The Diamond Category of a Locally Discrete Ordered Set.

Claus Michael Ringel

Let k be a field. Let I be a ordered set (what we call an ordered set is sometimes also said to be a **totally** ordered set or a chain). Let $k[I]$ be the incidence algebra of I; it is the k-algebra with basis the pairs (i,j) where $i,j \in I$ and $i \le j$, with multiplication $(i,j)(i',j') = (i,j')$ provided $j = i'$ and equal to zero, otherwise. In case I is an infinite set, then the algebra $k[I]$ does not have an identity element, but it always has sufficiently many idempotents, namely the pairs (i,i) with $i \in I$.

The aim of this note is to consider the following special case: Let T be an ordered set, let $I = T\vec{\times}\mathbb{Z}$ be the product $T \times \mathbb{Z}$ with the lexicographical ordering. We consider the category $\mathcal{D}(I)$ of all $k[I]$-modules which are finitely generated and finitely cogenerated and we will show that this category is a hereditary abelian k-category without non-zero projective objects, but with almost split sequences. In particular, $\mathcal{D}(I)$ is a hereditary abelian k-category with Serre duality as considered by Reiten and Van den Bergh [RV]. Recall that in this paper [RV], Reiten and Van den Bergh have classified all the hereditary abelian k-category with Serre duality which are derived equivalent to a noetherian one. The examples presented here are usually non-noetherian. In fact, we show (Corollary 2) that $\mathcal{D}(T\vec{\times}\mathbb{Z})$ is derived equivalent to a noetherian hereditary abelian category if and only if the cardinality of T is 1 or 2.

The categories of the form $\mathcal{D}(I)$ are of interest also for another reason: we may construct in this way nice abelian categories with arbitrary finite Krull-Gabriel dimension. Namely, we will show (Corollary 3) that the category $\mathcal{D}(T)$, where $T = \mathbb{Z}^n$ with lexicographical ordering, has Krull-Gabriel dimension $n-1$.

The author is indebted to Henning Krause and and Idun Reiten for questions and remarks which stimulated these investigations. In particular, Idun Reiten has pointed out that categories which are derived equivalent to the ones presented here, have already been studied in her joint work with M. Auslander [AR], see the final remark of the paper.

1. The serial representations of an ordered set.

Let $T \neq \emptyset$ be an ordered set. Let us construct the serial $k[T]$-modules (recall that a module is said to be *serial* provided its submodules are ordered). As usual, we call a subset $J \subseteq T$ an ideal provided for any $j \in J$ any element $i \in T$ with $i < j$ belongs to J. If J is an ideal of T, we may consider the vector space $M(J) = kJ$ with basis J as an $k[T]$-module with scalar multiplication $(i,i') * j = i$ in case $i' = j$ and zero otherwise (here, (i,i') is considered as a basis element of $k[T]$, thus we assume that $i \le i'$, whereas the letter j as well as the letter i on the right side of the equality sign are basis elements of M). Of course, if $I \subseteq J$ are ideals of T, then $M(I)$ is a submodule of $M(J)$. We denote $M(J/I) = M(J)/M(I)$. Note that we obtain a bijection between the submodules of $M(J/I)$ and the ideals I' with $I \subseteq I' \subseteq J$; in particular, all these modules $M(J/I)$ are serial, and any serial modules is obtained (up to isomorphism) in this way. We denote by $\mathcal{S}(T)$ the full subcategory of $\operatorname{Mod} k[T]$ given by all finite direct sums of serial modules.

Let us consider some maps and some extensions in the category $\mathcal{S}(T)$. If $I \subseteq I' \subseteq J$ are ideals of T, there is the canonical projection $M(J/I) \to M(J/I')$, and such a map will be denoted by π; if $I \subseteq J \subseteq J'$, there is the canonical inclusion $M(J/I) \to M(J'/I)$. it will be denoted by ι. In general, we have

$$\operatorname{Hom}(M(J/I), M(J'/I')) = \begin{cases} k & \text{in case} \quad I \subseteq I' \subset J \subseteq J', \\ 0 & \text{otherwise} \quad . \end{cases}$$

In case $I \subseteq I' \subset J \subseteq J'$, a non-zero map $M(J/I) \to M(J'/I')$ is given by the composition map in the following commutating diagram:

$$\begin{array}{ccccc} & & M(J'/I) & & \\ & \overset{\iota}{\nearrow} & & \overset{\pi}{\searrow} & \\ M(J/I) & & & & M(J'/I') \\ & \underset{\pi}{\searrow} & & \underset{\iota}{\nearrow} & \\ & & M(J/I') & & \end{array}$$

Of course, the commutative diagram yields also the following exact sequence

$$(*) \qquad 0 \to M(J/I) \xrightarrow{\left[\begin{smallmatrix}\iota \\ \pi\end{smallmatrix}\right]} M(J'/I) \oplus M(J/I') \xrightarrow{[\pi \ -\iota]} M(J'/I') \to 0$$

A similar exact sequence is obtained in case we deal with $I \subseteq I' = J \subseteq J'$:

$$0 \to M(J/I) \xrightarrow{\iota} M(J'/I) \xrightarrow{\pi} M(J'/I') \to 0.$$

Actually, we don't have to distinguish the two cases, the exact sequence $(*)$ exists for all quadruples I, I', J, J' with $I \subseteq I' \subseteq J \subseteq J'$, and reduces to the second one in case $I' = J$ so that $M(J'/I) = 0$. In general, the sequence $(*)$ is non-split provided $I \subset I'$ and $J \subset J'$.

The description of the Hom-sets of indecomposable objects has the following consequence: *For any ordered set T, the category $\mathcal{S}(T)$ is directed.*

We will use certain full subcategories $\mathcal{G}$ of $\mathcal{S}(T)$ which are constructed as follows: Let $\mathcal{I} = (I_0, I_1, \dots, I_m)$ be a chain of ideals of T, thus $I_0 \subset I_1 \subset \cdots \subset$

I_m. Denote by $\mathcal{G} = \mathcal{G}(\mathcal{I})$ the full subcategory of all $k[T]$-modules which have a finite filtration with factors of the form $N_r = M(I_r/I_{r-1})$, for $1 \le r \le m$. This subcategory $\mathcal{G}$ will be said to be the *grid* subcategory given by $\mathcal{I}$. Note that $\mathcal{G}$ is a full exact abelian subcategory and the objects N_r are simple objects in $\mathcal{G}$. Since $\dim_k \operatorname{Ext}^1(N_r, N_s) = 1$ for $s = r-1$ and 0 otherwise, we see that $\mathcal{G}$ is equivalent to the category of finitely generated B-modules, where B is a factor algebra of the path algebra kQ of the linearly oriented quiver Q of type A_m. The module $M(I_m/I_0)$ belongs to $\mathcal{G}$; it is indecomposable and has a filtration with all the factors $N_1, \dots, N_m$. This shows that actually $B = kQ$.

Let $M_1, \dots, M_n$ be serial $k[T]$-modules. Let $M_r = M(J_r/J'_r)$, for $1 \le r \le n$ and ideals $J'_r \subseteq J_r$ of T. Let $\{J_r, J'_r \mid 1 \le r \le n\} = \{I_0, \dots, I_m\} = \mathcal{I}$ with $I_0 \subset I_1 \subset \cdots \subset I_m$. Of course, $m \le 2n-1$. Let $\mathcal{G} = \mathcal{G}(\mathcal{I})$ be the grid subcategory given by $\mathcal{I}$, we call it the *grid subcategory generated by* $M_1, \dots, M_n$ and write $\mathcal{G} = \mathcal{G}(M_1, \dots, M_n)$.

These considerations may be interpreted as follows: *The category $\mathcal{S}(T)$ is a filtered union of grid subcategories, and any of these subcategories is equivalent to the category of* $\operatorname{mod} kQ$ *where Q is the linearly oriented quiver of type A_m for some finite m.* We will use this observation quite often.

Remark: As we have seen, the grid category $\mathcal{G}$ generated by the serial $k[T]$-modules $M_1, \dots, M_n$ is a full exact abelian subcategory of $\mathcal{S}(T)$ containing the given modules $M_1, \dots, M_n$, but it is not necessarily the smallest such subcategory. For example, if we start with ideals $J_0 \subset J_1 \subset J_2 \subset J_3$ and take $M_1 = M(J_3/J_0)$ and $M_2 = M(J_2/J_1)$, then the category of all direct sums of copies of M_1 and M_2 is a full exact abelian subcategory and contains both M_1 and M_2, but this is a proper subcategory of $\mathcal{G}(M_1, M_2)$.

2. The diamond category $\mathcal{D}(T)$ of an ordered set T.

Since (i,i) with $i \in T$ is a primitive idempotent of $A = k[T]$, the module $P(i) = A(i,i)$ is projective and serial. Of course, $P(i) = M(\langle i \rangle)$, where $\langle i \rangle$ is the ideal generated by i, namely the set of all elements $j \le i$. Also, let $\langle \hat{i} \rangle$ be the ideal of all elements $j < i$. Then $S(i) = M(\langle i \rangle / \langle \hat{i} \rangle)$ is a simple module and one obtains all the simple modules in this way.

For $i \le j$, let us denote $M(i,j) = M(\langle j \rangle / \langle \hat{i} \rangle)$. The module $M(i,j)$ has a simple top isomorphic to $S(j)$ and a simple socle isomorphic to $S(i)$. The following assertion is quite obvious:

Lemma 1. *The following assertions are equivalent for an A-module M.*
(i) *M is finitely generated and finitely cogenerated.*
(ii) *M is a finite direct sum of modules of the form $M(i,j)$.*

We denote by $\mathcal{D} = \mathcal{D}(T)$ the full subcategory of $\operatorname{mod} k[T]$ given by all modules which are finitely generated and finitely cogenerated (the modules $M(i,j)$

are said to be the *diamonds* of $\operatorname{mod} k[T]$). Note that $\mathcal{D}(T)$ is a full subcategory of $\mathcal{S}(T)$. In general, the category $\mathcal{D}(T)$ is not closed under kernels or cokernels.

Let I be an ordered set. Let $i < j$ in I be neighbors (i.e. there is no $t \in T$ with $j' < t < j$), then we write $j = i+1$ or $i = j-1$.

Lemma 2. *The following conditions are equivalent:*
(i) $\mathcal{D}(T)$ *is closed under kernels.*
(ii) *For any non-minimal element $j \in T$, there exists $j' < j$ such that j' and j are neighbors.*
(iii) *The $k[T]$-modules $M(i,j)$ are finitely presented, for all $i < j$ in T.*

Proof: (i) $\implies$ (ii): If $j \in T$ is non-minimal, let $i < j$. The kernel of $\pi\colon M(i,j) \to S(j)$ is $M(\langle \hat{j}\rangle/\langle \hat{i}\rangle)$. The latter module belongs to $\mathcal{D}(T)$ only in case $\langle \hat{j}\rangle$ is of the form $\langle j'\rangle$, but this means that $j' < j$ is a neighbor of j.
(ii) $\implies$ (iii) is clear: $M(i,j) = P(j)/P(i-1)$.
(iii) $\implies$ (i): Let $f\colon N \to N'$ be a map in $\mathcal{D}(T)$. It follows from (iii) that the kernel of f is finitely generated. As a submodule of N, the kernel is also finitely cogenerated. Thus, the kernel of f belongs to $\mathcal{D}(T)$.

3. Ordered sets which are locally discrete.

We say that I is *locally discrete* provided no element of I is an accumulation point: this means, first, that for any non-maximal element $i \in I$, the neighbor $i+1$ exists, and second, that for any non-minimal element $i \in I$, the neighbor $i-1$ exists.

The previous lemma together with its dual yields the following result.

Corollary 1. $\mathcal{D}(I)$ *is an exact abelian subcategory of* $\mathcal{S}(I)$ *if and only if I is locally discrete.*

Lemma 3. *Let $I \neq \emptyset$ be a locally discrete ordered set. The following conditions are equivalent:*
(i) *I has a minimal element.*
(ii) $\mathcal{D}(I)$ *has at least one indecomposable projective object.*
(iii) $\mathcal{D}(I)$ *has projective covers.*

Proof: (ii) $\implies$ (i): Let $M(i,j)$ be indecomposable projective. Assume i is not minimal. Then there is $i' < i$ and the canonical projection $\pi\colon M(i',j) \to M(i,j)$ is an epimorphism which does not split. This contradiction shows that i has to be minimal.
(i) $\implies$ (iii): Let t be minimal in I. Then the canonical projection $\pi\colon M(t,j) \to M(i,j)$ is a projective cover for any $i \leq j$.

Of course, there is also the dual assertion: *I has a maximal element, if and only if $\mathcal{D}(I)$ has at least one indecomposable injective object, if and only if $\mathcal{D}(I)$ has injective envelopes.*

Also we remark: If we start with a finite number of diamonds $M_1, \dots, M_n$, then the grid subcategory of $\mathcal{S}(I)$ generated by the modules $M_1, \dots, M_n$ is a subcategory of $\mathcal{D}(I)$. Namely, we write $M_r = M(j_r, j'_r)$, for $1 \le r \le n$, with elements $j_r \le j'_r$ in I, and consider the set $\{j_r - 1, j'_r \mid 1 \le r \le n\} = \{i_0, \dots, i_m\}$ with $i_0 < i_1 < \dots < i_m$. Then $\mathcal{G}(M_1, \dots, M_n)$ is the full subcategory of all $k[I]$-modules which have a filtration with factors $M(i_{r-1}+1, i_r)$, where $1 \le r \le m$.

Proposition 1. *Let I be a locally discrete ordered set. Then $\mathcal{D}(I)$ has Auslander-Reiten sequences and the Auslander-Reiten translation τ is given as follows: $\tau M(i,j) = M(i-1, j-1)$, for $i \le j$ and i not minimal.*

Remark. If $i \in I$ is minimal, and $i \le j$, then $M(i,j)$ is projective and its radical is $M(i, j-1)$ (using again the convention that $M(i, i-1) = 0$). If I has no minimal element, then $i \mapsto i-1$ is an injective map $I \to I$, and $\tau\colon \mathcal{D}(I) \to \mathcal{D}(I)$ is the induced functor. If I has neither a minimal nor a maximal element, then $i \mapsto i-1$ is bijective and the induced functor $\tau\colon \mathcal{D}(I) \to \mathcal{D}(I)$ will be an equivalence!

Proof. Let us assume that $i \le j$ and that i is not minimal. Then $M(i–1, j–1)$ is defined. Of course, we may consider the exact sequence

$$(*) \qquad 0 \longrightarrow M(i-1, j-1) \xrightarrow{\left[\begin{smallmatrix}\iota\\ \pi\end{smallmatrix}\right]} M(i-1, j) \oplus M(i, j-1) \xrightarrow{[\pi\ -\iota]} M(i,j) \longrightarrow 0,$$

we claim that this is an almost split sequence in $\mathcal{S}(I)$. Consider an indecomposable object N in $\mathcal{S}(I)$, and take $\mathcal{G} = \mathcal{G}(M(i,j), M(i-1, j-1), N)$. This category is equivalent to $\operatorname{mod} kQ$, where Q is a linearly ordered quiver of type A_m with $m \le 5$. The sequence $(*)$ lies in $\mathcal{G}$, and clearly is an almost split sequence in $\mathcal{G}$. Thus it has the desired lifting properties with respect to the maps $N \to M(i,j)$ and $M(i-1, j-1) \to N$.

4. The locally discrete ordered sets without extremal elements.

By definition, the *extremal* elements of an ordered sets are the elements which are minimal or maximal. Our further investigations will deal with locally discrete ordered sets without extremal elements. First, let us characterize these ordered sets by providing a construction.

Construction. Let $T \neq \emptyset$ be any ordered set and consider the product $I = T \times \mathbb{Z}$ with lexicographical ordering (we will write $I = T \overrightarrow{\times} \mathbb{Z}$), thus if $t, t' \in T$ and $z, z' \in \mathbb{Z}$, then

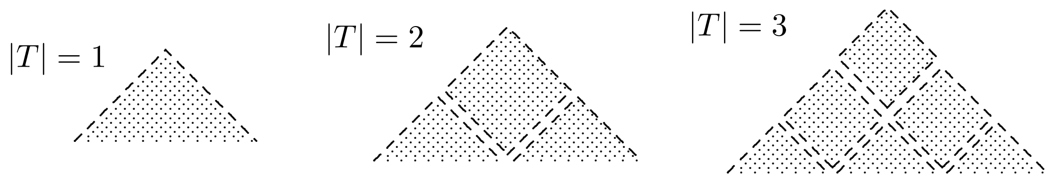
$$(t,z) < (t', z') \quad \iff \quad \text{either} \quad t < t' \quad \text{or} \quad t = t',\ z < z'.$$

Given $i = (t,z) \in I$ and $z' \in \mathbb{Z}$, we denote by $i + z' = (t, z+z')$. Note that the operation $+z'$ provides an automorphism of I.

We obtain in this way a locally discrete ordered set I with no extreme elements, and conversely, any locally discrete ordered set with no extreme elements is obtained in this way. Namely, given a locally discrete ordered set I with no extreme elements, call two elements i, i' of I equivalent (and write $i \sim i'$) provided the sets $\{j \mid i < j < i'\}$ and $\{j \mid i' < j < i\}$ are both finite (one will be empty), and let $T = I/\!\sim$. Then the ordered sets I and $T \overset{\rightarrow}{\times} \mathbb{Z}$ are isomorphic. On the other hand, any ordered set T is isomorphic to $(T \overset{\rightarrow}{\times} \mathbb{Z})/\!\sim$. This shows that dealing with a locally discrete ordered set I with no extreme elements, we always may assume $I = T \overset{\rightarrow}{\times} \mathbb{Z}$, where T is any ordered set (see also [Sch]).

The Auslander-Reiten quiver Γ of $\mathcal{D}(T \overset{\rightarrow}{\times} \mathbb{Z})$. Note that two modules $M((t,z),(t',z'))$ and $M((t_1,z_1),(t_1',z_1'))$ belong to the same component of Γ if and only if $t = t_1$ and $t' = t_1'$. Thus the components of Γ correspond bijectively to the pairs (t,t') of T with $t \le t'$. For $t \le t'$, we denote by $\Gamma_{t,t'}$ the component which contains all the modules of the form $M((t,z),(t',z'))$. There are two kinds of components: For $t = t'$, all these modules are of finite length and $\Gamma_{t,t'}$ is of the form $\mathbb{Z}A_\infty$. For $t < t'$, the modules $M((t,z),(t',z'))$ are neither artinian nor noetherian and the component $\Gamma_{t,t'}$ is of the form $\mathbb{Z}A_\infty^\infty$.

Let us sketch the structure of the Auslander-Reiten quivers in case $|T| \le 3$.



Observe that the second category (that for $|T| = 2$) is one which one knows very well: its derived category has been considered both in sections 6 and 7 of [R]

Corollary 2. *The category $\mathcal{D}(T \overset{\rightarrow}{\times} \mathbb{Z})$ is derived equivalent to a noetherian hereditary abelian category if and only if the cardinality of T is 1 or 2.*

Proof. As we have seen, the category $\mathcal{S}(I)$, and therefore also $\mathcal{D}(I)$ is directed, for any ordered set I. Now, let $I = T \overset{\rightarrow}{\times} \mathbb{Z}$, thus $\mathcal{D}(I)$ is abelian. Since $\mathcal{D}(I)$ is directed, also the derived category $D^b(\mathcal{D}(I))$ is directed. Now assume $\mathcal{D}(I)$ is derived equivalent to some noetherian hereditary abelian category $\mathcal{H}$. With $\mathcal{D}(I)$ also $\mathcal{H}$ is a k-category and satisfies Serre duality, thus $\mathcal{H}$ is one of the categories classified by Reiten and Van den Bergh [RV]. We have to see which of the categories listed under (a),(b),(c),(d) in [RV] are directed. In case (a), consider the category of finite dimensional representations of the linearly oriented quiver of type A_∞^∞, but this category is just the category $\mathcal{D}(\mathbb{Z})$, thus we deal with the case $|T| = 1$. In case (b), no category is directed. The case (d) has been discussed in [R], provided we deal with at least one ray. According to section 7 of [R], the directed categories are just those of bounded representation type. As we have seen, there are three essentially different cases. In one of these cases, we encounter Auslander-Reiten triangles with three middle terms, this is for $D^b(\mathcal{D}(I))$. In the remaining two cases $D^b(\mathcal{H})$ has 1 or 3 shift orbits of Auslander-Reiten components, and for $|T| = n$, the number of shift orbits for $D^b(\mathcal{D}(I))$ is $\binom{n}{2}$, thus $n = 1$ or $n = 2$. Of course, in case (d) without rays, we deal with the category of representations of a quiver Q without infinite paths: Again, only the cases where Q is of type A_∞ or A_∞^∞ are of interest, and the corresponding categories $D^b(\operatorname{mod} kQ)$ have 1 or 3 shift orbits of Auslander-Reiten components. Finally, the categories noted in case (c) are derived equivalent to categories already mentioned.

5. Krull-Gabriel dimension.

Recall that the existence and then the value of the *Krull-Gabriel dimension* of an abelian category $\mathcal{A}$ is defined inductively as follows: The zero category has dimension -1, and $\mathcal{A}$ is said to have Krull-Gabriel dimension $n+1$ provided the full subcategory $\mathcal{A}_0$ of objects in $\mathcal{A}$ of finite length is non-zero, and $\mathcal{A}/\!\!/\mathcal{A}_0$ has Krull-Gabriel dimension n. (Here, $\mathcal{B} = \mathcal{A}/\!\!/\mathcal{A}_0$ is an abelian category with an exact functor $\eta\colon \mathcal{A} \to \mathcal{B}$ such that any exact functor ϕ from $\mathcal{A}$ to an arbitrary abelian category which sends all finite length objects to zero, factors uniquely via η.) It is usually not easy to determine the structure of $\mathcal{A}/\!\!/\mathcal{A}_0$, the categories of the form $\mathcal{A} = \mathcal{D}(T\vec{\times}\mathbb{Z}\vec{\times}\mathbb{Z})$ seem to be a rare exception.

Proposition 2. *Let T be any ordered set. Then $\mathcal{D}(T\vec{\times}\mathbb{Z}\vec{\times}\mathbb{Z})/\!\!/\mathcal{D}(T\vec{\times}\mathbb{Z}\vec{\times}\mathbb{Z})_0$ is equivalent, as a category, to $\mathcal{D}(T\vec{\times}\mathbb{Z})$.*

Proof. We define an exact functor

$$\eta\colon \mathcal{D}(T\vec{\times}\mathbb{Z}\vec{\times}\mathbb{Z})/\!\!/\mathcal{D}(T\vec{\times}\mathbb{Z}\vec{\times}\mathbb{Z})_0 \longrightarrow \mathcal{D}(T\vec{\times}\mathbb{Z})$$

as follows: Let $t, t' \in T$, and $x, x', y, y' \in \mathbb{Z}$, with $(t, x, y) \le (t', x', y')$. Define

$$\eta M((t,x,y),(t',x',y')) = \begin{cases} M((t,x),(t',x'-1)) & \text{in case } t < t', \\ & \text{or } t = t' \text{ and } x < x', \\ 0 & \text{in case } t = t' \text{ and } x = x'. \end{cases}$$

The indecomposable modules of finite length are the modules $M((t,x,y),(t,x,y'))$ with $y \le y'$, thus we see that η vanishes precisely on the modules of finite length. In order to see that η is exact, we may restrict to a grid category $\mathcal{G} \subset \mathcal{D}$. In order to see that the restriction of η to $\mathcal{G}$ is exact, it is sufficient to show that the almost split sequences in $\mathcal{G}$ are sent to exact sequences, see [B]. Thus, we have to consider only the exact sequences in $\mathcal{D}$ with indecomposable end terms, they are of the form $(*)$, and it is straight forward to check that these sequences remain exact when we apply η.

Note that two indecomposable objects of $\mathcal{D}(T\vec{\times}\mathbb{Z}\vec{\times}\mathbb{Z})$ which are not of finite length have images under η which are isomorphic if and only if they belong to the same Auslander-Reiten component.

Finally, consider any exact functor ϕ from $\mathcal{D}(T\vec{\times}\mathbb{Z}\vec{\times}\mathbb{Z})$ to an abelian category which sends all finite length modules to zero. But this means that under ϕ all the maps in a given Auslander-Reiten component are sent to isomorphisms. It follows easily that ϕ factors through η.

Corollary 3. *Let $T = \mathbb{Z}^n$ with lexicographical ordering. Then $\mathcal{D}(T)$ has Krull-Gabriel dimension $n-1$.*

Remark. Idun Reiten has pointed out the relationship of these investigations to her joint work with M. Auslander, dealing with hereditary 1-Gorenstein dualizing k-varieties. In the paper [AR], Auslander and Reiten start with an ordered set I' which is locally discrete and which has a minimal element as well as a maximal element and they show that $\mathcal{D}(I') = \mathcal{S}(I')$ is such a hereditary 1-Gorenstein dualizing k-varieties ([AR], Theorem 4.7).

In general, we note the following: Call two ordered sets *rotational equivalent*, provided I has an ideal I_1 and I' has an ideal I_1' such that there are order isomorphisms $I_1 \simeq I' \setminus I_1'$ and $I \setminus I_1 \simeq I_1'$. *If I, I' are locally discrete ordered sets which are rotational equivalent, then the categories $\mathcal{D}(I)$ and $\mathcal{D}'(I')$ are derived equivalent.*

Of course, any locally discrete ordered set $I \neq \emptyset$ without a minimal or maximal element is rotational equivalent to a locally discrete ordered set I' with both a minimal and a maximal element. In this way, one sees that the categories $\mathcal{D}(I)$ considered here are derived equivalent to hereditary 1-Gorenstein dualizing k-varieties.

References

[AR] M. Auslander, I. Reiten: Stable equivalence of dualizing R-varieties. II. Hereditary dualizing R-varieties. Advances in Math. 17 (1975), 93-121.

[B] M.C.R. Butler: Grothendieck groups and almost split sequences. In: Integral Representations and Applications (ed. K.W. Roggenkamp). Springer LNM 882 (1981)

[RV] I. Reiten, M. Van den Bergh: Noetherian hereditary categories satisfying Serre duality. (Preprint)

[R] C.M. Ringel: A ray quiver construction of hereditary abelian categories with Serre duality. This volume.

[Sch] J. Schröer. On the Krull-Gabriel dimension of an algebra. Math. Z. 233 (2000), 287-303.

Claus Michael Ringel, Fakultät für Mathematik, Universität Bielefeld,

POBox 100 131, D-33 501 Bielefeld

ringel@mathematik.uni-bielefeld.de