B} -saturated.*

Proof. First, let us show that I_t with $t \geq 0$ is generated by \mathcal{B}_0 . We use induction on t . The assertion is trivial for $t = 0$. The module I_1 is obviously generated by $B(2), \dots, B(n)$ and $B(1, 2)$, thus by \mathcal{B}_0 . Next, 4.4 asserts that I_2 is generated by \mathcal{B}_0 . Thus, we can assume that $t \geq 3$.

Clearly, $\text{Hom}(B(1), I_s) \neq 0$ for any $s \geq 0$, thus $\text{Hom}(B(1), I_{t-2}) \neq 0$. Take a nonzero homomorphism $f: B(1) \rightarrow I_{t-2}$. Since $t - 2 \geq 1$, the map f has to be a monomorphism and its cokernel C will be a direct sum of preinjective modules of the form I_s with $s < t - 2$. We apply τ to the exact sequence

$$0 \rightarrow B(1) \rightarrow I_{t-2} \rightarrow C \rightarrow 0$$

and obtain the exact sequence

$$0 \rightarrow \tau B(1) \rightarrow I_t \rightarrow \tau C \rightarrow 0.$$

Now τC is a direct sum of modules of the form I_{s+2} with $s + 2 < t$. By induction, C is generated by \mathcal{B}_0 , thus 4.10 shows that I_t is generated by \mathcal{B}_0 .

Of course, all preinjective modules I_t satisfy $\text{Ext}^1(X, I_t) = 0$, for X regular. Since bristles are regular, we see that I_t is \mathcal{B} -saturated. \square

4.12. *Let M be an indecomposable module which is generated by bristles, but not a bristle. Then $\text{Hom}(M, B) = 0$ for any bristle B , thus τM is \mathcal{B} -saturated.*

Proof. Since M is generated by bristles, there are bristles B_1, \dots, B_t and a surjective map $f: \bigoplus B_i \rightarrow M$. Assume there exists a non-zero map $g: M \rightarrow B$. Then $gf_i \neq 0$ for some i . But a non-zero map between bristles is an isomorphism, thus g is a split monomorphism. Since M is indecomposable, g has to be an isomorphism, thus M is a bristle. This is the first assertion. The vector space isomorphism $\text{Ext}^1(B, \tau M) \simeq D \underline{\text{Hom}}(M, B)$ shows that $\text{Ext}^1(B, \tau M) = 0$ for any bristle B , thus τM is \mathcal{B} -saturated. \square

We also will need the dual assertion of 4.12:

4.13. *Let M be an indecomposable module which is cogenerated by bristles, but not a bristle. Then $\text{Hom}(B, M) = 0$ for any bristle B .*

4.14. Let us consider now elementary modules as defined by Crawley-Bovey and studied by Lukas and Kerner (see for example [3]): a regular module M is said to be *elementary* provided that for any non-zero regular submodule U of M , the factor module M/U is preinjective.

Let M be an elementary module with $\text{Hom}(B(1), M) \neq 0$. Then $\tau^t M$ is generated by \mathcal{B}_0 for all $t \geq 1$.

Proof. Let M be an elementary module with $\text{Hom}(B(1), M) \neq 0$. A non-zero homomorphism $f: B(1) \rightarrow M$ has to be injective, since M has no submodule isomorphic to $S(1)$. Since M is elementary, the cokernel C of f has to be preinjective. Thus, we deal with an exact sequence of the form

$$0 \rightarrow B(1) \rightarrow M \rightarrow C \rightarrow 0.$$

If we apply τ , we obtain the exact sequence

$$0 \rightarrow \tau B(1) \rightarrow \tau M \rightarrow \tau C \rightarrow 0.$$

Now τC is preinjective, thus according to 4.11, generated by \mathcal{B}_0 . According to 4.10, also τM is generated by \mathcal{B}_0 .

Now, the module τM is again elementary, and it has $\tau B(1)$ as a submodule. According to 4.6, $\tau B(1)$ has a submodule isomorphic to $B(1)$. It follows that $\text{Hom}(B(1), \tau M) \neq 0$. This shows, that τM satisfies again the assumption, thus also $\tau^2 M$ is generated by \mathcal{B}_0 . Altogether we see that we can use induction. \square

Since bristles are elementary modules, we see:

4.15. Corollary. *If B is a bristle, then the module $\tau^t B$ is generated by \mathcal{B}_0 , for all $t \geq 1$.*

4.16. Corollary. *If B is a bristle, then the module $\tau^{-t} B$ is cogenerated by bristles, for all $t \geq 1$.*

Corollary 4.16 follows from 4.15 by using duality.

4.17. *Any τ -orbit contains at most one bristle. If B, B' are bristles, then we have $\text{Hom}(\tau^t B, B') = 0$ for $t \geq 1$.*

Proof. Assume B and $\tau^t B$ are bristles, for some $t \geq 1$. This implies that $\binom{1}{1}$ is an eigenvector of Φ^t with eigenvalue 1, where Φ is the Coxeter transformation for $K(n)$. It is well-known (and easy to see) that 1 is not an eigenvalue of Φ^t for any $t \geq 1$. This yields the first assertion.

For the proof of the second assertion, let $t \geq 1$. We know that $\tau^t B$ is generated by bristles, thus there are maps $f_i: B_i \rightarrow \tau^t B$ with B_i a bristle, such that $(f_i)_i: \bigoplus B_i \rightarrow \tau^t B$ is surjective. Assume there exists a non-zero map $g: \tau^t B \rightarrow B'$. Then $gf_i \neq 0$ for some i . A non-zero map between bristles is an isomorphism, thus gf_i is an isomorphism and therefore $g: \tau^t B \rightarrow B'$ is split epi. Since $\tau^t B$ is indecomposable, we see that g is an isomorphism, thus $\tau^t B$ is a bristle. But the τ -orbit of τ cannot contain a second bristle. \square

4.18. *Let M be an elementary module with $\text{Hom}(B(1), M) \neq 0$. Then $\tau^t M$ is \mathcal{B} -saturated for all $t \geq 2$.*

Proof. Let M be an elementary module with $\text{Hom}(B(1), M) \neq 0$. Let $t \geq 2$. By 4.14 we know that $\tau^{t-1} M$ is generated by bristles. If we know that $\tau^{t-1} M$ is not a bristle, then we can apply 4.12 in order to see that $\tau^t M$ is \mathcal{B} -saturated.

Thus assume that $B = \tau^{t-1} M$ is a bristle. Then $M = \tau^{-t+1} B$ is cogenerated by bristles, according to Corollary 4.16. Since $\text{Hom}(B(1), M) = 0$, 4.13 shows that M has to be a bristle. But according to the first assertion 4.17, we know that any τ -orbit contains at most one bristle. \square

4.19. Corollary. *If B is a bristle, then $\tau^t B$ is \mathcal{B} -saturated for all $t \geq 2$.*

Note that the assumption $t \geq 2$ is essential, since τB is not \mathcal{B} -saturated: we obviously have $\text{Ext}^1(B, \tau B) \neq 0$.

4.20. Proof of Theorem 1.3. Part (a) has been established in 4.11. Thus it remains to deal with part (b). As we have mentioned already, it is sufficient to look at a module M which is regular.

First, let us consider the special case of an elementary module M . It has been shown by D. Baer that for any pair M', M'' of regular modules there is some number $s(M', M'')$ such that $\text{Hom}(M', \tau^s M'') \neq 0$ for all $s \geq s(M', M'')$ (see for example [3], 10.7). Thus $\text{Hom}(B(1), \tau^s M) \neq 0$ for all $s \geq s(B(1), M)$. According to 4.14 and 4.18, we know that $\tau^s M$ is generated by \mathcal{B}_0 and is \mathcal{B} -saturated for all $s \geq s(B(1), M) + 2$.

Now any regular module M has a filtration by elementary modules, say

$$0 = M_0 \subset M_1 \subset \cdots \subset M_m = M$$

with all M_i/M_{i-1} for $1 \leq i \leq m$ being elementary. Let

$$t(M) = \max_i s(B(1), M_i/M_{i-1}) + 2.$$

Let $t \geq t(M)$. The module $\tau^t M$ has the filtration

$$0 = \tau^t M_0 \subset \tau^t M_1 \subset \cdots \subset \tau^t M_m = \tau^t M$$

and the factors are $\tau^t M_i/\tau^t M_{i-1} = \tau^t(M_i/M_{i-1})$. These are elementary modules which are generated by \mathcal{B}_0 and are \mathcal{B} -saturated. Since the class of modules which are generated by \mathcal{B}_0 and \mathcal{B} -saturated is closed under extensions, it follows that also $\tau^t M$ is generated by \mathcal{B}_0 and \mathcal{B} -saturated. This completes the proof. \square

4.21. Corollary. *There is a regular module X which generates all the preinjective modules, for example $X = \bigoplus_{B \in \mathcal{B}_0} B$. There are also indecomposable regular modules which generate all preinjective modules.*

This provides an explicit answer to a question raised by Kerner in 1994 (Problem 10.9 in [3]): to exhibit a regular module X which generates all the preinjective modules. An existence proof for such a module was given by Kerner in [4].

Proof. The first assertion is a direct consequence of 1.3.

For the second assertion, let X be a suitable extensions $0 \rightarrow B(1) \rightarrow \overline{X} \rightarrow X \rightarrow 0$ with \overline{X} indecomposable (such an extension exists, since $\mathcal{B}_0 \cup \{B(1)\}$ is a set of pairwise non-isomorphic bristles, and pairwise non-isomorphic bristles are pairwise orthogonal, and $\text{Ext}^1(B, B') \neq 0$ for any pair B, B' of bristles). Such a module \overline{X} has dimension vector $(n+2, n+2)$. \square

4.22. Remark. Let us stress that the existence of a (finitely generated!) regular module which generates all preinjective modules is a special feature for the cases $n \geq 3$: It is a finiteness result in the wild cases which does not hold in the tame case $n = 2$, see Appendix C.

4.23. Addendum to the Main Theorem. Let \mathcal{B}' be a set of $n + 1$ bristles and let $t \geq 3$. The preinjective module I_t cannot be generated by \mathcal{B}' . Thus, the number $n + 2$ occurring in the Main Theorem is best possible.

Proof. First, let $t = 3$. We use the Euler form $\langle -, - \rangle$ on the Grothendieck group $K_0(\text{mod } \Lambda)$. Note that I_3 has the dimension vector $\mathbf{dim} I_3 = (n^3 - 2n, n^2 - 1)$. We have

$$\left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} n^3 - 2n \\ n^2 - 1 \end{pmatrix} \right\rangle = (n^3 - 2n) + (n^2 - 1) - n(n^2 - 1) = n^2 - n - 1.$$

Since for any bristle B , we have $\text{Ext}^1(B, I_3) = 0$, it follows that

$$\dim \text{Hom}(B, I_3) = n^2 - n - 1.$$

Let X be the direct sum of $n + 1$ pairwise non-isomorphic bristles. It follows that there is a right X -approximation of the form $f: X^t \rightarrow I_3$ with $t \leq n^2 - n - 1$. Now

$$\dim \text{top } X^t = (n + 1)t \leq (n + 1)(n^2 - n - 1) = n^3 - 2n - 1.$$

On the other hand,

$$\dim \text{top } I_3 = n^3 - 2n,$$

thus f cannot be surjective. This shows that X does not generate I_3 .

Now consider any $t \geq 3$. Assume that I_t is generated by \mathcal{B}' . Since I_t generates I_3 , it follows that \mathcal{B}' generates I_3 . But we have shown already that this is not possible. \square

4.24. Let us look at optimal ways to generate $\tau B(1)$, using bristle submodules.

Let \mathcal{B}' be a set of $n + 1$ bristles. If $\tau B(1)$ is generated by \mathcal{B}' , then $B(1)$ has to belong to \mathcal{B}' . The set \mathcal{B}'_1 given by $B(1)$ as well as the bristles $B(i, i + 1)$ for $1 \leq i \leq n$ generates $\tau B(1)$.

Proof. We have $\mathbf{dim} \tau B(1) = (n^2 - n - 1, n - 1)$. If B is a bristle, then

$$\left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} n^2 - n - 1 \\ n - 1 \end{pmatrix} \right\rangle = (n^2 - n - 1) + (n - 1) - n(n - 1) = n - 2.$$

Since bristles are bricks and pairwise orthogonal, the Auslander-Reiten formula shows that $\dim \text{Ext}^1(B(1), B(1)) = 1$ and $\dim \text{Ext}^1(B, B(1)) = 0$, provided B is not isomorphic to $B(1)$. It follows that $\dim \text{Hom}(B(1), \tau B(1)) = n - 1$, but $\dim \text{Hom}(B, B(1)) = n - 2$ in case B is not isomorphic to $B(1)$.

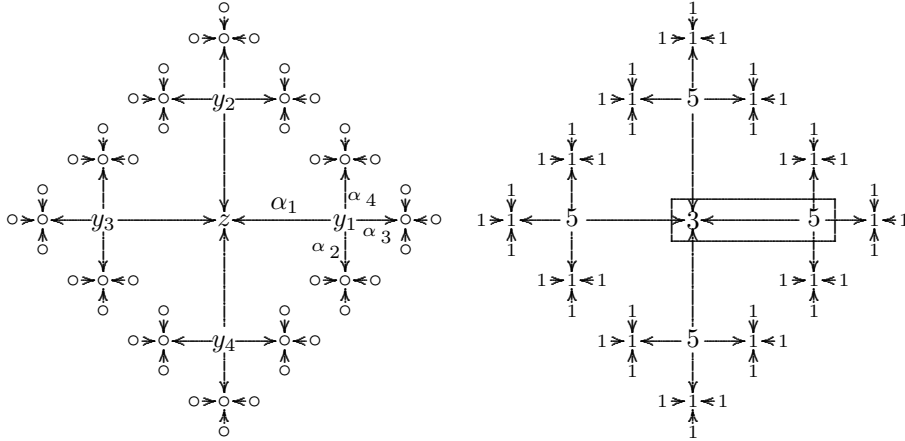
Let \mathcal{B}' be a set of bristles of cardinality $n + 1$ and assume that \mathcal{B}' does not contain $B(1)$. Since $\dim \text{Hom}(B, \tau B(1)) = n - 2$ for any $B \in \mathcal{B}'$, there is a right \mathcal{B}' -approximation of the form $f: X^t \rightarrow \tau B(1)$ with $t \leq n - 2$. We have $\dim \text{top } X^t = (n + 1)t \leq (n + 1)(n - 2) = n^2 - n - 2$, and $\dim \text{top } \tau B(1) = n^2 - n - 1$. As a consequence, f cannot be surjective. This shows that \mathcal{B}' cannot generate $\tau(B(1))$.

The proof of the second assertion is similar to the proofs presented in 4.4 and 4.6. \square

As we have seen in the proof, the existence of sufficiently many bristles in $\tau B(1)$ which are isomorphic to $B(1)$ is related to the non-vanishing of $\text{Ext}^1(B(1), \tau B(1))$. Of course, the only non-trivial extension is given by the middle term $\mu(B(1))$ of the Auslander-Reiten sequence ending in $B(1)$. We refer to the end of section 5.1 for a closer look at $\mu(B(1))$.

4.25. The use of the universal cover $\tilde{K}(n)$ of $K(n)$. We have used the universal cover $\tilde{K}(n)$ of $K(n)$ in order to exhibit bristle submodules of some n -Kronecker modules M .

(1) Any leaf x in the support of the covering module yields the bristle module $P(x)$ for $\tilde{K}(n)$, provided that x is a source. But there may be other bristle submodules of M of the form $\pi(U)$, where U is a $\tilde{K}(n)$ -bristle. Let us look at the covering module M for I_3 and, on the left, its support quiver:



Consider in the subspace M_{y_1} the intersection A of the three maps with labels $\alpha_1, \alpha_2, \alpha_3$. This subspace A is mapped under α_1 injectively to M_z , thus it generates a direct sum of two bristles of type $B(1)$ (see also (B.4) in the appendix).

Since $\dim \text{Hom}(B(1), I_3) = 11$, we see that we obtain sufficiently many bristles of type $B(1)$ by looking at the 9 leaves of type 1 and two of the bristles generated by elements of A .

(2) The universal cover X of I_2 displays all the bristles of type $B(i)$ of I_2 (with $1 \leq i \leq n$) in a nice way: as coming from bristles of \tilde{I}_2 given by leaves in Q . Namely, the number of leaves x in Q which are starting points of arrows with label α_i is $n-1$, and, as we have mentioned already in 4.5, we have $\dim \text{Hom}(B(i), I_2) = n-1$.

(3) The universal cover X of I_2 allows also to trace the bristles of type

$$B(1, 2, \dots, n) = (k, k; 1, 1, \dots, 1).$$

For $1 \leq j \leq n$, let $Y(j)$ be the sum of the submodules $P(x)$ of X , where x is a leaf and a neighbor of y_j . Then $\pi(Y(j))$ has dimension vector $\binom{n-1}{j}$, and $X/\sum Y(j)$ is a direct sum of $n-1$ copies of $S(z)$, and canonically isomorphic to X_z . On the one hand, we have $\dim \text{Hom}(B(1, 2, \dots, n), I_2) = n-1$, on the other hand $\text{Hom}(B(1, 2, \dots, n), Y(j)) = 0$ for all j . This shows that X has a submodule U isomorphic to $B(1, 2, \dots, n)^{n-1}$ such that the composition of the canonical maps $U \rightarrow X \rightarrow X/\sum Y(j)$ is surjective.

5. FURTHER CONSIDERATIONS.

5.1. Proposition. *Let M be an indecomposable bristled module, not a bristle and not simple projective. If τM is a bristled module, then the middle term $\mu(M)$ of the Auslander-Reiten sequence ending in M is a bristled module.*

If $n \geq 2$, then the middle term $\mu(B)$ of the Auslander-Reiten sequence ending in a bristle B is not a bristled module.

Proof. Let M be an indecomposable bristled module, not a bristle and not simple projective. Then M is not projective, thus we can consider the middle term $\mu(M)$ of the Auslander-Reiten sequence ending in M . Since the canonical map $\mu(M) \rightarrow M$ is right almost split, all the maps $B \rightarrow M$ with B a bristle can be lifted to $\mu(M)$. Together with the maps $B \rightarrow \tau(M)$, where B is a bristle, we obtain enough maps from bristles to $\mu(M)$ in order to see that $\mu(M)$ is generated by bristles.

Now consider $\mu(B)$, where B is a bristle and let $\epsilon: \mu(B) \rightarrow B$ be the canonical map. Assume that $\mu(B)$ is bristled. Then $\mu(B)$ is a factor module of a direct sum of bristles, thus there is a map $f: B' \rightarrow \mu(B)$ with B' a bristle such that $\epsilon f \neq 0$. But any non-zero map between bristles is an isomorphism, thus ϵf is an isomorphism and therefore ϵ is a split epimorphism. This is impossible, thus $\mu(B)$ cannot be bristled. \square

Let us have a closer look at $\mu(B)$, say for $B = B(1)$. It can be realized as a submodule of I_2 , as follows. As in section 4.4, we consider the universal cover $\tilde{K}(n)$ of $K(n)$ with push-down functor π and denote by X a representation of $\tilde{K}(n)$ with $\pi(X) = I_2$. Let X'' be the submodule of X generated by the submodules $Y(j)$ with $2 \leq j \leq n$ and the subspace X_z . Thus we have

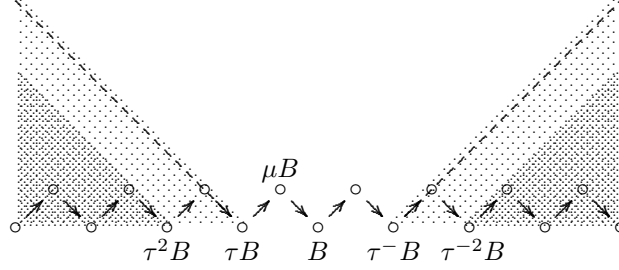
$$X' \subset X'' \subset X,$$

with X' as defined in 4.6 (and $\pi(X') = \tau B(1)$). In order to see that $\pi(X'') = \mu(B(1))$, we only have to observe that X'' is indecomposable and that X''/X' is an indecomposable length 2 module whose support is an arrow with label α_1 , thus $\pi(X''/X')$ is isomorphic to $B(1)$.

For example in case $n = 4$, the dimension vector of X'' looks as follows:

$$\begin{array}{c}
 X'' \\
 \begin{array}{ccccc}
 & & 1 & & \\
 & & \downarrow \alpha_4 & & \\
 & 1 & \xrightarrow{\alpha_1} & 1 & \xleftarrow{\alpha_3} & 1 \\
 & & & \uparrow \alpha_2 & & \\
 & & & 3 & \xrightarrow{\alpha_1} & 1 \\
 & 1 & \xrightarrow{\alpha_1} & 1 & \xleftarrow{\alpha_3} & 1 \\
 & & & \uparrow \alpha_2 & & \\
 & & & 1 & & \\
 & & & \downarrow \alpha_4 & & \\
 & & & 1 & \xrightarrow{\alpha_1} & 1 & \xleftarrow{\alpha_3} & 1 \\
 & & & & \uparrow \alpha_2 & & \\
 & & & & 1 & &
 \end{array}
 \end{array}$$

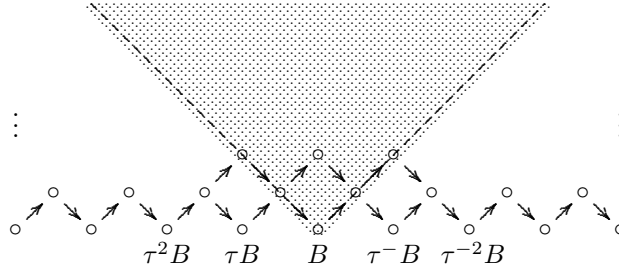
5.2. Let $n \geq 3$. The Auslander-Reiten component containing a bristle B has the following shape



On the left hand side, we see the shaded cocone of all modules with a path in the Auslander-Reiten quiver ending in τB , all these modules are bristled modules; the darker shading on the left marks the region of the modules which are in addition \mathcal{B} -saturated (the assertions follow from 4.15 and 4.19 and the closure under extensions).

On the right hand side, we see the shaded cone of all modules with a path in the Auslander-Reiten quiver starting at $\tau^- B$. By duality, all these modules are cogenerated by bristles. Here, the darker shading indicates that the corresponding modules M are in addition \mathcal{B} -cosaturated.

The remaining modules M in the component (those which are neither predecessors of τB nor successors of $\tau^- B$) all have bristle submodules as well as bristle factor modules (since the northeast arrows correspond to monomorphisms, the southeast arrows to epimorphisms); thus, with the exception of B itself, are neither bristled nor cobristled.



5.3. Let $n \geq 2$. If M is a module with a sectional path ending in a bristle B , then M is not \mathcal{B} -saturated. (In particular, if μB is the middle term of the Auslander-Reiten sequence ending in B , then μB is not \mathcal{B} -saturated.)

Proof. The epimorphism $M \rightarrow B$ yields a surjection $\text{Ext}^1(B, M) \rightarrow \text{Ext}^1(B, B)$. Calculating $\langle \mathbf{dim} B, \mathbf{dim} B \rangle$, we see that $\dim \text{Ext}^1(B, B) = n - 1$. \square

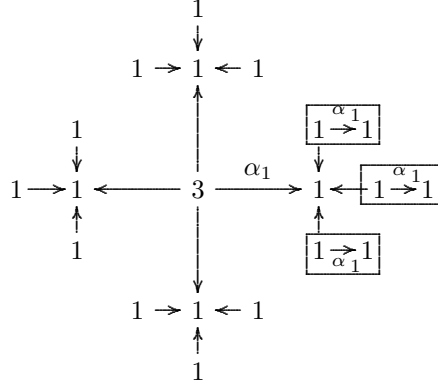
5.4. Actually, it is easy to see that for any bristle B , we have $\dim \text{Ext}^1(B, \mu B) = n - 1$. Namely, the Auslander-Reiten sequence ending in B yields an exact sequence

$$\text{Hom}(B, \mu B) \rightarrow \text{Hom}(B, B) \rightarrow \text{Ext}^1(\tau B, B) \rightarrow \text{Ext}^1(\mu B, B) \rightarrow \text{Ext}^1(B, B) \rightarrow 0.$$

The first map cannot be surjective, since otherwise the canonical projection $\mu B \rightarrow B$ would split. Since $\text{End}(B) = k$, we see that the first map is the zero map, thus the second map is injective. Again using that $\text{End}(B) = k$, we have $\dim \text{Ext}^1(\tau B, B) = 1$. Thus, the second map is an isomorphism and therefore the map $\text{Ext}^1(\mu B, B) \rightarrow$

$\text{Ext}^1(B, B)$ is bijective. As we have mentioned in the proof above, one may use the bilinear form $\langle -, - \rangle$ in order to see that $\dim \text{Ext}^1(B, B) = n - 1$. Thus, also $\dim \text{Ext}^1(\mu B, B) = n - 1$.

The following picture shows for $n = 4$ and $B = B(1)$ an indecomposable representation X''' of the universal cover $\tilde{K}(4)$ with submodule X'' such that $\pi(X'') = \mu(B(1))$ and $\pi(X'''/X'')$ is isomorphic to $B(1)^3$. Thus, the universal cover exhibits nicely a basis of $\text{Ext}^1(B(1), \mu(B(1)))$.



We have just seen that $\mu(B(1))$ is not \mathcal{B} -saturated. The following proposition shows that there are many modules, even many bristled modules, which are not \mathcal{B} -saturated.

5.5. Proposition. *An indecomposable \mathcal{B} -saturated module is simple or faithful.*

Proof. For $n = 1$, all indecomposable modules are simple or faithful. Thus, we may assume that $n \geq 2$. Let M be a non-faithful indecomposable n -Kronecker module. Let us assume that M is annihilated by α_n . We also assume that M is different from $S(1)$. The following lemma shows that M is not \mathcal{B} -saturated. \square

5.6. Lemma. *If M is a $K(n)$ -module which is annihilated by α_n , then*

$$\dim \text{Ext}^1(B(1), M) \geq \dim M_2.$$

Proof. We may consider both $B(1)$ and M also as $(n - 1)$ -Kronecker modules. We calculate $\langle \mathbf{dim} B(1), \mathbf{dim} M \rangle_t$ for $t = n$ and for $t = n - 1$. We have

$$\begin{aligned} \langle \mathbf{dim} B(1), \mathbf{dim} M \rangle_n &= \dim M_1 - (n - 1) \dim M_2, \\ \langle \mathbf{dim} B(1), \mathbf{dim} M \rangle_{n-1} &= \dim M_1 - (n - 2) \dim M_2, \end{aligned}$$

thus the difference is

$$D = \langle \mathbf{dim} B(1), \mathbf{dim} M \rangle_{n-1} - \langle \mathbf{dim} B(1), \mathbf{dim} M \rangle_n = \dim M_2.$$

Since there is no difference whether we calculate Hom for $K(n)$ or for $K(n - 1)$, the difference D is also

$$D = -\dim \text{Ext}_{K(n-1)}^1(B(1), M) + \dim \text{Ext}_{K(n)}^1(B(1), M).$$

Altogether, we see that

$$\dim \text{Ext}_{K(n)}^1(B(1), M) = \dim \text{Ext}_{K(n-1)}^1(B(1), M) + \dim M_2 \geq \dim M_2.$$

\square

APPENDIX A. THE BRISTLE VARIETY $\beta(M)$ OF A KRONECKER MODULE M .

We assume here that k is algebraically closed.

Let $M = (M_1, M_2, \alpha_1, \dots, \alpha_n)$ be a Kronecker module. First, we deal with the case where M has no direct summand of the form $S(1)$, thus the intersection of the kernels of the maps α_i is zero. Let $\mathbb{P}M_1$ be the projective space corresponding to M_1 , this is the set of all 1-dimensional subspaces of M_1 , endowed with the Zariski topology. If m is a non-zero element of M_1 , we write $\langle m \rangle$ for the k -subspace generated by m (this is an element of $\mathbb{P}M_1$). Let $\beta(M)$ be the subset of $\mathbb{P}M_1$ of all $\langle u \rangle$ such that u generates a bristle submodule in M_1 (or, equivalently, such that $\alpha_1(u), \dots, \alpha_n(u)$ generate a one-dimensional subspace of M_2). It is easy to see that $\beta(M)$ is closed in the Zariski topology, thus $\beta(M)$ is a projective variety, we call it the *bristle variety* of M .

For a general n -Kronecker module M , the *bristle variety* of M is by definition $\beta(M/M')$, where M' is the largest submodule of M generated by $S(1)$ (so that M/M' has no submodule of the form $S(1)$).

(A.1) Theorem. *Any projective variety occurs as a bristle variety.*

The proof of Theorem (A.1) is given in [6]. Note that this theorem is a variant of a sequence of results by Zimmermann-Huisgen, Hille, Reineke and Van den Bergh which concern the realization of projective varieties say as quiver Grassmannians. As a consequence of Theorem (A.1), the following is shown in [7]: *given any connected wild acyclic quiver Q with at least 3 vertices, then any projective variety occurs as a quiver Grassmannian for a suitable representation of Q .*

A Kronecker module $M = (M_1, M_2, \alpha_1, \dots, \alpha_n)$ is nothing else than a matrix pencil (as soon as we choose bases of the vector spaces M_1, M_2), and generators of bristle submodules can be considered as (generalized) eigenvectors. Indeed, if $n = 2$ and α_1 is the identity map of a vector space $V = M_1 = M_2$, then a submodule of the 2-Kronecker module $(V, V; 1, \phi)$ is a bristle of type $B_c = B(1 : c)$ if and only if any generator of U in M_1 is an eigenvector of ϕ with eigenvalue c .

APPENDIX B. BRISTLED MODULES FOR ARBITRARY ARTIN ALGEBRAS.

Let Λ be an artin algebra. Given a Λ -module M , let $|M|$ be its length. Here, we collect some properties of the bristled Λ -modules and show in which way the study of bristled Λ -modules can be reduced to the study of bristled Kronecker modules.

(B.1) Proposition. *If M is an indecomposable bristled Λ -module M then $|\text{top } M| \geq |\text{soc } M|$, with equality only in case M is simple or a bristle.*

Proof. A bristled module has Loewy-length at most 2. It follows that any indecomposable bristled Λ -module M is either simple or $\text{soc } M = \text{rad } M$. Of course, if M is simple, then $|\text{top } M| = 1 = |\text{soc } M|$.

Thus, let us assume that M is a bristled module with $\text{soc } M = \text{rad } M$. If $\text{top } M$ has length t , there is a surjective map $f: \bigoplus_{i=1}^t B_i \rightarrow M$ with bristles B_i . The kernel of f has to be contained in the socle of $\bigoplus B_i$, let s be its length. Then $|\text{soc } M| = t - s \leq t$, this is the first assertion. If $|\text{soc } M| = t$, then $s = 0$, thus f is an isomorphism. If M is indecomposable, then we must have that $t = 1$, thus M is a bristle. \square

Recall that a semi-simple Λ -module is said to be *homogeneous* provided that all its simple submodules are isomorphic.

(B.2) Proposition. *The socle of an indecomposable bristled module M is homogeneous.*

Proof. Let M be an indecomposable bristled module. Note that M has Loewy length at most 2, since M is bristled. If M has Loewy length 1, then M is simple, therefore nothing has to be shown. Thus, we assume that M has Loewy length 2. This means that $\text{rad } M = \text{soc } M$.

Let S_1, \dots, S_n be the simple Λ -modules and write $\text{soc } M = \bigoplus_{i=1}^n H_i$, where H_i is the sum of all submodules of M isomorphic to S_i .

By assumption, M is generated by bristles, thus M is the sum of images of some maps $B \rightarrow M$, where B is a bristle. Since $\text{rad } M = \text{soc } M$, it follows that M is generated by the images of some maps $B \rightarrow M$ which are in addition injective, thus M is the sum of submodules U_j which are isomorphic to bristles, say $M = \sum_{j \in J} U_j$. We may assume that J is minimal, or, equivalently, that $M/\text{soc } M = \bigoplus (U_j + \text{soc } M)/\text{soc } M$.

Let M_i be the sum of the submodules U_j such that $\text{soc } U_j$ is isomorphic to S_i . Now $\text{soc } U_j$ is a submodule of $\text{soc } M$ and isomorphic to S_i , thus $\text{soc } U_j \subseteq H_i$. It follows easily that $M = \bigoplus_{i=1}^n M_i$. Since we assume that M is indecomposable, we have $M = M_i$ for some i . Of course, the socle of M_i is just H_i , thus homogeneous. \square

(B.3) We say that a module M is the *socle amalgamation of the modules* M_1, \dots, M_t provided the modules M_1, \dots, M_t are submodules of M containing the socle $\text{soc } M$ of M and $M/\text{soc } M$ is the direct sum $\bigoplus_i M_i/\text{soc } M$.

It follows from (B.2) that *any indecomposable bristled module M is the socle amalgamation of bristled modules M_1, \dots, M_t , such that $\text{soc } M$ as well all the modules $M_i/\text{soc } M$ are homogeneous.*

Namely, $\overline{M} = M/\text{soc } M$ is semi-simple, thus we may write \overline{M} as a direct sum $\bigoplus_i \overline{M}_i$ of pairwise Hom-orthogonal homogeneous modules \overline{M}_i . Let M_i be the preimage of \overline{M}_i under the canonical projection $M \rightarrow \overline{M}$.

Thus, the first aim for understanding bristle modules seems to be to investigate bristled modules M such that both $\text{soc } M$ and $M/\text{soc } M$ are homogeneous.

In case Λ is a finite-dimensional k -algebra, where k is an algebraically closed field, it is sufficient to look just at the Kronecker algebras. Namely, we may assume that M is faithful and that Λ is basic. Under these assumptions, Λ is a basic radical-square-zero algebra with at most two simple modules. Since k is algebraically closed, only two cases are possible. If Λ has two simple modules, then Λ is already a Kronecker algebra. If Λ has only one simple module, then Λ is of the form $k[x_1, \dots, x_n]/(x_1, \dots, x_n)^2$, and therefore stably equivalent to the Kronecker algebra $kK(n)$. Under such a stable equivalence, the indecomposable bristled Λ -modules correspond bijectively to the indecomposable bristled $kK(n)$ -modules.

(B.4) Let us assume now that Λ is the path algebra of a finite acyclic quiver Q without multiple arrows. In this case, the support of a bristle B is an arrow. Attaching to any arrow α the thin indecomposable module $B(\alpha)$ with support α , we obtain a bijection between the arrows and the (isomorphism classes of the) bristles. There

is an easy numerical criterion to detect (at least some) bristle submodules of a kQ -module M : If $\alpha: y \rightarrow z$ is an arrow and $z_1 = z, z_2, \dots, z_t$ are the terminal vertices of the arrows starting in y , then $\dim \text{Hom}(B(\alpha), M) \geq \dim M_y - \sum_{i \geq 2} \dim M_{z_i}$. (For the proof, one just has to look at $\langle \mathbf{dim} B(\alpha), \mathbf{dim} M \rangle$.) Of course, if B has no submodule isomorphic to $S(y)$, then $\dim \text{Hom}(B(\alpha), M) = b$ implies that M has a submodule isomorphic to $B(\alpha)^b$.

In order to see that in general not all bristles are detected in this way, consider the following $\widetilde{\mathbb{D}}_5$ -quiver shown below on the left. There exists an indecomposable representation M with dimension vector presented below on the right, which has a submodule $B(\alpha)$, whereas $\dim M_y - \sum_{i \geq 2} \dim M_{z_i} = 0$.



(B.5) Using the universal covering $\widetilde{K}(n)$ of $K(n)$, we have shown in the paper that all the preinjective $K(n)$ -modules are bristled. But we should note that the non-injective preinjective $\widetilde{K}(n)$ -modules themselves are not bristled.

Proposition. *Let Y be an indecomposable preinjective $\widetilde{K}(n)$ -module. If Y is injective, then Y is bristled. If Y is not injective, then Y is not bristled.*

Proof. It is obvious that the indecomposable injective $\widetilde{K}(n)$ -modules are bristled. Thus, we only have to consider the remaining indecomposable preinjective $\widetilde{K}(n)$ -modules.

First, we consider the $\widetilde{K}(n)$ -module $X = X(z) = \widetilde{\tau}S(z)$, where z is a source of $\widetilde{K}(n)$ (and $\widetilde{\tau}$ the Auslander-Reiten translation for $\widetilde{K}(n)$); note that X has been discussed already in 4.4. As before, we denote by $\alpha_i: z \rightarrow y_i$ the arrows starting at z . The thin indecomposable representation of $\widetilde{K}(n)$ with support $\alpha_i: z \rightarrow y_i$ will be denoted by B_i . Note that the modules B_i are the only bristles with top $S(z)$.

We claim that $\text{Hom}(B_i, X) = 0$. Denote by Q' the full subquiver of $\widetilde{K}(n)$ with vertices z, y_1, \dots, y_n . The restriction X' of X to Q' is indecomposable (and not injective). Since B_i and all its factor modules are injective, when considered as representations of Q' , we see that $\text{Hom}(B_i, X') = 0$, thus also $\text{Hom}(B_i, X) = 0$.

If $\mathcal{B}(X)$ is the maximal bristled submodule of X , then the top of $\mathcal{B}(X)$ has no composition factor of the form $S(z)$. As a consequence, $\mathcal{B}(X)$ is a proper submodule of X . Thus X is not bristled.

If Y is an indecomposable preinjective representation of $\widetilde{K}(n)$ which is not injective, then Y has a factor module of the form $X(z)$ for some z . Since $X(z)$ is not bristled, also Y cannot be bristled. \square

APPENDIX C. A NON-FINITENESS ASSERTION FOR TAME ALGEBRAS.

(C.1) Proposition. *Let Λ be a tame hereditary artin algebra. There is no regular module which generates infinitely many indecomposable modules. In particular, there is no regular or preinjective module which generates all preinjective modules.*

Proof. Obviously, a preinjective module generates only finitely many indecomposable modules (and these are preinjective again).

If R is simple regular and $t \in \mathbb{N}$, then the only regular modules generated by $R[t]$ are modules in the tube of R of regular length at most t , thus only finitely many isomorphism classes. This shows that a regular module generates only finitely many indecomposable regular modules.

Finally, let R be indecomposable regular and M indecomposable preinjective. Then $\dim \operatorname{Hom}(R, M) \leq rs$ where r is the regular length of R and s is the maximal length of a simple regular module in the tube of R (use τ in order to reduce to the case that M is indecomposable injective; under this shift, r and s do not change). It follows that M is a factor module of R^{rs} , thus of bounded length. \square

(C.2) In the case of the 2-Kronecker quiver, one may strengthen the assertion as follows:

Proposition. *Let $\Lambda = kK(2)$ and M in $\operatorname{mod} \Lambda$. Then the following assertions are equivalent:*

- (i) M generates infinitely many indecomposable modules.
- (ii) M generates all indecomposable modules with the exception of at most finitely many preprojective modules.
- (iii) M has an indecomposable preprojective direct summand.

If the preprojective module P_t is a direct summand of M and t is minimal, then the only indecomposable modules which are not generated by M are the modules P_0, \dots, P_{t-1} .

The proof is left to the reader. \square

REFERENCES

- [1] Fahr, Ph., Ringel, C. M.: A partition formula for Fibonacci numbers. *Journal of Integer Sequences*, Vol. 11 (2008), Article 08.1.4.
- [2] Fahr, Ph., Ringel, C. M.: Categorification of the Fibonacci numbers using representations of quivers. *Journal of Integer Sequences*. Vol. 15 (2012), Article 12.2.1
- [3] Kerner, O.: Representations of wild quivers. In: *Representation Theory of Algebras and Related Topics*. CMS Conference Proceedings 19. Amer. Math. Soc. (1994), 65-107
- [4] Kerner, O.: More representations of wild quivers. *Contemp. Math.* 607 (2014), 35-55.
- [5] Ringel, C. M.: *Tame algebras and integral quadratic forms*. Springer Lecture Notes in Mathematics 1099. (1984).
- [6] Ringel, C. M.: The eigenvector variety of a matrix pencil. To appear in: *Linear Algebra and Appl.* arXiv:1703.04097.
- [7] Ringel, C. M.: Quiver Grassmannians for Wild Acyclic Quivers. arXiv:1703.08782.

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