# ON CONTRAVARIANTLY FINITE SUBCATEGORIES

**Claus Michael Ringel**

Abstract. Let $A$ be an artin algebra. Let $\mathcal{X}_1, \mathcal{X}_2$ be contravariantly finite, extension closed subcategories of $A$–mod, and assume $\mathrm{Ext}^1_A(\mathcal{X}_2, \mathcal{X}_1) = 0$. Then also the full subcategory of all $A$–modules $M$ which have a submodule $U \in \mathcal{X}_2$ such that $M/U \in \mathcal{X}_1$, is contravariantly finite and extension closed in $A$–mod.

Let $A$ be an artin algebra. We will consider (finitely generated left) $A$–modules, maps between $A$–modules will be written on the right hand of the argument, thus the composition of the maps $f : M_1 \to M_2$, $g : M_2 \to M_3$ will be denoted by $fg$. The category of all $A$–modules will be denoted by $A$–mod. All subcategories considered will be full and closed under isomorphisms, so usually we will describe subcategories by just specifying their objects (up to isomorphism).

Let $\mathcal{X}$ be a subcategory of $A$–mod. Recall that $\mathcal{X}$ is said to be *extension closed* provided for any exact sequence $0 \to X_2 \to E \to X_1 \to 0$ with $X_1, X_2 \in \mathcal{X}$, also $E \in \mathcal{X}$. Given an $A$–module $M$, a *right $\mathcal{X}$–approximation* of $M$ is a map $g : X \to M$ with $X \in \mathcal{X}$ such that for any map $h : X' \to M$ with $X' \in \mathcal{X}$, there is a map $h' : X' \to X$ such that $h = h'g$. In case every $A$–module has a right $\mathcal{X}$–approximation, $\mathcal{X}$ is said to be *contravariantly finite in* $A$–mod. We write $\mathrm{Ext}^1_A(\mathcal{X}, Y) = 0$ as an abbreviation for $\mathrm{Ext}^1_A(X, Y) = 0$ for all $X \in \mathcal{X}$, and we use corresponding notation in similar cases.

There is the following criterion:

**Proposition.** *Let $\mathcal{X}$ be an extension closed subcategory of $A$–mod. Then $\mathcal{X}$ is contravariantly finite in* $A$–mod *if and only if any $A$–module $M$ can be embedded into an $A$–module $\overline{M}$ such that $\overline{M}/M \in \mathcal{X}$ and* $\mathrm{Ext}^1_A(\mathcal{X}, \overline{M}) = 0$.

**Proof:** One direction is due to Auslander–Reiten [**AR**], the other one has been shown in [**R**], Lemma 2. For the convenience of the reader, we indicate the arguments of [**AR**], but we delete the functorial and homological interpretations of the individual steps.

1980 *Mathematics Subject Classification* (1985*Revision*): 16A64,16A46

So assume that $\mathcal{X}$ is contravariantly finite in $A$–mod, and let $M$ be an arbitrary $A$–module. According to Auslander–Smalø[**AS**], there is an embeddding $v: M \hookrightarrow M'$ with $M'/M \in \mathcal{X}$ such that for any embedding $w: M \hookrightarrow Y$ with $Y/M \in \mathcal{X}$, there is a map $f: Y \to M'$ with $wf = v$. Indeed, we just construct a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M & \xrightarrow{v} & M' & \longrightarrow & X' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow{\scriptstyle g} & & \\
0 & \longrightarrow & M & \longrightarrow & I & \longrightarrow & \Sigma M & \longrightarrow & 0
\end{array}$$

with exact rows, $I$ an injective $A$–module, and $g$ a right $\mathcal{X}$–approximation, starting with the lower row. Of course, we can assume that $v$ is an embedding, and one easily checks that $v$ has the desired property.

Recall that a map $y: M \to Y$ is called *left minimal* provided any endomorphism $e$ of $Y$ with $ye = y$ is an automorphism. We can decompose $M' = \overline{M} \oplus M''$ so that the image of $v$ is contained in $\overline{M}$, say $v = [u\,0]$ with an embedding $u: M \hookrightarrow \overline{M}$ which is left minimal. In this way, we obtain a left minimal embedding $u$ of $M$ into $\overline{M}$ with $\overline{M}/M \in \mathcal{X}$ and such that for any embedding $w: M \hookrightarrow Y$ with $Y/M \in \mathcal{X}$, there is a map $f: Y \to \overline{M}$ with $wf = u$.

In order to see that $\operatorname{Ext}^1_A(\mathcal{X}, \overline{M}) = 0$, consider an embedding $h: \overline{M} \hookrightarrow H$ with $H/\overline{M} \in \mathcal{X}$. We claim that $h$ splits. The cokernel $H/M$ of $uh: M \hookrightarrow H$ belongs to $\mathcal{X}$, since both $H/\overline{M}$ and $\overline{M}/M$ belong to $\mathcal{X}$, and $\mathcal{X}$ is extension closed. Thus, there is a map $f: H \to \overline{M}$ with $uhf = u$. But $u$ is minimal, thus $hf$ is an automorphism, and therefore $h$ is a split monomorphism. This completes the proof.

Consider now the following situation: Given subcategories $\mathcal{X}_1, \mathcal{X}_2$ of $A$–mod, let $\mathcal{X}_1 \int \mathcal{X}_2$ be the full subcategory of all $A$–modules $M$ which have a submodule $U$ belonging to $\mathcal{X}_2$ such that $M/U$ belongs to $\mathcal{X}_1$. One may wonder whether with $\mathcal{X}_1, \mathcal{X}_2$ also $\mathcal{X}_1 \int \mathcal{X}_2$ is contravariantly finite in $A$–mod. Using the criterion above, we are able to show:

**Theorem.** *Let $\mathcal{X}_1, \mathcal{X}_2$ be subcategories with $\operatorname{Ext}^1_A(\mathcal{X}_2, \mathcal{X}_1) = 0$. If both $\mathcal{X}_1, \mathcal{X}_2$ are extension closed and contravariantly finite in* $A$–mod, *then also $\mathcal{X}_1 \int \mathcal{X}_2$ is extension closed and contravariantly finite in* $A$–mod.

**Proof:** Let $\mathcal{X} = \mathcal{X}_1 \int \mathcal{X}_2$. In order to show that $\mathcal{X}$ is extension closed, let $M$ be an $A$–module with a submodule $U$ such that both $U$ and $M/U$ belong to $\mathcal{X}$. By definition, there are submodules $U' \subseteq U \subseteq U'' \subseteq M$ such that both $U', U''/U \in \mathcal{X}_2$ and both $U/U', M/U'' \in \mathcal{X}_1$. Since $\operatorname{Ext}^1_A(U''/U, U/U') = 0$, there is a submodule $M'$ with $U' \subseteq M' \subseteq U''$ such that $U''/M' \cong U/U'$ and $M'/U' \cong U''/U$. Since $\mathcal{X}_1$ is closed under extensions, and $M/U'', U''/M' \in \mathcal{X}_1$, also $M/M' \in \mathcal{X}_1$. Similarly, since $\mathcal{X}_2$ is closed under extensions, $M' \in \mathcal{X}_2$. Thus, $M$ belongs to $\mathcal{X}$.

In order to show that $\mathcal{X}$ is contravariantly finite in $A$–mod, we apply the Proposition. Let $M$ be any $A$–module. We want to show that $M$ can be embedded into an $A$–module $\overline{M}$ such that $\overline{M}/M \in \mathcal{X}$ and $\operatorname{Ext}^1_A(\mathcal{X}, \overline{M}) = 0$. Since $\mathcal{X}_2$ is extension closed and contravariantly finite in $A$–mod, there is an embedding $M \hookrightarrow Y$ such that $Y/M \in \mathcal{X}_2$ and $\operatorname{Ext}^1_A(\mathcal{X}_2, Y) = 0$. Since $\mathcal{X}_1$ is extension closed and contravariantly finite in $A$–mod, there is an embedding $Y \hookrightarrow \overline{M}$ such that $\overline{M}/Y \in \mathcal{X}_1$ and $\operatorname{Ext}^1_A(\mathcal{X}_1, \overline{M}) = 0$. Clearly, $\overline{M}/M \in \mathcal{X}$, since there is the submodule $Y/M \in \mathcal{X}_2$ and $\overline{M}/Y \in \mathcal{X}_1$. It remains to be seen that $\operatorname{Ext}^1_A(\mathcal{X}, \overline{M}) = 0$. We know already $\operatorname{Ext}^1_A(\mathcal{X}_1, \overline{M}) = 0$, thus we have to show that $\operatorname{Ext}^1_A(\mathcal{X}_2, \overline{M}) = 0$. Consider the exact sequence $0 \to Y \to \overline{M} \to \overline{M}/Y \to 0$. Since $\operatorname{Ext}^1_A(\mathcal{X}_2, Y) = 0$ and $\overline{M}/Y \in \mathcal{X}_1$, it follows that $\operatorname{Ext}^1_A(\mathcal{X}_2, \overline{M}) = 0$.

Let us stress that the operation $\int$ on subcategories is obviously associative, so given subcategories $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n$, the subcategory $\mathcal{X}_1 \int \mathcal{X}_2 \int \cdots \int \mathcal{X}_n$ consists of the modules $M$ which have a filtration $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_n$ such that $M_{i-1}/M_1 \in \mathcal{X}_i$ for all $1 \le i \le t$. Using induction, we immediately obtain the following result:

**Corollary 1.** *Let $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n$, be subcategories which are extension closed and contravariantly finite in $A$–mod. Assume that $\operatorname{Ext}^1_A(\mathcal{X}_j, \mathcal{X}_i) = 0$ for all $j > i$. Then also $\mathcal{X}_1 \int \mathcal{X}_2 \int \cdots \int \mathcal{X}_n$ is extension closed and contravariantly finite in $A$–mod.*

There is the dual notion of covariantly finite subcategories: Let $\mathcal{X}$ be a subcategory of $A$–mod. Given an $A$–module $M$, a *left $\mathcal{X}$–approximation* of $M$ is a map $f : M \to X$ with $X \in \mathcal{X}$ such that for any map $h : X \to X'$ with $X' \in \mathcal{X}$, there is a map $h' : X \to X'$ such that $h = fh'$. In case every $A$–module has a left $\mathcal{X}$–approximation, $\mathcal{X}$ is said to be *covariantly finite in* $A$–mod. And $\mathcal{X}$ is said to be *functorially finite in* $A$–mod provided $\mathcal{X}$ is both contravariantly and covariantly finite in $A$–mod. The dual assertion of Corollary 1 is the following:

**Corollary 2.** *Let $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_n$, be subcategories which are extension closed and covariantly finite in $A$–mod. Assume that*

$\operatorname{Ext}^1_A(\mathcal{X}_j,\mathcal{X}_i) = 0$ *for all* $j > i$. *Then also* $\mathcal{X}_1 \int \mathcal{X}_2 \int \cdots \int \mathcal{X}_n$ *is extension closed and covariantly finite in* $A$–mod.

## Applications

As first application, we will obtain Theorem 1 of [**R**].

Let $\Theta = \{\Theta(1),\dots,\Theta(n)\}$ be a finite set of $A$–modules with

$$\operatorname{Ext}^1_A(\Theta(j),\Theta(i)) = 0 \quad \text{for} \quad j \geq i.$$

We denote by $\mathcal{F}(\Theta)$ the full subcategory of $A$–mod of direct summands of modules having a filtration with factors in $\Theta$, thus, $M$ belongs to $\mathcal{F}(\Theta)$ if and only if $M$ has submodules $M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_t = M$ such that $M_{s-1}/M_s$ is isomorphic to a module in $\Theta$.

**Corollary.** *The subcategory $\mathcal{F}(\Theta)$ is functorially finite in $A$–mod.*

**Proof:** For any $1 \leq i \leq n$, let $\mathcal{X}_i$ be the subcategory of all modules which are direct sums of copies of $\Theta(i)$. Since $\operatorname{Ext}^1_A(\Theta(i),\Theta(i)) = 0$, we see that $\mathcal{X}_i$ is closed under extensions. Also, it is well-known and easy to see that $\mathcal{X}_i$ is functorially finite in $A$–mod (in order to obtain a right $\mathcal{X}_i$–approximation for a module $M$, take $[g_1,\dots,g_t] : M \to \Theta(i)^t$, where $g_1,\dots,g_t$ is a $k$–basis of $\operatorname{Hom}_A(M,\Theta(i))$, and similarly, one obtains a left $\mathcal{X}_i$–approximation). The assumption $\operatorname{Ext}^1_A(\Theta(j),\Theta(i)) = 0$ for $j > i$ yields $\operatorname{Ext}^1_A(\mathcal{X}_j,\mathcal{X}_i) = 0$ for $j > i$, thus we can apply Corollary 1 and Corollary 2 in order to conclude that $\mathcal{X} = \mathcal{X}_1 \int \mathcal{X}_2 \int \cdots \int \mathcal{X}_n$ is functorially finite in $A$–mod. But, of course, $\mathcal{X} = \mathcal{F}(\Theta)$.

As a second application, we obtain a recent result of Smalø[**S**].

Let $e \in A$ be an idempotent such that $eA(1-e) = 0$. Let $R = eAe$, and $S = (1-e)A(1-e)$. Note that we may write $A$ as a lower triangular matrix ring $A = \begin{bmatrix} R & 0 \\ T & S \end{bmatrix}$ with $T = (1-e)Ae$. We may (and will) consider both $R$–mod and $S$–mod as subcategories of $A$–mod, namely, we identify $R$–mod with the subcategory of all $A$–modules $M$ with $eM = M$, and $S$–mod with the subcategory of all $A$–modules $M$ with $eM = 0$. In this way, $R$–mod and $S$–mod are subcategories which are closed under submodules, factor modules and extensions, and thus they are functorially finite in $A$–mod. Given an $A$–module $M$, then $(1-e)M$ is always an $A$–submodule which belongs to $S$–mod, and $M/(1-e)M$ belongs to $R$–mod. In particular, $\operatorname{Ext}^1_A(S\text{–mod}, R\text{–mod}) = 0$. For, given an

$A$–module $M$ with a submodule $U$ such that $U$ belongs to $R$–mod and $M/U$ belongs to $S$–mod, then $(1-e)M$ is a direct complement to $U$.

Let $\mathcal{R}$ be a subcategory of $R$–mod, and $\mathcal{S}$ a subcategory of $S$–mod. Following Smalø[**S**], we denote $\mathcal{R} \int \mathcal{S}$ by $A\text{–mod}^{\mathcal{R}}_{\mathcal{S}}$.

**Corollary.** *Let $\mathcal{R}$ be an extension closed subcategory of $R$–*mod*, and let $\mathcal{S}$ be an extension closed subcategory of $S$–*mod*. If $\mathcal{R}$ is contravariantly finite in $R$–*mod*, and $\mathcal{S}$ is contravariantly finite in $S$–*mod*, then $A$–*mod$^{\mathcal{R}}_{\mathcal{S}}$ *is contravariantly finite in $A$–*mod*. If $\mathcal{R}$ is covariantly finite in $R$–*mod*, and $\mathcal{S}$ is covariantly finite in $S$–*mod*, then $A$–*mod$^{\mathcal{R}}_{\mathcal{S}}$ *is covariantly finite in $A$–*mod*.*

**Proof:** Clearly, a subcategory $\mathcal{R}$ of $R$–mod which is extension closed, or contravariantly finite, or covariantly finite in $R$–mod, has the same property even in $A$–mod. And similarly, a subcategory $\mathcal{S}$ of $S$–mod which is extension closed, or contravariantly finite, or covariantly finite in $S$–mod, has the same property even in $A$–mod. Also, as we have noted above, we have $\operatorname{Ext}^1_A(S\text{–mod}, R\text{–mod}) = 0$, thus $\operatorname{Ext}^1_A(\mathcal{S}, \mathcal{R}) = 0$.

Both results generalize a previous observation of Grecht [**G**], de la Pena and Simson [**PS**], and Vossieck [**V**] on prinjective modules. Recall that an $A$–module $M$ is called *prinjective,* provided it belongs to $A\text{–mod}^{\mathcal{P}(R)}_{\mathcal{I}(S)}$, where $\mathcal{P}(R)$ is the subcategory of projective $R$–modules, $\mathcal{I}(S)$ the subcategory of injective $S$–modules. Thus, $M$ is prinjective if and only if $(1-e)M$ is an injective $S$–module, and $M/(1-e)M$ is a projective $R$–module. Note that we have

$$A\text{–mod}^{\mathcal{P}(R)}_{\mathcal{I}(S)} = \mathcal{F}(\Theta),$$

with $\Theta(1), \ldots, \Theta(m)$ the indecomposable projective $R$–modules, and $\Theta(m+1), \ldots, \Theta(n)$ the indecomposable injective $S$–modules.

## Remark

This paper is written in English in order to be accessible to readers throughout the world, but we would like to stress that this does not mean that we support any imperialism. Indeed, we were shocked when we heard that the Iraki military machinery was going to bomb Washington in reaction to the US invasion in Grenada and Panama, but maybe we were misinformed by the nowadays even openly admitted censorship.

## References

[AR] Auslander, M. and Reiten, I.: Applications of contravariantly finite subcategories. Adv.Math.(To appear)

[AS] Auslander, M. and Smalø, S.: Almost split sequences in subcategories. J. Algebra **69** (1981), 426–454.

[PS] de la Pena, J. and Simson, D.: Prinjective modules, reflection functors, quadratic forms and almost split sequences. Trans.Amer.Math.Soc. (to appear)

[G] Grecht, R.: Kategorien von Moduln mit Unterräumen. Diplomarbeit. Zürich (1986)

[R] Ringel, C.M.: The category of modules with good filtrations over a quasi–hereditary algebra has almost split sequences. Math.Z. (to appear)

[S] Smalø, S.: Functorial finite subcategories over triangular matrix rings. Preprint 6/1989 Trondheim (to appear)

[V] Vossieck, D.: Representations de bifuncteurs et interpretation en termes de modules. C.R.Acad.Sci.Paris. 307, Ser. I (1988). 713–716

C.M. Ringel
Fakultät für Mathematik
Universität
D–4800 Bielefeld 1
West–Germany