The global dimension of the endomorphism ring of a generator-cogenerator for a hereditary artin algebra

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Abstract. Let Λ be a hereditary artin algebra and M a Λ-module which is both a generator and a cogenerator. We are going to describe the possibilities for the global dimension of $\mathrm{End}(M)$ in terms of the cardinalities of the Auslander-Reiten orbits of indecomposable Λ-modules.

Let Λ be an artin algebra. The modules to be considered will be left Λ-modules of finite length. Given a class $\mathcal{M}$ of modules we denote by $\mathrm{add}\,\mathcal{M}$ the class of modules which are direct summands of direct sums of modules in $\mathcal{M}$. The Auslander-Reiten translation will be denoted by τ. The τ-orbits to be considered will be those on the set of isomorphism classes of indecomposable modules.

Recall that a module M is called a *generator* if any projective module belongs to $\mathrm{add}\,M$; it is called a *cogenerator* if any injective module belongs to $\mathrm{add}\,M$. The endomorphism rings of modules which are both generators and cogenerators have attracted a lot of interest (Morita, Tachikawa, and many others, see for example [T]): these are just the artin algebras of dominant dimension at least 2. The relevance of the global dimension d of the endomorphism ring $\mathrm{End}(M)$ of such modules was stressed by M.Auslander [A]; in particular, he introduced the representation dimension of Λ as the smallest possible value d (provided Λ is not semisimple; for Λ semisimple, the representation dimension is defined to be 1).

The aim of this note is to determine the set of all possible values d in case Λ is hereditary.

Theorem. *Let Λ be a hereditary artin algebra and let d be either a natural number with $d \geq 3$ or else the symbol ∞. There exists a Λ-module M which is both a generator and a cogenerator such that the global dimension of* $\mathrm{End}(M)$ *is equal to d if and only if there exists a τ-orbit of cardinality at least d.*

Remark. Recall that Auslander has shown that Λ is representation-finite if and only if its representation dimension is at most 2. Thus, a representation-infinite artin algebra has no generator-cogenerator such that the global dimension of $\mathrm{End}(M)$ is equal to 2, but it has τ-orbits of cardinality at least 2. This shows that the assumption $d \geq 3$ cannot be omitted.

We will use the following criterion due to Auslander. Given modules M and X, denote by $\Omega_M(X)$ the kernel of a minimal right $\mathrm{add}\,M$-approximation $g_{MX}\colon M' \to X$ (this means that M' belongs to $\mathrm{add}\,M$, that any map $M \to X$ factors through g_{MX} and that g_{MX} is a right minimal map). We always will assume that M is a generator. Then any map g_{MX} is surjective; and $\Omega_M(X) = 0$ if and only if X belongs to $\operatorname{add} M$. Define inductively $\Omega_M^i(X)$ by $\Omega_M^0(X) = X$, and $\Omega_M^{i+1}(X) = \Omega_M(\Omega_M^i(X))$. By definition, the *$M$-dimension* M-dim X of X is the minimal value i such that $\Omega_M^i(X)$ belongs to $\operatorname{add} M$ (and is ∞ if no $\Omega_M^i(X)$ belongs to $\operatorname{add} M$).

Lemma (Auslander). *Let Λ be an artin algebra. Let M be a generator and a cogenerator and $d \geq 2$. The global dimension of* $\operatorname{End}(M)$ *is less or equal to d if and only if M-dim $X \leq d-2$ for any indecomposable Λ-module X.*

Proof: see [EIHS] or [CP], both being based on [A].

Proposition 1. *Let Λ be a hereditary artin algebra and let $d \geq 2$. Let M be a Λ-module which is both a generator and a cogenerator. Assume that one of the following conditions is satisfied:*
(1) $\tau^{d-2}N = 0$ *for any indecomposable non-injective module N in* $\operatorname{add} M$*, or*
(2) $\tau^{d-1}X = 0$ *for any indecomposable module X which does not belong to* $\operatorname{add} M$.
Then the global dimension of $\operatorname{End}(M)$ *is at most d.*

Proof. We have to show that the M-dimension of any indecomposable module X is at most $d-2$. If X belongs to $\operatorname{add} M$, then its M-dimension is zero. Thus we only have to consider indecomposable modules X which do not belong to $\operatorname{add} M$. In particular, X is not injective.

Case (1): We assume that $\tau^{d-2}N = 0$ for any indecomposable non-injective direct summand N of M. Here, we can assume that $d \geq 3$. For, if $d = 2$, then any indecomposable module in $\operatorname{add} M$ is injective, thus all projective modules are injective. But a hereditary algebra with this property is semisimple, thus also $\operatorname{End}(M)$ is semisimple.

By assumption, all the modules M_i are preprojective. We will use the predecessor relation on the set of (isomorphism classes of) indecomposable preprojective modules (if Z, Z' are indecomposable preprojective modules, then Z' is said to be a predecessor of Z provided there is a path from Z' to Z in the Auslander-Reiten quiver of Λ).

There is a non-split exact sequence

$$0 \to \Omega_M(X) \to \bigoplus M_i \to X \to 0$$

with indecomposable direct summands M_i of M. Note that none of the modules M_i is injective, since $\operatorname{Hom}(M_i, X) \neq 0$ and X is not injective.

If X' is an indecomposable direct summand of $\Omega_M(X)$, then X' is a predecessor of some M_i. By induction, we claim that any indecomposable direct summand Y of some $\Omega_M^t(X)$ is a predecessor of a non-zero module of the form $\tau^{t-1}M_i$ for some i. The case $t = 1$ has just been shown. Now assume the assertion is true for some t. Write $\Omega_M^t(X) = \bigoplus Z_j$ with indecomposable modules Z_j. Let Y be an indecomposable direct summand of

$$\Omega_M^{t+1}(X) = \Omega_M(\Omega_M^t(X)) = \Omega_M(\bigoplus Z_j) = \bigoplus \Omega_M(Z_j)$$

There is some j such that Y is a direct summand of $\Omega_M(Z_j)$. Note that Z_j does not belong to $\operatorname{add} M$, since otherwise $\Omega_M(Z_j) = 0$. There is an exact sequence

$$0 \to \Omega_M(Z_j) \to M' \xrightarrow{g} Z_j \to 0$$

(where g is a minimal right M-approximation) which is non-split, thus any indecomposable direct summand of $\Omega_M(Z_j)$, in particular Y, is a predecessor of τZ_j. By induction, Z_j is a predecessor of $\tau^{t-1}M_i$ for some M_i. Since Z_j is not projective, also $\tau^{t-1}M_i$ is not projective, thus $\tau^t M_i \neq 0$. Since Z_j is a predecessor of $\tau^{t-1}M_i$, it follows that τZ_j is a predecessor of the non-zero module $\tau^t M_i$. This completes the induction step.

For $t = d-2$, we see that any indecomposable direct summand of $\Omega_M^{d-2}(X)$ is a predecessor of some $\tau^{d-3}M_i$ with M_i an indecomposable non-injective direct summand of M. But these modules $\tau^{d-3}M_i$ are projective, thus also $\Omega_M^{d-2}(X)$ is projective and therefore in $\operatorname{add} M$. This completes the proof in the first case.

Case (2): Now we assume that $\tau^{d-1}X = 0$ for any indecomposable Λ-module which does not belong to $\operatorname{add} M$. We claim that any indecomposable direct summand Y of $\Omega_M^t(X)$ is a predecessor of $\tau^t X$. The proof is by induction, the case $t = 0$ being trivial. Thus, assume the condition is satisfied for some t and let Y' be an indecomposable direct summand of $\Omega_M^{t+1}(X)$. Write $\Omega_M^t(X) = \bigoplus Z_j$ with indecomposable modules Z_j, thus $\Omega_M^{t+1}(X) = \bigoplus \Omega_M(Z_j)$. It follows that Y' is a direct summand of some $\Omega_M(Z_j)$, thus a predecessor of τZ_j. Since, by induction, Z_j is a predecessor of $\tau^t X$, we see that Y' is a predecessor of $\tau^{t+1}X$, as required.

For $t = d-2$, we see that any indecomposable direct summand of $\Omega_M^{d-2}(X)$ is a predecessor of $\tau^{d-2}X$. Since $\tau^{d-1}X = 0$, we know that $\tau^{d-2}X$ is projective, thus also $\Omega_M^{d-2}(X)$ is projective and therefore in $\operatorname{add} M$. This completes the proof in the second case.

Proof of Theorem. We first deal with the case where d is a natural number.

Let M be a generator-cogenerator with $\operatorname{End}(M)$ having global dimension d and assume that all τ-orbits are of cardinality at most $d-1$. We want to apply Proposition 1 with d replaced by $d-1$. We show that the second assumption is valid: Let X be an indecomposable Λ-module which is not in $\operatorname{add} M$. In particular, X is not injective, thus $\tau^{-1}X$ is non-zero (and indecomposable). Since any τ-orbit has cardinality at most $d-1$, we must have $\tau^{d-2}X = 0$, since otherwise the sequence of modules $\tau^{-1}X,\ X,\ \ldots,\ \tau^{d-2}X$ provides d pairwise non-isomorphic indecomposable modules in a single τ-orbit. According to Proposition 1, the global dimension of $\operatorname{End}(M)$ is at most $d-1$, a contradiction.

The converse follows from the following construction:

Proposition 2. *Let Λ be a hereditary artin algebra, let $d \geq 3$ be a natural number. Let Z be an indecomposable non-injective module such that $\tau^{d-2}(Z)$ is simple and projective. Let*

$$0 \to \tau Z \to \bigoplus Y_j \to Z \to 0$$

be the Auslander-Reiten sequence ending in Z, with indecomposable modules Y_i.

Let $\mathcal{M}$ be the class of all indecomposable Λ modules which are projective or injective or of the form $\tau^i Y_j$ with $0 \leq i \leq d-3$. Let $\operatorname{add}\mathcal{M} = \operatorname{add} M$. *Then the global dimension of* $\operatorname{End}(M)$ *is precisely d.*

Proof. In order to see that the global dimension of $\operatorname{End}(M)$ is at most d, we use Proposition 1. Now the first assumption is satisfied. Namely, let N be an indecomposable module in $\mathcal{M}$ which is not injective. If N is projective, then $\tau N = 0$. Since $d \geq 3$, it follows that $\tau^{d-2}N = 0$. Otherwise, $N = \tau^i Y_j$ for some $0 \leq i \leq d-3$, thus it is sufficient to show that $\tau^{d-2}Y_j = 0$. Applying τ^{d-2} to an irreducible map $Y_j \to Z$, we see that either $\tau^{d-2}Y_j = 0$ or else $\tau^{d-2}Y_j$ is a proper predecessor of $\tau^{d-2}Z$. The latter is impossible, since we assume that $\tau^{d-2}Z$ is simple projective. Proposition 1 asserts that the global dimension of $\mathrm{End}(M)$ is at most d.

In order to show that the global dimension of $\mathrm{End}(M)$ is at least d, we show that the M-dimension of Z is equal to $d-2$. For $0 \leq i \leq d-3$, the Auslander-Reiten sequence ending in $\tau^i Z$ is of the form

$$0 \to \tau^{i+1}Z \to \bigoplus \tau^i Y_j \xrightarrow{g_i} \tau^i Z \to 0.$$

Since by construction all the modules $\tau^i Y_j$ with $0 \leq i \leq d-3$ belong to $\mathcal{M}$, we see that g_i is a right M-approximation, and, of course, also minimal. Now, for $0 \leq i \leq d-3$, the module $\tau^i Z$ does not belong to $\mathcal{M}$. This shows that

$$\Omega^i_M(Z) = \tau^i Z, \quad \text{for} \quad 0 \leq i \leq d-3,$$

and consequently, $M\text{-}\dim Z \geq d$.

For the proof of Theorem, we have to verify that the existence of a τ-orbit of cardinality at least d implies the existence of an indecomposable non-injective module Z such that $\tau^{d-2}(Z)$ is simple and projective.

Let X be indecomposable such that $\tau^{d-1}X \neq 0$. We can assume that X is preprojective (otherwise Λ is representation-infinite, and we can replace X by $\tau^{-d+1}P$ for some indecomposable projective Λ-module). Applying, if necessary some power of τ, we can assume that $\tau^{d-1}X \neq 0$, but $\tau^d X = 0$ Let $Y = \tau X$, then Y is indecomposable and not injective and $\tau^{d-2}Y \neq 0$, whereas $\tau^{d-1}Y = 0$. The module $\tau^{d-2}Y$ is projective (but not necessarily simple). Let S be a simple projective module with $\mathrm{Hom}(S, \tau^{d-2}Y) \neq 0$ and $Z = \tau^{-d+2}Y$. Then $\tau^{d-2}Z = S$ is obviously simple projective. And $\tau^{d-2}Z$ cannot be injective, since $\mathrm{Hom}(Z, \tau^{d-2}Y) \neq 0$ and $\tau^{d-2}Y$ is not injective.

If Λ is a representation-finite hereditary artin algebra and M is a generator and a cogenerator, then the global dimension of $\mathrm{End}(M)$ is finite. This is an immediate consequence of Proposition 1. We can argue also as follows: a representation-finite hereditary artin algebra is representation-directed, thus the quiver of the endomorphism ring of any Λ-module has to be directed.

Assume now that Λ is representation-infinite. Then there is a Λ-module N whose endomorphism ring is a division ring and such that $\mathrm{Ext}^1(N, N) \neq 0$. We show how such a module N can be used in order to construct a generator-cogenerator M such that $\mathrm{End}(M)$ has infinite global dimension.

Proposition 3. Λ *be a hereditary artin algebra, let* N *be a* Λ*-module whose endomorphism ring is a division ring and such that there is a non-split exact sequence*

$$0 \to N \xrightarrow{u} N' \xrightarrow{v} N \to 0.$$

Let $\mathcal{M}$ be the class of all indecomposable Λ modules which are projective or injective or isomorphic to N'. Let $\operatorname{add}\mathcal{M} = \operatorname{add} M$*. Then the global dimension of* $\operatorname{End}(M)$ *is infinite.*

Proof: First, observe that any map $f\colon N' \to N$ factors through v. Namely, $fu = 0$, since otherwise fu would be a non-zero endomorphism of N, thus invertible, and therefore u would be a split monomorphism. But $fu = 0$ implies that f factors through the cokernel v of u.

Let $p\colon P(N) \to N$ be a projective cover of N. We claim that

$$[v, p]\colon N' \oplus P(N) \to N$$

is a right $\mathcal{M}$-approximation (may-be not minimal). Since N is indecomposable and not injective, $\operatorname{Hom}(I, N) = 0$ for any injective module I. Since p is a projective cover, any map $P \to N$ with P projective factors through p. And we have already seen that any map $N' \to N$ factors through v.

Note that p itself is not a right $\mathcal{M}$-approximation, since the map $v\colon N' \to N$ cannot be factored through a projective module (otherwise N' would be projective). Thus, a minimal right $\mathcal{M}$-approximation of N is of the form $[v, p']\colon N' \oplus P' \to N$ with P' projective.

The kernel of $[v, p']$ is isomorphic to $N \oplus P'$. Namely, we start with the given exact sequence with maps u, v and consider the induced sequence given by the map p'. This yields the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{u} & N' & \xrightarrow{v} & N & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow{\scriptstyle p'} & & \\
0 & \longrightarrow & N & \longrightarrow & Z & \longrightarrow & P' & \longrightarrow & 0
\end{array}$$

The right square is a pullback and a pushout, thus there is a corresponding exact sequence

$$0 \to Z \to N' \oplus P' \xrightarrow{[v,p']} N \to 0,$$

this shows that the kernel of $[v, p']$ is isomorphic to Z. Since P' is projective, the exact sequence with middle term Z splits, thus Z is isomorphic to $N \oplus P'$.

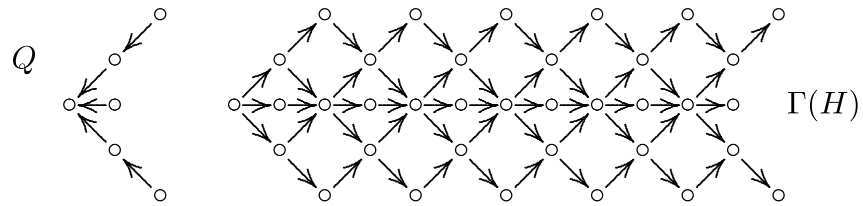
This shows that $\Omega_M(N) = N \oplus P'$. Inductively, we see that N is a direct summand of $\Omega_M^t(N)$ for all $t \geq 0$, thus the M-dimension of N is not finite. This completes the proof of Proposition 3, and also of Theorem.

Theorem shows that the possible values for the global dimension of a generator-cogenerator depend on the maximal length of the τ-orbits. Let us stress that the maximal length d of the τ-orbits depends not only on the Dynkin type of Λ, but on the given orientation. In fact, the following (optimal) bounds $d' \leq d \leq d''$ for the length of τ-orbits are well-known (for the simply laced cases, see [G]):

Dynkin type	A_n	B_n	C_n	D_{2m-1}	D_{2m}	E_6	E_7	E_8	F_4	G_2
d'	$\lceil \frac{n}{2} \rceil$	n	n	$2m-2$	$2m-1$	6	9	15	6	3
d''	n	n	n	$2m-1$	$2m-1$	8	9	15	6	3

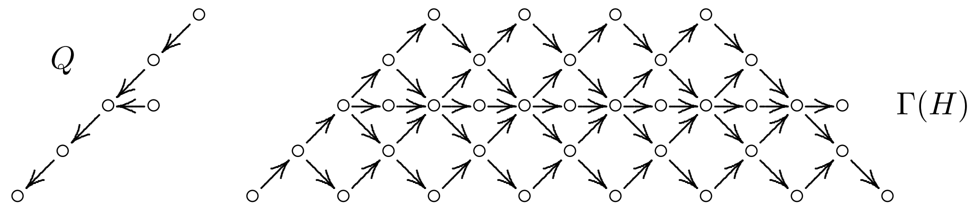
(Here, $\lceil \alpha \rceil$ denotes the minimal integer z with $\alpha \le z$.)

As an illustration, let us exhibit two hereditary algebras $H = kQ$ with Q a quiver of type E_6. First, we consider the subspace orientation, then $d = d' = 6$, since the Auslander-Reiten quiver $\Gamma(H)$ looks as follows:

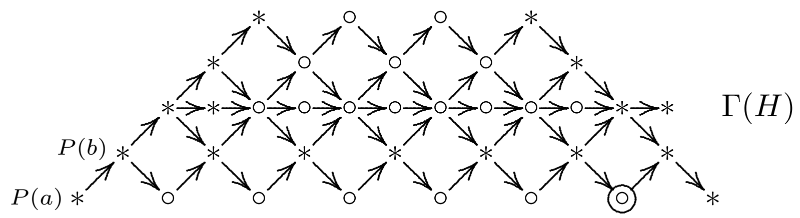


Here all τ-orbits have cardinality 6.

Second, consider an orientation with a path of length 4, so that $d = d'' = 8$:



The smallest generator-cogenerator M such that the global dimension of $\mathrm{End}(M)$ is equal to 8 is obtained by taking the direct sum of the indecomposable modules which are marked below by a star.



We denote by a the source of the quiver Q, and by b its neighboring vertex, then the encircled module $X = \tau^{-6}P(a)$ has the following M-resolution

$$0 \to P(a) \to P(b) \to \tau^{-1}P(b) \to \tau^{-2}P(b) \to \tau^{-3}P(b) \to \tau^{-4}P(b) \to \tau^{-5}P(b) \to X \to 0$$

(according to the proof of the Auslander lemma, we obtain in this way a projective resolution of a simple $\mathrm{End}(M)$-module S which shows that the projective dimension of S is 8.)

Of course, there are many additonal generator-cogenerators M' such that the global dimension of $\mathrm{End}(M')$ is equal to 8: just add to M summands from the τ-orbits of the indecomposable projective modules $P(c)$ with c different from the vertices a and b.

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