THE GLOBAL DIMENSION OF THE ENDOMORPHISM RING
OF A GENERATOR-COGENERATOR FOR A HEREDITARY
ARTIN ALGEBRA

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ABSTRACT. Let Λ be a hereditary Artin algebra and \( M \) a Λ-module that is both a generator and a cogenerator. We are going to describe the possibilities for the global dimension of \( \text{End}(M) \) in terms of the cardinalities of the Auslander–Reiten orbits of indecomposable Λ-modules.

RÉSUMÉ. Soit Λ une algèbre d’Artin héréditaire et \( M \) un Λ-module qui est un générateur-cogénérateur. Nous allons décrire toutes les possibilités pour la dimension globale de \( \text{End}(M) \) à l’aide des cardinalités des orbites d’Auslander–Reiten des Λ-modules indécomposables.

Let Λ be an Artin algebra. The modules to be considered will be left Λ-modules of finite length. Given a class \( \mathcal{M} \) of modules we denote by \( \text{add} \mathcal{M} \) the class of modules which are direct summands of direct sums of modules in \( \mathcal{M} \). The Auslander–Reiten translation will be denoted by \( \tau \). The \( \tau \)-orbits to be considered will be those on the set of isomorphism classes of indecomposable modules.

Recall that a module \( M \) is called a generator if any projective module belongs to \( \text{add} \mathcal{M} \); it is called a cogenerator if any injective module belongs to \( \text{add} \mathcal{M} \). The endomorphism rings of modules which are both generators and cogeners have attracted much interest (Morita, Tachikawa, and many others, see, for example, [6]): these are just the Artin algebras of dominant dimension at least 2. The relevance of the global dimension \( d \) of the endomorphism ring \( \text{End}(M) \) of such modules was stressed by M. Auslander [1]; in particular, he introduced the representation dimension of Λ as the smallest possible value of \( d \) (provided Λ is not semisimple; for Λ semisimple, the representation dimension is defined to be 1).

The aim of this note is to determine the set of all possible values of \( d \) in case Λ is hereditary.

THEOREM 1. Let Λ be a hereditary Artin algebra and let \( d \) be either a natural number with \( d \geq 3 \) or else the symbol \( \infty \). There exists a Λ-module \( M \) which is both a generator and a cogenerator such that the global dimension of \( \text{End}(M) \) is equal to \( d \) if and only if there exists a \( \tau \)-orbit of cardinality at least \( d \).

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Remark. Recall that Auslander has shown that $\Lambda$ is representation-finite if and only if its representation dimension is at most 2. Thus, a representation-infinite Artin algebra has no generator-cogenerator such that the global dimension of $\text{End}(M)$ is equal to 2, but it has $\tau$-orbits of cardinality at least 2. This shows that the assumption $d \geq 3$ cannot be omitted.

We will use the following criterion due to Auslander. Given modules $M$ and $X$, denote by $\Omega_M(X)$ the kernel of a minimal right $\text{add} M$-approximation $g_{MX}: M' \rightarrow X$ (this means that $M'$ belongs to $\text{add} M$, that any map $M \rightarrow X$ factors through $g_{MX}$, and that $g_{MX}$ is a right minimal map). We will always assume that $M$ is a generator. Then any map $g_{MX}$ is surjective, and $\Omega_M(X) = 0$ if and only if $X$ belongs to $\text{add} M$. Define inductively $\Omega^i_M(X)$ by $\Omega^0_M(X) = X$, and $\Omega^{i+1}_M(X) = \Omega_M(\Omega^i_M(X))$. By definition, the $M$-dimension $\text{M-dim} X$ of $X$ is the minimal value $i$ such that $\Omega^i_M(X)$ belongs to $\text{add} M$ (and is $\infty$ if no $\Omega^i_M(X)$ belongs to $\text{add} M$).

**Lemma 1 (Auslander).** Let $\Lambda$ be an Artin algebra. Let $M$ be a generator and a cogenerator and $d \geq 2$. The global dimension of $\text{End}(M)$ is less than or equal to $d$ if and only if $\text{M-dim} X \leq d - 2$ for any indecomposable $\Lambda$-module $X$.

**Proof.** See [3, 4], both being based on [1].

**Proposition 1.** Let $\Lambda$ be a hereditary Artin algebra and let $d \geq 2$. Let $M$ be a $\Lambda$-module which is both a generator and a cogenerator. Assume that one of the following conditions is satisfied:

(i) $\tau^{d-2} N = 0$ for any indecomposable non-injective module $N$ in $\text{add} M$;

(ii) $\tau^{d-1} X = 0$ for any indecomposable module $X$ which does not belong to $\text{add} M$.

Then the global dimension of $\text{End}(M)$ is at most $d$.

**Proof.** We must show that the $M$-dimension of any indecomposable module $X$ is at most $d - 2$. If $X$ belongs to $\text{add} M$, then its $M$-dimension is zero. Thus we only consider indecomposable modules $X$ which do not belong to $\text{add} M$. In particular, $X$ is not injective.

**Case (1).** We assume that $\tau^{d-2} N = 0$ for any indecomposable non-injective direct summand $N$ of $M$. Here, we can assume that $d \geq 3$. For, if $d = 2$, then any indecomposable module in $\text{add} M$ is injective, thus all projective modules are injective. But a hereditary algebra with this property is semisimple, thus also $\text{End}(M)$ is semisimple.

There is a non-split exact sequence $0 \rightarrow \Omega_M(X) \rightarrow \bigoplus M_i \rightarrow X \rightarrow 0$ with indecomposable direct summands $M_i$ of $M$. Note that none of the modules $M_i$ is injective, since $\text{Hom}(M_i, X) \neq 0$ and $X$ is not injective.

By assumption, all the modules $M_i$ are preprojective. We will use the predecessor relation on the set of (isomorphism classes of) indecomposable preprojective modules (if $Z, Z'$ are indecomposable preprojective modules, then $Z'$ is
said to be a predecessor of $Z$ provided there is a path from $Z'$ to $Z$ in the Auslander–Reiten quiver of $\Lambda$).

If $X'$ is an indecomposable direct summand of $\Omega_M(X)$, then $X'$ is a predecessor of some $M_i$. By induction, we claim that any indecomposable direct summand $Y$ of some $\Omega_M^t(X)$ is a predecessor of a non-zero module of the form $\tau^{t-1}M_i$ for some $i$. The case $t = 1$ has just been shown. Now assume the assertion is true for some $t$. Write $\Omega_M^t(X) = \bigoplus Z_j$ with indecomposable modules $Z_j$. Let $Y$ be an indecomposable direct summand of

$$\Omega_M^{t+1}(X) = \Omega_M(\Omega_M^t(X)) = \Omega_M(\bigoplus Z_j) = \bigoplus \Omega_M(Z_j)$$

There is some $j$ such that $Y$ is a direct summand of $\Omega_M(Z_j)$. Note that $Z_j$ does not belong to $add M$, since otherwise $\Omega_M(Z_j) = 0$. There is an exact sequence

$$0 \to \Omega_M(Z_j) \to M' \overset{g}{\to} Z_j \to 0,$$

(where $g$ is a minimal right $M$-approximation) which is non-split, thus any indecomposable direct summand of $\Omega_M(Z_j)$, in particular $Y$, is a predecessor of $\tau Z_j$. By induction, $Z_j$ is a predecessor of $\tau^{t-1}M_i$ for some $M_i$. Since $Z_j$ is not projective, $\tau^{t-1}M_i$ too is not projective, thus $\tau^tM_i \neq 0$. Since $Z_j$ is a predecessor of $\tau^{t-1}M_i$, it follows that $\tau Z_j$ is a predecessor of the non-zero module $\tau^tM_i$. This completes the induction step.

For $t = d - 2$, we see that any indecomposable direct summand of $\Omega_M^{d-2}(X)$ is a predecessor of some $\tau^{d-3}M_i$ with $M_i$ an indecomposable non-injective direct summand of $M$. But these modules $\tau^{d-3}M_i$ are projective, thus also $\Omega_M^{d-2}(X)$ is projective and therefore in $add M$. This completes the proof in the first case.

CASE (2). Now we assume that $\tau^{d-1}X = 0$ for any indecomposable $\Lambda$-module which does not belong to $add M$. We claim that any indecomposable direct summand $Y$ of $\Omega_M(X)$ is a predecessor of $\tau^tX$. The proof is by induction, the case $t = 0$ being trivial. Thus, assume the condition is satisfied for some $t$ and let $Y'$ be an indecomposable direct summand of $\Omega_M^{t+1}(X)$. Write $\Omega_M^t(X) = \bigoplus Z_j$ with indecomposable modules $Z_j$, thus $\Omega_M^{t+1}(X) = \bigoplus \Omega_M(Z_j)$. It follows that $Y'$ is a direct summand of some $\Omega_M(Z_j)$, thus a predecessor of $\tau Z_j$. Since, by induction, $Z_j$ is a predecessor of $\tau^tX$, we see that $Y'$ is a predecessor of $\tau^{t+1}X$, as required.

For $t = d - 2$, we see that any indecomposable direct summand of $\Omega_M^{d-2}(X)$ is a predecessor of $\tau^{d-2}X$. Since $\tau^{d-1}X = 0$, we know that $\tau^{d-2}X$ is projective, thus also $\Omega_M^{d-2}(X)$ is projective and therefore in $add M$. This completes the proof in the second case.

Proof of Theorem 1  We first deal with the case where $d$ is a natural number.

Let $M$ be a generator-cogenerator with $\text{End}(M)$ having global dimension $d$, and assume that all $\tau$-orbits are of cardinality at most $d - 1$. We want to apply
Proposition 1 with $d$ replaced by $d - 1$. We show that the second assumption is valid. Let $X$ be an indecomposable $\Lambda$-module which is not in $\text{add} M$. In particular, $X$ is not injective, thus $\tau^{-1}X$ is non-zero (and indecomposable). Since any $\tau$-orbit has cardinality at most $d - 1$, we must have $\tau^{d-2}X = 0$, since otherwise the sequence of modules $\tau^{-1}X, X, \ldots, \tau^{d-2}X$ provides $d$ pairwise non-isomorphic indecomposable modules in a single $\tau$-orbit. According to Proposition 1, the global dimension of $\text{End}(M)$ is at most $d - 1$, a contradiction.

The converse follows from the following construction.

**Proposition 2.** Let $\Lambda$ be a hereditary Artin algebra, and let $d \geq 3$ be a natural number. Let $Z$ be an indecomposable non-injective module such that $\tau^{d-2}(Z)$ is simple and projective. Let

$0 \to \tau Z \to \bigoplus Y_j \to Z \to 0$

be the Auslander–Reiten sequence ending in $Z$, with indecomposable modules $Y_j$.

Let $M$ be the class of all indecomposable $\Lambda$ modules which are projective or injective or of the form $\tau^i Y_j$ with $0 \leq i \leq d - 3$. Let $\text{add} M = \text{add} M$. Then the global dimension of $\text{End}(M)$ is precisely $d$.

**Proof.** In order to see that the global dimension of $\text{End}(M)$ is at most $d$, we use Proposition 1. Now the first assumption is satisfied. Namely, let $N$ be an indecomposable module in $M$ which is not injective. If $N$ is projective, then $\tau N = 0$. Since $d \geq 3$, it follows that $\tau^{d-2}N = 0$. Otherwise, $N = \tau^i Y_j$ for some $0 \leq i \leq d - 3$, thus it is sufficient to show that $\tau^{d-2} Y_j = 0$. Applying $\tau^{d-2}$ to an irreducible map $Y_j \to Z$, we see that either $\tau^{d-2} Y_j = 0$ or else $\tau^{d-2} Y_j$ is a proper predecessor of $\tau^{d-2}Z$. The latter is impossible, since we assume that $\tau^{d-2}Z$ is simple projective. Proposition 1 asserts that the global dimension of $\text{End}(M)$ is at most $d$.

In order to show that the global dimension of $\text{End}(M)$ is at least $d$, we show that the $M$-dimension of $Z$ is equal to $d - 2$. For $0 \leq i \leq d - 3$, the Auslander–Reiten sequence ending in $\tau^i Z$ is of the form

$0 \to \tau^{i+1} Z \to \bigoplus \tau^i Y_j \xrightarrow{g_i} \tau^i Z \to 0$.

Since by construction all the modules $\tau^i Y_j$ with $0 \leq i \leq d - 3$ belong to $M$, we see that $g_i$ is a right $M$-approximation, and, of course, also minimal. Now, for $0 \leq i \leq d - 3$, the module $\tau^i Z$ does not belong to $M$. This shows that

$\Omega^i_M(Z) = \tau^i Z$, for $0 \leq i \leq d - 3$,

and consequently, $M$-$\dim Z \geq d$. □

For the proof of Theorem 1, we must verify that the existence of a $\tau$-orbit of cardinality at least $d$ implies the existence of an indecomposable non-injective module $Z$ such that $\tau^{d-2}(Z)$ is simple and projective.
Let $X$ be indecomposable such that $\tau^{-d}X \neq 0$. We can assume that $X$ is preprojective (otherwise $\Lambda$ is representation-infinite, and we can replace $X$ by $\tau^{-d+1}P$ for some indecomposable projective $\Lambda$-module). Applying, if necessary some power of $\tau$, we can assume that $\tau^{-d}X \neq 0$, but $\tau^dX = 0$. Let $Y = \tau X$. Then $Y$ is indecomposable and not injective and $\tau^{d-2}Y \neq 0$, whereas $\tau^{d-1}Y = 0$. The module $\tau^{d-2}Y$ is projective (but not necessarily simple). Let $S$ be a simple projective module with $\text{Hom}(S, \tau^{d-2}Y) \neq 0$ and $Z = \tau^{-d+2}Y$. Then $\tau^dZ = S$ is obviously simple projective. And $\tau^{d-2}Z$ cannot be injective, since $\text{Hom}(Z, \tau^{d-2}Y) \neq 0$ and $\tau^{d-2}Y$ is not injective.

If $\Lambda$ is a representation-finite hereditary Artin algebra and $M$ is a generator and a cogenerator, then the global dimension of $\text{End}(M)$ is finite. This is an immediate consequence of Proposition 1. We can also argue as follows: a representation-finite hereditary Artin algebra is representation-directed, thus the quiver of the endomorphism ring of any $\Lambda$-module has to be directed.

Assume now that $\Lambda$ is representation-infinite. Then there is a $\Lambda$-module $N$ whose endomorphism ring is a division ring and such that $\text{Ext}^1(N, N) \neq 0$. We show how such a module $N$ can be used in order to construct a generator-cogenerator $M$ such that $\text{End}(M)$ has infinite global dimension.

**Proposition 3.** Let $\Lambda$ be a hereditary Artin algebra, let $N$ be a $\Lambda$-module whose endomorphism ring is a division ring and such that there is a non-split exact sequence

$$0 \to N \xrightarrow{u} N' \xrightarrow{v} N \to 0.$$ 

Let $\mathcal{M}$ be the class of all indecomposable $\Lambda$ modules which are projective or injective or isomorphic to $N'$. Let $\text{add}\mathcal{M} = \text{add}M$. Then the global dimension of $\text{End}(M)$ is infinite.

**Proof.** First, observe that any map $f : N' \to N$ factors through $v$. Namely, $fu = 0$, since otherwise $fu$ would be a non-zero endomorphism of $N$, thus invertible, and therefore $u$ would be a split monomorphism. But $fu = 0$ implies that $f$ factors through the cokernel $v$ of $u$.

Let $p : P(N) \to N$ be a projective cover of $N$. We claim that

$$[v, p] N' \oplus P(N) \to N$$

is a right $\mathcal{M}$-approximation (perhaps not minimal). Since $N$ is indecomposable and not injective, $\text{Hom}(I, N) = 0$ for any injective module $I$. Since $p$ is a projective cover, any map $P \to N$ with $P$ projective factors through $p$. And we have already seen that any map $N' \to N$ factors through $v$.

Note that $p$ itself is not a right $\mathcal{M}$-approximation, since the map $v : N' \to N$ cannot be factored through a projective module (otherwise $N'$ would be projective). Thus, a minimal right $\mathcal{M}$-approximation of $N$ is of the form $[v, p'] : N' \oplus P' \to N$ with $P'$ projective.

The kernel of $[v, p']$ is isomorphic to $N \oplus P'$. Namely, we start with the given exact sequence with maps $u, v$ and consider the induced sequence given by the
map $p'$. This yields the following commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \longrightarrow & N & \xrightarrow{u} & N' & \xrightarrow{v} & N & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & Z & \longrightarrow & P' & \longrightarrow & 0
\end{array}
$$

The right square is a pullback and a pushout, thus there is a corresponding exact sequence

$$0 \to Z \to N' \oplus P' \xrightarrow{[v,p']} N \to 0.$$ 

This shows that the kernel of $[v,p']$ is isomorphic to $Z$. Since $P'$ is projective, the exact sequence with middle term $Z$ splits, thus $Z$ is isomorphic to $N \oplus P'$.

This shows that $\Omega_M(N) = N \oplus P'$. Inductively, we see that $N$ is a direct summand of $\Omega^t_M(N)$ for all $t \geq 0$, thus the $M$-dimension of $N$ is not finite. This completes the proof of Proposition 3, and also of Theorem 1. □

Theorem 1 shows that the possible values for the global dimension of the endomorphism ring of a generator-cogenerator depend on the maximal length of the $\tau$-orbits. Let us stress that the maximal length $d$ of the $\tau$-orbits depends not only on the Dynkin type of $\Lambda$, but on the given orientation. In fact, the following (optimal) bounds $d' \leq d \leq d''$ for the length of $\tau$-orbits are well known (for the simply laced cases, see [5]):

<table>
<thead>
<tr>
<th>Dynkin type</th>
<th>$A_n$</th>
<th>$B_n$</th>
<th>$C_n$</th>
<th>$D_{2m-1}$</th>
<th>$D_{2m}$</th>
<th>$E_6$</th>
<th>$E_7$</th>
<th>$E_8$</th>
<th>$F_4$</th>
<th>$G_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d'$</td>
<td>$\lfloor \frac{n}{2} \rfloor$</td>
<td>$n$</td>
<td>$n$</td>
<td>$2m-2$</td>
<td>$2m-1$</td>
<td>$6$</td>
<td>$9$</td>
<td>$15$</td>
<td>$6$</td>
<td>$3$</td>
</tr>
<tr>
<td>$d''$</td>
<td>$n$</td>
<td>$n$</td>
<td>$n$</td>
<td>$2m-1$</td>
<td>$2m-1$</td>
<td>$8$</td>
<td>$9$</td>
<td>$15$</td>
<td>$6$</td>
<td>$3$</td>
</tr>
</tbody>
</table>

(Here, $[\alpha]$ denotes the minimal integer $z$ with $\alpha \leq z$.)

As an illustration, let us exhibit two hereditary algebras $H = kQ$ with $Q$ a quiver of type $E_6$. First, we consider the subspace orientation, then $d = d' = 6$, since the Auslander–Reiten quiver $\Gamma(H)$ looks as follows:

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Here all $\tau$-orbits have cardinality 6.
Second, consider an orientation with a path of length 4, so that $d = d'' = 8$:

The smallest generator-cogenerator $M$ such that the global dimension of $\text{End}(M)$ is equal to 8 is obtained by taking the direct sum of the indecomposable modules which are marked below by a star.

We denote by $a$ the source of the quiver $Q$, and by $b$ its neighboring vertex, then the encircled module $X = \tau^{-6} P(a)$ has the following $M$-resolution

$$0 \to P(a) \to P(b) \to \tau^{-1} P(b) \to \tau^{-2} P(b) \to \tau^{-3} P(b) \to \tau^{-4} P(b) \to \tau^{-5} P(b) \to X \to 0.$$

(According to the proof of the Auslander lemma, we obtain in this way a projective resolution of a simple $\text{End}(M)$-module $S$ which shows that the projective dimension of $S$ is 8.)

Of course, there are many additional generator-cogenerators $M'$ such that the global dimension of $\text{End}(M')$ is equal to 8: just add to $M$ summands from the $\tau$-orbits of the indecomposable projective modules $P(c)$ with $c$ different from the vertices $a$ and $b$.

**References**


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