Morphisms determined by objects:

The case of modules over artin algebras.

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Abstract. Let Λ be an artin algebra. In his Philadelphia Notes, M. Auslander showed that any homomorphism between Λ-modules is right determined by a Λ-module C, but a formula for C which he wrote down has to be modified. The paper presents corresponding counter-examples, but also provides a quite short proof of Auslander's assertion that any homomorphism is right determined by a module. Using the same methods, we describe the minimal right determiner of a morphism, as discussed in the book by Auslander, Reiten and Smalø. In addition, we look at the role of indecomposable projective direct summands of a minimal right determiner and provide a detailed analysis of the kernel-determined morphisms: these are those morphisms which are right determined by a module without any non-zero projective direct summand. In this way, we answer a question raised in the book by Auslander, Reiten and Smalø. What we encounter is an intimate relationship to the vanishing of Ext^2.

Let Λ be an artin algebra, the modules which we consider are finitely generated left Λ-modules. A morphism $\alpha\colon X \to Y$ of Λ-modules is said to be *right determined* by a Λ-module C provided the following condition is satisfied: given any morphism $\alpha'\colon X' \to Y$ such that $\alpha'\phi$ factors through α for any $\phi\colon C \to X'$, then α' itself factors through α. This definition is due to Auslander; the papers [A1] and [A2] are devoted to this concept. One of the main assertions of Auslander claims that any morphism $\alpha\colon X \to Y$ is right determined by $C = \operatorname{Tr} D(K) \oplus P(Q)$, see [A2], Theorem 2.6; here K is the kernel, Q the cokernel of α, and $\operatorname{Tr}(M)$ denotes the transpose, $D(M)$ the dual and $P(M)$ the projective cover of a module M.

The aim of this note is to show that this assertion is not correct as stated (in contrast to the weaker statements Theorem 3.17 (b) of [A1] and Corollary XI.1.4 in [ARS]). In section 1, we will present corresponding examples. The assertion has to be slightly modified: not the projective cover of Q is relevant, but the projective cover of the **socle** $\operatorname{soc} Q$ of Q.

Theorem 1. *Let $\alpha\colon X \to Y$ be a morphism. Let K be the kernel of α and Q the cokernel of α. Then α is right determined by* $\operatorname{Tr} D(K) \oplus P(\operatorname{soc} Q)$.

The modification of Auslander's treatment is formulated in Lemma 1 below (this should replace [A2] Lemma 2.1.b). Auslander's proof is somewhat hidden in two rather long papers, but there is a second treatment of this topic in the book by Auslander, Reiten, Smalø [ARS], see the last chapter. Still we feel that it may be appreciated if we provide a complete (and quite short) direct proof of Theorem 1. This will be done in section 2. In section 3 we will use the same methods in order to describe the minimal right determiner $T(\alpha)$ of α, as it was introduced in [ARS]. In section 4 we will discuss the following question: given a simple submodule S of $\operatorname{Cok}(\alpha)$, when is $P(S)$ a direct summand of $T(\alpha)$? The final section 5 is devoted to a detailed analysis

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of the structure of those maps α which are right determined by $\operatorname{Tr} D(K)$, with K the kernel of α, or, equivalently, by a module without an indecomposable projective direct summand. The problem of characterizing this class was raised in [ARS].

Auslander's theory of morphisms being determined by modules has to be considered as an exciting frame for working with the category of Λ-modules. What Auslander has achieved is a clear description of the poset structure of this category as well as a blueprint for interrelating individual modules and families of modules. We refer to the survey [R] which outlines the general setting and shows the wealth of these ideas by exhibiting many examples.

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1. Two Examples.

Example 1. Consider the quiver of type $\mathbb{A}_3$ with linear orientation, say with simple modules indexed by $1, 2, 3$, such that $S(1)$ is projective, $S(3)$ is injective. Let $\alpha\colon S(1) \to P(3)$ be the inclusion map, thus the kernel is zero, and the projective cover of the cokernel is again $P(3)$. We claim that α is not right determined by $C = P(3)$. Consider the inclusion map $\alpha'\colon P(2) \to P(3)$. Obviously, α' cannot be factored through α. However, we have $\operatorname{Hom}(C, P(2)) = 0$, and the only map $\phi\colon C \to P(2)$ (the zero-map) has as composition with α' the zero-map $C \to P(3)$. But the zero-map $C \to P(3)$ factors through α, trivially.

Example 2. Actually, an even easier example is given by the quiver $\mathbb{A}_2$, but here we deal with α being a zero map (some may consider this as a degenerate case, thus we presented first another example). Denote the two simple modules by $S(1)$ and $S(2)$, with $S(1)$ being projective, $S(2)$ being injective. We take as α the zero-map $0 \to P(2)$, its cokernel is $P(2)$ and already projective. But α is not right determined by $C = P(2)$, since the inclusion map $\alpha'\colon S(1) \to P(2)$ does not factor through α (after all, α is zero), whereas for any map $\phi\colon C \to S(1)$ (there is only the zero map) the composition $\alpha'\phi$ factors through α.

Remark. Let us stress that Auslander's claim is correct in case Λ is commutative, or, more generally, in case all the arrows of the quiver of Λ are loops. Namely, in this case (and only in this case) $\operatorname{add} P(M) = \operatorname{add} P(\operatorname{soc} M)$ for any Λ-module M.

2. The proof of Theorem 1.

We start with the necessary amendment to Auslander's treatment.

Given an indecomposable projective module P, we always will denote the inclusion map $\operatorname{rad} P \to P$ by ι, the projection $P \to P/\operatorname{rad} P$ by π.

Lemma 1. *Let $\alpha\colon X \to Y$ be a morphism with image $\alpha(X)$. Let $\alpha'\colon X' \to Y$ be a morphism. Assume that for any simple submodule S of the cokernel $Q = \operatorname{Cok}(\alpha)$ and any map $\phi\colon P(S) \to X'$ with $\alpha'\phi(\operatorname{rad} P(S)) \subseteq \alpha(X)$, the map $\alpha'\phi$ factors through α. Then the image of α' is contained in $\alpha(X)$.*

Proof. We assume that the image of α' is not contained $\alpha(X)$ and want to derive a contradiction. Let us denote by $\gamma\colon Y \to Q$ the cokernel map for α. By assumption, $\gamma\alpha' \neq 0$. Let U be the image of $\gamma\alpha'$, with epimorphism $\epsilon\colon X' \to U$ and inclusion map $\mu\colon U \to Q$, thus $\mu\epsilon = \gamma\alpha'$. Since U is non-zero, we may consider a simple submodule S of U, say with inclusion map $\nu\colon S \to U$. Of course, S is a simple submodule of Q. Let $\pi\colon P(S) \to S$ be a projective cover of S. Since $P(S)$ is projective and ϵ is an epimorphism, we can lift $\nu\pi$ and obtain a map $\phi\colon P(S) \to X'$ with $\epsilon\phi = \nu\pi$. Note that

$$\gamma\alpha'\phi = \mu\epsilon\phi = \mu\nu\pi.$$

Since $\pi\iota = 0$, it follows that

$$\gamma\alpha'\phi\iota = \mu\nu\pi\iota = 0.$$

This shows that the image of $\alpha'\phi\iota$ is contained in the kernel of γ, but this is $\alpha(X)$. In this way, we see that $\alpha'\phi(\operatorname{rad} P(S)) \subseteq \alpha(X)$.

Thus, we are in the situation mentioned in the statement of the Lemma: there is given a map $\phi\colon P(S) \to X'$, such that $\alpha'\phi(\operatorname{rad} P(S)) \subseteq \alpha(X)$ and by the assumption of the Lemma, we know that the map $\alpha'\phi$ factors through α, say $\alpha'\phi = \alpha\phi'$ for some $\phi'\colon P(S) \to X$. Therefore

$$\mu\nu\pi = \gamma\alpha'\phi = \gamma\alpha\phi' = 0,$$

since γ is the cokernel of α. But $\mu\nu$ is a monomorphism, therefore $\pi = 0$, a contradiction.

Let us continue, as promised, with the complete proof of Theorem 1. The only prerequisite which we will use is the existence of almost split sequences. To be precise: we will need for any indecomposable non-injective module M a non-split short exact sequence

$$0 \to M \xrightarrow{\sigma} N \xrightarrow{\rho} \operatorname{Tr} D(M) \to 0,$$

such that for any map $\zeta\colon M \to N'$ which is not a split monomorphism, there is $\zeta'\colon N \to N'$ with $\zeta = \zeta'\sigma$.

Lemma 2. *Let $\alpha\colon X \to Y$ be a morphism with kernel K and image $\alpha(X)$. Let $\alpha'\colon X' \to Y$ be a morphism with image contained in $\alpha(X)$. Assume that for any map $\phi\colon \operatorname{Tr} D(K) \to X'$, the composition $\alpha'\phi$ factors through α. Then α' factors through α.*

Remark. Given a morphism $\alpha\colon X \to Y$, we may try to split off non-zero direct summands of X which lie in the kernel of α. If this is not possible, then α is said to be *right minimal.* In general, we may write $X = X_0 \oplus X_1$ with X_0 contained in the kernel of α and such that $\alpha|X_1$ is right minimal; then we call the kernel of $\alpha|X_1$ the *intrinsic kernel* of α (note that it is unique up to isomorphism). An indecomposable direct summand L of the kernel of α is a direct summand of the intrinsic kernel, if and only if the composition of the embeddings $L \subseteq K \subseteq X$ is not a split monomorphism.

It will be of interest in section 3 that one may replace in Lemma 2 the kernel K by the intrinsic kernel K', thus the assertion of Lemma 2 can be strengthened as follows: *Assume that for any map $\phi\colon \operatorname{Tr} D(K') \to X'$, the composition $\alpha'\phi$ factors through α. Then α' factors through α.*

Proof of Lemma 2 (and its strengthening). We may assume that $Y = \alpha(X)$, thus there is given the exact sequence η with epimorphism $\alpha\colon X \to Y$

and kernel $\mu\colon K \to X$. We form the induced exact sequence η' with respect to α', thus there is the following commutative diagram with exact rows:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & K & \xrightarrow{\mu} & X & \xrightarrow{\alpha} & Y & \longrightarrow & 0 & \qquad & \eta \\
 & & \| & & \uparrow{\scriptstyle \beta'} & & \uparrow{\scriptstyle \alpha'} & & & & \\
0 & \longrightarrow & K & \xrightarrow{\nu} & W & \xrightarrow{\beta} & X' & \longrightarrow & 0 & \qquad & \eta'
\end{array}$$

If η' is a split exact sequence, then α' factors through α.

Let us assume that α' does not factor through α, in order to derive a contradiction, again. Thus η' is not a split exact sequence. Write $K = \bigoplus K_i$ with indecomposable modules K_i and projection maps $\pi_i\colon K \to K_i$. Since η' does not split, there is some index i such that the exact sequence induced from η' by the map π_i does not split. This means that we have the following commutative diagram with exact rows which do not split:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & K & \xrightarrow{\nu} & W & \xrightarrow{\beta} & X' & \longrightarrow & 0 & \qquad & \eta' \\
 & & {\scriptstyle \pi_i}\downarrow & & {\scriptstyle \pi_i'}\downarrow & & \| & & & & \\
0 & \longrightarrow & K_i & \xrightarrow{\nu_i} & W_i & \xrightarrow{\beta_i} & X' & \longrightarrow & 0 & \qquad & \eta_i'
\end{array}$$

Let us add here, that K_i has to be a direct summand of the intrinsic kernel of α. This observation is necessary in order to see that the remark made above is justified.

Since $\nu_i\colon K_i \to W_i$ is a monomorphism which does not split, we see that K_i cannot be injective, thus there is an almost split sequence

$$0 \to K_i \xrightarrow{\sigma_i} V_i \xrightarrow{\rho_i} \operatorname{Tr} D(K_i) \to 0,$$

and ν_i can be factored as $\nu_i = \nu_i'\sigma_i$ for some $\nu_i'\colon V_i \to W_i$. Thus we obtain the following commutative square on the left, and therefore also the map $\phi : \operatorname{Tr} D(K_i) \to X'$ with a commutative square on the right:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & K_i & \xrightarrow{\nu_i} & W_i & \xrightarrow{\beta_i} & X' & \longrightarrow & 0 & \qquad & \eta_i' \\
 & & \| & & \uparrow{\scriptstyle \nu_i'} & & \uparrow{\scriptstyle \phi} & & & & \\
0 & \longrightarrow & K_i & \xrightarrow{\sigma_i} & V_i & \xrightarrow{\rho_i} & \operatorname{Tr} D(K_i) & \longrightarrow & 0 & \qquad & \omega_i
\end{array}$$

By assumption, the map $\phi\alpha'\colon \operatorname{Tr} D(K_i) \to Y$ factors through α, that means there is $\phi'\colon \operatorname{Tr} D(K_i) \to X$ with $\alpha\phi' = \alpha'\phi$. Now, W is the pullback of α, α', thus there is a map $\phi''\colon \operatorname{Tr} D(K_i) \to W$ such that $\beta\phi'' = \phi$ and $\beta'\phi'' = \phi'$. It follows that

$$\phi = \beta\phi'' = \beta_i\pi_i'\phi''.$$

But if ϕ factors through β_i, then the exact sequence ω_i induced from η_i' by ϕ has to split. This is a contradiction, since ω_i is an Auslander-Reiten sequence, thus non-split.

Proof of Theorem 1. Let $\alpha\colon X \to Y$ be a morphism with kernel K and cokernel Q and let $C = \operatorname{Tr} D(K) \oplus P(\operatorname{soc} Q)$. Let $\alpha'\colon X' \to Y$ be a morphism such that $\alpha'\phi$ factors through α for any map $\phi\colon C \to X'$.

If S is a simple submodule of Q, then $P(S)$ is a direct summand of $P(\operatorname{soc} Q)$, thus of C. Thus, for any map $\phi\colon P(S) \to X'$, the composition $\alpha'\phi$ factors through α. Lemma 1 asserts that the image of α' is contained in the image of α. Now we use that $\operatorname{Tr} D(K)$ is a direct summand of C, thus for any map $\phi\colon \operatorname{Tr} D(K) \to X'$, the composition $\alpha'\phi$ factors through α. Thus we can apply Lemma 2 in order to see that α' factors through α. This shows that α is right determined by C.

Example 3. Let us add an example which may be illuminating, albeit it is extremely special. Let Λ be the path algebra of a finite directed quiver. Let b be a vertex of the quiver and assume that there are s arrows starting in b, say $b \to a_i$ with $1 \le i \le s$, and that there are t arrows ending in b, say $c_j \to b$ with $1 \le j \le t$. For any vertex x, we denote by $S(x)$ the simple module with support x, by $P(x)$ the projective cover of $S(x)$, by $I(x)$ the injective envelope of $S(x)$.

Let α be a non-zero map $X = P(b) \to I(b) = Y$, this is the homomorphism which we want to look at. Note that the image of α is $S(x)$. The kernel of α is the radical of $P(b)$, thus the direct sum of the modules $P(a_i)$ with $1 \le i \le s$. The cokernel of α is the factor module of $I(b)$ modulo its socle, thus it is the direct sum of the modules $I(c_j)$ with $1 \le j \le t$. The projective cover of the socle of $I(c_j)$ is $P(c_j)$. Altogether we see: the theorem asserts that α is right determined by the module

$$C = \bigoplus_{i=1}^{s} \operatorname{Tr} D(P(a_i)) \oplus \bigoplus_{j=1}^{t} P(c_j).$$

But this module C is precisely the middle term of the almost split sequence starting in $P(b)$.

This should not come as a surprise. Namely, let X' be an indecomposable module and assume that there is a non-zero map $\alpha'\colon X' \to Y = I(b)$. Then there is a map $\beta' : P(b) \to X'$ with composition $\alpha'\beta' = \alpha$. Now either β' is invertible so that α' factors through α, or else β' is not invertible and α' does not factor through α. In the latter case, β' factors through the minimal left almost split map $\gamma\colon P(b) \to C$ starting in $P(b)$, this means that there is some $\phi\colon C \to X'$ with $\beta' = \phi\gamma$. But if we look at the composition of ϕ and α', then one should be aware that no non-zero map $C \to I(b)$ factors through α.

3. Minimal right determiners.

Taking into account the Remark after Lemma 2, the Theorem we discuss can be strengthened as follows: *Any morphism $X \to Y$ is right determined by $\operatorname{Tr} D(K') \oplus P(\operatorname{soc} Q)$,* where K' is the intrinsic kernel and Q the cokernel of α. But one can do even better.

Let us call a module $T = T(\alpha)$ a *minimal right determiner* for α, provided T right determines α and is a direct summand of any module C which right determines α. According to [ARS], Proposition XI.2.4, a minimal right determiner for α exists and is the direct sum of all modules N which almost factor through α, one from each isomorphism class. The aim of this section is to present a proof of this result using the considerations of section 2.

We recall from [ARS] that an indecomposable module N is said to *almost factor through* $\alpha\colon X \to Y$ provided there is a morphism $\eta\colon N \to Y$ which does not factor through α whereas for any radical map $\psi\colon M \to N$, the composition $\eta\psi$ factors through α. Obviously, the latter condition can be replaced by the condition that the map $\eta\rho$ factors through α, where ρ is the minimal right almost split map ending in N. Thus an indecomposable module N almost factors though α provided there exists a commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\ \rho\ } & N \\ {\scriptstyle\eta'}\big\downarrow & & \big\downarrow{\scriptstyle\eta} \\ X & \xrightarrow{\ \alpha\ } & Y \end{array}$$

such that η does not factor through α (with ρ minimal right almost split). Note that in case $N = P$ is (indecomposable) projective, the minimal right almost split map ending in P is just the map $\iota\colon \operatorname{rad} P \to P$.

Lemma 3. *Let P be an indecomposable projective module which almost factors through a map α. Then P is the projective cover of a simple submodule of* $\operatorname{Cok}(\alpha)$.

Proof. Let $\eta\colon P \to Y$ be a map which does not factor through $\alpha\colon X \to Y$, whereas $\eta\iota$ factors through α. Consider the image U of η in Y and the factor module $S = (U + \alpha(X))/\alpha(X) \subseteq Y/\alpha(X) = \operatorname{Cok}(\alpha)$. Since $\eta(\operatorname{rad} P) \subseteq \alpha(X)$, we see that S is either simple or zero. But if $S = 0$, then $\eta(P) \subseteq \alpha(X)$ and the projectivity of P implies that η factors through α. Since this is not the case, S is simple and η provides an epimorphism $P \to S$.

Lemma 4. *Let $\alpha\colon X \to Y$ be a morphism. Let K' be the intrinsic kernel of α and P the direct sum of all indecomposable projective modules which almost factor through α, one from each isomorphism class. Then α is right determined by* $\operatorname{Tr} D(K') \oplus P$.

Proof: Let $\alpha'\colon X' \to Y$ be a morphism which does not factor through α. We have to find an indecomposable module C which is either of the form $\operatorname{Tr} D(L)$, where L is a direct summand of K' or a projective module which almost factors through α, and a morphism $\phi\colon C \to X'$ such that $\alpha'\phi$ does not factor through α. According to the strengthened Lemma 2, such a pair C, ϕ exists if the image of α' is contained in the image $\alpha(X)$ of α.

Thus we can assume that the image of α' is not contained in $\alpha(X)$. According to Lemma 1, there is a simple submodule S of the cokernel Q of α and a map $\phi\colon P(S) \to X'$ with $\alpha'\phi(\operatorname{rad} P(S)) \subseteq \alpha(X)$ such that $\alpha'\phi$ does not factor through α. Write $\alpha = \alpha_2\alpha_1$ with inclusion map $\alpha_2\colon \alpha(X) \to Y$. Using this notation, $\alpha'\phi\iota = \phi'\alpha_2$ for some ϕ' (the restriction of ϕ). If $\phi' = \alpha_1\phi''$, then $\iota\alpha'\phi = \alpha_2\alpha_1\phi'' = \alpha\phi''$ together with the fact that $\alpha'\phi$ does not factor through α shows that $P(S)$ almost factors through α, thus $P(S), \phi$ is the required pair.

Finally, we have to consider the case where ϕ' does not factor through α_1. But then $\alpha_2\phi'$ does not factor through α (namely, $\alpha_2\phi' = \alpha\psi$ shows that $\alpha_2\phi' = \alpha\psi = \alpha_2\alpha_1\psi$, but α_2 is injective, thus $\phi' = \alpha_1\psi$). Now $\alpha_2\phi'$ is a morphism with image in $\alpha(X)$, thus as in the first part of the proof, there is an indecomposable direct summand C of K' and a map $\eta\colon C \to \operatorname{rad} P$ such that $\alpha_2\phi'\eta$ does not factor through α. If we rewrite the composition $\alpha_2\phi'\eta = \alpha'\phi\iota\eta = \alpha'(\phi\iota\eta)$, then we see that we have achieved what we want, namely the pair $C, \phi\iota\eta$.

It remains to be seen that we have obtained in this way a minimal right determiner for α, at least up to multiplicities.

Lemma 5. *Assume that α is right determined by a module C. Let L be an indecomposable direct summand of the intrinsic kernel of α. Then L is not injective,* $\operatorname{Tr} D(L)$ *is isomorphic to a direct summand of C, and* $\operatorname{Tr} D(L)$ *almost factors through α.*

Proof: Let K be the kernel of α, say with inclusion map $\mu\colon K \to X$. Since L is a direct summand of K, there is L' with $K = L \oplus L'$, and we denote by $\mu'\colon L \to K$ the embedding. And we write $\alpha = \alpha_2\alpha_1$ with $\alpha_1\colon X \to \alpha(X)$ surjective, and $\alpha_2\colon \alpha(X) \to Y$ the inclusion map. Since $\mu\mu'$ is an embedding which does not split, we see that K is not injective, thus there is an almost split sequence

$$0 \to L \xrightarrow{\sigma} M \xrightarrow{\rho} \operatorname{Tr} D(L) \to 0,$$

and we can lift the map $\mu\mu'$ to M: there is a map $\mu''\colon M \to X$ with $\mu''\sigma = \mu\mu'$. Since ρ is the cokernel of σ, there is a map $\eta\colon \operatorname{Tr} D(L) \to Y$ such that $\eta\rho = \alpha\mu''$, thus we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{\sigma} & M & \xrightarrow{\rho} & \operatorname{Tr} D(L) & \longrightarrow & 0 \\
 & & {\scriptstyle\mu'}\downarrow & & {\scriptstyle\mu''}\downarrow & & \downarrow{\scriptstyle\eta} & & \\
0 & \longrightarrow & K & \xrightarrow{\mu} & X & \xrightarrow{\alpha} & Y & &
\end{array}.$$

We claim that η does not factor through α. In order to prove this, we recall that L is a direct summand of K, say $K = L \oplus L'$, and we form the induced exact sequence the given Auslander-Reiten sequence with the split monomorphism $\mu'\colon L \to K = L \oplus L'$. The induced sequence is the direct sum of the Auslander-Reiten sequence and a sequence of the form $0 \to L' \to L' \to 0 \to 0$, in particular non-split, see the diagram below. Since $\mu''\sigma = \mu\mu'$, we obtain a map $\beta\colon M \oplus L' \to X$ and then a map $\beta'\colon \operatorname{Tr} D(L) \to \alpha(X)$ such that the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & L & \xrightarrow{\sigma} & M & \xrightarrow{\rho} & \operatorname{Tr} D(L) & \longrightarrow & 0 \\
 & & {\scriptstyle\mu'}\downarrow & & {\scriptstyle\mu''}\downarrow & & \| & & \\
0 & \longrightarrow & K & \xrightarrow{\mu} & M \oplus L' & \longrightarrow & Y & \longrightarrow & 0 \\
 & & \| & & {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\beta'} & & \\
0 & \longrightarrow & K & \xrightarrow{\mu} & X & \xrightarrow{\alpha_1} & \alpha(X) & \longrightarrow & 0
\end{array}$$

Note that a comparison with the diagram above shows that $\eta = \alpha_2\beta'$. From the diagram we see that the horizontal middle sequence is induced from the lower sequence by β'. Since the horizontal middle sequence does not split, we see that β' does not factor through α_1. Now assume that η factors through α, say $\eta = \alpha\zeta$ for some $\zeta\colon \operatorname{Tr} D(L) \to X$. Then

$$\alpha_2\alpha_1\zeta = \alpha\zeta = \eta = \alpha_2\beta',$$

implies that $\alpha_1\zeta = \beta'$, since α_2 is injective. But we know already that β' does not factor through α_1, thus η does not factor through α, as we wanted to show.

Since C right determines α, and $\eta\colon \operatorname{Tr} D(L) \to Y$ does not factor through α, there has to exist a morphism $\phi\colon C \to \operatorname{Tr} D(L)$ such that also $\eta\phi$ cannot be factored through α. Now again we use that the upper sequence is an Auslander-Reiten sequence. Assume that ϕ is not split epi. Then there is $\phi'\colon C \to M$ such that $\rho\phi' = \phi$, and therefore

$$\eta\phi = \eta\rho\phi' = \alpha\beta\mu''\phi'$$

is a factorization of $\eta\phi$ through α, a contradiction. This shows that ϕ is split epi, thus $\operatorname{Tr} D(L)$ is isomorphic to a direct summand of C.

Finally, we see that $\operatorname{Tr} D(L)$ almost factors through α, since there is the diagram

$$\begin{array}{ccc} M & \xrightarrow{\rho} & \operatorname{Tr} D(L) \\ {\scriptstyle \mu''}\downarrow & & \downarrow{\scriptstyle \eta} \\ X & \xrightarrow{\alpha} & Y \end{array}$$

and η does not factor through α.

Lemma 6. *Assume that α is right determined by a module C. Let P be an indecomposable projective which almost factors through α. Then P is isomorphic to a direct summand of C.*

Proof: There exists a commutative diagram

$$\begin{array}{ccc} \operatorname{rad} P & \xrightarrow{\iota} & P \\ \downarrow & & \downarrow{\scriptstyle \eta} \\ X & \xrightarrow{\alpha} & Y \end{array}$$

such that η does not factor through α. Since C right determines α, there must exist $\phi\colon C \to P$ such that also $\eta\phi$ does not factor through α. Now ϕ does not map into $\operatorname{rad} P$, since otherwise $\eta\phi$ would factor through α. But this means that ϕ is surjective and therefore a split epimorphism.

Theorem 2. *Let $\alpha\colon X \to Y$ be given. Let T be the direct sum of modules of the form $\operatorname{Tr} D(L)$, where L is an indecomposable direct summand of the intrinsic kernel of α and of the indecomposable projective modules which almost factor through α, one from each isomorphism class. Then T is a minimal right determiner for α.*

Proof. This is a direct consequence of the Lemmata 4, 5 and 6.

Corollary 1. *Let $\alpha\colon X \to Y$ be given. A non-projective indecomposable module N almost factors through α if and only if $N = \operatorname{Tr} D(L)$ for some indecomposable direct summand L of the intrinsic kernel of α.*

Proof. On the one hand, we have seen in Lemma 5 that the modules of the form $\operatorname{Tr} D(L)$ almost factor through α. On the other hand, it is clear that an indecomposable module which almost factors through α is a direct summand of any right determiner for α (see for example [ARS] Lemma XI.2.1), thus of $T(\alpha)$.

Corollary 2. *Let $\alpha\colon X \to Y$ be given. An indecomposable module N almost factors through α if and only if it is a direct summand of $T(\alpha)$.*

4. The indecomposable projective direct summands of $T(\alpha)$.

Theorem 2 shows that $T(\alpha)$ has two kinds of indecomposable direct summands: First of all, there are those of the form $\operatorname{Tr} D(L)$, where L is any direct summand of the intrinsic kernel of α, and clearly they are never projective. Second, there may be indecomposable projective modules. Here we want to discuss these latter summands.

Recall that if S is a simple module such that $P(S)$ is a direct summand of $T(\alpha)$, then, according to Lemma 3, S is a simple submodule of $\operatorname{Cok}(\alpha)$. But the converse does not hold. *Not every module $P(S)$ with S a simple submodule of* $\operatorname{Cok}(\alpha)$ *almost factors through* α.

Example 4. This example has been exhibited in the book of Auslander, Reiten, Smalø [ARS], after Proposition XI.1.6. Let Λ be a local uniserial ring with the unique simple module S, and let $\alpha\colon P \to Y$ be a morphism with P the indecomposable projective module and Y also indecomposable. If $P = P(S)$ almost factors through α, then $\alpha = 0$, and therefore α is right determined by $\operatorname{Tr} D(\operatorname{Ker}(\alpha))$.

Actually, for any artin algebra with global dimension at least 2 there do exist corresponding examples, as the following basic observation shows:

Example 5. *Let $\delta\colon P_1 \to P_0$ be a minimal presentation of a simple module S. If $P(S)(= P_0)$ almost factors through δ, then δ is injective,* thus the projective dimension of S is at most 1. Proof: Write $\delta = \iota\epsilon$, where $\iota\colon \operatorname{rad} P_0 \to P_9$ is the inclusion map. If P_0 almost factors through δ, there is $\eta\colon P_0 \to P_0$ not factoring through δ and $\eta'\colon \operatorname{rad} P_0 \to P_1$ such that $\eta\iota = \delta\eta'$, whereas η does not factor through δ. Then δ does not map into $\operatorname{rad} P_0$, therefore η has to be invertible, and $\eta\iota = \iota\epsilon\eta'$ implies that $\iota = \eta^{-1}\iota\epsilon\eta'$, thus $1_{\operatorname{rad} P_0} = \epsilon\eta'$. But this means that ϵ is split epimorphism, thus an isomorphism (since it is a projective cover).

Here are three sufficient conditions for $P(S)$ to be a direct summand of $T(\alpha)$.

Proposition 1. *Let $\alpha\colon X \to Y$ be a monomorphism with cokernel Q. If S is a simple submodule of Q, then $P(S)$ almost factors through α.*

Proof. We may assume that α is an inclusion map. Since S is a submodule of Y/X, there is a map $\eta\colon P(S) \to Y$, such that the composition of η with $Y \to Y/X$ maps onto S. But then $\eta(\operatorname{rad} P(S)) \subseteq X$. Thus $P(S)$ almost factors through α.

Proposition 2. *Let $\alpha\colon X \to Y$ be a morphism. If S is a simple submodule of Y with $S \cap \alpha(X) = 0$, then $P(S)$ almost factors through α.*

Proof: Let S is a simple submodule of Y, and let $\eta\colon P(S) \to Y$ be a morphism with image S. Then $\eta\iota = 0$. Thus the following diagram commutes:

$$\begin{array}{ccc} \operatorname{rad} P(S) & \xrightarrow{\ \iota\ } & P(S) \\ {\scriptstyle 0}\big\downarrow & & \big\downarrow{\scriptstyle \eta} \\ X & \xrightarrow{\ \alpha\ } & Y \end{array} .$$

Since $S \cap \alpha(X) = 0$, we see that η does not factor through α.

Proposition 3. *Let $\alpha\colon X \to Y$ be a morphism with cokernel Q. Let S be a simple submodule of Q. If the projective dimension of S is at most 1, then $P(S)$ almost factors through α.*

Proof. Let $\pi\colon P(S) \to S$ be a projective cover and $\nu\colon S \to Q$ the inclusion map. Let $\gamma\colon Y \to Q$ be the cokernel map. The projectivity of $P(S)$ yields a map $\eta\colon P(S) \to Y$ such that $\gamma\eta = \nu\pi$. Here, we denote the projection $Y \to Y/\alpha(X) = Q$ by γ. Then $\gamma\eta\iota = \nu\pi\iota = 0$, thus η maps $\operatorname{rad} P(S)$ into $\alpha(X)$. This shows that we have the following commutative diagram

$$\begin{array}{ccc} \operatorname{rad} P(S) & \xrightarrow{\ \iota\ } & P(S) \\ {\scriptstyle \eta'}\big\downarrow & & \big\downarrow{\scriptstyle \eta} \\ \alpha(X) & \xrightarrow{\ \alpha_2\ } & Y \end{array} ,$$

as before we write $\alpha = \alpha_2\alpha_1$ where $\alpha_2\colon \alpha(X) \to Y$ is the canonical inclusion of $\alpha(X) = \alpha(X)$ into Y. Since the projective dimension of S is at most 1, we know that $\operatorname{rad} P(S)$ is projective, thus we can lift η' and obtain $\eta''\colon \operatorname{rad} P(S) \to X$ with $\alpha_1\eta'' = \eta'$, thus there is the commutative diagram

$$\begin{array}{ccc} \operatorname{rad} P(S) & \xrightarrow{\iota} & P(S) \\ {\scriptstyle\eta''}\big\downarrow & & \big\downarrow{\scriptstyle\eta} \\ X & \xrightarrow{\alpha} & Y \end{array} .$$

Of course, η does not factor through α since $\gamma\eta \neq 0$.

It follows that for a hereditary artin algebra, the projective cover $P(S)$ of any simple submodule of $\operatorname{Cok}(\alpha)$ is a direct summand of $T(\alpha)$.

Finally, there is the following characterization:

Proposition 4. *Let S be a simple module. Then $P(S)$ is a direct summand of $T(\alpha)$ if and only if there exists a module J with submodule X and $J/X = S$ and a morphism $\widetilde{\alpha}\colon J \to Y$ such that its restriction to X is α and the kernels of α and $\widetilde{\alpha}$ coincide.*

The condition that the kernels of α and $\widetilde{\alpha}$ coincide is equivalent to the condition that the image of α is properly contained in the image of $\widetilde{\alpha}$.

Proof: First, let us assume that there exists a module J with submodule X and $J/X = S$ and a morphism $\widetilde{\alpha}\colon J \to Y$ such that its restriction to X is α and such that the image of α is a properly contained in the image of $\widetilde{\alpha}$. Denote the projection map $J \to J/X = S$ by ϵ. Let $\pi\colon P(S) \to S$ be a projective cover and lift it to J, thus we obtain $\pi'\colon P(S) \to J$ such that $\epsilon\pi' = \pi$. Since $\epsilon\pi'(\operatorname{rad} P(S)) = \pi(\operatorname{rad} P(S)) = 0$, we have $\pi'(\operatorname{rad} P(S)) \subseteq X$. Let us denote by $\pi''\colon \operatorname{rad} P(S) \to X$ the restriction of π' to $\operatorname{rad} P(S)$. Then the diagram

$$\begin{array}{ccc} \operatorname{rad} P(S) & \xrightarrow{\iota} & P(S) \\ {\scriptstyle\pi''}\big\downarrow & & \big\downarrow{\scriptstyle\widetilde{\alpha}\pi'} \\ X & \xrightarrow{\alpha} & Y \end{array}$$

commutes, since $\widetilde{\alpha}|X = \alpha$.

It remains to be seen that $\widetilde{\alpha}\pi'$ does not factors through α. Assume for the contrary that $\widetilde{\alpha}\pi' = \alpha\zeta$, for some map $\zeta\colon P(S) \to X$. Now $J = X + \pi'(P(S))$, thus

$$\begin{aligned} \widetilde{\alpha}(J) &= \widetilde{\alpha}(X + \pi'(P(S))) = \alpha(X) + \widetilde{\alpha}\pi'(P(S)) \\ &= \alpha(X) + \alpha\zeta(P(S)) = \alpha(X), \end{aligned}$$

contrary to our assumption.

Conversely, assume that $P(S)$ almost factors through α, thus we have a diagram of the following form

$$\begin{array}{ccc} \operatorname{rad} P(S) & \xrightarrow{\iota} & P(S) \\ {\scriptstyle\eta'}\big\downarrow & & \big\downarrow{\scriptstyle\eta} \\ X & \xrightarrow{\alpha} & Y \end{array}$$

and η does not factor through α, thus the image of η is not contained in the image of α. Starting with the exact sequence with monomorphism ι, we form the sequence induced by η' and obtain the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{rad} P(S) & \xrightarrow{\iota} & P(S) & \xrightarrow{\pi} & S & \longrightarrow & 0 \\
 & & {\scriptstyle \eta'}\big\downarrow & & \big\downarrow{\scriptstyle \eta''} & & \big\| & & \\
0 & \longrightarrow & X & \xrightarrow{\iota'} & J & \longrightarrow & S & \longrightarrow & 0
\end{array}$$

Since $\eta\iota = \alpha\eta'$, there is a map $\widetilde{\alpha}\colon J \to Y$ such that $\alpha = \widetilde{\alpha}\iota'$ and $\eta = \widetilde{\alpha}\eta''$. Thus, we see that α has an extension $\widetilde{\alpha}$ to J. Since $\eta = \widetilde{\alpha}\eta''$, the image of η is contained in the image of $\widetilde{\alpha}$. This shows that the image of $\widetilde{\alpha}$ cannot be equal to the image of α, since otherwise the image of η would be contained in the image of α, in contrast to our assumption. This concludes the proof.

Proposition 4 (but also already Proposition 3) show that the obstructions for the projective cover $P(S)$ of a simple submodule of $\operatorname{Cok}(\alpha)$ to be a direct summand of $T(\alpha)$ are elements of Ext^2, namely the equivalence classes of the exact sequences

$$(*) \qquad 0 \to K \to X \to J \to S \to 0,$$

where K is the kernel of α and $J = \gamma^{-1}(S)$ (here γ is the cokernel map $Y \to \operatorname{Cok}(\alpha)$) and where the composition of the map $X \to J$ with the inclusion map $J \to Y$ is just α. Thus we have:

Corollary. *Let $\alpha\colon X \to Y$ be a morphism with kernel K and cokernel Q. If S is a submodule of Q and $\operatorname{Ext}^2(S, K) = 0$, then $P(S)$ is a direct summand of $T(\alpha)$.*

5. Kernel-determined morphisms.

Since any morphism α is right determined by the direct sum of the module $\operatorname{Tr} D(\operatorname{Ker}(\alpha))$ and a projective module P, one may ask for a characterization of those morphisms α for which one of these two summands already right determines α.

First, let us deal with the morphisms which are right determined by a projective module. Here, the answer is well-known and easy to obtain: *A morphism α is right determined by a projective module if and only if α is injective* (see Theorem 1 and Lemma 5).

Also, *an inclusion map $X \to Y$ is right determined by the projective module P, if and only if P generates the socle of Y/X.* (If P generates the socle of Y/X, then P right determines α according to Theorem 1. Conversely, assume that P right determines α, and let S be a simple submodule of Y/X. According to Proposition 1, $P(S)$ almost factors through α, thus Theorem 2 asserts that $P(S)$ is a direct summand of P. This shows that P generates the socle of Y/X.) There is the following consequence: *If we fix a projective module $P \neq 0$, and consider any module X, then there are morphisms $\alpha\colon X \to Y$ with Y of arbitrarily large length, such that α is right determined by P* (just take the inclusion maps of the form $X \to Y$ with Y the direct sum of X and arbitrarily many copies of $P/\operatorname{rad}(P)$). If Λ is representation-infinite, then there are even such examples with Y indecomposable.

The second case are the morphisms α which are right determined by $\operatorname{Tr} D(\operatorname{Ker}(\alpha))$, we call them *kernel-determined* morphisms. This is the topic of the considerations in this section. Note that the problem of characterizing these maps has been raised in [ARS], 368-369.

Lemma 7. *Let α be a morphism. The following conditions are equivalent:*
(i) *α is right determined by $\operatorname{Tr} D(K)$, where K is the kernel of α.*
(ii) *α is right determined by $\operatorname{Tr} D(K')$, where K' is the intrinsic kernel of α.*
(iii) *α is right determined by a module C without an indecomposable projective direct summand.*

Proof. Clearly (ii) $\implies$ (i) $\implies$ (iii). Now assume (iii). According to Theorem 2, any indecomposable projective module P which almost factors through α is a direct summand of C, thus there are no such modules P. Using again Theorem 2, we see that (ii) is satisfied.

Note that α is kernel-determined if and only if the equivalent conditions of lemma 7 are satisfied. Let us first show that for a kernel-determined morphism $\alpha\colon X \to Y$, the length of Y is bounded by a number which only depends on X. We denote by $|M|$ the length of the module M.

Lemma 8. *If $\alpha\colon X \to Y$ is kernel-determined, then Y is an essential extensions of $\alpha(X)$; in particular, $|Y| \leq q|X|$ where q is the maximal length of an indecomposable injective module.*

If Y is an essential extension of $N = \alpha(X)$, then we may assume that Y is a submodule of $I(N)$ with $N \subseteq Y$.

Proof of lemma 8. According to Proposition 2, there is no simple submodule S of Y with $S \cap \alpha(S) = 0$, this jut means that Y is an essential extension of $\alpha(X)$. Thus Y can be considered as a submodule of the injective envelope I of $\alpha(X)$. But then $|I| \leq q|\alpha(X)| \leq q|X|$.

Given a module M, let $\overline{M}$ be a module having M as an essential submodule with $\overline{M}/M$ semisimple and such that $\overline{M}$ is of maximal possible length; we call $\overline{M}$ a *small envelope* of M. We can construct $\overline{M}$ as follows:

$$\overline{M} = \omega^{-1}(\operatorname{soc} I(M)/M),$$

where $I(M)$ is an injective envelope of M and $\omega\colon I(M) \to I(M)/M$ is the canonical projection map (thus, if necessary, we will assume that $\overline{M}$ is a submodule of $I(M)$ which contains M). Clearly, any homomorphism $\phi\colon M \to N$ gives rise to an extension $\overline{\phi}\colon \overline{M} \to \overline{N}$ (by this we mean a homomorphism whose restriction to M is just ϕ). Let us stress that usually $\overline{\phi}$ is not uniquely determined (the construction $M \mapsto \overline{M}$ is not functorial). But there is the following unicity result which is of interest for the further considerations:

Lemma 9. *Let $\epsilon\colon X \to N$ be an epimorphism, and choose an injective envelope $I(N)$ of N. Then there is a (uniquely determined) submodule $N \subseteq I_\epsilon(N) \subseteq I(N)$ with the following property: If $\overline{X}$ is a small envelope of X and $\overline{\epsilon}\colon \overline{X} \to I(N)$ is an extension of ϵ, then $\overline{\epsilon}(\overline{X}) = I_\epsilon(N)$.*

Proof: If we deal with two extensions of ϵ, say $\epsilon_1, \epsilon_2 : \overline{X} \to I(N)$, then the difference $\epsilon_2 - \epsilon_1$ vanishes on X and its image is a semisimple module. But any semisimple submodule of $I(N)$ is contained in N and $N = \epsilon(X) \subseteq \epsilon_1(\overline{X})$. Thus, $\epsilon_2 = \epsilon_1 + (\epsilon_2 - \epsilon_1)$ shows that

$$\epsilon_2(\overline{X}) \subseteq \epsilon_1(\overline{X}) + (\epsilon_2 - \epsilon_1)(\overline{X}) \subseteq \epsilon_1(\overline{X}) + N \subseteq \epsilon_1(\overline{X}).$$

Of course, by symmetry we also have $\epsilon_2(\overline{X}) \subseteq \epsilon_1(\overline{X})$, and therefore equality.

Clearly, the submodule $I_\epsilon(N)$ incorporates the information about the vanishing in Ext^2 of the exact sequences of the form $(*)$, where $K \to X$ is the kernel map for $\epsilon\colon X \to N$.

Theorem 3. *Let $\epsilon\colon X \to N$ be an epimorphism. Consider a submodule $N \subseteq Y \subseteq I(N)$ and denote by $\nu\colon N \to Y$ the inclusion map. Let $\alpha = \nu\epsilon$. Then $\alpha\colon X \to Y$ is kernel-determined if and only if $Y \cap I_\epsilon(N) = N$.*

Proof. We fix some notation. Let $D = Y \cap I_\epsilon(N)$. Let $\nu'\colon N \to D$, $\nu''\colon D \subseteq Y$, $\nu'''\colon Y \to I(N)$, $\kappa\colon D \to I_\epsilon(N)$, and $\mu\colon X \to \overline{X}$ be the inclusion maps. Thus we have $\nu = \nu''\nu$.

The inclusion map $\kappa\nu'\colon X \to I_\epsilon(N)$ is part of the following commutativity relation:

$$\kappa\nu'\epsilon = \overline{\epsilon}_1\mu, \tag{1}$$

where we denote by $\overline{\epsilon}_1$ the epimorphism part of an extension $\overline{\epsilon}$ of ϵ.

First, let us assume that $\nu'\colon N \subset D = Y \cap I_\epsilon(N)$ is a proper inclusion. Then there exists an indecomposable projective module P and a homomorphism $\eta\colon P \to D$ such that the image of η does not lie inside N. Now $\overline{\epsilon}_1\colon X \to I_\epsilon(N)$ is surjective, thus we can lift the map $\kappa\eta\colon P(S) \to I_\epsilon(N)$ to $\overline{X}$ and obtain $\eta'\colon P(S) \to \overline{X}$ such that

$$\overline{\epsilon}_1\eta' = \kappa\eta \tag{2}$$

Also note that $\eta'\iota$ maps into the radical of $\overline{X}$, thus into X. This shows that there is $\eta''\colon \operatorname{rad} P(S) \to X$ such that

$$\mu\eta'' = \eta'\iota. \tag{3}$$

Altogether, we deal with the following diagram:

$$\begin{array}{ccccc}
\operatorname{rad} P(S) & \xrightarrow{\ \iota\ } & P(S) & \xrightarrow{\ \eta\ } & D \\
{\scriptstyle \eta''}\downarrow & (3) & \downarrow{\scriptstyle \eta'} & (2) & \\
X & \xrightarrow{\ \mu\ } & \overline{X} & & \downarrow{\scriptstyle \kappa} \\
{\scriptstyle \epsilon}\downarrow & (1) & & \searrow{\scriptstyle \overline{\epsilon}_1} & \\
N & & \xrightarrow[\kappa\nu']{\qquad\qquad} & & I_\epsilon(N)
\end{array}$$

Using the three equalities (1), (3), (2), we see:

$$\kappa\nu'\epsilon\eta'' = \overline{\epsilon}_1\mu\eta'' = \overline{\epsilon}_1\eta'\iota = \kappa\eta\iota.$$

but κ is injective, thus $\nu'\epsilon\eta'' = \eta\iota$, and therefore

$$\alpha\eta'' = \nu''\nu'\epsilon\eta'' = \nu''\eta\iota.$$

This asserts that the following diagram commutes

$$\begin{array}{ccc}
\operatorname{rad} P & \xrightarrow{\ \iota\ } & P \\
{\scriptstyle \eta''}\downarrow & & \downarrow{\scriptstyle \nu''\eta} \\
X & \xrightarrow{\ \alpha\ } & Y
\end{array}$$

Since by construction the right map $\nu''\eta$ does not map into N, it does not factor through ϵ, thus also not through $\alpha = \nu''\nu'\epsilon$, therefore we see that P almost factors through α. But this shows that α is not kernel-determined.

Conversely, let us assume that $\alpha = \nu''\nu\epsilon$ is not kernel-determined, thus there is an indecomposable projective module P and a commutative diagram

$$\begin{array}{ccc} \operatorname{rad} P & \xrightarrow{\iota} & P \\ {\scriptstyle\psi'}\downarrow & & \downarrow{\scriptstyle\psi} \\ X & \xrightarrow{\alpha} & Y \end{array}$$

such that ψ does not factor through $\alpha = \nu\epsilon$, thus $\psi(P)$ is not contained in N. Let us form a pushout of ι and ψ', say

$$\begin{array}{ccc} \operatorname{rad} P & \xrightarrow{\iota} & P \\ {\scriptstyle\psi'}\downarrow & & \downarrow{\scriptstyle\psi''} \\ X & \xrightarrow{\iota'} & J \end{array}$$

we obtain a map $\beta\colon J \to Y$ such that $\beta\psi'' = \psi$ and $\beta\iota' = \alpha$. Since Y is a submodule of $I(N)$, the image $\beta(J)$ of β is a submodule of Y, thus of $I(N)$.

Let us show that ι' does not split and its cokernel is simple. The cokernel of ι' is isomorphic to the cokernel of ι, thus simple.

Let us show that *the kernel of β is just $\iota(K)$,* where K is the kernel of α. Since $\beta\iota = \alpha$, we see that $\iota(K)$ is contained in the kernel of β, thus it remains to show that $|\operatorname{Ker}(\beta)| \le |\operatorname{Ker}(\alpha)|$ (note that ι is injective). Since $\alpha = \beta\iota$, the image N of α is contained in the image of β. This must be a proper inclusion. Otherwise, we use $\psi = \beta\psi''$ in order to obtain that $\operatorname{Im}(\psi) \subseteq \operatorname{Im}(\beta) = \operatorname{Im}(\alpha) = N$, a contradiction. Thus $|\operatorname{Im}(\beta)| \ge |\operatorname{Im}(\alpha)| + 1$. Therefore

$$|\operatorname{Ker}(\beta)| = |J| - |\operatorname{Im}(\beta)| \le |X| + 1 - |\operatorname{Im}(\alpha)| - 1 = |\operatorname{Ker}(\alpha)|.$$

It follows that ι' does not split. Otherwise we have $J = \iota(X) \oplus S$. Now the kernel of β is $\iota(K) = \iota(K) \oplus 0$, and therefore β would provide an embedding of $X/K \oplus S$ into Y. However, by assumption, Y is an essential extension of $N = X/K$, a contradiction.

Thus we have shown that ι' is a monomorphism with simple cokernel, and it does not split. Therefore, we may assume that J is a submodule of $\overline{X}$, If we compose β with $\nu'''\colon Y \to I(N)$, we obtain the following commutative square

$$\begin{array}{ccc} X & \xrightarrow{\iota'} & J \\ {\scriptstyle\epsilon}\downarrow & & \downarrow{\scriptstyle\nu'''\beta} \\ N & \xrightarrow{\nu'''\nu} & I(N) \end{array}$$

which shows that $\nu'''\beta$ is part of an extension $\overline{\epsilon}\colon \overline{X} \to I(N)$ of ϵ. As a consequence, the image of β is contained in $I_\epsilon(N)$. But the image $\beta(J)$ of β is also a submodule of Y, that $\beta(J) \subseteq D$.

Since $\beta\psi'' = \psi$, the image of β contains the image of ψ, thus $\beta(J)$ is not contained in N.

Altogether we see that $\beta(J) \subseteq Y \cap I_\epsilon(N)$, and $\beta(J) \not\subseteq N$, thus $Y \cap I_\epsilon(N) \neq N$. This completes the proof.

Example 4, continued. Again, let Λ be a local uniserial ring. Let X, Y be indecomposable Λ-modules and $\alpha\colon X \to Y$ a morphism. We have noted above that if X is projective and $\alpha \neq 0$, then α is kernel-determined. On the other hand, if α is surjective, then again, α is kernel-determined.

But also the converse is true: If $\alpha\colon X \to Y$ is kernel-determined, then either $\alpha \neq 0$ and X is projective, or else α is surjective. Here is the proof: Assume that α is kernel-determined. According to Proposition 2, we must have $\alpha \neq 0$. Assume that X is not projective, thus also not injective. Write $\alpha = \nu\epsilon$, where ϵ is surjective and $\nu\colon N \to Y$ is the inclusion of a non-zero submodule N of Y. Since X is not injective, X is a proper submodule of $\overline{N}$. Let $\overline{\epsilon}\colon \overline{X} \to \overline{N}$ be an extension of ϵ. Then also $\overline{\epsilon}$ is surjective. But this means that $I_\epsilon(N) = \overline{N}$, and therefore Theorem 3 asserts that $Y = N$, thus α is surjective.

Corollary. *Let $\epsilon\colon X \to N$ be an epimorphism and $N \subseteq Y$ an inclusion map with semisimple cokernel such that the composition $X \to N \to Y$ is kernel-determined. Then there is an inclusion map $Y \to Z$ such that the composition $X \to Y \to Z$ has semisimple cokernel, is kernel-determined and satisfies*

$$|Z| = |N| + |\overline{N}| - |I_\epsilon(N)|.$$

In particular, the length of Z only depends on ϵ.

Proof: We can assume that Y is a submodule of $\overline{N}$. Choose $N \subseteq Z \subseteq \overline{N}$ maximal with $Z \cap I_\epsilon(N) = N$. According to Theorem 3, the composition $X \to Y \to Z$ (which is the composition of ϵ and the inclusion map $N \to Z$) is kernel-determined. The maximality of Z implies that $Z + I_\epsilon(N) = \overline{N}$. The stated equality comes from the formula

$$|Z| + |\overline{\epsilon}(X)| = |Z \cap \overline{\epsilon}(X)| + |Z + \overline{\epsilon}(X)|.$$

Summary. The kernel-determined morphisms can be characterized as suitable prolongations of epimorphisms. Here, we call the composition $X \to Y \to Z$ a *prolongation* of $X \to Y$ provided the map $Y \to Z$ is an inclusion map; the prolongation is said to be *proper* provided the map $Y \to Z$ is a proper inclusion map.

(a) Any epimorphism $X \to N$ is kernel-determined.

(b) If the map $X \to Y$ has a prolongation $X \to Y \to Y'$ which is kernel-determined, then $X \to Y$ is kernel-determined and $Y \to Y'$ is an essential extension.

(c) Let $X \to N$ be an epimorphism, and $N \subseteq Y \subseteq I(N)$. If $X \to N \to Y \cap \overline{N}$ is kernel-determined, also $X \to N \to Y$ is kernel-determined.

(d) Any kernel-determined map $X \to Y$ has a maximal kernel-determined prolongation $X \to Y \to Y'$.

(e) If $X \to N$ is an epimorphism, and $N \subseteq Y \subseteq I(N)$, then $X \to N \to Y$ is kernel-determined if and only if $Y \cap I_\epsilon(N) = N$.

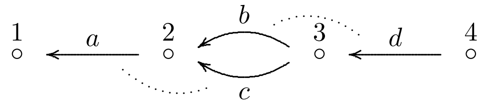
(f) If $X \to N$ is an epimorphism and $X \to N \to Y$ is a maximal kernel-determined prolongation, then

$$|\operatorname{soc}(Y/N)| = |\operatorname{soc}(I(N)/N)| - |I_\epsilon(N)/N|;$$

in particular, the length of $\operatorname{soc}(Y/N)$ is determined by ϵ.

Thus, if $X \to N$ is an epimorphism and $X \to N \to Y$ and $X \to N \to Y'$ are maximal kernel-determined prolongations, then $\mathrm{soc}(Y/N)$ and $\mathrm{soc}(Y'/N)$ have the same length, but Y and Y' may have different length, as the following example shows:

Example 6. Consider the representations of the following quiver with relations over the field k:



We denote the simple, projective, or injective module corresponding to the vertex x by $S(x), P(x), I(x)$, respectively. The full subquiver with vertices $2, 3$ is the Kronecker quiver, the representations with support in this subquiver will be said to be Kronecker modules. The 2-dimensional indecomposable Kronecker module which is annihilated by $\lambda_1 b + \lambda_2 c$ (not both λ_1, λ_2 equal to zero) will be denoted by $R(\lambda_1 b + \lambda_2 c)$. For example, $I(1)/S(1) = R(c)$ and $\mathrm{rad}\, P(4) = R(b)$.

Let $X = P(2)$ and $N = S(2)$ and $\epsilon\colon X \to N$ the canonical projection $P(2) \to S(2)$. Then $\overline{X} = I(1)$ is indecomposable with composition factors $S(1), S(2), S(3)$. The module $\overline{N}$ has length 3, namely one composition factor $S(2)$ and two composition factors $S(3)$, it is just the indecomposable injective Kronecker module of length 3 and $I_\epsilon(N) = R(c)$.

In view of Theorem 3, we are interested in the submodules Y of $\overline{N}$ which satisfy $Y \cap I_\epsilon(N) = N$, thus $Y \cap R(c) = N$. Besides N itself, these are the Kronecker modules of the from $R(b + \lambda c)$ with $\lambda \in k$. The modules $Z = R(b + \lambda c)$ provide the maximal kernel-determined prolongations $X \to Y \to Z$ of $X \to N$ inside $\overline{N}$.

Now only the map $X \to N \to R(b)$ has a proper kernel-determined prolongation, namely $X \to R(b) \to P(4)$. The other maps $X \to N \to R(b + \lambda c)$ with $\lambda \neq 0$ have no proper kernel-determined prolongation.

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