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The first Brauer–Thrall conjecture

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Abstract. Let Λ be an artin algebra and M a Λ-module. We show: If M is not the direct sum of copies of a finite number of indecomposable modules of finite length, then M has indecomposable submodules as well as indecomposable factor modules of arbitrarily large finite length. This improves the assertion of the first Brauer–Thrall conjecture as established by Roiter in 1968: Any artin algebra with infinitely many isomorphism classes of indecomposable modules of finite length has indecomposable modules of arbitrarily large finite length.

Key words. Artin algebra, modules of finite length, indecomposability, Krull–Remak–Schmidt–Azumaya theorem, Gabriel–Roiter measure, Gabriel–Roiter comeasure, Brauer–Thrall conjectures.

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Let Λ be an artin algebra. The modules to be considered are left Λ-modules, and not necessarily of finite length. A module M is said to be *of finite type,* provided M is the direct sum of (arbitrarily many) copies of a finite number of indecomposable modules of finite length.

Theorem 1. *A module M which is not of finite type contains indecomposable submodules of arbitrarily large finite length.*

Remarks. **(1)** Recall that Λ is said to be *representation-finite* provided there are only finitely many isomorphism classes of indecomposable Λ-modules of finite length, otherwise Λ is called *representation-infinite.* The first Brauer–Thrall conjecture asserts that *a representation-infinite artin algebra has indecomposable submodules of arbitrarily large finite length.* The conjecture was solved by Roiter [5] in 1968. The theorem can be seen as a strengthening: Assume that there are infinitely many isomorphism classes of indecomposable modules M_i; take the direct sum $M = \bigoplus M_i$. The (Krull–Remak–Schmidt–)Azumaya theorem shows that M is not of finite type, thus we obtain indecomposable modules of arbitrarily large finite length as submodules of this particular module M.

(2) In order to provide a proof of the first Brauer–Thrall conjecture, it is sufficient to deal with the case where there are infinitely many isomorphism classes of indecomposable modules M_i of a fixed length and consider the direct sum $M = \bigoplus M_i$. This case has been discussed in [2], Appendix A. There, we have shown that there are large indecomposable modules of finite length which are cogenerated by M (but they are not necessarily submodules of M, but only of the direct sum of countably many copies of M).

Note 1: [2, Appendix A] (here and in the following, analogous cases)?

The essential part of the present proof will consist in dealing precisely with the special case of M being the direct sum of infinitely many pairwise non-isomorphic indecomposable modules of equal length.

(3) A weaker assertion is the "direct sum theorem" (4.3) of [3]: *Assume that M is a module with only finitely many isomorphism classes of indecomposable submodules of finite length. Then M is a direct sum of finite length modules* (and thus obviously of finite type). The first part of the proof presented below will incorporate the corresponding arguments used in [3].

(4) Observe that the "direct sum theorem" (and thus our theorem) implies the following result (see [6], Corollary 9.5, or [4], and also [1], Corollary 4.8): *If an artin-algebra Λ is of finite type, then any Λ-module is of finite type.* We should stress that the converse implication is an obvious consequence of the (Krull–Remak–Schmidt)–Azumaya theorem.

Theorem 2. *A module M which is not of finite type has indecomposable factor modules of arbitrarily large finite length.*

Remarks. (5) Let us assume that M is a Λ-module which is not of finite type. This implies that Λ is representation-infinite, thus according to Auslander ([1], see also [2]) there are indecomposable Λ-modules of infinite length. We have shown that M contains indecomposable submodules and factor modules of arbitrarily large finite length, but M may not contain an indecomposable submodule or factor module of infinite length, as the following examples show:

Let Λ be the Kronecker algebra. Let P_i, be the preprojective indecomposable modules, let Q_i be the preinjective indecomposable modules, such that $\mathrm{Hom}(P_i, P_{i+1})$ and $\mathrm{Hom}(Q_{i+1}, Q_i)$ are non-zero, for all $i \in \mathbb{N}$.

First, consider $M = \bigoplus_{i\in\mathbb{N}} P_i$. We show that any indecomposable submodule U of M is of finite length. Let $M_j = \bigoplus_{j\le i} P_i$. Assume U is any submodule. If U is contained in all M_j, then $U = 0$. Thus assume that U is contained in M_j, but not in M_{j+1}. We get a non-zero map $U \to M_j/M_{j+1} = P_j$. According to [5], U splits off a direct summand of the form P_i with $i \le j$.

Second, let $M' = \bigoplus_{i\in\mathbb{N}} Q_i$. We claim that any indecomposable factor module X of M' is of finite length. Let $p : M' \to X$ be the projection map. Let $M'_j = \bigoplus_{i\le j} Q_i$. Since M' is the union of these submodules, there has to be some j such that the restriction of p to M'_j is non-zero. But $\mathrm{Hom}(M'_j, X) \ne 0$ implies that X has an indecomposable preinjective direct summand, see [5]. Since we assume that X is indecomposable, it follows that X is of finite length.

We do not know whether there may exist a module which is not of finite type such that all its indecomposable submodules as well as all its indecomposable factor modules are of finite length.

(6) For the sake of completeness, let us note that for any representation-infinite artin algebra, there always do exist also modules M of **finite** type which have indecomposable submodules of arbitrarily large finite length and indecomposable factor modules of arbitrarily large finite length. For example, let P be any generator (for example $P = {}_\Lambda\Lambda$), and M the direct sum of countably many copies of P. Then M is of finite type and any Λ-module of finite length occurs as a factor module of M. Similarly, if

I is a cogenerator (for example $I = D(\Lambda_\Lambda)$) and M the direct sum of countably many copies of I, then M is of finite type and any Λ-module of finite length occurs as a submodule of M.

Proof of Theorem 1. We assume that the module M is not of finite type and show the existence of indecomposable submodules of arbitrarily large length. Thus, assume that the indecomposable submodules of M of finite length are of bounded length, thus there are only finitely many possible Gabriel–Roiter measures, see [2] and [3]. Assume that the indecomposable submodules of M of finite length have Gabriel–Roiter measure $\gamma_1 < \gamma_2 < \cdots < \gamma_s$. We show by induction on s that M is of finite type. The case $s = 1$ is well-known and easy to see: if any indecomposable submodule of M of finite length is simple, then M has to be semi-simple, thus of finite type.

Note 2: *If* instead of *if*?

Assume now that $s \geq 2$. Consider a submodule M' of M which is a direct sum of modules of Gabriel–Roiter measure γ_s, and maximal with this property. If M' is of finite type, then [3], theorem 4.2 asserts that M' is Σ-pure injective in $\mathcal{D}(\gamma_s)$, and of course M' is a pure submodule of M, thus M' is a direct summand of M, say $M = M' \oplus M''$ for some module M''. However, the indecomposable submodules of M'' of finite length have Gabriel–Roiter measure $\gamma_1, \ldots, \gamma_{s-1}$ (note that γ_s cannot occur by the maximality of M'), thus by induction M'' is of finite type. Then also $M = M' \oplus M''$ is of finite type.

Thus we can assume that there is a submodule $M^1 = \bigoplus_{i \geq 1} M_i$ of M which is an infinite direct sum of pairwise non-isomorphic indecomposable modules M_i with Gabriel–Roiter measure γ_s, indexed over $\mathbb{N}$. For any $r \in \mathbb{N}$, let $M^r = \bigoplus_{i \geq r} M_i$.

The modules M_i have all the same length, say length t. Let $\mathcal{U}_r$ be the set of isomorphism classes of indecomposable submodules of M^r of length at most $t - 1$.

(a) *The set $\mathcal{U}_r$ is finite for almost all r.* Otherwise, choose inductively pairwise non-isomorphic submodules U_j of M^1 of length at most $t - 1$ such that $U = \sum_{j \in \mathbb{N}} U_j$ is the direct sum of the modules U_j. (Namely, assume we have found $U_1, \ldots, U_s$ with $U' = \bigoplus_{j=1}^{s} U_j \subseteq M^1$, then $U' \subseteq \bigoplus_{i=1}^{r-1} M_i$ for some r. If $\mathcal{U}_r$ is infinite, we find inside M^r an indecomposable submodule U_{s+1} of length at most $t - 1$ which is not isomorphic to any of the $U_1, \ldots, U_s$. Since $\bigoplus_{i=1}^{r-1} M_i$ and M^r intersect in zero, we see that $\sum_{j=1}^{s+1} U_j$ is a direct sum.) As a submodule of M, all the indecomposable submodules of U of finite length have Gabriel–Roiter measure γ_i with $1 \leq i \leq s$ and actually γ_s does not occur as a Gabriel–Roiter measure (since such a submodule would be a direct summand of U, impossible). By induction, U has to be of finite type – but by construction, $U = \bigoplus_{j \in \mathbb{N}} U_j$ is not of finite type.

Let $\mathcal{U} = \bigcap_r \mathcal{U}_r$. As we have seen, this is a finite set of isomorphism classes, and of course non-empty. There is some r' with $\mathcal{U} = \mathcal{U}_{r'}$ and without loss of generality, we can assume that $r' = 1$ (replacing M^1 by $M^{r'}$). Thus we deal with the following situation: $M^1 = \bigoplus_{i \geq 1} M_i$ is an infinite direct sum of pairwise non-isomorphic indecomposable modules M_i with Gabriel–Roiter measure γ_s, and any indecomposable submodule of M^1 of length at most $t - 1$ is also a submodule of $M^r = \bigoplus_{i \geq r} M_i$ for any r.

(b) *Any indecomposable module of length at most $t - 1$ and cogenerated by M^1 is isomorphic to a submodule of M^1.* Assume that N is of length at most $t - 1$ and cogenerated by M^1, thus there is a finite number of maps $\pi : N \to M_i$ such that the kernels of these maps intersect in zero. These maps π cannot be surjective, since N is of length at most $t-1$, whereas M_i is of length t. If we decompose the images $\pi(N)$ of these maps, we obtain indecomposable submodules N_j of M_i of length at most $t-1$, and such submodules N_j occur frequently inside M^1, namely inside M^r, for any r. This shows that N is a submodule of M^1.

(c) In particular, we see that there are only finitely many isomorphism classes of modules which are cogenerated by M^1 and of length at most $t-1$. Let S be the direct sum of all the simple modules. As in [2], we consider the class $\mathcal{N}$ of all indecomposable modules cogenerated by $M^1 \oplus S$ and not isomorphic to any M_i. Clearly, this class is again closed under cogeneration and still finite. For any module M_i, let $f^{\mathcal{N}} M_i$ be the maximal factor module of M_i which belongs to $\operatorname{add} \mathcal{N}$. Since M_i does not belong to $\mathcal{N}$, we see that $f^{\mathcal{N}} M_i$ is a module of length at most $t-1$ and cogenerated by $M^1 \oplus S$, thus there are only finitely many possibilities. It follows that there is a module Q in $\operatorname{add} N$ such that $f^{\mathcal{N}} M_i = Q$ for infinitely many i. Without loss of generality, we even may assume that $f^{\mathcal{N}} M_i = Q$ for all i (by deleting the remaining factors). For any module M_i, fix a projection $q_i : M_i \to Q$ and let K be the kernel of the map $(f_i)_i : M^1 \to Q$. Roiter's coamalgamation lemma (see [2]) asserts that K has no direct summand isomorphic to M_i, thus no submodule of Gabriel–Roiter measure γ_s (since M^1 belongs to $\mathcal{D}(\gamma_s)$ and M_i is relative injective in $\mathcal{D}(\gamma_s)$). By induction we see that K has to be of finite type. But this contradicts the Ext-Lemma [2]: for any extension of the form

$$0 \to K \to X \to Q \to 0$$

with Q of finite length and K of infinite length and of finite type, the modules K and X will have common indecomposable direct summands. For $X = M^1$, the indecomposable direct summands have Gabriel–Roiter measure γ_s, but K has not even a submodule of measure γ_s. □

Proof of Theorem 2. Here we will need the Gabriel–Roiter comeasure. Now the Gabriel–Roiter measure concerns the existence of chains of indecomposable submodules of a given finite length module X, similarly, the comeasure measures the existence of chains of indecomposable factor modules of X, see [2]. In order to define the comeasure of the (left) Λ-module X of finite length, we just look at the dual DX of X, this a right Λ-module, thus a left Λ^{op}-module. By definition, the comeasure of X is $\gamma^*(X) = -\gamma(DX)$, where $\gamma(DX)$ is the Gabriel–Roiter measure of DX (as a Λ^{op}-module). Note that the minus sign is used so that the natural order of the (here now negative) rational numbers corresponds to the categorical structure of $\operatorname{mod} \Lambda$; in particular, if $f : X \to Y$ is an epimorphism between finite length modules, then $\gamma^*(X) \leq \gamma^*(Y)$.

We consider now a Λ-module M and we assume that the indecomposable factor modules of M of finite length are of bounded length, thus there are only finitely many possible Gabriel–Roiter comeasures. Assume that the indecomposable factor modules of M of finite length have Gabriel–Roiter comeasures $\delta_s < \delta_{s-1} < \cdots < \delta_1$. We show by induction on s that M is of finite type. The case $s = 1$ is well-known and easy to see: if any indecomposable factor module of M of finite length is simple, then M has to be semi-simple, thus of finite type (here we use that Λ is perfect).

Assume now that $s \geq 2$ and let $\delta = \delta_s$. Thus M has a factor module of finite length with comeasure δ, and any factor module of M of finite length has comeasure $\delta' \geq \delta$.

(a) *Let X be indecomposable of finite length with comeasure δ. Then any epimorphism $M \to X$ splits.* Namely, let $f : M \to X$ be an epimorphism with kernel U. We show that U is a pure submodule. Let $U' \subseteq U$ be a submodule with U/U' of finite length. Then the canonical map $M/U' \to M/U$ splits, according to the main property of the comeasure. According to [5] this means that U is a pure submodule. A pure submodule of finite colength is always a direct summand.

(b) *Let X be indecomposable of finite length with comeasure δ. Then $M = M' \oplus U$ where, on the one hand, U is a direct sum of copies of X, whereas, on the other hand, there is no surjective map $M' \to X$.* Proof. Let U be a maximal pure submodule of M which is a direct sum of copies of X (one obtains such a U by transfinite induction, splitting off copies of X; the existence of a maximal submodule of this kind comes from Zorn's Lemma). Since X is Σ-algebraically compact, we see that U is even a direct summand, say $M = U \oplus M'$. Assume that there exists an epimorphism $M' \to X$. Then by (a), this epimorphism splits, thus $M' = X \oplus M''$, and $M = U \oplus M' = U \oplus X \oplus M''$. But, this contradicts the maximality of U.

(c) We consider now pairwise non-isomorphic indecomposable factor modules of M of comeasure δ_s. If possible, we construct inductively submodules

$$M = U_0 \supset U_1 \supset U_2 \supset \cdots$$

such that the factors U_{i-1}/U_i are indecomposable with comeasure δ_s and pairwise non-isomorphic. If this process stops, then (b) asserts that we can write M as a direct sum of copies of finitely many indecomposables with comeasure δ_s and a module M' which has no factor module with comeasure δ_s. Since for M', the number of comeasures of factor modules has decreased by 1, we can use induction: By induction, M' is of finite type, thus also M is of finite type.

Thus consider the case where the sequence does not stop. Let $X_i = U_{i-1}/U_i$, then (a) shows that we get a pure submodule $V = \bigoplus X_i$ in M. Consider the Λ^{op}-module $V' = \bigoplus X_i^*$. This is a Λ^{op}-module which is not of finite type. Fix a natural number b. According to Theorem 1, there is an indecomposable submodule N of V' of length greater than b. Now N is a finite length submodule of V', thus a submodule of $\bigoplus_{i=1}^t X_i^*$ for some t. Dualizing, we see that $\bigoplus_{i=1}^t X_i$ has N^* as factor module. But $\bigoplus_{i=1}^t X_i$ is a direct summand of M, thus a factor module of M, therefore N^* is a factor module of M, and of length $|N^*| = |N| > b$. This contradicts the assumption that the factor modules of M are of bounded length! Thus the case where the sequence does not stop, cannot occur. This completes the proof. □

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