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REPRESENTATION THEORY OF ALGEBRAS I: MODULES

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Abstract. This is the first part of a planned book "Introduction to Representation Theory of Algebras". These notes are based on a 3 Semester Lecture Course on Representation Theory held by the first author at the University of Bielefeld from Autumn 1993 and on an (ongoing) 4 Semester Lecture Course by the second author at the University of Bonn from Autumn 2006.

Preliminary version

Contents

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1. Introduction

1.1. **About this course.** This is the first part of notes for a lecture course "Introduction to Representation Theory". As a prerequisite only a good knowledge of Linear Algebra is required. We will focus on the representation theory of quivers and finite-dimensional algebras.

The intersection between the content of this course and a classical Algebra course just consists of some elementary ring theory. We usually work over a fixed field K. Field extensions and Galois theory do not play a role.

This part contains an introduction to general module theory. We prove the classical theorems of Jordan-Hölder and Krull-Remak-Schmidt, and we develop the representation theory of semisimple algebras. (But let us stress that in this course, semisimple representations carry the label "boring and not very interesting".) We also start to investigate short exact sequences of modules, pushouts, pullbacks and properties of Auslander Reiten sequences. Some first results on the representation theory of path algebras (or equivalently, the representation theory of quivers) are presented towards the end of this first part. We study the Jacobson radical of an algebra, decompositions of the regular representation of an algebra, and also describe the structure of semisimple algebras (which is again regarded as boring). Furthermore, we develop the theory of projective modules.

As you will notice, this first part of the script concentrates on modules and algebras. But what we almost do not study yet are *modules over algebras*. (An exception are semisimple modules and projective modules. Projective modules will be important later on when we begin to study homological properties of algebras and modules.)

Here are some topics we will discuss in this series of lecture courses:

- Representation theory of quivers and finite-dimensional algebras
- Homological algebra
- Auslander-Reiten theory
- Knitting of preprojective components
- Tilting theory
- Derived and triangulated categories
- Covering theory
- Categorifications of cluster algebras
- Preprojective algebras
- Ringel-Hall algebras, (dual)(semi) canonical bases of quantized enveloping algebras
- Quiver representations and root systems of Kac-Moody Lie algebras
- Homological conjectures
- Tame and wild algebras
- Functorial filtrations and applications to the representation theory of clans and biserial algebras
- Gabriel-Roiter measure
- Degenerations of modules

- Decomposition theory for irreducible components of varieties of modules

1.2. **Notation and conventions.** Throughout let K be a (commutative) field. Set $K^* = K \setminus \{0\}$. Sometimes we will make additional assumptions on K. (For example, we often assume that K is algebraically closed.)

Typical examples of fields are $\mathbb{Q}$ (the field of rational numbers), $\mathbb{R}$ (the real numbers), $\mathbb{C}$ (the complex numbers), the finite fields $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ where p is a prime number. The field $\mathbb{C}$ is algebraically closed.

Let $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ be the natural numbers (including 0).

All vector spaces will be K-vector spaces, and all linear maps are assumed to be K-linear.

If I is a set, we denote its cardinality by $|I|$. If I' is subset of I we write $I' \subseteq I$. If additionally $I' \neq I$ we also write $I' \subset I$.

For a set M let $\mathrm{Abb}(M, M)$ be the set of maps $M \to M$. By 1_M (or id_M) we denote the map defined by $1_M(m) = m$ for all $m \in M$. Given maps $f\colon L \to M$ and $g\colon M \to N$, we denote the composition by $gf\colon L \to N$. Sometimes we also write $g \circ f$ instead of gf.

1.3. **Acknowledgements.** We thank Tim Eickmann, Alexander Ivanov, Julian Kühlshammer, Nicola Pace and Jeff Serbus for typo hunting.

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Part 1. Modules I: J-Modules

2. Basic terminology

2.1. J-modules. Our aim is to study modules over algebras. Before defining what this means, we introduce a very straightforward notion of a module which does not involve an algebra:

Let J be a set (finite or infinite). This set is our "index set", and in fact only the cardinality of J is of interest to us. If J is finite, then we often take $J = \{1, \dots, n\}$. We also fix a field K.

A J-**module** is given by $(V, \phi_j)_{j \in J}$ where V is a vector space and for each $j \in J$ we have a linear map $\phi_j \colon V \to V$.

Often we just say "module" instead of J-module, and we might say "Let V be a module." without explicitly mentioning the attached linear maps ϕ_j.

For a natural number $m \geq 0$ an m-**module** is by definition a J-module where $J = \{1, \dots, m\}$.

2.2. Isomorphisms of J-modules. Two J-modules $(V, \phi_j)_j$ and $(W, \psi_j)_j$ are **isomorphic** if there exists a vector space isomorphism $f \colon V \to W$ such that

$$f\phi_j = \psi_j f$$

for all $j \in J$.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\phi_j"'] & W \arrow[d, "\psi_j"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

The **dimension** of a J-module $(V, \phi_j)_j$ is just the dimension of the vector space V.

Matrix version: If V and W are finite-dimensional, choose a basis $v_1, \dots, v_n$ of V and a basis $w_1, \dots, w_n$ of W. Assume that the isomorphism $f \colon V \to W$ is represented by a matrix F (with respect to the chosen bases), and let Φ_j and Ψ_j be a corresponding matrices of ϕ_j and ψ_j, respectively. Then $F\Phi_j = \Psi_j F$ for all j, i.e. $F^{-1}\Psi_j F = \Phi_j$ for all j.

If two modules V and W are isomorphic we write $V \cong W$.

2.3. Submodules. Let $(V, \phi_j)_j$ be a module. A subspace U of V is a **submodule** of V if $\phi_j(u) \in U$ for all $u \in U$ and all $j \in J$. Note that the subspaces 0 and V are always submodules of V. A submodule U of V is a **proper submodule** if $U \subset V$, i.e. $U \neq V$.

End of Lecture 1

Example: Let
$$\phi = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$
Then the 1-module (K^2, ϕ) has exactly three submodules, two of them are proper submodules.

Matrix version: If V is finite-dimensional, choose a basis $v_1, \dots, v_n$ of V such that $v_1, \dots, v_s$ is a basis of U. Let $\phi_{j,U} \colon U \to U$ be the linear map defined by $\phi_{j,U}(u) = \phi_j(u)$ for all $u \in U$. Observe that $(U, \phi_{j,U})_j$ is again a J-module. Then the matrix Φ_j of ϕ_j (with respect to this basis) is of the form
$$\Phi_j = \begin{pmatrix} A_j & B_j \\ 0 & C_j \end{pmatrix}.$$
In this case A_j is the matrix of $\phi_{j,U}$ with respect to the basis $v_1, \dots, v_s$.

Let V be a vector space and X a subset of V, then $\langle X \rangle$ denotes the subspace of V generated by X. This is the smallest subspace of V containing X. Similarly, for elements $x_1, \dots, x_n$ in V let $\langle x_1, \dots, x_n \rangle$ be the subspace generated by the x_i.

Let I be a set, and for each $i \in I$ let U_i be a subspace of V. Then the sum $\sum_{i \in I} U_i$ is defined as the subspace $\langle X \rangle$ where $X = \bigcup_{i \in I} U_i$.

Let $V = (V, \phi_j)_j$ be a module, and let X be a subset of V. The intersection $U(X)$ of all submodules U of V with $X \subseteq U$ is the **submodule generated by** X. We call X a **generating set** of $U(X)$. If $U(X) = V$, then we say that V is **generated by** X.

Lemma 2.1. *Let X be a subset of a module V. Define a sequence of subspaces U_i of V as follows: Let U_0 be the subspace of V which is generated by X. If U_i is defined, let*
$$U_{i+1} = \sum_{j \in J} \phi_j(U_i).$$
Then
$$U(X) = \sum_{i \geq 0} U_i.$$

Proof. Set
$$U_X = \sum_{i \geq 0} U_i.$$
One can easily check that U_X is a submodule of V, and of course U_X contains X. Thus $U(X) \subseteq U_X$. Vice versa, one can show by induction that every submodule U with $X \subseteq U$ contains all subspaces U_i, thus U also contains U_X. Therefore $U_X \subseteq U(X)$. □

Let now c be a cardinal number. We say that a module V is c-**generated**, if V can be generated by a set X with cardinality at most c. A module which is generated by a finite set is called **finitely generated**.

By $\aleph_0$ we denote the smallest infinite cardinal number. We call V **countably generated** if V can be generated by a countable set. In other words, V is countably generated if and only if V is $\aleph_0$-generated.

If V can be generated by just one element, then V is a **cyclic module**.

A generating set X of a module V is called a **minimal generating set** if there exists no proper subset X' of X which generates V. If Y is a finite generating set of V, then there exists a subset $X \subseteq Y$, which is a minimal generating set of V.

Warning: Not every module has a minimal generating set. For example, let V be a vector space with basis $\{e_i \mid i \in \mathbb{N}_1\}$, and let $\phi\colon V \to V$ be the endomorphism defined by $\phi(e_i) = e_{i-1}$ for all $i \geq 2$ and $\phi(e_1) = 0$. Then every generating set of the module $N(\infty) = (V, \phi)$ is infinite.

Lemma 2.2. *If V is a finitely generated module, then every generating set of V contains a finite generating set.*

Proof. Let $X = \{x_1, \dots, x_n\}$ be a finite generating set of V, and let Y be an arbitrary generating set of V. As before we have

$$V = U(Y) = \sum_{i \geq 0} U_i.$$

We have $x_j = \sum_{i \geq 0} u_{ij}$ for some $u_{ij} \in U_i$ and all but finitely many of the u_{ij} are zero. Thus there exists some $N \geq 0$ such that $x_j = \sum_{i=0}^{N} u_{ij}$ for all $1 \leq j \leq n$. Each element in U_i is a finite linear combination of elements of the form $\phi_{j_i} \cdots \phi_{j_1}(y)$ for some $j_1, \dots, j_i \in J$ and $y \in Y$. This yields the result. □

Warning: Finite minimal generating sets of a module V do not always have the same cardinality: Let $V = M_2(K)$ be the vector space of 2×2-matrices, and take the module given by V together with all linear maps $A\colon V \to V$, $A \in M_2(K)$. Then $\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)\}$ and $\{\left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)\}$ are minimal generating sets of V.

Lemma 2.3. *A module V is finitely generated if and only if for each family U_i, $i \in I$ of submodules of V with $V = \sum_{i \in I} U_i$ there exists a finite subset $L \subseteq I$ such that $V = \sum_{i \in L} U_i$.*

Proof. Let $x_1, \dots, x_n$ be a generating set of V, and let U_i be submodules with $V = \sum_{i \in I} U_i$. Then each element x_l lies in a finite sum $\sum_{i \in I(l)} U_i$. This implies $V = \sum_{l=1}^{n} \sum_{i \in I(l)} U_i$.

Vice versa, let X be an arbitrary generating set of V. For $x \in X$ let U_x be the cyclic submodule generated by x. We get $V = \sum_{x \in X} U_x$. If there exists a finite subset $Y \subseteq X$ with $V = \sum_{x \in Y} U_x$, then Y is a generating set of V. □

2.4. **Factor modules.** Let U be a submodule of V. Recall that

$$V/U = \{\overline{v} = v + U \mid v \in V\}$$

and $v+U = v'+U$ if and only if $v-v' \in U$. Define $\overline{\phi_j}\colon V/U \to V/U$ by $\overline{\phi_j}(v+U) = \phi_j(v)+U$. This is well defined since U is a submodule.

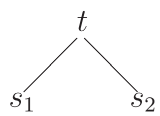
Then $(V/U, \overline{\phi_j})_j$ is a J-module, the **factor module** corresponding to U.

Matrix version: In the situation of Section 2.3, we have that $v_{s+1}+U, \dots, v_n+U$ is a basis of V/U and the matrix of $\overline{\phi_j}$ with respect to this basis is C_j.

2.5. **The lattice of submodules.** A **partially ordered set** (or **poset**) is given by $(S, \leq)$ where S is a set and $\leq$ is a relation on S, i.e. $\leq$ is **transitive** ($s_1 \leq s_2 \leq s_3$ implies $s_1 \leq s_3$), **reflexive** ($s_1 \leq s_1$) and **anti-symmetric** ($s_1 \leq s_2$ and $s_2 \leq s_1$ implies $s_1 = s_2$).

One can try to visualize a partially ordered set $(S, \leq)$ using its **Hasse diagram**: This is an oriented graph with vertices the elements of S, and one draws an arrow $s \to t$ if $s < t$ and if $s \leq m \leq t$ implies $s = m$ or $m = t$. Ususally one tries to draw the diagram with arrows pointing upwards and then one forgets the orientation of the arrows and just uses unoriented edges.

For example, the following Hasse diagram describes the partially ordered set $(S, \leq)$ with three elements s_1, s_2, t with $s_i < t$ for $i = 1, 2$, and s_1 and s_2 are not comparable in $(S, \leq)$.



For a subset $T \subseteq S$ an **upper bound** for T is some $s \in S$ such that $t \leq s$ for all $t \in T$. A **supremum** s_0 of T is a smallest upper bound, i.e. s_0 is an upper bound and if s is an upper bound then $s_0 \leq s$.

Similarly, define a **lower bound** and an **infimum** of T.

End of Lecture 2

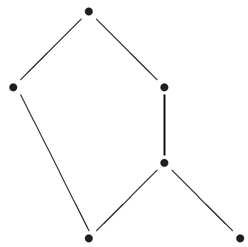
The poset $(S, \leq)$ is a **lattice** if for any two elements $s, t \in S$ there is a supremum and an infimum of $T = \{s, t\}$. In this case write $s + t$ (or $s \cup t$) for the supremum and $s \cap t$ for the infimum.

One calls $(S, \leq)$ a **complete lattice** if there is a supremum and infimum for every subset of S.

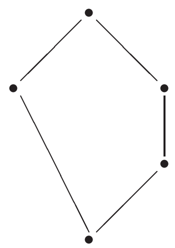
Example: The natural numbers $\mathbb{N}$ together with the usual ordering form a lattice, but this lattice is not complete. For example, the subset $\mathbb{N}$ itself does not have a supremum in $\mathbb{N}$.

A lattice $(S, \leq)$ is called **modular** if
$$s_1 + (t \cap s_2) = (s_1 + t) \cap s_2$$
for all elements $s_1, s_2, t \in S$ with $s_1 \leq s_2$.

This is not a lattice:



This is a complete lattice, but it is not modular:



The following lemma is straightforward:

Lemma 2.4. *Sums and intersections of submodules are again submodules.*

Lemma 2.5. *Let $(V, \phi_j)_j$ be a module. Then the set of all submodules of V is a complete lattice where $U_1 \leq U_2$ if $U_1 \subseteq U_2$.*

Proof. Straightforward: The supremum of a set $\{U_i \mid i \in I\}$ of submodules is $\sum_{i \in I} U_i$, and the infimum is $\bigcap_{i \in I} U_i$. □

Lemma 2.6 (Dedekind)**.** *Let U_1, U_2, W be submodules of a module V such that $U_1 \subseteq U_2$. Then we have*

$$U_1 + (W \cap U_2) = (U_1 + W) \cap U_2.$$

Proof. It is sufficient to proof the statement for subspaces of vector spaces. The inclusion $\subseteq$ is obvious. For the other inclusion let $u \in U_1$, $w \in W$ and assume $u + w \in U_2$. Then $w = (u + w) - u$ belongs to U_2 and thus also to $W \cap U_2$. Thus $u + w \in U_1 + (W \cap U_2)$. □

Thus the lattice of submodules of a module is modular.

2.6. **Examples.** **(a)**: Let K be a field, and let $V = (K^2, \phi, \psi)$ be a 2-module where

$$\phi = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} \text{ and } \psi = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $c_1 \neq c_2$. By e_1 and e_2 we denote the canonical basis vectors of K^2. The module V is *simple*, i.e. V does not have any non-zero proper submodule. The 1-module

(K^2, ϕ) has exactly two non-zero proper submodules. Let

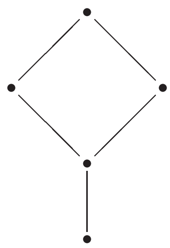
$$\theta = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then (K^2, ϕ, θ) has exactly one non-zero proper submodule, namely $U = \langle e_1 \rangle$. We have $U \cong (K, c_1, 0)$, and $V/U \cong (K, c_2, 0)$. In particular, U and V/U are not isomorphic.

(b): Let

$$\phi = \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \text{ and } \psi = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

with pairwise different c_i. Then the lattice of submodules of (K^3, ϕ, ψ) looks like this:

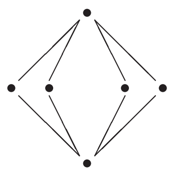


The non-zero proper submodules are $\langle e_3 \rangle$, $\langle e_1, e_3 \rangle$ and $\langle e_2, e_3 \rangle$.

(c): Let

$$\phi = \begin{pmatrix} c_1 & 0 & & \\ 0 & c_2 & & \\ & & c_1 & 0 \\ & & 0 & c_2 \end{pmatrix} \text{ and } \psi = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$$

with $c_1 \neq c_2$. If $K = \mathbb{F}_3$, then the lattice of submodules of (K^4, ϕ, ψ) looks like this:



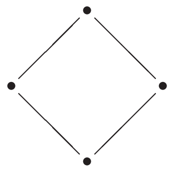
The non-zero proper submodules are $\langle e_1, e_2 \rangle$, $\langle e_3, e_4 \rangle$, $\langle e_1 + e_3, e_2 + e_4 \rangle$ and $\langle e_1 + 2e_3, e_2 + 2e_4 \rangle$.

(d):

Let

$$\phi = \begin{pmatrix} c_1 & 0 & & \\ 0 & c_2 & & \\ & & c_3 & 0 \\ & & 0 & c_4 \end{pmatrix} \text{ and } \psi = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}$$

with pairwise different c_i. Then the lattice of submodules of (K^4, ϕ, ψ) looks like this:



The non-zero proper submodules are $\langle e_1, e_2 \rangle$ and $\langle e_3, e_4 \rangle$.

2.7. **Decompositons and direct sums of modules.** Let $(V, \phi_j)_j$ be a module, and let U_1 and U_2 be submodules of V. If $U_1 \cap U_2 = 0$ and $U_1 + U_2 = V$, then this is called a **direct decomposition** of V, and we say $(V, \phi_j)_j$ is the **direct sum** of the submodules U_1 and U_2. In this case we write $V = U_1 \oplus U_2$.

A submodule U of V is a **direct summand** of V if there exists a submodule U' such that $U \oplus U' = V$. In this case we say that U' is a **direct complement** of U in V.

Matrix version: Assume that V is finite-dimensional. Choose a basis $v_1, \dots, v_s$ of U_1 and a basis $v_{s+1}, \dots, v_n$ of U_2. Then the matrix Φ_j of ϕ_j with respect to the basis $v_1, \dots, v_n$ of V is of the form

$$\Phi_j = \begin{pmatrix} A_j & 0 \\ 0 & B_j \end{pmatrix}$$

where A_j and B_j are the matrices of ϕ_{j,U_1} and ϕ_{j,U_2}, respectively.

Vice versa, let $(V, \phi_j)_j$ and $(W, \psi_j)_j$ be modules. Define

$$(V, \phi_j)_j \oplus (W, \psi_j)_j = (V \oplus W, \phi_j \oplus \psi_j)_j$$

where

$$V \oplus W = V \times W = \{(v, w) \mid v \in V, w \in W\}$$

and $(\phi_j \oplus \psi_j)(v, w) = (\phi_j(v), \psi_j(w))$.

In this case $V \oplus W$ is the direct sum of the submodules $V \oplus 0$ and $0 \oplus W$.

On the other hand, if $(V, \phi_j)_j$ is the direct sum of two submodules U_1 and U_2, then we get an isomorphism

$$U_1 \oplus U_2 \to V$$

defined by $(u_1, u_2) \mapsto u_1 + u_2$.

A module $(V, \phi_j)_j$ is **indecomposable** if the following hold:

- $V \neq 0$,
- Let U_1 and U_2 be submodules of V with $U_1 \cap U_2 = 0$ and $U_1 + U_2 = V$, then $U_1 = 0$ or $U_2 = 0$.

If $(V, \phi_j)_j$ is not indecomposable, then it is called **decomposable**.

More generally, we can construct direct sums of more than two modules, and we can look at direct decompositions of a module into a direct sum of more than two modules. This is defined in the obvious way. For modules $(V_i, \phi_j^{(i)})_j$, $1 \leq i \leq t$ we write

$$(V_1, \phi_j^{(1)})_j \oplus \cdots \oplus (V_t, \phi_j^{(t)})_j = \bigoplus_{i=1}^{t} (V_i, \phi_j^{(i)})_j.$$

2.8. **Products of modules.** Let I be a set, and for each $i \in I$ let V_i be a vector space. The **product** of the vector spaces V_i is by definition the set of all sequences $(v_i)_{i \in I}$ with $v_i \in V_i$. We denote the product by

$$\prod_{i \in I} V_i.$$

With componentwise addition and scalar multiplication, this is again a vector space. The V_i are called the factors of the product. For linear maps $f_i \colon V_i \to W_i$ with $i \in I$ we define their product

$$\prod_{i \in I} f_i \colon \prod_{i \in I} V_i \to \prod_{i \in I} W_i$$

by $\left(\prod_{i \in I} f_i\right)((v_i)_i) = (f_i(v_i))_i$. Obviously, $\bigoplus_{i \in I} V_i$ is a subspace of $\prod_{i \in I} V_i$. If I is a finite set, then $\prod_{i \in I} V_i = \bigoplus_{i \in I} V_i$.

Now for each $i \in I$ let $V_i = (V_i, \phi_j^{(i)})_j$ be a J-module. Then the **product** of the modules V_i is defined as

$$(V, \phi_j)_j = \prod_{i \in I} V_i = \prod_{i \in I} (V_i, \phi_j^{(i)})_j = \left(\prod_{i \in I} V_i, \prod_{i \in I} \phi_j^{(i)} \right)_j.$$

Thus V is the product of the vector spaces V_i, and ϕ_j is the product of the linear maps $\phi_j^{(i)}$.

2.9. **Examples: Nilpotent endomorphisms.** Sometimes one does not study all J-modules, but one assumes that the linear maps associated to the elements in J satisfy certain relations. For example, if J just contains one element, we could study all J-modules (V, f) such that $f^n = 0$ for some fixed n. Or, if J contains two elements, then we can study all modules (V, f, g) such that $fg = gf$.

Assume $|J| = 1$. Thus a J-module is just (V, ϕ) with V a vector space and $\phi \colon V \to V$ a linear map. We additionally assume that ϕ is nilpotent, i.e. $\phi^m = 0$ for some m and that V is finite-dimensional. We denote this class of modules by $\mathcal{N}^{\text{f.d.}}$.

We know from LA that there exists a basis $v_1, \ldots, v_n$ of V such that the corresponding matrix Φ of ϕ is of the form

$$\Phi = \begin{pmatrix} J(\lambda_1) & & & \\ & J(\lambda_2) & & \\ & & \ddots & \\ & & & J(\lambda_t) \end{pmatrix}$$

where $J(\lambda_i)$, $1 \leq i \leq t$ is a $\lambda_i \times \lambda_i$-matrix of the form

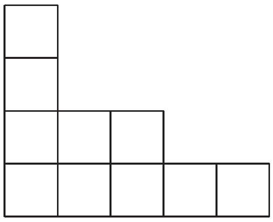
$$J(\lambda_i) = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}$$

for some partition $\lambda = (\lambda_1, \ldots, \lambda_t)$ of n.

A **partition** of some $n \in \mathbb{N}$ is a sequence $\lambda = (\lambda_1, \ldots, \lambda_t)$ of integers with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t \geq 1$ and $\lambda_1 + \cdots + \lambda_t = n$.

Example: The partitions of 4 are (4), $(3, 1)$, $(2, 2)$, $(2, 1, 1)$ and $(1, 1, 1, 1)$.

One can visualize partitons with the help of **Young diagrams**: For example the Young diagram of the partition $(4, 2, 2, 1, 1)$ is the following:



Let $e_1, \ldots, e_m$ be the standard basis of K^m where

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, e_m = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

To each partition $\lambda = (\lambda_1, \ldots, \lambda_t)$ of n we associate a module

$$N(\lambda) = \bigoplus_{i=1}^{t} N(\lambda_i) = (K^n, \phi_\lambda)$$

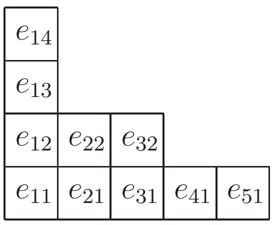
where for $m \in \mathbb{N}$ we have

$$N(m) = (K^m, \phi_m)$$

with ϕ_m the endomorphism defined by $\phi_m(e_j) = e_{j-1}$ for $2 \leq j \leq m$ and $\phi_m(e_1) = 0$. In other words, the matrix of ϕ_m with respect to the basis $e_1, \ldots, e_m$ is $J(m)$.

End of Lecture 3

We can visualize $N(\lambda)$ with the help of Young diagrams. For example, for $\lambda = (4, 2, 2, 1, 1)$ we get the following diagram:



Here the vectors
$$\{e_{ij} \mid 1 \leq i \leq 5, 1 \leq j \leq \lambda_i\}$$
denote a basis of
$$K^{10} = K^4 \oplus K^2 \oplus K^2 \oplus K^1 \oplus K^1.$$
Let $\phi_\lambda\colon K^{10} \to K^{10}$ be the linear map defined by $\phi_\lambda(e_{ij}) = e_{ij-1}$ for $2 \leq j \leq \lambda_i$ and $\phi_\lambda(e_{i1}) = 0$. Thus $N(\lambda) = (K^{10}, \phi_\lambda)$.

So ϕ_λ operates on the basis vectors displayed in the boxes of the Young diagram by mapping them to the vector in the box below if there is a box below, and by mapping them to 0 if there is no box below.

The matrix of ϕ_λ with respect to the basis
$$e_{11}, e_{12}, e_{13}, e_{14}, e_{21}, e_{22}, e_{31}, e_{32}, e_{41}, e_{51}$$
is
$$\begin{pmatrix} J(4) &&&& \\ & J(2) &&& \\ && J(2) && \\ &&& J(1) & \\ &&&& J(1) \end{pmatrix} = \begin{pmatrix} 0&1&&&&&&&& \\ &0&1&&&&&&& \\ &&0&1&&&&&& \\ &&&0&&&&&& \\ &&&&0&1&&&& \\ &&&&&0&&&& \\ &&&&&&0&1&& \\ &&&&&&&0&& \\ &&&&&&&&0& \\ &&&&&&&&&0 \end{pmatrix}.$$

Similarly, for an arbitrary partition $\lambda = (\lambda_1, \ldots, \lambda_t)$ of n we will work with a basis $\{e_{ij} \mid 1 \leq i \leq t, 1 \leq j \leq \lambda_i\}$ of K^n, and we define a linear map $\phi_\lambda\colon K^n \to K^n$ by $\phi_\lambda(e_{ij}) = e_{ij-1}$ for $2 \leq j \leq \lambda_i$ and $\phi_\lambda(e_{i1}) = 0$. For simplicity, define $e_{i0} = 0$ for all i.

Theorem 2.7. *For every module (V, ϕ) with V an n-dimensional vector space and ϕ a nilpotent linear map $V \to V$ there exists a unique partition λ of n such that*
$$(V, \phi) \cong N(\lambda).$$

Proof. Linear Algebra (Jordan Normal Form). □

Now let $\lambda = (\lambda_1, \ldots, \lambda_t)$ be a again a partition of n, and let $x \in N(\lambda) = (K^n, \phi)$. Thus
$$x = \sum_{i,j} c_{ij} e_{ij}$$
for some $c_{ij} \in K$. We want to compute the submodule $U(x) \subseteq N(\lambda)$ generated by x:

We get
$$\phi(x) = \phi\left(\sum_{i,j} c_{ij} e_{ij}\right) = \sum_{i,j} c_{ij}\phi(e_{ij}) = \sum_{i,j:j\geq 2} c_{ij} e_{ij-1}.$$
Similarly, we can easily write down $\phi^2(x)$, $\phi^3(x)$, etc. Now let r be maximal such that $c_{ir} \neq 0$ for some i. This implies $\phi^{r-1}(x) \neq 0$ but $\phi^r(x) = 0$. It follows that the vectors $x, \phi(x), \ldots, \phi^{r-1}(x)$ generate $U(x)$ as a vector space, and we see that $U(x)$ is isomorphic to $N(r)$.

For example, the submodule $U(e_{ij})$ of $N(\lambda)$ is isomorphic to $N(j)$ and the corresponding factor module $N(\lambda)/U(e_{ij})$ is isomorphic to

$$N(\lambda_i - j) \oplus \bigoplus_{a \neq i} N(\lambda_a).$$

Let us look at a bit closer at the example $\lambda = (3,1)$:

$$\begin{array}{|c|c}
\cline{1-1}
e_{13} & \\
\cline{1-1}
e_{12} & \\
\hline
e_{11} & \multicolumn{1}{c|}{e_{21}} \\
\hline
\end{array}$$

We get

$$\begin{aligned}
U(e_{21}) &\cong N(1), & N(3,1)/U(e_{21}) &\cong N(3), \\
U(e_{11}) &\cong N(1), & N(3,1)/U(e_{11}) &\cong N(2,1), \\
U(e_{12}) &\cong N(2), & N(3,1)/U(e_{12}) &\cong N(1,1), \\
U(e_{13}) &\cong N(3), & N(3,1)/U(e_{13}) &\cong N(1), \\
U(e_{12}+e_{21}) &\cong N(2), & N(3,1)/U(e_{12}+e_{21}) &\cong N(2).
\end{aligned}$$

Let us check the last of these isomorphisms: Let $x = e_{12} + e_{21} \in N(3,1) = (K^4, \phi)$. We get $\phi(x) = e_{11}$ and $\phi^2(x) = 0$. It follows that $U(x)$ is isomorphic to $N(2)$. Now as a vector space, $N(3,1)/U(x)$ is generated by the residue classes $\overline{e_{13}}$ and $\overline{e_{12}}$. We have $\phi(e_{13}) = e_{12}$ and $\phi(e_{12}) = e_{11}$. In particular, $\phi(e_{12}) \in U(x)$. Thus $N(3,1)/U(x) \cong N(2)$.

2.10. **Exercises. 1**: Let W and U_i, $i \in I$ be a set of submodules of a module $(V, \phi_j)_j$ such that for all $k, l \in I$ we have $U_k \subseteq U_l$ or $U_k \supseteq U_l$. Show that

$$\sum_{i \in I} U_i = \bigcup_{i \in I} U_i$$

and

$$\bigcup_{i \in I} (W \cap U_i) = W \cap \left(\bigcup_{i \in I} U_i \right).$$

2: Let K be a field and let $V = (K^4, \phi, \psi)$ be a module such that

$$\phi = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \lambda_3 & \\ & & & \lambda_4 \end{pmatrix}$$

with pairwise different $\lambda_i \in K$. How can the lattice of submodules of V look like?

3: Which of the following lattices can be the lattice of submodules of a 4-dimensional module of the form (V, ϕ, ψ)? In each case you can work with a field K of your choice.

Of course it is better if you find examples which are independent of the field, if this is possible.

(see the pictures distributed during the lecture)

4: Classify all submodules U of $V = N(2,1), N(3,1), N(2,2)$ and determine in each case the isomorphism class of U and of the factor module V/U.

For $K = \mathbb{F}_2$ and $K = \mathbb{F}_3$ draw the corresponding Hasse diagrams.

Let $K = \mathbb{F}_p$ with p a prime number, and let λ and μ be partitions. How many submodules U of V with $U \cong N(\lambda)$ and $V/U \cong N(\mu)$ are there?

5: Let U be a maximal submodule of a module V, and let W be an arbitrary submodule of V. Show that either $W \subseteq U$ or $U + W = V$.

6: Find two 2×2-matrices A and B with coefficients in K such that (K^2, A, B) has exactly 4 submodules.

7: Show: If V is a 2-dimensional module with at least 5 submodules, then every subspace of V is a submodule.

8: Let V be a 2-dimensional module with at most 4 submodules. Show that V is cyclic.

---

3. Homomorphisms between modules

3.1. **Homomorphisms.** Let $(V, \phi_j)_j$ and $(W, \psi_j)_j$ be two modules. A linear map $f\colon V \to W$ is a **homomorphism** (or **module homomorphism**) if

$$f\phi_j = \psi_j f$$

for all $j \in J$.

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\phi_j"'] & W \arrow[d, "\psi_j"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

We write $f\colon (V, \phi_j)_j \to (W, \psi_j)_j$ or just $f\colon V \to W$. An injective homomorphism is also called a **monomorphism**, and a surjective homomorphism is an **epimorphism**. A homomorphism which is injective and surjective is an **isomorphism**, compare Section 2.2.

If $f\colon (V, \phi_j)_j \to (W, \psi_j)_j$ is an isomorphism, then the inverse $f^{-1}\colon W \to V$ is also a homomorphism, thus also an isomorphism: We have

$$f^{-1}\psi_j = f^{-1}\psi_j f f^{-1} = f^{-1} f \phi_j f^{-1} = \phi_j f^{-1}.$$

For modules $(U, \mu_j)_j$, $(V, \phi_j)_j$, $(W, \psi_j)_j$ and homomorphisms $f\colon U \to V$ and $g\colon V \to W$ the composition $gf\colon U \to W$ is again a homomorphism.

Here is a trivial example of a homomorphism: Let $(V, \phi_j)_j$ be a module, and let U be a submodule of V. Then the map $\iota\colon U \to V$ defined by $\iota(u) = u$ is a homomorphism, which is called the (**canonical**) **inclusion**.

Similarly, the map $\pi\colon V \to V/U$ defined by $\pi(v) = v + U$ is a homomorphism, which is called the (**canonical**) **projection**.

If $f\colon (V, \phi_j)_j \to (W, \psi_j)_j$ is a homomorphism, then define

$$\operatorname{Ker}(f) = \{v \in V \mid f(v) = 0\},$$

the **kernel** of f, and

$$\operatorname{Im}(f) = \{f(v) \mid v \in V\},$$

the **image** of f. Furthermore, $\operatorname{Cok}(f) = W/\operatorname{Im}(f)$ is the **cokernel** of f.

One can easily check that $\operatorname{Ker}(f)$ is a submodule of V: For $v \in \operatorname{Ker}(f)$ and $j \in J$ we have $f\phi_j(v) = \psi_j f(v) = \psi_j(0) = 0$.

Similarly, $\operatorname{Im}(f)$ is a submodule of W: For $v \in V$ and $j \in J$ we have $\psi_j f(v) = f\phi_j(v)$, thus $\psi_j f(v)$ is in $\operatorname{Im}(f)$.

For a homomorphism $f\colon V \to W$ let $f_1\colon V \to \operatorname{Im}(f)$ defined by $f_1(v) = f(v)$ (the only difference between f and f_1 is that we changed the target module of f from W to $\operatorname{Im}(f)$), and let $f_2\colon \operatorname{Im}(f) \to W$ be the canonical inclusion. Then f_1 is an epimorphism and f_2 a monomorphism, and we get $f = f_2 f_1$. In other words, every homomorphism is the composition of an epimorphism followed by a monomorphism.

Let V and W be J-modules. For homomorphisms $f, g\colon V \to W$ define

$$f + g\colon V \to W$$

by $(f + g)(v) = f(v) + g(v)$. This is again a homomorphism. Similarly, for $c \in K$ we can define

$$cf\colon V \to W$$

by $(cf)(v) = cf(v)$, which is also a homomorphism. Thus the set of homomorphisms $V \to W$ forms a subspace of the vector space $\operatorname{Hom}_K(V, W)$ of linear maps from V to W. This subspace is denoted by $\operatorname{Hom}_J(V, W)$ and sometimes we just write $\operatorname{Hom}(V, W)$.

A homomorphism $V \to V$ is also called an **endomorphism**. The set $\operatorname{Hom}_J(V, V)$ of endomorphisms is denoted by $\operatorname{End}_J(V)$ or just $\operatorname{End}(V)$. This is a K-algebra with multiplication given by the composition of endomorphims. One often calls $\operatorname{End}(V)$ the **endomorphism algebra** (or the **endomorphism ring**) of V.

3.2. **Definition of a ring.** A **ring** is a set R together with two maps $+\colon R \times R \to R$, $(a, b) \mapsto a + b$ (the **addition**) and $\cdot\colon R \times R \to R$, $(a, b) \mapsto ab$ (the **multiplication**) such that the following hold:

- Associativity of addition: $(a + b) + c = a + (b + c)$ for all $a, b, c \in R$,
- Commutativity of addition: $a + b = b + a$ for all $a, b \in R$,
- Existence of a 0-element: There exists exactly one element $0 \in R$ with $a + 0 = a$ for all $a \in R$,
- Existence of an additive inverse: For each $a \in R$ there exists exactly one element $-a \in R$ such that $a + (-a) = 0$,
- Associativity of multiplication: $(ab)c = a(bc)$ for all $a, b, c \in R$,
- Existence of a 1-element: There exists exactly one element $1 \in R$ with $1a = a1 = a$ for all $a \in R$,
- Distributivity: $(a + b)c = ac + bc$ and $a(b + c) = ab + ac$ for all $a, b, c \in R$.

A ring R is **commutative** if $ab = ba$ for all $a, b \in R$.

3.3. **Definition of an algebra.** A K-**algebra** is a K-vector space A together with a map $\cdot\colon A \times A \to A$, $(a, b) \mapsto ab$ (the **multiplication**) such that the following hold:

- Associativity of multiplication: $(ab)c = a(bc)$ for all $a, b, c \in A$;
- Existence of a 1-element: There exists an element 1 which satisfies $1a = a1 = a$ for all $a \in A$;
- Distributivity: $a(b + c) = ab + ac$ and $(a + b)c = ac + ac$ for all $a, b, c \in A$;
- Compatibility of multiplication and scalar multiplication: $\lambda(ab) = (\lambda a)b = a(\lambda b)$ for all $\lambda \in K$ and $a, b \in A$.

The element 1 is uniquely determined and we often also denoted it by 1_A.

In other words, a K-algebra is a ring A, which is also a K-vector space such that additionally $\lambda(ab) = (\lambda a)b = a(\lambda b)$ for all $\lambda \in K$ and $a, b \in A$.

In contrast to the definition of a field, the definitions of a ring and an algebra do not require that the element 0 is different from the element 1. Thus there is a ring and an algebra which contains just one element, namely $0 = 1$. If $0 = 1$, then $R = \{0\}$.

3.4. **Homomorphism Theorems.**

Theorem 3.1 (Homomorphism Theorem)**.** *If V and W are J-modules, and if $f\colon V \to W$ is a homomorphism, then f induces an isomorphism*

$$\overline{f}\colon V/\operatorname{Ker}(f) \to \operatorname{Im}(f)$$

defined by $\overline{f}(v + \operatorname{Ker}(f)) = f(v)$.

Proof. One easily shows that $\overline{f}$ is well defined, and that it is a homomorphism. Obviously $\overline{f}$ is injective and surjective, and thus an isomorphism. $\square$

Remark: The above result is very easy to prove. Nevertheless we call it a Theorem, because of its importance.

We derive some consequences from Theorem 3.1:

Corollary 3.2 (First Isomorphism Theorem)**.** *If $U_1 \subseteq U_2$ are submodules of a module V, then*

$$V/U_2 \cong (V/U_1)/(U_2/U_1).$$

Proof. Note that U_2/U_1 is a submodule of V/U_1. Thus we can build the factor module $(V/U_1)/(U_2/U_1)$. Let

$$V \to V/U_1 \to (V/U_1)/(U_2/U_1)$$

be the composition of the two canonical projections. This homomorphism is obviously surjective and its kernel is U_2. Now we use Theorem 3.1. □

Corollary 3.3 (Second Isomorphism Theorem)**.** *If U_1 and U_2 are submodules of a module V, then*

$$U_1/(U_1 \cap U_2) \cong (U_1 + U_2)/U_2.$$

Proof. Let

$$U_1 \to U_1 + U_2 \to (U_1 + U_2)/U_2$$

be the composition of the inclusion $U_1 \to U_1 + U_2$ and the projection $U_1 + U_2 \to (U_1 + U_2)/U_2$. This homomorphism is surjective (If $u_1 \in U_1$ and $u_2 \in U_2$, then $u_1 + u_2 + U_2 = u_1 + U_2$ is the image of u_1.) and its kernel is $U_1 \cap U_2$ (An element $u_1 \in U_1$ is mapped to 0 if and only if $u_1 + U_2 = U_2$, thus if and only if $u_1 \in U_1 \cap U_2$.). □

$$\begin{tikzcd}[column sep=small] & U_1 + U_2 \arrow[dl, dash] \arrow[dr, dash] & \\ U_1 \arrow[dr, dash] & & U_2 \arrow[dl, dash] \\ & U_1 \cap U_2 & \end{tikzcd}$$

In particular, if $U_1 \subset U_2$ and W are submodules of a module V, then the above results yield the isomorphisms

$$\begin{aligned} (U_2 \cap W)/(U_1 \cap W) &\cong (U_1 + U_2 \cap W)/U_1, \\ U_2/(U_1 + U_2 \cap W) &\cong (U_2 + W)/(U_1 + W). \end{aligned}$$

The module $(U_1 + U_2 \cap W)/U_1$ is a submodule of U_2/U_1, and $U_2/(U_1 + W \cap U_2)$ is the corresponding factor module.

$$\begin{tikzcd}[column sep=small]
 & U_2 \arrow[d, dash] & \\
 & U_1 + U_2 \cap W \arrow[dl, dash] \arrow[dr, dash] & \\
U_1 \arrow[dr, dash] & & U_2 \cap W \arrow[dl, dash] \\
 & U_1 \cap W &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & U_2 + W \arrow[dl, dash] \arrow[dr, dash] & \\
U_1 + W \arrow[dr, dash] & & U_2 \arrow[dl, dash] \\
 & U_1 + U_2 \cap W &
\end{tikzcd}$$

End of Lecture 4

3.5. **Homomorphisms between direct sums.** Let

$$V = \bigoplus_{j=1}^{n} V_j$$

be a direct sum of modules. By

$$\iota_{V,j} \colon V_j \to V$$

we denote the canonical inclusion and by

$$\pi_{V,j} \colon V \to V_j$$

the canonical projection. (Each $v \in V$ is of the form $v = \sum_{j=1}^{n} v_j$ where the $v_j \in V_j$ are uniquely determined. Then $\pi_{V,j}(v) = v_j$.) These maps are all homomorphisms. They satisfy

$$\begin{aligned}
\pi_{V,j} \circ \iota_{V,j} &= 1_{V_j}, \\
\pi_{V,i} \circ \iota_{V,j} &= 0 \quad \text{if } i \neq j, \\
\sum_{j=1}^{n} \iota_{V,j} \circ \pi_{V,j} &= 1_V.
\end{aligned}$$

Now let V and W be modules, which are a finite direct sum of certain submodules, say

$$V = \bigoplus_{j=1}^{n} V_j \quad \text{and} \quad W = \bigoplus_{i=1}^{m} W_i.$$

If $f \colon V \to W$ is a homomorphism, define

$$f_{ij} = \pi_{W,i} \circ f \circ \iota_{V,j} \colon V_j \to W_i$$

We can write $f \colon V \to W$ in matrix form

$$f = \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & & \vdots \\ f_{m1} & \cdots & f_{mn} \end{bmatrix}$$

and we can use the usual matrix calculus: Let us write elements $v \in V$ and $w \in W$ as columns

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \quad \text{and} \quad w = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}$$

with $v_j \in V_j$ and $w_i \in W_i$. If $f(v) = w$ we claim that

$$\begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & & \vdots \\ f_{m1} & \cdots & f_{mn} \end{bmatrix} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^n f_{1j}(v_j) \\ \vdots \\ \sum_{j=1}^n f_{mj}(v_j) \end{bmatrix} = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix}.$$

Namely, if $v \in V$ we get for $1 \leq i \leq m$

$$\begin{aligned} \sum_{j=1}^n f_{ij}(v_j) &= \sum_{j=1}^n (\pi_{W,i} \circ f \circ \iota_{V,j})(v_j) \\ &= \left(\pi_{W,i} \circ f \circ \left(\sum_{j=1}^n \iota_{V,j} \circ \pi_{V,j} \right) \right)(v) \\ &= (\pi_{W,i} \circ f)(v) = w_i. \end{aligned}$$

The first term is the matrix product of the ith row of the matrix of f with the column vector v, the last term is the ith entry in the column vector w.

Vice versa, if $f_{ij} \colon V_j \to W_i$ with $1 \leq j \leq n$ and $1 \leq i \leq m$ are homomorphisms, then we obtain with

$$\sum_{i,j} \iota_{W,i} \circ f_{ij} \circ \pi_{V,j}$$

a homomorphism $f \colon V \to W$, and of course we can write f as a matrix

$$f = \begin{bmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & & \vdots \\ f_{m1} & \cdots & f_{mn} \end{bmatrix}.$$

The composition of such morphisms given by matrices can be realized via matrix multiplication.

If A is a matrix, we denote its transpose by ${}^tA$. In particular, we can write the column vector v we looked at above as $v = {}^t[v_1, \ldots, v_n]$.

Now $f \mapsto (f_{ij})_{ij}$ defines an isomorphism of vector spaces

$$\operatorname{Hom}_J \left(\bigoplus_{j=1}^n V_j, \bigoplus_{i=1}^m W_i \right) \to \bigoplus_{j=1}^n \bigoplus_{i=1}^m \operatorname{Hom}_J(V_j, W_i).$$

In particular, for every module X we obtain isomorphisms of vector spaces

$$\operatorname{Hom}_J \left(X, \bigoplus_{i=1}^m W_i \right) \to \bigoplus_{i=1}^m \operatorname{Hom}_J(X, W_i)$$

and

$$\mathrm{Hom}_J\left(\bigoplus_{j=1}^n V_j, X\right) \to \bigoplus_{j=1}^n \mathrm{Hom}_J(V_j, X).$$

3.6. **Idempotents and direct decompositions.** An element r in a ring R is an **idempotent** if $r^2 = r$. We will see that idempotents in endomorphism rings of modules play an important role.

Let $V = U_1 \oplus U_2$ be a direct decomposition of a module V. Thus U_1 and U_2 are submodules of V such that $U_1 \cap U_2 = 0$ and $U_1 + U_2 = V$. Let $\iota_i \colon U_i \to V$ and $\pi_i \colon V \to U_i$ be the corresponding inclusions and projections. We can write these homomorphisms in matrix form

$$\iota_1 = {}^t[1\ \ 0], \quad \iota_2 = {}^t[0\ \ 1], \quad \pi_1 = [1\ \ 0], \quad \pi_2 = [0\ \ 1].$$

Define $e_1 = \iota_1\pi_1$ and $e_2 = \iota_2\pi_2$. Then both e_1 and e_2 are idempotents in the endomorphism ring $\mathrm{End}(V)$ of V. (For example, $e_1^2 = \iota_1\pi_1\iota_1\pi_1 = \iota_1 1_{U_1}\pi_1 = e_1$.) Set

$$e(U_1, U_2) = e_1.$$

Proposition 3.4. *Let V be a J-module. If we associate to an idempotent $e \in \mathrm{End}(V)$ the pair $(\mathrm{Im}(e), \mathrm{Ker}(e))$, then we obtain a bijection between the set of all idempotents in $\mathrm{End}_J(V)$ and the set of pairs (U_1, U_2) of submodules of V such that $V = U_1 \oplus U_2$.*

Proof. Above we associated to a direct decompositon $V = U_1 \oplus U_2$ the idempotent $e_1 = \iota_1\pi_1 \in \mathrm{End}(V)$. This idempotent is uniquely determined by the following two properties: For all $u_1 \in U_1$ we have $e_1(u_1) = u_1$, and for all $u_2 \in U_2$ we have $e_1(u_2) = 0$. From e_1 we can easily obtain the above direct decomposition: We have $U_1 = \mathrm{Im}(e_1)$ and $U_2 = \mathrm{Ker}(e_1)$.

Vice versa, let $e \in \mathrm{End}(V)$ be an idempotent. Define $U_1 = \mathrm{Im}(e)$ and $U_2 = \mathrm{Ker}(e)$. Of course U_1 and U_2 are submodules of V. We also get $U_1 \cap U_2 = 0$: If $x \in U_1 \cap U_2$, then $x \in U_1 = \mathrm{Im}(f)$, thus $x = e(y)$ for some y, and $x \in U_2 = \mathrm{Ker}(e)$, thus $e(x) = 0$. Since $e^2 = e$ we obtain $x = e(y) = e^2(y) = e(x) = 0$.

Finally, we show that $U_1 + U_2 = V$: If $v \in V$, then $v = e(v) + (v - e(v))$ and $e(v) \in \mathrm{Im}(e) = U_1$. Furthermore, $e(v - e(v)) = e(v) - e^2(v) = 0$ shows that $v - e(v) \in \mathrm{Ker}(e) = U_2$.

Thus our idempotent e yields a direct decomposition $V = U_1 \oplus U_2$. Since $e(u_1) = u_1$ for all $u_1 \in U_1$ and $e(u_2) = 0$ for all $u_2 \in U_2$, we see that e is the idempotent corresponding to the direct decomposition $V = U_1 \oplus U_2$. □

The endomorphism ring $\mathrm{End}(V)$ of a module V contains of course always the idempotents 0 and 1. Here 0 corresponds to the direct decomposition $V = 0 \oplus V$, and 1 corresponds to $V = V \oplus 0$.

If e is an idempotent in a ring, then $1 - e$ is also an idempotent. (Namely $(1-e)^2 = 1 - e - e + e^2 = 1 - e$.)

If the idempotent $e \in \mathrm{End}(V)$ corresponds to the pair (U_1, U_2) with $V = U_1 \oplus U_2$, then $1-e$ corresponds to (U_2, U_1). (One easily checks that $\mathrm{Im}(1-e) = \mathrm{Ker}(e)$ and $\mathrm{Ker}(1-e) = \mathrm{Im}(e)$.)

Corollary 3.5. *For a module V the following are equivalent:*

- *V is indecomposable;*
- *$V \neq 0$, and 0 and 1 are the only idempotents in $\mathrm{End}(V)$.*

Later we will study in more detail the relationship between idempotents in endomorphism rings and direct decompositions.

3.7. **Split monomorphisms and split epimorphisms.** Let V and W be modules. An injective homomorphism $f\colon V \to W$ is called **split monomorphism** if $\mathrm{Im}(f)$ is a direct summand of W. A surjective homomorphism $f\colon V \to W$ is a **split epimorphism** if $\mathrm{Ker}(f)$ is a direct summand of V.

Lemma 3.6. *Let $f\colon V \to W$ be a homomorphism. Then the following hold:*

(i) *f is a split monomorphism if and only if there exists a homomorphism $g\colon W \to V$ such that $gf = 1_V$;*
(ii) *f is a split epimorphism if and only if there exists a homomorphism $h\colon W \to V$ such that $fh = 1_W$.*

Proof. Assume first that f is a split monomorphism. Thus $W = \mathrm{Im}(f) \oplus C$ for some submodule C of W. Let $\iota\colon \mathrm{Im}(f) \to W$ be the inclusion homomorphism, and let $\pi\colon W \to \mathrm{Im}(f)$ be the projection with kernel C. Let $f_0\colon V \to \mathrm{Im}(f)$ be defined by $f_0(v) = f(v)$ for all $v \in V$. Thus $f = \iota f_0$. Of course, f_0 is an isomorphism. Define $g = f_0^{-1}\pi\colon W \to V$. Then we get

$$gf = (f_0^{-1}\pi)(\iota f_0) = f_0^{-1}(\pi\iota)f_0 = f_0^{-1}f_0 = 1_V.$$

Vice versa, assume there is a homomorphism $g\colon W \to V$ such that $gf = 1_V$. Set $e = fg$. This is an endomorphism of W, and we have

$$e^2 = (fg)(fg) = f(gf)g = f1_Vg = e,$$

thus e is an idempotent. In particular, the image of e is a direct summand of W. But it is easy to see that $\mathrm{Im}(e) = \mathrm{Im}(f)$: Since $e = fg$ we have $\mathrm{Im}(e) \subseteq \mathrm{Im}(f)$, and $f = f1_V = fgf = ef$ yields the other inclusion $\mathrm{Im}(f) \subseteq \mathrm{Im}(e)$. Thus $\mathrm{Im}(f)$ is a direct summand of W.

This proves part (i) of the statement. We leave part(ii) as an exercise. □

3.8. **Exercises. 1**: Prove part (ii) of the above lemma.

2: Let K be a field of characteristic 0. For integers $i, j \in \mathbb{Z}$ with $i \leq j$ let $M(i,j)$ be the 2-module (K^{j-i+1}, Φ, Ψ) where

$$\Phi = \begin{pmatrix} i &&&& \\ & i+1 &&& \\ && \ddots && \\ &&& j-1 & \\ &&&& j \end{pmatrix} \text{ and } \Psi = \begin{pmatrix} 0 & 1 &&& \\ & 0 & 1 && \\ && \ddots & \ddots & \\ &&& 0 & 1 \\ &&&& 0 \end{pmatrix}.$$

Compute $\mathrm{Hom}(M(i,j), M(k,l))$ for all integers $i \leq j$ and $k \leq l$.

3: Let

$$V = (K^2, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}).$$

Show: $\mathrm{End}(V)$ is the set of matrices of the form $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ with $a, b \in K$.

Compute the idempotents in $\mathrm{End}(V)$.

Compute all direct sum decompositions $V = V_1 \oplus V_2$, with V_1 and V_2 submodules of V.

4: Let

$$V = (K^3, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}).$$

Show: $\mathrm{End}(V)$ is the set of matrices of the form

$$\begin{pmatrix} a & 0 & 0 \\ b & a & 0 \\ c & 0 & a \end{pmatrix}$$

with $a, b, c \in K$.

Use this to show that V is indecomposable.

Show that V is not simple.

5: Let V and W be J-modules. We know that $V \times W$ is again a J-module.

Let $f\colon V \to W$ be a module homomorphism, and let

$$\Gamma_f = \{(v, f(v)) \mid v \in V\}$$

be the *graph of f*.

Show: The map $f \mapsto \Gamma_f$ defines a bijection between $\mathrm{Hom}_J(V, W)$ and the set of submodules $U \subseteq V \times W$ with $U \oplus (0 \times W) = V \times W$.

6: Let

$$V = (K^3, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}).$$

Compute $\mathrm{End}(V)$ (as a set of 3×3-matrices).

Determine all idempotents e in $\mathrm{End}(V)$.

Determine all direct sum decompositions $V = V_1 \oplus V_2$ (with drawings in case $K = \mathbb{R}$).

Describe the map $e \mapsto (\mathrm{Im}(e), \mathrm{Ker}(e))$ (where e runs through the set of idempotents in $\mathrm{End}(V)$).

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4. **Examples of infinite dimensional 1-modules**

4.1. **The module $N(\infty)$.** Let V be a K-vector space with basis $\{e_i \mid i \geq 1\}$. Define a K-linear endomorphism $\phi\colon V \to V$ by $\phi(e_1) = 0$ and $\phi(e_i) = e_{i-1}$ for all $i \geq 2$.

We want to study the 1-module

$$N(\infty) := (V, \phi).$$

We clearly have a chain of submodules

$$N(0) \subset N(1) \subset \cdots \subset N(i) \subset N(i+1) \subset \cdots$$

of $N(\infty)$ where $N(0) = 0$, and $N(i)$ is the submodule with basis $e_1, \ldots, e_i$ where $i \geq 1$. Clearly,

$$N(\infty) = \bigcup_{i \in \mathbb{N}_0} N(i).$$

The following is clear and will be used in the proof of the next lemma: Every submodule of a J-module is a sum of cyclic modules.

Lemma 4.1. *The following hold:*

(i) *The $N(i)$ are the only proper submodules of $N(\infty)$;*
(ii) *The $N(i)$ are cyclic, but $N(\infty)$ is not cyclic;*
(iii) *$N(\infty)$ is indecomposable.*

Proof. First we determine the cyclic submodules: Let $x \in V$. Thus there exists some n such that $x \in N(n)$ and

$$x = \sum_{i=1}^{n} a_i e_i.$$

If $x = 0$, the submodule $U(x)$ generated by x is just $N(0) = 0$. Otherwise, $U(x)$ is equal to $N(i)$ where i the the maximal index $1 \leq j \leq n$ such that $a_j \neq 0$. Note that the module $N(\infty)$ itself is therefore not cyclic.

Now let U be any submodule of V. It follows that U is a sum of cyclic modules, thus

$$U = \sum_{i \in I} N(i)$$

for some $I \subseteq \mathbb{N}_0$. If I is finite, we get $U = N(\max\{i \in I\})$, otherwise we have $U = N(\infty)$. In particular, this implies that $N(\infty)$ is indecomposable. $\square$

A J-module V is **uniform** if for any non-zero submodules U_1 and U_2 one has $U_1 \cap U_2 \neq 0$. It follows from the above considerations that $N(\infty)$ is a uniform module.

4.2. **Polynomial rings.** This section is devoted to study some interesting and important examples of modules arising from the polynomial ring $K[T]$ in one variable T.

As always, K is a field. Recall that the characteristic $\mathrm{char}(K)$ is by definition the minimum n such that the n-fold sum $1 + 1 + \cdots + 1$ of the identity element of K is zero, if such a minimum exists, and $\mathrm{char}(K) = 0$ otherwise. One easily checks that $\mathrm{char}(K)$ is either 0 or a prime number.

The elements in $K[T]$ are of the form

$$f = \sum_{i=0}^{m} a_i T^i$$

with $a_i \in K$ for all i and $m \geq 0$. We set $T^0 = 1$. One calls f **monic** if $a_n = 1$ where n is the maximal $1 \leq i \leq m$ such that $a_i \neq 0$. If $f \neq 0$, then the **degree** of f is the maximum of all i such that $a_i \neq 0$. Otherwise the degree of f is $-\infty$.

By $\mathcal{P}$ we denote the set of monic, irreducible polynomials in $K[T]$. For example, if $K = \mathbb{C}$ we have $\mathcal{P} = \{T - c \mid c \in \mathbb{C}\}$.

Exercise: Determine $\mathcal{P}$ in case $K = \mathbb{R}$. (Hint: All irreducible polynomials over $\mathbb{R}$ have degree at most 2.)

Note that $\{1, T^1, T^2, \ldots\}$ is a basis of the K-vector space $K[T]$.

Let

$$T\cdot \colon K[T] \to K[T]$$

be the K-linear map which maps a polynomial f to Tf. In particular, T^i is mapped to T^{i+1}.

Another important K-linear map is

$$\frac{d}{dT} \colon K[T] \to K[T]$$

which maps a polynomial $\sum_{i=0}^{m} a_i T^i$ to its derivative

$$\frac{d}{dT}(f) = \sum_{i=1}^{m} a_i i T^{i-1}.$$

Of course, in the above expression, i stands for the i-fold sum $1 + 1 + \cdots + 1$ of the identity 1 of K. Thus, if $\mathrm{char}(K) = p > 0$, then $i = 0$ in K if and only if i is divisible by p. In particular $\frac{d}{dT}(T^{np}) = 0$ for all $n \geq 0$.

We know that every polynomial p can be written as a product

$$p = cp_1p_2 \cdots p_t$$

where c is a constant (degree 0) polynomial, and the p_i are monic irreducible polynomials. The polynomials p_i and c are uniquely determined up to reordering.

4.3. **The module $(K[T], \frac{d}{dT})$.** We want to study the 1-module

$$V := (K[T], \frac{d}{dT}).$$

Let V_n be the submodule of polynomials of degree $\leq n$ in V. With respect to the basis $1, T^1, \ldots, T^n$ we get

$$V_n \cong (K^{n+1}, \begin{pmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & \ddots & \ddots & \\ & & & 0 & n \\ & & & & 0 \end{pmatrix}).$$

Exercise: If $\mathrm{char}(K) = 0$, then

$$(K^{n+1}, \begin{pmatrix} 0 & 1 & & & \\ & 0 & 2 & & \\ & & \ddots & \ddots & \\ & & & 0 & n \\ & & & & 0 \end{pmatrix}) \cong (K^{n+1}, \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}).$$

Proposition 4.2. *We have*

$$(K[T], \frac{d}{dT}) \cong \begin{cases} N(\infty) & \textit{if } \mathrm{char}(K) = 0, \\ \bigoplus_{i \in \mathbb{N}_0} N(p) & \textit{if } \mathrm{char}(K) = p. \end{cases}$$

Proof. Define a K-linear map

$$f : (K[T], \frac{d}{dT}) \to N(\infty)$$

by $T^i \mapsto i! \cdot e_{i+1}$ where $i! := i(i-1) \cdots 1$ for $i \geq 1$. Set $0! = 1$. We have

$$f\left(\frac{d}{dT}(T^i)\right) = f(iT^{i-1}) = if(T^{i-1}) = i(i-1)! \cdot e_i = i! \cdot e_i.$$

On the other hand,

$$\phi(f(T^i)) = \phi(i! \cdot e_{i+1}) = i! \cdot e_i.$$

This implies that the diagram

$$\begin{tikzcd} K[T] \arrow[r, "f"] \arrow[d, "\frac{d}{dT}"'] & N(\infty) \arrow[d, "\phi"] \\ K[T] \arrow[r, "f"] & N(\infty) \end{tikzcd}$$

commutes, and therefore f is a homomorphism of 1-modules. If $\mathrm{char}(K) = 0$, then f is an isomorphism with inverse

$$f^{-1} \colon e_{i+1} \mapsto \frac{1}{i!} \cdot T^i$$

where $i \geq 0$.

Now assume $\mathrm{char}(K) = p > 0$. We get $i! = 0$ if and only if $i \geq p$.

The 1-module

$$(W, \phi) := \bigoplus_{i \in \mathbb{N}_0} N(p)$$

has as a basis $\{e_{ij} \mid i \in \mathbb{N}_0, 1 \leq j \leq p\}$ where

$$\phi(e_{ij}) = \begin{cases} 0 & \text{if } j = 1, \\ e_{i,j-1} & \text{otherwise.} \end{cases}$$

Define a K-linear map

$$f \colon W \to K[T]$$

by

$$e_{ij} \mapsto \frac{1}{(j-1)!} T^{ip+j-1}.$$

Since $j \leq p$ we know that p does not divide $(j-1)!$, thus $(j-1)! \neq 0$ in K. One easily checks that f defines a vector space isomorphism.

Exercise: Prove that

$$f(\phi(e_{ij})) = \frac{d}{dT}(f(e_{ij}))$$

and determine f^{-1}.

We get that f is an isomorphism of 1-modules. □

4.4. **The module $(K[T], T\cdot)$.** Next, we want to study the 1-module

$$V := (K[T], T\cdot).$$

Let $a = \sum_{i=0}^{n} a_i T^i$ be a polynomial in $K[T]$. The submodule $U(a)$ of V generated by a is

$$(a) := U(a) = \{ab \mid b \in K[T]\}.$$

We call (a) the **principal ideal generated by** a.

Proposition 4.3. *All ideals in the ring $K[T]$ are principal ideals.*

Proof. Look it up in any book on Algebra. □

In other words: Each submodule of V is of the form (a) for some $a \in K[T]$.

Now it is easy to check that $(a) = (b)$ if and only if $a|b$ and $b|a$ if and only if there exists some $c \in K^*$ with $b = ca$. (For polynomials p and q we write $p|q$ if $q = pf$ for some $f \in K[T]$.)

It follows that for submodules (a) and (b) of V we have

$$(a) \cap (b) = \text{l.c.m.}(a, b)$$

and

$$(a) + (b) = \text{g.c.d.}(a, b).$$

Here l.c.m.(a, b) denotes the lowest common multiple, and g.c.d.(a, b) is the greatest common divisor.

Let $R = K[T]$ be the polynomial ring in one variable T, and let $a_1, \ldots, a_n$ be elements in R.

Lemma 4.4 (Bézout). *Let $R = K[T]$ be the polynomial ring in one variable T, and let $a_1, \ldots, a_n$ be elements in R. There exists a greatest common divisor d of $a_1, \ldots, a_n$, and there are elements r_i in R such that*

$$d = \sum_{i=1}^{n} r_i a_i.$$

It follows that d is the greatest common divisor of elements $a_1, \ldots, a_n$ in $K[T]$ if and only if the ideal $(a_1, \ldots, a_n)$ generated by the a_i is equal to the ideal (d) generated by d.

The greatest common divisor of elements $a_1, \ldots, a_n$ in $K[T]$ is 1 if and only if there exists elements $r_1, \ldots, r_n$ in $K[T]$ such that

$$1 = \sum_{i=1}^{n} r_i a_i.$$

Let $\mathcal{P}$ be the set of monic irreducible polynomials in $K[T]$. Recall that every polynomial $p \neq 0$ in $K[T]$ can be written as

$$p = c p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$$

where $c \in K^*$, $e_i \geq 1$ and the p_i are pairwise different polynomials in $\mathcal{P}$. Furthermore, c, the e_i and the p_i are uniquely determined (up to reordering).

If $b|a$ then there is an epimorphism

$$K[T]/(a) \to K[T]/(b)$$

defined by $p + (a) \mapsto p + (b)$.

Now let p be a non-zero polynomial with

$$p = c p_1^{e_1} p_2^{e_2} \cdots p_t^{e_t}$$

as above.

Proposition 4.5 (Chinese Reminder Theorem). *There is an isomorphism of 1-modules*

$$K[T]/(p) \to \prod_{i=1}^{t} K[T]/(p_i^{e_i}).$$

Proof. We have $p_i^{e_i}|p$ and therefore there is an epimorphism (of 1-modules)

$$\pi_i\colon K[T]/(p) \to K[T]/(p_i^{e_i}).$$

This induces a homomorphism

$$\pi\colon K[T]/(p) \to \prod_{i=1}^{t} K[T]/(p_i^{e_i})$$

defined by $\pi(a) = (\pi_1(a), \dots, \pi_t(a))$. Clearly, $a \in \operatorname{Ker}(\pi)$ if and only if $\pi_i(a) = 0$ for all i if and only if $p_i^{e_i}|a$ for all i if and only if $p|a$. This implies that π is injective.

For a polynomial a of degree n we have $\dim K[T]/(a) = n$, and the residue classes of $1, T, \dots, T^{n-1}$ form a basis of $K[T]/(a)$.

In particular, $\dim K[T]/(p_i^{e_i}) = \deg(p_i^{e_i})$ and

$$\prod_{i=1}^{t} \dim K[T]/(p_i^{e_i}) = \deg(p).$$

Thus for dimension reasons we get that π must be also surjective. $\square$

Exercises: Let p be an irreducible polynomial in $K[T]$.

Show: The module $(K[T]/(p), T\cdot)$ is a simple 1-module, and all simple 1-modules (over a field K) are isomorphic to a module of this form.

Show: The submodules of the factor module $K[T]/(p^e)$ are

$$0 = (p^e)/(p^e) \subset (p^{e-1})/(p^e) \subset \cdots \subset (p)/(p^e) \subset K[T]/(p^e),$$

and we have

$$((p^i)/(p^e))/((p^{i+1})/(p^e)) \cong (p^i)/(p^{i+1}) \cong K[T]/(p).$$

Special case: The polynomial T is an irreducible polynomial in $K[T]$, and one easily checks that the 1-modules $(K[T]/(T^e), T\cdot)$ and $N(e)$ are isomorphic.

Notation: Let $p \in \mathcal{P}$ be a monic, irreducible polynomial in $K[T]$. Set

$$N\binom{n}{p} = (K[T]/(p^n), T\cdot).$$

This is a cyclic and indecomposable 1-module. The modules $N\binom{1}{p}$ are the only simple 1-modules (up to isomorphism).

Exercise: If $p = T - c$ for some $c \in K$, then we have

$$N\binom{n}{p} \cong (K^n, \Phi := \begin{pmatrix} c & 1 & & & \\ & c & 1 & & \\ & & \ddots & \ddots & \\ & & & c & 1 \\ & & & & c \end{pmatrix}).$$

The residue classes of the elements $(T-c)^i$, $0 \le i \le n-1$ form a basis of $N\binom{n}{p}$. We have

$$T \cdot (T-c)^i = (T-c)^{i+1} + c(T-c)^i.$$

The module

$$(K^n, \begin{pmatrix} c & 1 & & & \\ & c & 1 & & \\ & & \ddots & \ddots & \\ & & & c & 1 \\ & & & & c \end{pmatrix})$$

has as a basis the canonical basis vectors $e_1, \dots, e_n$. We have $\Phi(e_1) = ce_1$ and $\Phi(e_i) = ce_i + e_{i-1}$ if $i \geq 2$. Then

$$f \colon (T-c)^i \mapsto e_{n-i}$$

for $i \geq 0$ yields an isomorphism of 1-modules: One easily checks that

$$f(T \cdot (T-c)^i) = \Phi(f((T-c)^i))$$

for all $i \geq 0$.

Conclusion: If we can determine the set $\mathcal{P}$ of irreducible polynomials in $K[T]$, then one has quite a good description of the submodules and also the factor modules of $(K[T], T\cdot)$. But of course, describing $\mathcal{P}$ is very hard (or impossible) if the field K is too complicated.

4.5. **The module** $(K(T), T\cdot)$**.** Let $K(T)$ be the **ring of rational functions** in one variable T. The elements of $K(T)$ are of the form $\frac{p}{q}$ where p and q are polynomials in $K[T]$ wit $q \neq 0$. Furthermore, $\frac{p}{q} = \frac{p'}{q'}$ if and only if $pq' = qp'$. Copying the ususal rules for adding and multiplying fractions, $K(T)$ becomes a ring (it is even a K-algebra). Clearly, all non-zero elements in $K(T)$ have an inverse, thus $K(T)$ is also a field. It contains $K[T]$ as a subring, the embedding given by $p \mapsto \frac{p}{1}$.

Set $K[T] = (K[T], T\cdot)$ and $K(T) = (K(T), T\cdot)$.

Obviously, $K[T]$ is a submodule of $K(T)$. But there are many other interesting submodules:

For $p \in \mathcal{P}$, set

$$K[T, p^{-1}] = \left\{ \frac{q}{p^n} \mid q \in K[T], n \in \mathbb{N}_0 \right\} \subset K(T).$$

For example, if $p = T$, we can think of the elements of $K[T, T^{-1}]$ as linear combinations

$$\sum_{i \in \mathbb{Z}} a_i T^i$$

with only finitely many of the a_i being non-zero. Here we write $T^{-m} = \frac{1}{T^m}$ for $m \geq 1$.

4.6. **Exercises.** **1**: Show: The module $K[T, T^{-1}]/K[T]$ is isomorphic to $N(\infty)$. Its basis are the residue classes of $T^{-1}, T^{-2}, \dots$.

2: Let $K[T]$ be the vector space of polynomials in one variable T with coefficients in a field K, and let $\frac{d}{dT}$ be the differentiation map, i.e if $p = \sum_{i=0}^{n} a_i T^i$ is a polynomial, then

$$\frac{d}{dT}(p) = \sum_{i=1}^{n} a_i i T^{i-1}.$$

Show that the 1-module $(K[T], \frac{d}{dT})$ is indecomposable if $\mathrm{char}(K) = 0$.

Write $(K[T], \frac{d}{dT})$ as a direct sum of indecomposable modules if $\mathrm{char}(K) > 0$.

3: Let $T\cdot$ be the map which sends a polynomial p to Tp.

Show that the 2-module $(K[T], \frac{d}{dT}, T\cdot)$ is simple and that $K \cong \mathrm{End}(K[T], \frac{d}{dT}, T\cdot)$ if $\mathrm{char}(K) = 0$.

Compute $\mathrm{End}(K[T], \frac{d}{dT}, T\cdot)$ if $\mathrm{char}(K) > 0$.

Show that $(K[T], \frac{d}{dT}, T\cdot)$ is not simple in case $\mathrm{char}(K) > 0$.

For endomorphisms f and g of a vector space let $[f, g] = fg - gf$ be its commutator.

Show that $[T\cdot, \frac{d}{dT}] = 1$.

4: Let K be a field, and let $\mathcal{P}$ be the set of monic irreducible polynomials in $K[T]$. For $a \in K(T)$ set

$$K[T]a = \{fa \mid f \in K[T]\} \subset K(T).$$

For every $p \in \mathcal{P}$ let $K[T, p^{-1}]$ be the subalgebra of $K(T)$ generated by T and p^{-1}. In other words,

$$K[T, p^{-1}] = \{\frac{q}{p^n} \mid q \in K[T], n \in \mathbb{N}_0\} \subset K(T).$$

a: Show: The modules $K[T]p^{-n}/K[T]$ and $K[T]/(p^n)$ are isomorphic. Use this to determine the submodules of $K[T]p^{-n}/K[T]$.

b: If U is a proper submodule of $K[T]p^{-n}/K[T]$, then $U = K[T]p^{-n}/K[T]$ for some $n \in \mathbb{N}_0$.

c: We have

$$K(T) = \sum_{p \in \mathcal{P}} K[T, p^{-1}].$$

Let

$$\iota_p \colon K[T, p^{-1}]/K[T] \to K(T)/K[T]$$

be the inclusion.

Show: The homomorphism

$$\iota = \bigoplus_{p \in \mathcal{P}} \iota_p \colon \sum_{p \in \mathcal{P}} (K[T, p^{-1}]/K[T]) \to K(T)/K[T]$$

is an isomorphism.

d: Determine the submodules of $K(T)/K[T]$.

5. Semisimple modules and their endomorphism rings

Some topics discussed in this section are also known as "Artin-Wedderburn Theory". Just open any book on Algebra.

5.1. Semisimple modules.

A module V is **simple** (or **irreducible**) if $V \neq 0$ and the only submodules are 0 and V.

A module V is **semisimple** if V is a direct sum of simple modules.

End of Lecture 5

A proper submodule U of a module V is called a **maximal submodule** of V, if there does not exist a submodule U' with $U \subset U' \subset V$. It follows that a submodule $U \subseteq V$ is maximal if and only if the factor module V/U is simple.

Theorem 5.1. *For a module V the following are equivalent:*

(i) *V is semisimple;*
(ii) *V is a sum of simple modules;*
(iii) *Every submodule of V is a direct summand.*

The proof of Theorem 5.1 uses the Axiom of Choice:

Axiom 5.2 (Axiom of Choice)**.** *Let $f: I \to L$ be a surjective map of sets. Then there exists a map $g: L \to I$ such that $fg = 1_L$.*

Let I be a partially ordered set. A subset C of I is a **chain** in I if for all $c, d \in C$ we have $c \leq d$ or $d \leq c$. An equivalent formulation of the Axiom of Choice is the following:

Axiom 5.3 (Zorn's Lemma)**.** *Let I be a non-empty partially ordered set. If for every chain in I there exists a supremum, then I contains a maximal element.*

This is not surprising: The implication (ii) $\Longrightarrow$ (i) yields the existence of a basis of a vector space. (We just look at the special case $J = \emptyset$. Then J-modules are just vector spaces. The simple J-modules are one-dimensional, and every vector space is a sum of its one-dimensional subspaces, thus condition (ii) holds.)

Proof of Theorem 5.1. The implication (i) $\Longrightarrow$ (ii) is obvious. Let us show (ii) $\Longrightarrow$ (iii): Let V be a sum of simple submodules, and let U be a submodule of V. Let $\mathcal{W}$ be the set of submodules W of V with $U \cap W = 0$. Together with the inclusion $\subseteq$, the set $\mathcal{W}$ is a partially ordered set. Since $0 \in \mathcal{W}$, we know that $\mathcal{W}$ is non-empty.

If $\mathcal{W}' \subseteq \mathcal{W}$ is a chain, then

$$W' = \bigcup_{W \in \mathcal{W}'} W$$

belongs to $\mathcal{W}$: If $x \in U \cap W'$, then x belongs to some W in $\mathcal{W}'$, and therefore $x \in U \cap W = 0$.

Now Zorn's Lemma 5.3 says that $\mathcal{W}$ contains a maximal element. So let $W \in \mathcal{W}$ be maximal. We know that $U \cap W = 0$. On the other hand, we show that $U + W = V$: Since V is a sum of simple submodules, it is enough to show that each simple submodule of V is containd in $U + W$. Let S be a simple submodule of V. If we assume that S is not contained in $U + W$, then $(U + W) \cap S$ is a proper submodule of S. Since S is simple, we get $(U + W) \cap S = 0$, and therefore $U \cap (W + S) = 0$: If $u = w + s$ with $u \in U$, $w \in W$ and $s \in S$, then $u - w = s \in (U + W) \cap S = 0$. Thus $s = 0$ and $u = w \in U \cap W = 0$.

This implies that $W + S$ belongs to $\mathcal{W}$. The maximality of W in $\mathcal{W}$ yields that $W = W + S$ and therefore we get $S \subseteq W$, which is a contradiction to our assumption $S \not\subseteq U + W$. Thus we see that $U + W = V$. So W is a direct complement of U in V.

(iii) $\Longrightarrow$ (ii): Let $\mathcal{S}$ be the set of submodules of V, which are a sum of simple submodules of V. We have $0 \in \mathcal{S}$. (We can think of 0 as the sum over an empty set of simple submodules of V.)

Together with the inclusion $\subseteq$, the set $\mathcal{S}$ forms a partially ordered set. Since 0 belongs to $\mathcal{S}$, we know that $\mathcal{S}$ is non-empty.

If $\mathcal{S}'$ is a chain in $\mathcal{S}$, then

$$\bigcup_{U \in \mathcal{S}'} U$$

belongs to $\mathcal{S}$. Zorn's Lemma tells us that $\mathcal{S}$ contains a maximal element. Let U be such a maximal element.

We claim that $U = V$: Assume there exists some $v \in V$ with $v \notin U$. Let $\mathcal{W}$ be the set of submodules W of V with $U \subseteq W$ and $v \notin W$. Again we interpret $\mathcal{W}$ together with the inclusion $\subseteq$ as a partially ordered set. Since $U \in \mathcal{W}$, we know that $\mathcal{W}$ is non-empty, and if $\mathcal{W}'$ is a chain in $\mathcal{W}$, then

$$\bigcup_{W \in \mathcal{W}'} W$$

belongs to $\mathcal{W}$. Zorn's Lemma yields a maximal element in $\mathcal{W}$, say W. Let W' be the submodule generated by W and v. Since $v \notin W$, we get $W \subset W'$. On the other hand, if X is a submodule with $W \subseteq X \subset W'$, then v cannot be in X, since W' is generated by W and v. Thus X belongs to $\mathcal{W}$, and the maximality of W implies $W = X$. Thus we see that W is a maximal submodule of W'. Condition (iii) implies that W has a direct complement C. Let $C' = C \cap W'$. We have $W \cap C' = W \cap (C \cap W') = 0$, since $W \cap C = 0$. Since the submodule lattice of a module is modular (and since $W \subseteq W'$), we get

$$W + C' = W + (C \cap W') = (W + C) \cap W' = V \cap W' = W'.$$

This implies

$$W'/W = (W + C')/W \cong C'/(W \cap C') = C'.$$

Therefore C' is simple.

$$\begin{tikzcd}[column sep=small]
 & & V \arrow[dl, dash] \arrow[dr, dash] & \\
 & W' \arrow[dl, dash] \arrow[dr, dash] & & C \arrow[dl, dash] \\
W \arrow[dr, dash] & & C' \arrow[dl, dash] & \\
 & 0 & &
\end{tikzcd}$$

Because U is a sum of simple modules, we get that $U + C'$ is a sum of simple modules, thus it belongs to $\mathcal{S}$. Now $U \subset U + C'$ yields a contradiction to the maximality of U in $\mathcal{S}$.

(ii) $\Longrightarrow$ (i): We show the following stronger statement:

Lemma 5.4. *Let V be a module, and let $\mathcal{U}$ be the set of simple submodules U of V. If $V = \sum_{U \in \mathcal{U}} U$, then there exists a subset $\mathcal{U}' \subseteq \mathcal{U}$ such that $V = \bigoplus_{U \in \mathcal{U}'} U$.*

Proof. A subset $\mathcal{U}'$ of $\mathcal{U}$ is called *independent*, if the sum $\sum_{U \in \mathcal{U}'} U$ is a direct sum. Let $\mathcal{T}$ be the set of independent subsets of $\mathcal{U}$, together with the inclusion of sets $\subseteq$ this is a partially ordered set. Since the empty set belongs to $\mathcal{T}$ we know that $\mathcal{T}$ is non-empty. If $\mathcal{T}'$ is a chain in $\mathcal{T}$, then

$$\bigcup_{\mathcal{U}' \in \mathcal{T}'} \mathcal{U}'$$

is obviously in $\mathcal{T}$. Thus by Zorn's Lemma there exists a maximal element in $\mathcal{T}$. Let $\mathcal{U}'$ be such a maximal element. Set

$$W = \sum_{U \in \mathcal{U}'} U.$$

Since $\mathcal{U}'$ belongs to $\mathcal{T}$, we know that this is a direct sum. We claim that $W = V$: Otherwise there would exist a submodule U in $\mathcal{U}$ with $U \not\subseteq W$, because V is the sum of the submodules in $\mathcal{U}$. Since U is simple, this would imply $U \cap W = 0$. Thus the set $\mathcal{U}' \cup \{U\}$ is independent and belongs to $\mathcal{T}$, a contradiction to the maximality of $\mathcal{U}'$ in $\mathcal{T}$. $\square$

This finishes the proof of Theorem 5.1. $\square$

Here is an important consequence of Theorem 5.1:

Corollary 5.5. *Submodules and factor modules of semisimple modules are semisimple.*

Proof. Let V be a semisimple module. If W is a factor module of V, then $W = V/U$ for some submodule U of V. Now U has a direct complement C in V, and C is isomorphic to W. Thus every factor module of V is isomorphic to a submodule of V. Therefore it is enough to show that all submodules of V are semisimple.

$$\begin{tikzcd} & V \arrow[dl, dash] \arrow[dr, dash] & \\ U \arrow[dr, dash] & & C \arrow[dl, dash] \\ & 0 & \end{tikzcd}$$

Let U be submodule of V. We check condition (iii) for U: Every submodule U' of U is also a submodule of V. Thus there exists a direct complement C of U' in V. Then $C \cap U$ is a direct complement of U' in U.

$$\begin{tikzcd} & V \arrow[dl, dash] \arrow[dr, dash] & & \\ C \arrow[dr, dash] & & U \arrow[dl, dash] \arrow[dr, dash] & \\ & C \cap U \arrow[dr, dash] & & U' \arrow[dl, dash] \\ & & 0 & \end{tikzcd}$$

Of course $U' \cap (C \cap U) = 0$, and the modularity yields $U' + (C \cap U) = (U' + C) \cap U = V \cap U = U$. $\square$

Let V be a semisimple module. For every simple module S let V_S be the sum of all submodules U of V such that $U \cong S$. The submodule V_S depends only on the isomorphism class $[S]$ of S. Thus we obtain a family $(V_S)_{[S]}$ of submodules of V which are indexed by the isomorphism classes of simple modules. The submodules V_S are called the **isotypical components** of V.

Proposition 5.6. *Let V be a semisimple module. Then the following hold:*

- $V = \bigoplus_{[S]} V_S$;
- *If V' is a submodule of V, then $V'_S = V' \cap V_S$;*
- *If W is another semisimple module and $f\colon V \to W$ is a homomorphism, then $f(V_S) \subseteq W_S$.*

Proof. First, we show the following: If U is a simple submodule of V, and if $\mathcal{W}$ is a set of simple submodules of V such that $V = \sum_{W \in \mathcal{W}} W$, then $U \cong W$ for some

$W \in \mathcal{W}$: Since $V = \sum_{W \in \mathcal{W}} W$, there is a subset $\mathcal{W}'$ of $\mathcal{W}$ such that $V = \bigoplus_{W \in \mathcal{W}'} W$. For every $W \in \mathcal{W}'$ let $\pi_W \colon V \to W$ be the corresponding projection. Let

$$\iota \colon U \to V = \bigoplus_{W \in \mathcal{W}'} W$$

be the inclusion homomorphism. If U and W are not isomorphic, then $\pi_W \circ \iota = 0$. Since $\iota \neq 0$ there must be some $W \in \mathcal{W}'$ with $\pi_W \circ \iota \neq 0$. Thus U and W are isomorphic.

Since V is semisimple, we have $V = \sum_{[S]} V_S$. To show that this sum is direct, let us look at a fixed isomorphism class $[S]$. Let $\mathcal{T}$ be the set of all isomorphism classes of simple modules different from $[S]$. Define

$$U = V_S \cap \sum_{[T] \in \mathcal{T}} V_T.$$

Since U is a submodule of V, we know that U is semisimple. Thus U is generated by simple modules. If U' is a simple submodule of U, then U' is isomorphic to S, because U and therefore also U' are submodules of V_S. On the other hand, since U' is a submodule of $\sum_{[T] \in \mathcal{T}} V_T$, we get that U' is isomorphic to some T with $[T] \in \mathcal{T}$, a contradiction. Thus U contains no simple submodules, and therefore $U = 0$.

If V' is a submodule of V, then we know that V' is semisimple. Obviously, we have $V'_S \subseteq V' \cap V_S$. On the other hand, every simple submodule of $V' \cap V_S$ is isomorphic to S and therefore contained in V'_S. Since $V' \cap V_S$ is generated by simple submodules, we get $V' \cap V_S \subseteq V'_S$.

Finally, let W be also a semisimple module, and let $f \colon V \to W$ be a homomorphism. If U is a simple submodule of V_S, then $U \cong S$. Now $f(U)$ is either 0 or again isomorphic to S. Thus $f(U) \subseteq W_S$. Since V_S is generated by its simple submodules, we get $f(V_S) \subseteq W_S$. □

5.2. **Endomorphism rings of semisimple modules.** A **skew field** is a ring D (with 1) such that every non-zero element in D has a multiplicative inverse.

Lemma 5.7 (Schur (Version 1)). *Let S be a simple module. Then the endomorphism ring* End(S) *is a skew field.*

Proof. We know that End(S) is a ring. Let $f \colon S \to S$ be an endomorphism of S. It follows that Im(f) and Ker(f) are submodules of S. Since S is simple we get either Ker$(f) = 0$ and Im$(f) = S$, or we get Ker$(f) = S$ and Im$(f) = 0$. In the first case, f is an isomorphism, and in the second case $f = 0$. Thus every non-zero element in End(S) is invertible. □

Let us write down the following reformulation of Lemma 5.7:

Lemma 5.8 (Schur (Version 2)). *Let S be a simple module. Then every endomorphism $S \to S$ is either 0 or an isomorphism.*

End of Lecture 6

Let V be a semisimple module, and as before let V_S be its isotypical components. We have

$$V = \bigoplus_{[S]} V_S,$$

and every endomorphism f of V maps V_S to itself. Let $f_S\colon V_S \to V_S$ be the homomorphism obtained from f via restriction to V_S, i.e. $f_S(v) = f(v)$ for all $v \in V_S$. Then $f \mapsto (f_S)_{[S]}$ defines an algebra isomorphism

$$\mathrm{End}(V) \to \prod_{[S]} \mathrm{End}(V_S).$$

Products of rings: Let I be an index set, and for each $i \in I$ let R_i be a ring. By

$$\prod_{i \in I} R_i$$

we denote the **product** of the rings R_i. Its elements are the sequences $(r_i)_{i\in I}$ with $r_i \in R_i$, and the addition and multiplication is defined componentwise, thus $(r_i)_i + (r_i')_i = (r_i + r_i')_i$ and $(r_i)_i \cdot (r_i')_i = (r_i r_i')_i$.

The above isomorphism tells us, that to understand $\mathrm{End}(V)$, we only have to understand the rings $\mathrm{End}(V_S)$. Thus assume $V = V_S$. We have

$$V = \bigoplus_{i \in I} S$$

for some index set I. The structure of $\mathrm{End}(V)$ only depends on the skew field $D = \mathrm{End}(S)$ and the cardinality $|I|$ of I.

If I is finite, then $|I| = n$ and $\mathrm{End}(V)$ is just the ring $M_n(D)$ of $n \times n$-matrices with entries in D.

If I is infinite, we can interpret $\mathrm{End}(V)$ as an "infinite matrix ring": Let $M_I(D)$ be the **ring of column finite matrices**: Let R be a ring. Then the elements of $M_I(R)$ are double indexed families $(r_{ij})_{ij}$ with $i, j \in I$ and elements $r_{ij} \in R$ such that for every j only finitely many of the r_{ij} are non-zero. Now one can define the multiplication of two such column finite matrices as

$$(r_{ij})_{ij} \cdot (r'_{st})_{st} = \left(\sum_{j \in I} r_{ij} r'_{jt} \right)_{it}.$$

The addition is defined componentwise. (This definition makes also sense if I is finite, where we get the usual matrix ring with rows and columns indexed by the elements in I and not by $\{1, \ldots, n\}$ as usual.)

Lemma 5.9. *For every index set I and every finitely generated module W we have*

$$\mathrm{End}\left(\bigoplus_{i \in I} W \right) \cong M_I(\mathrm{End}(W)).$$

Proof. Let $\iota_j \colon W \to \bigoplus_{i\in I} W$ be the canonical inclusions, and let $\pi_j \colon \bigoplus_{i\in I} W \to W$ be the canonical projections. We map

$$f \in \operatorname{End}\left(\bigoplus_{i\in I} W\right)$$

to the double indexed family $(\pi_i \circ f \circ \iota_j)_{ij}$. Since W is finitely generated, the image of every homomorphism $f \colon W \to \bigoplus_{i\in I} W$ is contained in a submodule $\bigoplus_{i\in I'} W$ where I' is a finite subset of I. This yields that the matrix $(\pi_i \circ f \circ \iota_j)_{ij}$ is column finite. □

5.3. **Exercises.** **1**: Let K be an algebraically closed field.

Classify the simple 1-modules (V, ϕ).

Classify the 2-dimensional simple 2-modules (V, ϕ, ψ).

For every $n \geq 1$ construct an n-dimensional simple 2-module (V, ϕ, ψ).

2: Show that every simple 1-module is finite-dimensional.

Show: If K is algebraically closed, then every simple 1-module is 1-dimensional.

Show: If $K = \mathbb{R}$, then every simple 1-module is 1- or 2-dimensional.

3: Let (V, ϕ_1, ϕ_2) be a 2-module with $V \neq 0$ and $[\phi_1, \phi_2] = 1$.

Show: If $\operatorname{char}(K) = 0$, then V is infinite dimensional.

Hint: Assume V is finite-dimensional, and try to get a contradiction. You could work with the *trace* (of endomorphisms of V). Which endomorphisms does one have to look at?

4: Let $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \in M(2, \mathbb{C})$. Find a matrix $B \in M(2, \mathbb{C})$ such that $(\mathbb{C}^2, A, B)$ is simple.

5: Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \in M(2, \mathbb{C})$. Show that there does not exist a matrix $B \in M(2, \mathbb{C})$ such that $(\mathbb{C}^2, A, B)$ is simple.

6: Let $V = (V, \phi_j)_{j\in J}$ be a finite-dimensional J-module such that all ϕ_j are diagonalizable.

Show: If $[\phi_i, \phi_j] = 0$ for all $i, j \in J$ and if V is simple, then V is 1-dimensional.

6. Socle and radical of a module

6.1. **Socle of a module.** The **socle** of a module V is by definition the sum of all simple submodules of V. We denote the socle of V by $\mathrm{soc}(V)$. Thus $\mathrm{soc}(V)$ is semisimple and every semisimple submodule U of V is contained in $\mathrm{soc}(V)$.

Let us list some basic properties of socles:

Lemma 6.1. *We have $V = \mathrm{soc}(V)$ if and only if V is semisimple.*

Proof. Obvious. □

Lemma 6.2. $\mathrm{soc}(\mathrm{soc}(V)) = \mathrm{soc}(V)$.

Proof. Obvious. □

Lemma 6.3. *If $f: V \to W$ is a homomorphism, then $f(\mathrm{soc}(V)) \subseteq \mathrm{soc}(W)$.*

Proof. The module $f(\mathrm{soc}(V))$ is isomorphic to a factor module of $\mathrm{soc}(V)$, thus it is semisimple. As a semisimple submodule of W, we know that $f(\mathrm{soc}(V))$ is contained in $\mathrm{soc}(W)$. □

Lemma 6.4. *If U is a submodule of V, then $\mathrm{soc}(U) = U \cap \mathrm{soc}(V)$.*

Proof. Since $\mathrm{soc}(U)$ is semisimple, it is a submodule of $\mathrm{soc}(V)$, thus of $U \cap \mathrm{soc}(V)$. On the other hand, $U \cap \mathrm{soc}(V)$ is semisimple, since it is a submodule of $\mathrm{soc}(V)$. Because $U \cap \mathrm{soc}(V)$ is a semisimple submodule of U, we get $U \cap \mathrm{soc}(V) \subseteq \mathrm{soc}(U)$. □

Lemma 6.5. *If V_i with $i \in I$ are modules, then*

$$\mathrm{soc}\left(\bigoplus_{i\in I} V_i\right) = \bigoplus_{i \in I} \mathrm{soc}(V_i).$$

Proof. Let $V = \bigoplus_{i\in I} V_i$. Every submodule $\mathrm{soc}(V_i)$ is semisimple, thus it is contained in $\mathrm{soc}(V)$. Vice versa, let U be a simple submodule of V, and let $\pi_i: V \to V_i$ be the canonical projections. Then $\pi_i(U)$ is either 0 or simple, thus it is contained in $\mathrm{soc}(V_i)$. This implies $U \subseteq \bigoplus_{i\in I} \mathrm{soc}(V_i)$. The simple submodules of V generate $\mathrm{soc}(V)$, thus we also have $\mathrm{soc}(V) \subseteq \bigoplus_{i\in I} \mathrm{soc}(V_i)$. □

6.2. **Radical of a module.** The socle of a module V is the largest semisimple submodule. One can ask if every module has a largest semisimple factor module.

For $|J| = 1$ the example $V = (K[T], T\cdot)$ shows that this is not the case: For every irreducible polynomial p in $K[T]$, the module $K[T]/(p)$ is simple with basis the residue classes of $1, T, T^2, \dots, T^{m-1}$ where m is the degree of the polynomial p.

Now assume that $W = K[T]/U$ is a largest semisimple factor module of V. This would imply $U \subseteq (p)$ for every irreducible polynomial p. Since

$$\bigcap_{p \in \mathcal{P}} (p) = 0,$$

we get $U = 0$ and therefore $W = K[T]$. Here $\mathcal{P}$ denotes the set of all irreducible polynomials in $K[T]$. But V is not at all semisimple. Namely V is indecomposable and not simple. In fact, V does not contain any simple submodules.

Recall: A submodule U of a module V is called a **maximal submodule** if $U \subset V$ and if $U \subseteq U' \subset V$ implies $U = U'$.

By definition the **radical** of V is the intersection of all maximal submodules of V. The radical of V is denoted by $\operatorname{rad}(V)$.

Note that $\operatorname{rad}(V) = V$ if V does not contain any maximal submodule. For example, $\operatorname{rad}(N(\infty)) = N(\infty)$.

The factor module $V/\operatorname{rad}(V)$ is called the **top** of V and is denoted by $\operatorname{top}(V)$.

Lemma 6.6. *Let V be a module. The radical of V is the intersection of all submodules U of V such that V/U is semisimple.*

Proof. Let $r(V)$ be the intersection of all submodules U of V such that V/U is semisimple. Clearly, we get $r(V) \subseteq \operatorname{rad}(V)$. To get the other inclusion $\operatorname{rad}(V) \subseteq r(V)$, let U be a submodule of V with V/U semisimple. We can write V/U as a direct sum of simple modules S_i, say $V/U = \bigoplus_{i \in I} S_i$. For every $i \in I$ let U_i be the kernel of the projection $V \to V/U \to S_i$. This is a maximal submodule of V, and therefore we know that $\operatorname{rad}(V) \subseteq U_i$. Since $U = \bigcap_{i \in I} U_i$, we get $\operatorname{rad}(V) \subseteq U$ which implies $\operatorname{rad}(V) \subseteq r(V)$. □

Note that in general the module $V/\operatorname{rad}(V)$ does not have to be semisimple: If $V = (K[T], T\cdot)$, then from the above discussion we get $V/\operatorname{rad}(V) = V$ and V is not semisimple. However, if V is a "module of finite length", then the factor module $V/\operatorname{rad}(V)$ is semisimple. This will be discussed in Part 3, see in particular Lemma 14.9.

Let us list some basic properties of the radical of a module:

Lemma 6.7. *We have $\operatorname{rad}(V) = 0$ if and only if 0 can be written as an intersection of maximal submodules of V.*

Lemma 6.8. *If U is a submodule of V with $U \subseteq \operatorname{rad}(V)$, then $\operatorname{rad}(V/U) = \operatorname{rad}(V)/U$. In particular, $\operatorname{rad}(V/\operatorname{rad}(V)) = 0$.*

Proof. **Exercise**. □

Lemma 6.9. *If $f\colon V \to W$ is a homomorphism , then $f(\operatorname{rad}(V)) \subseteq \operatorname{rad}(W)$.*

Proof. We show that $f(\mathrm{rad}(V))$ is contained in every maximal submodule of W: Let U be a maximal submodule of W. If $f(V) \subseteq U$, then we get of course $f(\mathrm{rad}(V)) \subseteq U$. Thus, assume $f(V) \not\subseteq U$. It is easy to see that $U \cap f(V) = f(f^{-1}(U))$.

$$\begin{tikzcd}[column sep=small]
& W \arrow[dl, dash] \arrow[dr, dash] & \\
U \arrow[dr, dash] & & f(V) \arrow[dl, dash] \\
& f(f^{-1}(U)) &
\end{tikzcd}$$

Thus

$$W/U \cong f(V)/f(f^{-1}(U)) \cong V/f^{-1}(U)$$

is simple, and therefore $f^{-1}(U)$ is a maximal submodule of V and contains $\mathrm{rad}(V)$. So we proved that $f(\mathrm{rad}(V)) \subseteq f(f^{-1}(U))$ for all maximal submodules U of W. Since $\mathrm{rad}(V) \subseteq f^{-1}(U)$ for all such U, we get

$$f(\mathrm{rad}(V)) \subseteq ff^{-1}(\mathrm{rad}(W)) \subseteq \mathrm{rad}(W).$$

□

Lemma 6.10. *If U is a submodule of V, then $(U + \mathrm{rad}(V))/U \subseteq \mathrm{rad}(V/U)$.*

Proof. **Exercise**. □

In Lemma 6.10 there is normally no equality: Let $V = (K[T], T\cdot)$ and $U = \langle T^2 \rangle = (T^2)$. We have $\mathrm{rad}(V) = 0$, but $\mathrm{rad}(V/U) = (T)/(T^2) \neq 0$.

Lemma 6.11. *If V_i with $i \in I$ are modules, then*

$$\mathrm{rad}\left(\bigoplus_{i\in I} V_i\right) = \bigoplus_{i\in I} \mathrm{rad}(V_i).$$

Proof. Let $V = \bigoplus_{i\in I} V_i$, and let $\pi_i \colon V \to V_i$ be the canonical projections. We have $\pi_i(\mathrm{rad}(V)) \subseteq \mathrm{rad}(V_i)$, and therefore $\mathrm{rad}(V) \subseteq \bigoplus_{i\in I} \mathrm{rad}(V_i)$. Vice versa, let U be a maximal submodule of V. Let U_i be the kernel of the composition $V_i \to V \to V/U$ of the obvious canonical homomorphisms. We get that either U_i is a maximal submodule of V_i or $U_i = V_i$. In both cases we get $\mathrm{rad}(V_i) \subseteq U_i$. Thus $\bigoplus_{i\in I} \mathrm{rad}(V_i) \subseteq U$. Since $\mathrm{rad}(V)$ is the intersection of all maximal submodules of V, we get $\bigoplus_{i\in I} \mathrm{rad}(V_i) \subseteq \mathrm{rad}(V)$. □

6.3. **Exercises.** **1**: Show: If the submodules of a finite-dimensional module V form a chain (i.e. if for all submodules U_1 and U_2 of V we have $U_1 \subseteq U_2$ or $U_2 \subseteq U_1$), then U is cyclic.

2: Assume $\mathrm{char}(K) = 0$. Show: The submodules of the 1-module $(K[T], \frac{d}{dT})$ form a chain, but $(K[T], \frac{d}{dT})$ is not cyclic.

3: For $\lambda \in K$ let $J(\lambda, n)$ be the Jordan block of size $n \times n$ with eigenvalue λ. For $\lambda_1 \neq \lambda_2$ in K, show that the 1-module $(K^n, J(\lambda_1, n)) \oplus (K^m, J(\lambda_2, m))$ is cyclic.

End of Lecture 7

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Part 2. Short Exact Sequences

7. Digression: Categories

This section gives a quick introduction to the concept of categories.

7.1. **Categories.** A **category** $\mathcal{C}$ consists of objects and morphisms, the objects form a class, and for any objects X and Y there is a set $\mathcal{C}(X,Y)$, the set of **morphisms** from X to Y. Is f such a morphism, we write $f\colon X \to Y$. For all objects X,Y,Z in $\mathcal{C}$ there is a **composition map**

$$\mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \to \mathcal{C}(X,Z), \quad (g,f) \mapsto gf,$$

which satisfies the following properties:

- For any object X there is a morphism $1_X\colon X \to X$ such that $f1_X = f$ and $1_X g = g$ for all morphisms $f\colon X \to Y$ and $g\colon Z \to X$.
- The composition of morphisms is associative: For $f\colon X \to Y$, $g\colon Y \to Z$ and $h\colon Z \to A$ we assume $(hg)f = h(gf)$.

For morphisms $f\colon X \to Y$ and $g\colon Y \to Z$ we call $gf\colon X \to Z$ the **composition** of f and g.

A morphism $f\colon X \to Y$ is an **isomorphism** if there exists a morphism $g\colon Y \to X$ such that $gf = 1_X$ and $fg = 1_Y$.

When necessary, we write $\mathrm{Ob}(\mathcal{C})$ for the class of objects in $\mathcal{C}$. However for an object X, we often just say "X lies in $\mathcal{C}$" or write "$X \in \mathcal{C}$".

Remark: Note that we speak of a "class" of objects, and not of sets of objects, since we want to avoid set theoretic difficulties: For example the J-modules do not form a set, otherwise we would run into contradictions. (Like: "The set of all sets.")

If $\mathcal{C}'$ and $\mathcal{C}$ are categories with $\mathrm{Ob}(\mathcal{C}') \subseteq \mathrm{Ob}(\mathcal{C})$ and $\mathcal{C}'(X,Y) \subseteq \mathcal{C}(X,Y)$ for all objects $X,Y \in \mathcal{C}'$ such that the compositions of morphisms in $\mathcal{C}'$ coincide with the compositions in $\mathcal{C}$, then $\mathcal{C}'$ is called a **subcategory** of $\mathcal{C}$. In case $\mathcal{C}'(X,Y) = \mathcal{C}(X,Y)$ for all $X,Y \in \mathcal{C}'$, one calls $\mathcal{C}'$ a **full subcategory** of $\mathcal{C}$.

We only look at K-**linear categories**: We assume additionally that the morphism sets $\mathcal{C}(X,Y)$ are K-vector spaces, and that the composition maps

$$\mathcal{C}(Y,Z) \times \mathcal{C}(X,Y) \to \mathcal{C}(X,Z)$$

are K-bilinear. In K-linear categories we often write $\mathrm{Hom}(X,Y)$ instead of $\mathcal{C}(X,Y)$.

By $\mathrm{Mod}(K)$ we denote the category of K-vector spaces. Let $\mathrm{mod}(K)$ be the category of finite-dimensional K-vector spaces.

7.2. **Functors.** Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **covariant functor** $F\colon \mathcal{C} \to \mathcal{D}$ associates to each object $X \in \mathcal{C}$ an object $F(X) \in \mathcal{D}$, and to each morphism

$f\colon X \to Y$ in $\mathcal{C}$ a morphism $F(f)\colon F(X) \to F(Y)$ in $\mathcal{D}$ such that the following hold:

- $F(1_X) = 1_{F(X)}$ for all objects $X \in \mathcal{C}$;
- $F(gf) = F(g)F(f)$ for all morphisms f, g in $\mathcal{C}$ such that their composition gf is defined.

By a **functor** we always mean a covariant functor. A trivial example is the following: If $\mathcal{C}'$ is a subcategory of $\mathcal{C}$, then the inclusion is a functor.

Similarly, a **contravariant functor** $F\colon \mathcal{C} \to \mathcal{D}$ associates to any object $X \in \mathcal{C}$ an object $F(X) \in \mathcal{D}$, and to each morphism $f\colon X \to Y$ in $\mathcal{C}$ a morphism $F(f)\colon F(Y) \to F(X)$ such that the following hold:

- $F(1_X) = 1_{F(X)}$ for all objects $X \in \mathcal{C}$;
- $F(gf) = F(f)F(g)$ for all morphisms f, g in $\mathcal{C}$ such that their composition gf is defined.

Thus if we deal with contravariant functors, the order of the composition of morphisms is reversed.

End of Lecture 8

If $\mathcal{C}$ and $\mathcal{D}$ are K-linear categories, then a covariant (resp. contravariant) functor $F\colon \mathcal{C} \to \mathcal{D}$ is K-**linear**, if the map $\mathcal{C}(X,Y) \to \mathcal{D}(F(X), F(Y))$ (resp. $\mathcal{C}(X,Y) \to \mathcal{D}(F(Y), F(X))$) defined by $f \mapsto F(f)$ is K-linear for all objects $X, Y \in \mathcal{C}$.

In Section 8 we will see examples of functors.

7.3. **Equivalences of categories.** Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. Then F is called **full**, if for all objects $X, Y \in \mathcal{C}$ the map $\mathcal{C}(X,Y) \to \mathcal{D}(F(X), F(Y))$, $f \mapsto F(f)$ is surjective, and F is **faithful** if these maps are all injective. If every object $X' \in \mathcal{D}$ is isomorphic to an object $F(X)$ for some $X \in \mathcal{C}$, then F is **dense**.

A functor which is full, faithful and dense is called an **equivalence** (of categories). If $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence, then there exists an equivalence $G\colon \mathcal{D} \to \mathcal{C}$ such that for all objects $C \in \mathcal{C}$ the objects C and $GF(C)$ are isomorphic, and for all objects $D \in \mathcal{D}$ the objects D and $FG(D)$ are isomorphic.

If $F\colon \mathcal{C} \to \mathcal{D}$ is an equivalence of categories such that $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$, $X \to F(X)$ is bijective, then F is called an **isomorphism** (of categories). If F is such an isomorphism, then there exists a functor $G\colon \mathcal{D} \to \mathcal{C}$ such that $C = GF(C)$ for all objects $C \in \mathcal{C}$ and $D = FG(D)$ for all objects $D \in \mathcal{D}$. Then G is obviously also an isomorphism. Isomorphisms of categories are very rare. In most constructions which yield equivalences F of categories, it is difficult to decide if F sends two isomorphic objects $X \neq Y$ to the same object.

7.4. **Module categories.** Given a class $\mathcal{M}$ of J-modules, which is closed under isomorphisms and under finite direct sums. Then $\mathcal{M}$ (together with the homomorphisms between the modules in $\mathcal{M}$) is called a **module category**.

(Thus we assume the following: If $V \in \mathcal{M}$ and if $V \cong V'$, then $V' \in \mathcal{M}$. Also, if $V_1, \dots, V_t$ are modules in $\mathcal{M}$, then $V_1 \oplus \cdots \oplus V_t \in \mathcal{M}$.)

If we say that $f: X \to Y$ is a homomorphism in $\mathcal{M}$, then this means that both modules X and Y lie in $\mathcal{M}$ (and that f is a homomorphism).

The module category of all J-modules is denoted by $\mathcal{M}(J)$. Thus $\mathrm{Mod}(K) = \mathcal{M}(\emptyset)$. For $J = \{1, \dots, n\}$ set $\mathcal{M}(n) := \mathcal{M}(J)$.

7.5. **Exercises. 1**: For $c \in K$ let $\mathcal{N}_c$ be the module category of 1-modules (V, ϕ) with $(\phi - c1_V)^m = 0$ for some m. Show that all module categories $\mathcal{N}_c$ are isomorphic (as categories) to $\mathcal{N} := \mathcal{N}_0$.

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8. **Hom-functors and exact sequences**

Let V, W, X, Y be modules, and let $f: V \to W$ and $h: X \to Y$ be homomorphisms. For $g \in \mathrm{Hom}_J(W, X)$ we define a map

$$\mathrm{Hom}_J(f, h): \mathrm{Hom}_J(W, X) \to \mathrm{Hom}_J(V, Y), g \mapsto hgf.$$

It is easy to check that $\mathrm{Hom}_J(f, h)$ is a linear map of vector spaces: For $g, g_1, g_2 \in \mathrm{Hom}_J(W, X)$ and $c \in K$ we have

$$h(g_1 + g_2)f = hg_1f + hg_2f \quad \text{and} \quad h(cg)f = c(hgf).$$

If $V = W$ and $f = 1_V$, then instead of $\mathrm{Hom}_J(1_V, h)$ we mostly write

$$\mathrm{Hom}_J(V, h): \mathrm{Hom}_J(V, X) \to \mathrm{Hom}_J(V, Y),$$

thus by definition $\mathrm{Hom}_J(V, h)(g) = hg$ for $g \in \mathrm{Hom}_J(V, X)$. If $X = Y$ and $h = 1_X$, then instead of $\mathrm{Hom}_J(f, 1_X)$ we write

$$\mathrm{Hom}_J(f, X): \mathrm{Hom}_J(W, X) \to \mathrm{Hom}_J(V, X),$$

thus $\mathrm{Hom}_J(f, X)(g) = gf$ for $g \in \mathrm{Hom}_J(W, X)$.

Typical examples of functors are Hom-functors: Let $\mathcal{M}$ be a module category, which consists of J-modules. Each J-module $V \in \mathcal{M}$ yields a functor

$$\mathrm{Hom}_J(V, -): \mathcal{M} \to \mathrm{mod}(K)$$

which associate to any module $X \in \mathcal{M}$ the vector space $\mathrm{Hom}_J(V, X)$ and to any morphism $h: X \to Y$ in $\mathcal{M}$ the morphism $\mathrm{Hom}_J(V, h): \mathrm{Hom}_J(V, X) \to \mathrm{Hom}_J(V, Y)$ in $\mathrm{mod}(K)$.

Similarly, every object $X \in \mathcal{M}$ yields a contravariant functor

$$\operatorname{Hom}_J(-, X)\colon \mathcal{M} \to \operatorname{mod}(K).$$

Let U, V, W be modules, and let $f\colon U \to V$ and $g\colon V \to W$ be homomorphisms. If $\operatorname{Im}(f) = \operatorname{Ker}(g)$, then (f, g) is called an **exact sequence**. Mostly we denote such an exact sequence in the form

$$U \xrightarrow{f} V \xrightarrow{g} W.$$

We also say, the sequence is **exact at** V. Given such a sequence with $U = 0$, exactness implies that g is injective. (For $U = 0$ we have $\operatorname{Im}(f) = 0 = \operatorname{Ker}(g)$, thus g is injective.) Similarly, if $W = 0$, exactness yields that f is surjective. (For $W = 0$ we have $\operatorname{Ker}(g) = V$, but $\operatorname{Im}(f) = V$ means that f is surjective.)

Given modules V_i with $0 \le i \le t$ and homomorphisms $f_i\colon V_{i-1} \to V_i$ with $1 \le i \le t$, then the sequence $(f_1, \dots, f_t)$ is an **exact sequence** if

$$\operatorname{Im}(f_{i-1}) = \operatorname{Ker}(f_i)$$

for all $2 \le i \le t$. Also here we often write

$$V_0 \xrightarrow{f_1} V_1 \xrightarrow{f_2} \cdots \xrightarrow{f_t} V_t.$$

Typical examples of exact sequences can be obtained as follows: Let V and W be modules and let $g\colon V \to W$ be a homomorphism. Let $\iota\colon \operatorname{Ker}(g) \to V$ be the inclusion, and let $\pi\colon W \to \operatorname{Cok}(g)$ be the projection. Then the sequence

$$0 \to \operatorname{Ker}(g) \xrightarrow{\iota} V \xrightarrow{g} W \xrightarrow{\pi} \operatorname{Cok}(g) \to 0$$

is exact. (Recall that $\operatorname{Cok}(g) = W/\operatorname{Im}(g)$.)

Vice versa, if we have an exact sequence of the form

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W$$

then f is injective and $\operatorname{Im}(f) = \operatorname{Ker}(g)$. Similarly, if

$$U \xrightarrow{g} V \xrightarrow{h} W \to 0$$

is an exact sequence, then h is surjective and $\operatorname{Im}(g) = \operatorname{Ker}(h)$.

Lemma 8.1. *Let $0 \to U \xrightarrow{f} V \xrightarrow{g} W$ be an exact sequence of J-modules. Then $gf = 0$, and for every homomorphism $b\colon X \to V$ with $gb = 0$ there exists a unique homomorphism $b'\colon X \to U$ with $b = fb'$.*

$$\begin{tikzcd}
 & & X \arrow[dl, dashed, "b'"'] \arrow[d, "b"] & \\
0 \arrow[r] & U \arrow[r, "f"] & V \arrow[r, "g"] & W
\end{tikzcd}$$

Proof. Of course we have $gf = 0$. Let now $b\colon X \to V$ be a homomorphism with $gb = 0$. This implies that $\mathrm{Im}(b) \subseteq \mathrm{Ker}(g)$. Set $U' = \mathrm{Ker}(g)$, and let $\iota\colon U' \to V$ be the inclusion. Thus $b = \iota b_0$ for some homomorphism $b_0\colon X \to U'$. There is an isomorphism $f_0\colon U \to U'$ with $f = \iota f_0$. If we define $b' = f_0^{-1} b_0$, then we obtain

$$fb' = (\iota f_0)(f_0^{-1} b_0) = \iota b_0 = b.$$

We still have to show the uniqueness of b': Let $b''\colon X \to U$ be a homomorphism with $fb'' = b$. Then the injectivity of f implies $b' = b''$. □

There is the following reformulation of Lemma 8.1:

Lemma 8.2. *Let $0 \to U \xrightarrow{f} V \xrightarrow{g} W$ be an exact sequence of J-modules. Then for every J-module X, the sequence*

$$0 \to \mathrm{Hom}_J(X,U) \xrightarrow{\mathrm{Hom}_J(X,f)} \mathrm{Hom}_J(X,V) \xrightarrow{\mathrm{Hom}_J(X,g)} \mathrm{Hom}_J(X,W)$$

is exact. ("$\mathrm{Hom}_J(X,-)$ is a left exact functor.")

Proof. We have $\mathrm{Hom}_J(X,g) \circ \mathrm{Hom}_J(X,f) = 0$: For any homomorphism $a\colon X \to U$ we get

$$(\mathrm{Hom}_J(X,g) \circ \mathrm{Hom}_J(X,f))(a) = gfa = 0.$$

This implies $\mathrm{Im}(\mathrm{Hom}_J(X,f)) \subseteq \mathrm{Ker}(\mathrm{Hom}_J(X,g))$.

Vice versa, let $b \in \mathrm{Ker}(\mathrm{Hom}_J(X,g))$. Thus $b\colon X \to V$ is a homomorphism with $gb = 0$. We know that there exists some $b'\colon X \to U$ with $fb' = b$. Thus $\mathrm{Hom}_J(X,f)(b') = fb' = b$. This shows that $b \in \mathrm{Im}(\mathrm{Hom}_J(X,f))$. The uniqueness of b' means that $\mathrm{Hom}_J(X,f)$ is injective. □

Here are the corresponding dual statements of the above lemmas:

Lemma 8.3. *Let $U \xrightarrow{f} V \xrightarrow{g} W \to 0$ be an exact sequence of J-modules. Then $gf = 0$, and for every homomorphism $c\colon V \to Y$ with $cf = 0$ there exists a unique homomorphism $c'\colon W \to Y$ with $c = c'g$.*

$$\begin{tikzcd}
U \arrow[r, "f"] & V \arrow[r, "g"] \arrow[d, "c"'] & W \arrow[r] \arrow[dl, dashed, "c'"'] & 0 \\
 & Y & &
\end{tikzcd}$$

Proof. **Exercise**. □

And here is the corresponding reformulation of Lemma 8.2:

Lemma 8.4. *Let $U \xrightarrow{f} V \xrightarrow{g} W \to 0$ be an exact sequence of J-modules. Then for every J-module X, the sequence*

$$0 \to \mathrm{Hom}_J(W,Y) \xrightarrow{\mathrm{Hom}_J(g,Y)} \mathrm{Hom}_J(V,Y) \xrightarrow{\mathrm{Hom}_J(f,Y)} \mathrm{Hom}_J(U,Y)$$

is exact. ("$\mathrm{Hom}_J(-,Y)$ *is a left exact contravariant functor.")*

Proof. **Exercise**. □

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9. **Equivalences of short exact sequences**

9.1. **Short exact sequences.** An exact sequence of the form

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

is called a **short exact sequence**. This sequence **starts in** U and **ends in** W. Its **middle term** is V and its **end terms** are U and W. For such a short exact sequence we often write (f,g) instead of $(0,f,g,0)$.

Two short exact sequences

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

and

$$0 \to U \xrightarrow{f'} V' \xrightarrow{g'} W \to 0$$

are **equivalent** if there exists a homomorphism $h\colon V \to V'$ such that the following diagram is commutative:

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r, "f"] \arrow[d, equal] & V \arrow[r, "g"] \arrow[d, "h"] & W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U \arrow[r, "f'"] & V' \arrow[r, "g'"] & W \arrow[r] & 0
\end{tikzcd}$$

Remark: The expression "commutative diagram" means the following: Given are certain modules and between them certain homomorphisms. One assumes that for any pair of paths which start at the same module and also end at the same module, the compositions of the corresponding homomorphisms coincide. It is enough to check that for the smallest subdiagrams. For example, in the diagram appearing in the next lemma, commutativity means that $bf = f'a$ and $cg = g'b$. (And therefore also $cgf = g'f'a$.) In the above diagram, commutativity just means $hf = f'$ and $g = g'h$. We used the homomorphisms 1_U and 1_W to obtain a nicer looking diagram. Arranging such diagrams in square form has the advantage that we can speak about rows and columns of a diagram. A frequent extra assumption is that certain columns or rows are exact. In this lecture course, we will see many more commutative diagrams!

Lemma 9.1. *Let*

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "f"] \arrow[d, "a"] & V \arrow[r, "g"] \arrow[d, "b"] & W \arrow[r] \arrow[d, "c"] & 0 \\ 0 \arrow[r] & U' \arrow[r, "f'"] & V' \arrow[r, "g'"] & W' \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram with exact rows. If a and c are isomorphisms, then b is also an isomorphism.

End of Lecture 9

Proof. First, we show that b is injective: If $b(v) = 0$ for some $v \in V$, then $cg(v) = g'b(v) = 0$. This implies $g(v) = 0$ since c is an isomorphism. Thus v belongs to $\mathrm{Ker}(g) = \mathrm{Im}(f)$. So $v = f(u)$ for some $u \in U$. We get $f'a(u) = bf(u) = b(v) = 0$. Now $f'a$ is injective, which implies $u = 0$ and therefore $v = f(u) = 0$.

Second, we prove that b is surjective: Let $v' \in V'$. Then $c^{-1}g'(v') \in W$. Since g is surjective, there is some $v \in V$ with $g(v) = c^{-1}g'(v')$. Thus $cg(v) = g'(v')$. This implies

$$g'(v' - b(v)) = g'(v') - g'b(v) = g'(v') - cg(v) = 0.$$

So $v' - b(v)$ belongs to $\mathrm{Ker}(g') = \mathrm{Im}(f')$. Therefore there exists some $u' \in U'$ with $f'(u') = v' - b(v)$. Let $u = a^{-1}(u')$. Because $f'(u') = f'a(u) = bf(u)$, we get $v' = f'(u') + b(v) = b(f(u) + v)$. Thus v' is in the image of b. So we proved that b is an isomorphism. □

The method used in the proof of the above lemma is called "Diagram chasing".

Lemma 9.1 shows that equivalence of short exact sequences is indeed an equivalence relation on the set of all short exact sequences starting in a fixed module U and ending in a fixed module W:

Given two short exact exact sequences (f, g) and (f', g') like in the assumption of Lemma 9.1. If there exists a homomorphism $h\colon V \to V'$ such that $hf = f'$ and $g = g'h$, then h^{-1} satisfies $h^{-1}f' = f$ and $g' = gh^{-1}$. This proves the symmetry of the relation.

If there is another short exact sequence (f'', g'') with $f''\colon U \to V''$ and $g''\colon V'' \to W$ and a homomorphism $h'\colon V' \to V''$ such that $h'f' = f''$ and $g' = g''h'$, then $h'h\colon V \to V''$ is a homomorphism with $h'hf = f''$ and $g = g''h'h$. This shows our relation is transitive.

Finally, (f, g) is equivalent to itself, just take $h = 1_V$. Thus the relation is reflexive.

A short exact sequence

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

is a **split exact sequence** (or **splits**) if $\mathrm{Im}(f)$ is a direct summand of V. In other words, the sequence splits if f is a split monomorphism, or (equivalently) if g is a split epimorphism. (Remember that $\mathrm{Im}(f) = \mathrm{Ker}(g)$.)

Lemma 9.2. *A short exact sequence $0 \to U \to V \to W \to 0$ splits if and only if it is equivalent to the short exact sequence*

$$0 \to U \xrightarrow{\iota_1} U \oplus W \xrightarrow{\pi_2} W \to 0,$$

where ι_1 is the inclusion of U into $U \oplus W$, and π_2 is the projection from $U \oplus W$ onto W with kernel U.

Proof. Let (f, g) with $f\colon U \to V$ and $g\colon V \to W$ be a short exact sequence. If it splits, then f is a split monomorphism. Thus there exists some $f'\colon V \to U$ with $f'f = 1_U$. So

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r, "f"] \arrow[d, equal] & V \arrow[r, "g"] \arrow[d, "\left[\begin{smallmatrix} f' \\ g \end{smallmatrix}\right]"] & W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U \arrow[r, "\iota_1"] & U \oplus W \arrow[r, "\pi_2"] & W \arrow[r] & 0
\end{tikzcd}$$

is a commutative diagram: If we write $\iota_1 = {}^t[1, 0]$ and $\pi_2 = [0, 1]$, then we see that ${}^t[f', g]f = {}^t[1, 0] = \iota_1$ and $g = [0, 1] \circ {}^t[f', g] = \pi_2 \circ {}^t[f', g]$. Thus (f, g) is equivalent to (ι_1, π_2).

Vice versa, assume that (f, g) and (ι_1, π_2) are equivalent. Thus there exists some $h\colon V \to U \oplus W$ such that $hf = \iota_1$ and $g = \pi_2 h$. Let π_1 be the projection from $U \oplus W$ onto U with kernel W. Then $\pi_1 hf = \pi_1 \iota_1 = 1_U$. Thus f is a split monomorphism. □

9.2. **Exercises. 1**: Let

$$\begin{tikzcd}
0 \arrow[r] & V_1 \arrow[r, "f_1"] & V \arrow[r, "f_2"] \arrow[d, "a"] & V_2 \arrow[r] & 0 \\
0 \arrow[r] & W_1 \arrow[r, "g_1"] & W \arrow[r, "g_2"] & W_2 \arrow[r] & 0
\end{tikzcd}$$

be a diagram of J-modules with exact rows.

Show: There exists a homomorphism $a_1\colon V_1 \to W_1$ with $af_1 = g_1 a_1$ if and only if there exists a homomorphism $a_2\colon V_2 \to W_2$ with $g_2 a = a_2 f_2$.

2: Let

$$\begin{tikzcd}
V_1 \arrow[r, "f_1"] \arrow[d, "a_1"] & V_2 \arrow[r, "f_2"] \arrow[d, "a_2"] & V_3 \arrow[r, "f_3"] \arrow[d, "a_3"] & V_4 \arrow[r, "f_4"] \arrow[d, "a_4"] & V_5 \arrow[d, "a_5"] \\
W_1 \arrow[r, "g_1"] & W_2 \arrow[r, "g_2"] & W_3 \arrow[r, "g_3"] & W_4 \arrow[r, "g_4"] & W_5
\end{tikzcd}$$

be a commutative diagramm of J-modules with exact rows.

Show: If a_1 is an epimorphism, and if a_2 and a_4 are monomorphisms, then a_3 is a monomorphism.

If a_5 is a monomorphism, and if a_2 and a_4 are epimorphisms, then a_3 is an epimorphism.

If a_1, a_2, a_4, a_5 are isomorphisms, then a_3 is an isomorphism.

3: Let

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

be a short exact sequence of J-modules.

Show: The exact sequence (f, g) splits if and only if for all J-modules X the sequence

$$0 \to \mathrm{Hom}_J(X, U) \xrightarrow{\mathrm{Hom}_J(X,f)} \mathrm{Hom}_J(X, V) \xrightarrow{\mathrm{Hom}_J(X,g)} \mathrm{Hom}_J(X, W) \to 0$$

is exact. (By the results we obtained so far, it is enough to show that $\mathrm{Hom}_J(X, g)$ is surjective for all X.)

4: If the sequence

$$0 \to U_i \xrightarrow{f_i} V_i \xrightarrow{g_i} U_{i+1} \to 0$$

is exact for all $i \in \mathbb{Z}$, then the sequence

$$\cdots \to V_{i-1} \xrightarrow{f_i g_{i-1}} V_i \xrightarrow{f_{i+1} g_i} V_{i+1} \to \cdots$$

is exact.

5: Construct an example of a short exact sequence

$$0 \to U \to U' \oplus W \to W \to 0$$

such that $U \not\cong U'$.

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10. **Pushout and pullback**

10.1. **Pushout.** Let U, V_1, V_2 be modules, and let $f_1 : U \to V_1$ and $f_2 : U \to V_2$ be homomorphisms.

$$\begin{tikzcd} & V_1 \\ U \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \\ & V_2 \end{tikzcd}$$

Define

$$W = V_1 \oplus V_2 / \{(f_1(u), -f_2(u)) \mid u \in U\}$$

and $g_i : V_i \to W$ where $g_1(v_1) = \overline{(v_1, 0)}$ and $g_2(v_2) = \overline{(0, v_2)}$. Thus g_i is the composition of the inclusion $V_i \to V_1 \oplus V_2$ followed by the projection $V_1 \oplus V_2 \to W$.

One calls W (or more precisely W together with the homomorphisms g_1 and g_2) the **pushout** (or **fibre sum**) of f_1 and f_2.

So W is the cokernel of the homomorphism ${}^t[f_1, -f_2]\colon U \to V_1 \oplus V_2$, and $[g_1, g_2]\colon V_1 \oplus V_2 \to W$ is the corresponding projection. We get an exact sequence

$$U \xrightarrow{{}^t[f_1,-f_2]} V_1 \oplus V_2 \xrightarrow{[g_1,g_2]} W \to 0.$$

Obviously,

$$U \xrightarrow{{}^t[f_1,f_2]} V_1 \oplus V_2 \xrightarrow{[g_1,-g_2]} W \to 0$$

is also an exact sequence.

Proposition 10.1 (Universal property of the pushout)**.** *For the module W and the homomorphisms $g_1\colon V_1 \to W$ and $g_2\colon V_2 \to W$ as defined above the following hold: We have $g_1f_1 = g_2f_2$, and for every module X together with a pair $(h_1\colon V_1 \to X, h_2\colon V_2 \to X)$ of homomorphisms such that $h_1f_1 = h_2f_2$ there exists a uniquely determined homomorphism $h\colon W \to X$ such that $h_1 = hg_1$ and $h_2 = hg_2$.*

$$\begin{tikzcd}
& V_1 \arrow[dr, "g_1"] \arrow[drr, "h_1"] & & \\
U \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & W \arrow[r, dashed, "h"] & X \\
& V_2 \arrow[ur, "g_2"] \arrow[urr, "h_2"'] & &
\end{tikzcd}$$

Proof. Of course $g_1f_1 = g_2f_2$. If we have $h_1f_1 = h_2f_2$ for some homomorphisms $h_i\colon V_i \to X$, then we can write this as

$$[h_1, h_2] \begin{bmatrix} f_1 \\ -f_2 \end{bmatrix} = 0.$$

This implies that the homomorphism $[h_1, h_2]$ factorizes through the cokernel of ${}^t[f_1, -f_2]$. In other words there is a homomorphism $h\colon W \to X$ such that

$$[h_1, h_2] = h[g_1, g_2].$$

But this means that $h_1 = hg_1$ and $h_2 = hg_2$. The factorization through the cokernel is unique, thus h is uniquely determined. $\square$

More generally, let $f_1\colon U \to V_1$, $f_2\colon U \to V_2$ be homomorphisms. Then a pair $(g_1\colon V_1 \to W, g_2\colon V_2 \to W)$ is called a **pushout** of (f_1, f_2), if the following hold:

- $g_1f_1 = g_2f_2$;
- For all homomorphisms $h_1\colon V_1 \to X$, $h_2\colon V_2 \to X$ such that $h_1f_1 = h_2f_2$ there exists a unique(!) homomorphism $h\colon W \to X$ such that $hg_1 = h_1$ and $hg_2 = h_2$.

Lemma 10.2. *Let $f_1\colon U \to V_1$, $f_2\colon U \to V_2$ be homomorphisms, and assume that $(g_1\colon V_1 \to W, g_2\colon V_2 \to W)$ and also $(g_1'\colon V_1 \to W', g_2'\colon V_2 \to W')$ are pushouts of (f_1, f_2). Then there exists an isomorphism $h\colon W \to W'$ such that $hg_1 = g_1'$ and $hg_2 = g_2'$. In particular, $W \cong W'$.*

Proof. **Exercise**. □

End of Lecture 10

10.2. **Pullback.** Let V_1, V_2, W be modules, and let $g_1 \colon V_1 \to W$ and $g_2 \colon V_2 \to W$ be homomorphisms.

$$\begin{tikzcd}
V_1 \arrow[dr, "g_1"] & \\
& W \\
V_2 \arrow[ur, "g_2"'] &
\end{tikzcd}$$

Define

$$U = \{(v_1, v_2) \in V_1 \oplus V_2 \mid g_1(v_1) = g_2(v_2)\}.$$

One easily checks that U is a submodule of $V_1 \oplus V_2$. Define $f_i \colon U \to V_i$ by $f_i(v_1, v_2) = v_i$. Thus f_i is the composition of the inclusion $U \to V_1 \oplus V_2$ followed by the projection $V_1 \oplus V_2 \to V_i$. One calls U (or more precisely U together with the homomorphisms f_1 and f_2) the **pullback** (or **fibre product**) of g_1 and g_2. So U is the kernel of the homomorphism $[g_1, -g_2] \colon V_1 \oplus V_2 \to W$ and ${}^t[f_1, f_2] \colon U \to V_1 \oplus V_2$ is the corresponding inclusion. We get an exact sequence

$$0 \to U \xrightarrow{{}^t[f_1,f_2]} V_1 \oplus V_2 \xrightarrow{[g_1,-g_2]} W.$$

Of course, also

$$0 \to U \xrightarrow{{}^t[f_1,-f_2]} V_1 \oplus V_2 \xrightarrow{[g_1,g_2]} W.$$

is exact.

Proposition 10.3 (Universal property of the pullback)**.** *For the module U and the homomorphisms $f_1 \colon U \to V_1$ and $f_2 \colon U \to V_2$ as defined above the following hold: We have $g_1 f_1 = g_2 f_2$, and for every module Y together with a pair $(h_1 \colon Y \to V_1, h_2 \colon Y \to V_2)$ of homomorphisms such that $g_1 h_1 = g_2 h_2$ there exists a uniquely determined homomorphism $h \colon Y \to U$ such that $h_1 = f_1 h$ and $h_2 = f_2 h$.*

$$\begin{tikzcd}
& & V_1 \arrow[dr, "g_1"] & \\
Y \arrow[urr, "h_1"] \arrow[r, dashed, "h"] \arrow[drr, "h_2"'] & U \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & W \\
& & V_2 \arrow[ur, "g_2"'] &
\end{tikzcd}$$

Proof. **Exercise**. □

More generally, let $g_1 \colon V_1 \to W$, $g_2 \colon V_2 \to W$ be homomorphisms. Then a pair $(f_1 \colon U \to V_1, f_2 \colon U \to V_2)$ is called a **pullback** of (g_1, g_2), if the following hold:

- $g_1 f_1 = g_2 f_2$;
- For all homomorphisms $h_1 \colon Y \to V_1$, $h_2 \colon Y \to V_2$ such that $g_1 h_1 = g_2 h_2$ there exists a unique(!) homomorphism $h \colon Y \to U$ such that $f_1 h = h_1$ and $f_2 h = h_2$.

Lemma 10.4. *Let $g_1 \colon V_1 \to W$, $g_2 \colon V_2 \to W$ be homomorphisms, and assume that $(f_1 \colon U \to V_1, f_2 \colon U \to V_2)$ and also $(f_1' \colon U' \to V_1, f_2' \colon U' \to V_2)$ are pullbacks of (g_1, g_2). Then there exists an isomorphism $h \colon U' \to U$ such that $f_1 h = f_1'$ and $f_2 h = f_2'$. In particular, $U \cong U'$.*

Proof. **Exercise**. □

Since the pushout of a pair $(f_1 \colon U \to V_1, f_2 \colon U \to V_2)$ (resp. the pullback of a pair $(g_1 \colon V_1 \to W, g_2 \colon V_2 \to W)$) is uniquely determined up to a canonical isomorphism, we speak of “the pushout” of (f_1, f_2) (resp. “the pullback” of (g_1, g_2)).

10.3. **Properties of pushout and pullback.**

Lemma 10.5. *Let $(g_1 \colon V_1 \to W, g_2 \colon V_2 \to W)$ be the pushout of homomorphisms $(f_1 \colon U \to V_1, f_2 \colon U \to V_2)$, and let $(f_1' \colon U' \to V_1, f_2' \colon U' \to V_2)$ be the pullback of (g_1, g_2). Then the uniquely determined homomorphism $h \colon U \to U'$ with $f_1 = f_1' h$ and $f_2 = f_2' h$ is surjective. If ${}^t[f_1, f_2]$ is injective, then h is an isomorphism, and (f_1, f_2) is a pullback of (g_1, g_2).*

$$\begin{tikzcd}
 & & V_1 \arrow[dr, "g_1"] & \\
U \arrow[urr, "f_1"] \arrow[drr, "f_2"'] \arrow[r, dashed, "h"] & U' \arrow[ur, "f_1'"'] \arrow[dr, "f_2'"] & & W \\
 & & V_2 \arrow[ur, "g_2"'] &
\end{tikzcd}$$

Proof. **Exercise**. □

Lemma 10.6. *Let $(f_1 \colon U \to V_1, f_2 \colon U \to V_2)$ be the pullback of homomorphisms $(g_1 \colon V_1 \to W, g_2 \colon V_2 \to W)$, and let $(g_1' \colon V_1 \to W', g_2' \colon V_2 \to W')$ be the pushout of (f_1, f_2). Then the uniquely determined homomorphism $h \colon W' \to W$ with $g_1 = h g_1'$ and $g_2 = h g_2'$ is injective. If $[g_1, g_2]$ is surjective, then h is an isomorphism, and (g_1, g_2) is the pushout of (f_1, f_2).*

$$\begin{tikzcd} & V_1 \arrow[dr, "g_1'"] \arrow[drr, "g_1"] & & \\ U \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & W' \arrow[r, dashed, "h"] & W \\ & V_2 \arrow[ur, "g_2'"] \arrow[urr, "g_2"'] & & \end{tikzcd}$$

Proof. **Exercise**. □

Lemma 10.7. *Let $(g_1\colon V_1 \to W, g_2\colon V_2 \to W)$ be the pushout of a pair $(f_1\colon U \to V_1, f_2\colon U \to V_2)$. If f_1 is injective, then g_2 is also injective.*

Proof. Assume $g_2(v_2) = 0$ for some $v_2 \in V_2$. By definition $g_2(v_2)$ is the residue class of $(0, v_2)$ in W, thus there exists some $u \in U$ with $(0, v_2) = (f_1(u), -f_2(u))$. If we assume that f_1 is injective, then $0 = f_1(u)$ implies $u = 0$. Thus $v_2 = -f_2(u) = 0$. □

Lemma 10.8. *Let $(f_1\colon U \to V_1, f_2\colon U \to V_2)$ be the pullback of a pair $(g_1\colon V_1 \to W, g_2\colon V_2 \to W)$. If g_1 is surjective, then f_2 is also surjective.*

Proof. Let $v_2 \in V_2$. If we assume that g_1 is surjective, then for $g_2(v_2) \in W$ there exists some $v_1 \in V_1$ such that $g_1(v_1) = g_2(v_2)$. But then $u = (v_1, v_2)$ belongs to U, and therefore $f_2(u) = v_2$. □

Pushouts are often used to construct bigger modules from given modules. If V_1, V_2 are modules, and if U is a submodule of V_1 and of V_2, then we can construct the pushout of the inclusions $f_1\colon U \to V_1$, $f_2\colon U \to V_2$. We obtain a module W and homomorphisms $g_1\colon V_1 \to W$, $g_2\colon V_2 \to W$ with $g_1 f_1 = g_2 f_2$.

$$\begin{tikzcd} & V_1 \arrow[dr, "g_1"] & \\ U \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & & W \\ & V_2 \arrow[ur, "g_2"'] & \end{tikzcd}$$

Since f_1 and f_2 are both injective, also g_1 and g_2 are injective. Also (up to canonical isomorphism) (f_1, f_2) is the pullback of (g_1, g_2).

10.4. **Induced exact sequences.** Let

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

be a short exact sequence, and let $a\colon U \to X$ be any homomorphism. We construct the pushout $(a'\colon V \to P, f'\colon X \to P)$ of $(f\colon U \to V, a\colon U \to X)$. Since the homomorphisms $g\colon V \to W$ and $0\colon X \to W$ satisfy the equation $gf = 0 = 0a$, there is a homomorphism $g'\colon P \to W$ with $g'a' = g$ and $g'f' = 0$. Thus we obtain the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "f"] \arrow[d, "a"] & V \arrow[r, "g"] \arrow[d, "a'"] & W \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & X \arrow[r, "f'"] & P \arrow[r, "g'"] & W \arrow[r] & 0 \end{tikzcd}$$

and we claim that (f', g') is again a short exact sequence, which we call the **(short exact) sequence induced** by a. We write $a_*(f, g) = (f', g')$.

Proof. Since f is injective, we know that f' is also injective. Since $g = g'a'$ is surjective, also g' is surjective. By construction $g'f' = 0$, thus $\mathrm{Im}(f') \subseteq \mathrm{Ker}(g')$. We have to show that also the other inclusion holds: Let $\overline{(v, x)} \in \mathrm{Ker}(g')$ where $v \in V$ and $x \in X$. Thus

$$0 = g'(\overline{(v, x)}) = g'(a'(v) + f'(x)) = g'(a'(v)) = g(v).$$

Since (f, g) is an exact sequence, there is some $u \in U$ with $f(u) = v$. This implies

$$f'(x + a(u)) = \overline{(0, x + a(u))} = \overline{(v, x)},$$

because $(v, x) - (0, x + a(u)) = (v, -a(u)) = (f(u), -a(u))$. □

Dually, let $b\colon Y \to W$ be any homomorphism. We take the pullback $(g''\colon Q \to Y, b''\colon Q \to Y)$ of $(b\colon Y \to W, g\colon V \to W)$. Since the homomorphisms $0\colon U \to Y$ and $f\colon U \to V$ satisfy $b0 = 0 = gf$, there exists a homomorphism $f''\colon U \to Q$ with $g''f'' = 0$ and $b''f'' = f$. Again we get a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "f''"] \arrow[d, equal] & Q \arrow[r, "g''"] \arrow[d, "b''"] & Y \arrow[r] \arrow[d, "b"] & 0 \\ 0 \arrow[r] & U \arrow[r, "f"] & V \arrow[r, "g"] & W \arrow[r] & 0, \end{tikzcd}$$

and similarly as before we can show that (f'', g'') is again a short exact sequence. We write $b^*(f, g) = (f'', g'')$, and call this the **(short exact) sequence induced** by b.

Lemma 10.9. *Let*

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

be a short exact sequence. Then the following hold:

(i) *If $a\colon U \to X$ is a homomorphism, then there exists a homomorphism*
$$a''\colon V \to X$$
with $a = a''f$ if and only if the induced sequence $a_*(f, g)$ splits;

(ii) *If $b\colon Y \to W$ is a homomorphism, then there exists a homomorphism*
$$b''\colon Y \to V$$
with $b = gb''$ if and only if the induced sequence $b^*(f, g)$ splits.

$$\begin{tikzcd}
& & & Y \arrow[d, "b"] \arrow[dl, dashed, "b''"'] & \\
0 \arrow[r] & U \arrow[r, "f"] \arrow[d, "a"'] & V \arrow[r, "g"] \arrow[dl, dashed, "a''"] & W \arrow[r] & 0 \\
& X & & &
\end{tikzcd}$$

Proof. Let $a\colon U \to X$ be a homomorphism. We obtain a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r, "f"] \arrow[d, "a"] & V \arrow[r, "g"] \arrow[d, "a'"] & W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r, "f'"] & P \arrow[r, "g'"] & W \arrow[r] & 0
\end{tikzcd}$$

The lower sequence is by definition $a_*(f,g)$. If this sequence splits, then there exists some $f''\colon P \to X$ such that $f''f' = 1_X$. Define $a'' = f''a'$. Then $a''f = f''a'f = f''f'a = a$. Vice versa, let $a''\colon V \to X$ be a homomorphism with $a''f = a$. Since $a''f = 1_X a$, the universal property of the pushout shows that there exists a homomorphism $h\colon P \to X$ such that $a'' = ha'$, $1_X = hf'$.

$$\begin{tikzcd}
& V \arrow[dr, "a'"] \arrow[drr, "a''"] & & \\
U \arrow[ur, "f"] \arrow[dr, "a"'] & & P \arrow[r, dashed, "h"] & X \\
& X \arrow[ur, "f'"] \arrow[urr, "1_X"'] & &
\end{tikzcd}$$

In particular, f' is a split monomorphism. Thus the sequence $(f',g') = a_*(f,g)$ splits.

The second part of the lemma is proved dually. □

End of Lecture 11

Lemma 10.10. *Let*

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r, "f"] \arrow[d, "a"] & V \arrow[r, "g"] \arrow[d, "a''"] & W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r, "f''"] & P' \arrow[r, "g''"] & W \arrow[r] & 0
\end{tikzcd}$$

be a commutative diagram with exact rows. Then the pair (a'', f'') *is a pushout of* (f, a).

Proof. We construct the induced exact sequence $a_*(f,g) = (f',g')$: Let $(a'\colon V \to P, f'\colon X \to P)$ be the pushout of (f,a). For $g'\colon P \to W$ we have $g = g'a'$.

Since $a''f = f''a$ there exists some homomorphism $h\colon P \to P'$ with $a'' = ha'$ and $f'' = hf'$. We claim that $g' = g''h$: This follows from the uniqueness of the factorization through a pushout, because we know that

$$g'a' = g = g''a'' = g''ha'$$

and

$$g'f' = 0 = g''f'' = g''hf'.$$

Thus we have seen that h yields an equivalence of the short exact sequences (f', g') and (f'', g''), In particular, h has to be an isomorphism. But if h is an isomorphism, then the pair $(a'' = ha', f'' = hf')$ is a pushout of (f, a), since by assumption (a', f') is a pushout of (f, a). □

We leave it as an exercise to prove the corresponding dual of the above lemma:

Lemma 10.11. *Let*

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "f'''"] \arrow[d, equal] & Q' \arrow[r, "g'''"] \arrow[d, "b'''"] & Y \arrow[r] \arrow[d, "b"] & 0 \\ 0 \arrow[r] & U \arrow[r, "f"] & V \arrow[r, "g"] & W \arrow[r] & 0 \end{tikzcd}$$

be a commutative diagram with exact rows. Then the pair (b''', g''') is a pullback of (g, b).

10.5. **Examples.** Let

$$0 \to N(2) \xrightarrow{f} N(3) \xrightarrow{g} N(1) \to 0$$

be a short exact sequence, and let $h\colon N(2) \to N(1)$ be a homomorphism. As before, $N(m)$ is the m-dimensional 1-module (K^m, ϕ) with basis $e_1, \dots, e_m$, $\phi(e_1) = 0$ and $\phi(e_i) = e_{i-1}$ for all $2 \le i \le m$. We will fix such bases for each m, and display the homomorphisms f, g and h as matrices: For example, let

$$f = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } g = [0, 0, 1].$$

For $h = [0, 1]$, the induced sequence $h_*(f, g)$ is of the form

$$0 \to N(1) \to N(2) \to N(1) \to 0.$$

For $h = [0, 0]$, the induced sequence $h_*(f, g)$ is of the form

$$0 \to N(1) \to N(1) \oplus N(1) \to N(1) \to 0.$$

Similarly as above let

$$0 \to N(3) \xrightarrow{f} N(4) \xrightarrow{g} N(1) \to 0$$

be the obvious canonical short exact sequence, and let $h\colon N(3) \to N(2)$ be the homomorphism given by the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus h is the canonical epimorphism from $N(3)$ onto $N(2)$. Now one can check that the pushout of (f, h) is isomorphic to $N(3)$.

On the other hand, if h is given by the matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then the pushout of (f, h) is isomorphic to $N((2,1)) = N(2) \oplus N(1)$.

10.6. **Exercises.** **1**: Let

$$0 \to U \xrightarrow{f_1} V_1 \xrightarrow{g_1} W \to 0$$

and

$$0 \to U \xrightarrow{f_2} V_2 \xrightarrow{g_2} W \to 0$$

be equivalent short exact sequences of J-modules, and let $a\colon U \to X$ be a homomorphism. Show that the two short exact sequences $a_*(f_1, g_1)$ and $a_*(f_2, g_2)$ are equivalent.

2: Recall: For any partition λ, we defined a 1-module $N(\lambda)$. Let

$$0 \to N(n) \xrightarrow{f_1} N(2n) \xrightarrow{g_1} N(n) \to 0$$

be the short exact sequence with f_1 the canonical inclusion and g_1 the canonical projection, and let

$$\eta: \quad 0 \to N(n) \xrightarrow{f_2} N(\lambda) \xrightarrow{g_2} N(n) \to 0$$

be a short exact sequence with $\lambda = (\lambda_1, \lambda_2)$.

Show: There exists some homomorphism $a\colon N(n) \to N(n)$ such that $a_*(f_1, g_1) = \eta$.

3: Let

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

be a short exact sequence of J-modules, and let $a\colon U \to X$, $a'\colon X \to X'$, $b\colon Y \to W$, $b'\colon Y' \to Y$ be homomorphisms of J-modules. Show:

- The induced sequences $(a'a)_*(f, g)$ and $a'_*(a_*(f, g))$ are equivalent;
- The induced sequences $(bb')^*(f, g)$ and $(b')^*(b^*(f, g))$ are equivalent;
- The induced sequences $a_*(b^*(f, g))$ and $b^*(a_*(f, g))$ are equivalent.

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11. Irreducible homomorphisms and Auslander-Reiten sequences

11.1. **Irreducible homomorphisms.** Let $\mathcal{M}$ be a module category. A homomorphism $f\colon V \to W$ in $\mathcal{M}$ is **irreducible** (in $\mathcal{M}$) if the following hold:

- f is not a split monomorphism;

- f is not a split epimorphism;
- For any factorization $f = f_2 f_1$ in $\mathcal{M}$, f_1 is a split monomorphism or f_2 is a split epimorphism.

Note that any homomorphism $f\colon V \to W$ has many factorizations $f = f_2 f_1$ with f_1 a split monomorphism or f_2 a split epimorphism: Let C be any module in $\mathcal{M}$, and let $g\colon V \to C$ and $h\colon C \to W$ be arbitrary homomorphisms. Define $f_1 = {}^t[1, 0]\colon V \to V \oplus C$ and $f_2 = [f, h]\colon V \oplus C \to W$. Then $f = f_2 f_1$ with f_1 a split monomorphism.

Similarly, define $f_1' = {}^t[f, g]\colon V \to W \oplus C$ and $f_2' = [1, 0]\colon W \oplus C \to W$. Then $f = f_2' f_1'$ with f_2' a split epimorphism. We could even factorize f as $f = f_2'' f_1''$ with f_1'' a split monomorphism and f_2'' a split epimorphism: Take $f_1'' = {}^t[1, 1, 0]\colon V \to V \oplus V \oplus W$ and $f_2'' = [0, f, 1]\colon V \oplus V \oplus W \to W$.

Thus the main point is that the third condition in the above definition applies to ALL factorizations $f = f_2 f_1$ of f.

The notion of an irreducible homomorphism makes only sense if we talk about a certain fixed module category $\mathcal{M}$.

Examples: Let V and W be non-zero modules in a module category $\mathcal{M}$. Examples of homomorphisms which are NOT irreducible are $0 \to 0$, $0 \to W$, $V \to 0$, $0\colon V \to W$, $1_V\colon V \to V$. (Recall that the submodule 0 of a module is always a direct summand.)

Lemma 11.1. *Assume that $\mathcal{M}$ is a module category which is closed under images (i.e. if $f\colon U \to V$ is a homomorphism in $\mathcal{M}$, then $\mathrm{Im}(f)$ is in $\mathcal{M}$). Then every irreducible homomorphism in $\mathcal{M}$ is either injective or surjective.*

Proof. Assume that $f\colon U \to V$ is a homomorphism in $\mathcal{M}$ which is neither injective nor surjective, and let $f = f_2 f_1$ where $f_1\colon U \to \mathrm{Im}(f)$ is the homomorphism defined by $f_1(u) = f(u)$ for all $u \in U$, and $f_2\colon \mathrm{Im}(f) \to V$ is the inclusion homomorphism. Then f_1 is not a split monomorphism (it is not even injective), and f_2 is not a split epimorphism (it is not even surjective). $\square$

11.2. **Auslander-Reiten sequences and Auslander-Reiten quivers.** Again let $\mathcal{M}$ be a module category. An exact sequence

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

with $U, V, W \in \mathcal{M}$ is an **Auslander-Reiten sequence** in $\mathcal{M}$ if the following hold:

(i) The homomorphisms f and g are irreducible in $\mathcal{M}$;
(ii) Both modules U and W are indecomposable.

(We will see that for many module cateogories assumption (ii) is not necessary: If $\mathcal{M}$ is closed under kernels of surjective homomorphisms and $f\colon U \to V$ is an injective homomorphism which is irreducible in $\mathcal{M}$, then $\mathrm{Cok}(f)$ is indecomposable. Similarly, if $\mathcal{M}$ is closed under cokernels of injective homomorphisms and $g\colon V \to W$ is a surjective homomorphism which is irreducible in $\mathcal{M}$, then $\mathrm{Ker}(g)$ is indecomposable.)

Let (Γ_0, Γ_1) and (Γ_0, Γ_2) be two "quivers" with the same set Γ_0 of vertices, but with disjoint sets Γ_1 and Γ_2 of arrows. Then $(\Gamma_0, \Gamma_1, \Gamma_2)$ is called a **biquiver**. The arrows in Γ_1 are the 1-arrows and the arrows in Γ_2 the 2-arrows. To distinguish these two types of arrows, we usually draw dotted arrows for the 2-arrows. (Thus a **biquiver** Γ is just an oriented graph with two types of arrows: The set of vertices is denoted by Γ_0, the "1-arrows" are denoted by Γ_1 and the "2-arrows" by Γ_2.)

Let $\mathcal{M}$ be a module category, which is closed under direct summands. Then the **Auslander-Reiten quiver** of $\mathcal{M}$ is a biquiver $\Gamma_{\mathcal{M}}$ which is defined as follows: The vertices are the isomorphism classes of indecomposable modules in $\mathcal{M}$. For a module V we often write $[V]$ for its isomorphism class. There is a 1-arrow $[V] \to [W]$ if and only if there exists an irreducible homomorphism $V \to W$ in $\mathcal{M}$, and there is a 2-arrow from $[W]$ to $[U]$ if and only if there exists an Auslander-Reiten sequence

$$0 \to U \to V \to W \to 0.$$

The Auslander-Reiten quiver is an important tool which helps to understand the structure of a given module category.

Later we will modify the above definition of an Auslander-Reiten quiver and also allow more than one arrow between two given vertices.

11.3. **Properties of irreducible homomorphisms.** We want to study irreducible homomorphisms in a module category $\mathcal{M}$ in more detail.

For this we assume that $\mathcal{M}$ is closed under kernels of surjective homomorphisms, that is for every surjective homomorphism $g\colon V \to W$ in $\mathcal{M}$, the kernel $\mathrm{Ker}(g)$ belongs to $\mathcal{M}$. In particular, this implies the following: If $g_1\colon V_1 \to W$, $g_2\colon V_2 \to W$ are in $\mathcal{M}$, and if at least one of these homomorphisms g_i is surjective, then also the pullback of (g_1, g_2) is in $\mathcal{M}$.

Lemma 11.2 (Bottleneck Lemma)**.** *Let $\mathcal{M}$ be a module category which is closed under kernels of surjective homomorphisms. Let*

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

be a short exact sequence in $\mathcal{M}$, and assume that f is irreducible in $\mathcal{M}$. If $g'\colon V' \to W$ is any homomorphism, then there exists a homomorphism $b_1\colon V' \to V$ with $gb_1 = g'$, or there exists a homomorphism $b_2\colon V \to V'$ with $g'b_2 = g$.

$$\begin{tikzcd} & & & V' \arrow[dl, dashed, "b_1"'] \arrow[d, "g'"] & \\ 0 \arrow[r] & U \arrow[r, "f"] & V \arrow[r, "g"] & W \arrow[r] & 0 \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} & & & V' \arrow[d, "g'"] & \\ 0 \arrow[r] & U \arrow[r, "f"] & V \arrow[ur, dashed, "b_2"] \arrow[r, "g"] & W \arrow[r] & 0 \end{tikzcd}$$

The name "bottleneck" is motivated by the following: Any homomorphism with target W either factors through g or g factors through it. So everything has to pass through the "bottleneck" g.

Proof. The induced sequence $(g')^*(f,g)$ looks as follows:

$$\begin{tikzcd} 0 \arrow[r] & U \arrow[r, "f_1"] \arrow[d, equal] & P \arrow[r, "g_1"] \arrow[d, "f_2"] & V' \arrow[r] \arrow[d, "g'"] & 0 \\ 0 \arrow[r] & U \arrow[r, "f"] & V \arrow[r, "g"] & W \arrow[r] & 0 \end{tikzcd}$$

The module P is the pullback of (g,g'), thus P belongs to $\mathcal{M}$. We obtain a factorization $f = f_2 f_1$ in $\mathcal{M}$. By our assumption, f is irreducible in $\mathcal{M}$, thus f_1 is a split monomorphism or f_2 is a split epimorphism. In the second case, there exists some $f_2' \colon V \to P$ such that $f_2 f_2' = 1_V$. Therefore for $b_2 := g_1 f_2'$ we get

$$g' b_2 = g' g_1 f_2' = g f_2 f_2' = g 1_V = g.$$

On the other hand, if f_1 is a split monomorphism, then the short exact sequence (f_1, g_1) splits, and it follows that g_1 is a split epimorphism. We obtain a homomorphism $g_1' \colon V' \to P$ with $g_1 g_1' = 1_{V'}$. For $b_1 := f_2 g_1'$ we get

$$g b_1 = g f_2 g_1' = g' g_1 g_1' = g' 1_{V'} = g'.$$

$\square$

Corollary 11.3. *Let $\mathcal{M}$ be a module category which is closed under kernels of surjective homomorphisms. If*

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

is a short exact sequence in $\mathcal{M}$ with f irreducible in $\mathcal{M}$, then W is indecomposable.

End of Lecture 12

Proof. Let $W = W_1 \oplus W_2$, and let $\iota_i \colon W_i \to W$ be the inclusions. We assume that $W_1 \neq 0 \neq W_2$. Thus none of these two inclusions is surjective. By the Bottleneck Lemma, there exist homomorphisms $c_i \colon W_i \to V$ with $g c_i = \iota_i$. (If there were homomorphisms $c_i' \colon V \to W_i$ with $\iota_i c_i' = g$, then g and therefore also ι_i would be surjective, a contradiction.)

Let $C = \operatorname{Im}(c_1) + \operatorname{Im}(c_2) \subseteq V$. We have $\operatorname{Im}(f) \cap C = 0$: If $f(u) = c_1(w_1) + c_2(w_2)$ for some $u \in U$ and $w_i \in W_i$, then

$$0 = gf(u) = gc_1(w_1) + gc_2(w_2) = \iota_1(w_1) + \iota_2(w_2)$$

and therefore $w_1 = 0 = w_2$.

On the other hand, we have $\operatorname{Im}(f) + C = V$: If $v \in V$, then $g(v) = \iota_1(w_1') + \iota_2(w_2')$ for some $w_i' \in W_i$. This implies $g(v) = gc_1(w_1') + gc_2(w_2')$, thus $v - c_1(w_1') - c_2(w_2')$ belongs to $\operatorname{Ker}(g)$ and therefore to $\operatorname{Im}(f)$. If we write this element in the form $f(u')$ for some $u' \in U$, then $v = f(u') + c_1(w_1') + c_2(w_2')$.

Altogether, we see that $\mathrm{Im}(f)$ is a direct summand of V, a contradiction since we assumed f to be irreducible. $\square$

Corollary 11.4. *Let $\mathcal{M}$ be a module category which is closed under kernels of surjective homomorphisms. If*

$$0 \to U_1 \xrightarrow{f_1} V_1 \xrightarrow{g_1} W \to 0$$

$$0 \to U_2 \xrightarrow{f_2} V_2 \xrightarrow{g_2} W \to 0$$

are two Auslander-Reiten sequences in $\mathcal{M}$, then there exists a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & U_1 \arrow[r, "f_1"] \arrow[d, "a"] & V_1 \arrow[r, "g_1"] \arrow[d, "b"] & W \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & U_2 \arrow[r, "f_2"] & V_2 \arrow[r, "g_2"] & W \arrow[r] & 0 \end{tikzcd}$$

with a and b isomorphisms.

Proof. Since f_1 is irreducible, there exists a homomorphism $b\colon V_1 \to V_2$ with $g_1 = g_2 b$ or a homomorphism $b'\colon V_2 \to V_1$ with $g_2 = g_1 b'$. For reasons of symmetry, it is enough to consider only one of these cases. Let us assume that there exists $b\colon V_1 \to V_2$ with $g_1 = g_2 b$. This implies the existence of a homomorphism $a\colon U_1 \to U_2$ with $bf_1 = f_2 a$. (Since $g_2 b f_1 = 0$, we can factorize bf_1 through the kernel of g_2.) Thus we constructed already a commutative diagram as in the statement of the corollary.

It remains to show that a and b are isomorphisms: Since g_1 is irreducible, and since g_2 is not a split epimorphism, the equality $g_1 = g_2 b$ implies that b is a split monomorphism. Thus there is some $b'\colon V_2 \to V_1$ with $b'b = 1_{V_1}$. We have $b' f_2 a = b' b f_1 = f_1$, and since f_1 is irreducible, a is a split monomorphism or $b' f_2$ is a split epimorphism. Assume $b' f_2\colon U_2 \to V_1$ is a split epimorphism. We know that U_2 is indecomposable and that $V_1 \neq 0$, thus $b' f_2$ has to be an isomorphism. This implies that f_2 is a split monomorphism, a contradiction. So we conclude that $a\colon U_2 \to U_1$ is a split monomorphism. Now U_2 is indecomposable and $U_1 \neq 0$, thus a is an isomorphism. This implies that b also has to be an isomorphism. $\square$

Corollary 11.5. *Let $\mathcal{M}$ be a module category which is closed under direct summands and under kernels of surjective homomorphisms. Let*

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

be an Auslander-Reiten sequence in $\mathcal{M}$. If Y is a module in $\mathcal{M}$ which can be written as a finite direct sum of indecomposable modules, and if $h\colon Y \to W$ is a homomorphism which is not a split epimorphism, then there exists a homomorphism $h'\colon Y \to V$ with $gh' = h$.

Proof. We first assume that Y is indecomposable. By the Bottleneck Lemma, instead of h' there could exist a homomorphism $g'\colon V \to Y$ with $g = hg'$. But g is irreducible and h is not a split epimorphism. Thus g' must be a split monomorphism. Since

Y is indecomposable and $V \neq 0$, this implies that g' is an isomorphism. Thus $h = g(g')^{-1}$.

Now let $Y = \bigoplus_{i=1}^t Y_i$ with Y_i indecomposable for all i. As usual let $\iota_s \colon Y_s \to \bigoplus_{i=1}^t Y_i$ be the inclusion homomorphisms. Set $h_i = h\iota_i$. By our assumptions, h is not a split epimorphism, thus the same is true for h_i. Thus we know that there are homomorphisms $h_i' \colon Y_i \to V$ with $gh_i' = h_i$. Then $h' = [h_1', \dots, h_t']$ satisfies $gh' = h$. $\square$

Now we prove the converse of the Bottleneck Lemma (but note the different assumption on $\mathcal{M}$):

Lemma 11.6 (Converse Bottleneck Lemma). *Let $\mathcal{M}$ be a module category which is closed under cokernels of injective homomorphisms. Let*

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

be a non-split short exact sequence in $\mathcal{M}$ such that the following hold: For every homomorphism $g' \colon V' \to W$ in $\mathcal{M}$ there exists a homomorphism $b_1 \colon V' \to V$ with $gb_1 = g'$, or there exists a homomorphism $b_2 \colon V \to V'$ with $g'b_2 = g$. Then it follows that f is irreducible in $\mathcal{M}$.

End of Lecture 13

Proof. Let $f = f_2 f_1$ be a factorization of f in $\mathcal{M}$. Thus $f_1 \colon U \to V'$ for some V' in $\mathcal{M}$. The injectivity of f implies that f_1 is injective as well. Let $g_1 \colon V' \to W'$ be the cokernel map of f_1. By assumption W' belongs to $\mathcal{M}$. Since $gf_2f_1 = gf = 0$, we can factorize gf_2 through g_1. Thus we obtain $g' \colon W' \to W$ with $g'g_1 = gf_2$. Altogether we constructed the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r, "f"] & V \arrow[r, "g"] & W \arrow[r] & 0 \\
0 \arrow[r] & U \arrow[r, "f_1"] \arrow[u, equal] & V' \arrow[r, "g_1"] \arrow[u, "f_2"'] & W' \arrow[r] \arrow[u, "g'"'] & 0
\end{tikzcd}$$

It follows that the pair (f_2, g_1) is the pullback of (g, g'). Our assumption implies that for g' there exists a homomorphism $b_1 \colon W' \to V$ with $gb_1 = g'$ or a homomorphism $b_2 \colon V \to W'$ with $g'b_2 = g$.

If b_1 exists with $gb_1 = g' = g'1_{W'}$, then the pullback property yields a homomorphism $h \colon W' \to V'$ with $b_1 = f_2 h$ and $1_{W'} = g_1 h$. In particular we see that g_1 is a split epimorphism, and therefore f_1 is a split monomorphism.

$$\begin{tikzcd}
 & & V \arrow[dr, "g"] & \\
W' \arrow[urr, "b_1"] \arrow[r, dashed, "h"] \arrow[drr, "1_{W'}"'] & V' \arrow[ur, "f_2"'] \arrow[dr, "g_1"] & & W \\
 & & W' \arrow[ur, "g'"'] &
\end{tikzcd}$$

In the second case, if b_2 exists with $g1_V = g = g'b_2$, we obtain a homomorphism $h'\colon V \to V'$ with $1_V = f_2h'$ and $b_2 = g_1h'$. Thus f_2 is a split epimorphism.

$$\begin{tikzcd}
 & & V \arrow[dr, "g"] & \\
V \arrow[urr, "1_V"] \arrow[r, dashed, "h'"] \arrow[drr, "b_2"'] & V' \arrow[ur, "f_2"] \arrow[dr, "g_1"'] & & W \\
 & & W' \arrow[ur, "g'"'] &
\end{tikzcd}$$

□

11.4. **Dual statements.** Let us formulate the corresponding dual statements:

Lemma 11.7 (Bottleneck Lemma)**.** *Let $\mathcal{M}$ be a module category which is closed under cokernels of injective homomorphisms. Let*

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

be a short exact sequence in $\mathcal{M}$, and assume that g is irreducible in $\mathcal{M}$. If $f'\colon U \to V'$ is any homomorphism, then there exists a homomorphism $a_1\colon V \to V'$ with $a_1f = f'$, or there exists a homomorphism $a_2\colon V' \to V$ with $a_2f' = f$.

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r, "f"] \arrow[d, "f'"'] & V \arrow[r, "g"] \arrow[dl, dashed, "a_1"] & W \arrow[r] & 0 \\
 & V' & & &
\end{tikzcd}
\quad \text{or} \quad
\begin{tikzcd}
0 \arrow[r] & U \arrow[r, "f"] \arrow[d, "f'"'] & V \arrow[r, "g"] & W \arrow[r] & 0 \\
 & V' \arrow[ur, dashed, "a_2"'] & & &
\end{tikzcd}$$

Corollary 11.8. *Let $\mathcal{M}$ be a module category which is closed under cokernels of injective homomorphisms. If*

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

is a short exact sequence in $\mathcal{M}$, and if g is irreducible in $\mathcal{M}$, then U is indecomposable.

Corollary 11.9. *Let $\mathcal{M}$ be a module category which is closed under cokernels of injective homomorphisms. If*

$$0 \to U \xrightarrow{f_1} V_1 \xrightarrow{g_1} W_1 \to 0$$

$$0 \to U \xrightarrow{f_2} V_2 \xrightarrow{g_2} W_2 \to 0$$

are two Auslander-Reiten sequences in $\mathcal{M}$, then there exists a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r, "f_1"] \arrow[d, equal] & V_1 \arrow[r, "g_1"] & W_1 \arrow[r] & 0 \\
0 \arrow[r] & U \arrow[r, "f_2"] & V_2 \arrow[r, "g_2"] \arrow[u, "b"'] & W_2 \arrow[r] \arrow[u, "c"'] & 0
\end{tikzcd}$$

with b and c isomorphisms.

Corollary 11.10. *Let $\mathcal{M}$ be a module category which is closed under direct summands and under cokernels of injective homomorphisms. Let*

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

be an Auslander-Reiten sequence in $\mathcal{M}$. If X is a module in $\mathcal{M}$ which can be written as a finite direct sum of indecomposable modules, and if $h\colon U \to X$ is a homomorphism which is not a split monomorphism, then there exists a homomorphism $h'\colon V \to X$ with $h'f = h$.

Lemma 11.11 (Converse Bottleneck Lemma)**.** *Let $\mathcal{M}$ be a module category which is closed under kernels of surjective homomorphisms. Let*

$$0 \to U \xrightarrow{f} V \xrightarrow{g} W \to 0$$

be a non-split short exact sequence in $\mathcal{M}$ such that the following hold: For every homomorphism $f'\colon U \to V'$ in $\mathcal{M}$ there exists a homomorphism $a_1\colon V \to V'$ with $a_1 f = f'$, or there exists a homomorphism $a_2\colon V' \to V$ with $a_2 f' = f$. Then it follows that g is irreducible in $\mathcal{M}$.

The proofs of these dual statements are an exercise.

11.5. **Examples: Irreducible maps in $\mathcal{N}^{f.d.}$.** In this section let

$$\mathcal{M} := \mathcal{N}^{\text{f.d.}}$$

be the module category of all 1-modules (V, ϕ) with V finite-dimensional and ϕ nilpotent.

Recall that we denoted the indecomposable modules in $\mathcal{M}$ by $N(n)$ where $n \geq 1$. Let us also fix basis vectors $e_1, \ldots, e_n$ of $N(n) = (V, \phi)$ such that $\phi(e_i) = e_{i-1}$ for $2 \leq i \leq n$ and $\phi(e_1) = 0$.

By

$$\iota_n\colon N(n) \to N(n+1)$$

we denote the canonical inclusion (defined by $\iota_n(e_n) = e_n$), and let

$$\pi_{n+1}\colon N(n+1) \to N(n)$$

be the canonical projection (defined by $\pi_{n+1}(e_{n+1}) = e_n$). For $n > t$ let

$$\pi_{n,t} := \pi_{t+1} \circ \cdots \circ \pi_{n+1} \circ \pi_n \colon N(n) \to N(t),$$

and for $t < m$ set

$$\iota_{t,m} := \iota_{m-1} \circ \cdots \circ \iota_{t+1} \circ \iota_t \colon N(t) \to N(m).$$

Finally, let $\pi_{n,n} = \iota_{n,n} = 1_{N(n)}$.

Lemma 11.12. *For $m, n \geq 1$ the following hold:*

(i) *Every injective homomorphism $N(n) \to N(n+1)$ is irreducible (in $\mathcal{M}$);*
(ii) *Every surjective homomorphism $N(n+1) \to N(n)$ is irreducible (in $\mathcal{M}$).*
(iii) *If $f\colon N(n) \to N(m)$ is irreducible (in $\mathcal{M}$), then either $m = n+1$ or $n = m+1$, and f is either injective or surjective.*

Proof. Let $h\colon N(n) \to N(n+1)$ be an injective homomorphism. Clearly, h is neither a split monomorphism nor a split epimorphism. Let $h = gf$ where $f\colon N(n) \to N(\lambda)$ and $g\colon N(\lambda) \to N(n+1)$ are homomorphisms with $\lambda = (\lambda_1, \ldots, \lambda_t)$ a partition. (Recall that the isomorphism classes of objects in $\mathcal{M}$ are parametrized by partitions of natural numbers.) Thus

$$f = {}^t[f_1, \ldots, f_t]\colon N(n) \to \bigoplus_{i=1}^{t} N(\lambda_i)$$

and

$$g = [g_1, \ldots, g_t]\colon \bigoplus_{i=1}^{t} N(\lambda_i) \to N(n+1)$$

with $f_i\colon N(n) \to N(\lambda_i)$ and $g_i\colon N(\lambda_i) \to N(n+1)$ homomorphisms and

$$h = gf = \sum_{i=1}^{t} g_i f_i.$$

Since h is injective, we have $h(e_1) \neq 0$. Thus there exists some i with $g_i f_i(e_1) \neq 0$. This implies that $g_i f_i$ is injective, and therefore f_i is injective.

If $\lambda_i > n+1$, then $g_i(e_1) = 0$, a contradiction. (Note that $g_i f_i(e_1) \neq 0$ implies $g_i(e_1) \neq 0$).) Thus λ_i is either n or $n+1$. If $\lambda_i = n$, then f_i is an isomorphism, if $\lambda_i = n+1$, then g_i is an isomorphism. In the first case, set

$$f' = [0, \ldots, 0, f_i^{-1}, 0, \ldots, 0]\colon N(\lambda) \to N(n).$$

We get $f'f = 1_{N(n)}$, thus f is a split monomorphism.

In the second case, set

$$g' = {}^t[0, \ldots, 0, g_i^{-1}, 0, \ldots, 0]\colon N(n+1) \to N(\lambda).$$

It follows that $gg' = 1_{N(n+1)}$, so g is a split epimorphism. This proves part (i). Part (ii) is proved similarly.

Next, let $f\colon N(n) \to N(m)$ be an irreducible homomorphism. We proved already before that every irreducible homomorphism has to be either injective or surjective. If $m \geq n+2$, then f factors through $N(n+1)$ as $f = f_2 f_1$ where f_1 is injective but not split, and f_2 is not surjective, a contradiction. Similarly, if $m \leq n-2$, then f factors through $N(n-1)$ as $f = f_2 f_1$ where f_1 is not injective, and f_2 is surjective but not split, again a contradiction. This proves (iii). □

Lemma 11.13. *For $m, n \geq 1$ the following hold:*

- *Every non-invertible homomorphism $N(n) \to N(m)$ in $\mathcal{M}$ is a linear combination of compositions of irreducible homomorphisms;*
- *Every endomorphism $N(n) \to N(n)$ in $\mathcal{M}$ is a linear combination of $1_{N(n)}$ and of compositions of irreducible homomorphisms.*

Proof. Let $f\colon N(n) \to N(m)$ be a homomorphism with

$$f(e_n) = \sum_{i=1}^{t} a_i e_i$$

with $a_t \neq 0$. It follows that $n \geq t$ and $m \geq t$, and dim Im$(f) = t$. Let

$$g = \iota_{t,m} \circ \pi_{n,t}\colon N(n) \to N(m).$$

Now it is easy to check that

$$\dim \operatorname{Im}(f - a_t g) \leq t - 1.$$

We see that $f - a_t g$ is not an isomorphism, thus by induction assumption it is a linear combination of compositions of irreducible homomorphisms in $\mathcal{M}$. Also, g is either $1_{N(n)}$ (in case $n = m$) or it is a composition of irreducible homomorphisms. Thus $f = a_t g + (f - a_t g)$ is of the required form. $\square$

Thus we determined all irreducible homomorphisms between indecomposable modules in $\mathcal{M}$. So we know how the 1-arrows of the Auslander-Reiten quiver of $\mathcal{M}$ look like. We still have to determine the Auslander-Reiten sequences in $\mathcal{M}$ in order to get the 2-arrows as well.

11.6. **Exercises.** Use the Converse Bottleneck Lemma to show that for $n \geq 1$ the short exact sequence

$$0 \to N(n) \xrightarrow{\left[\begin{smallmatrix}\iota_n \\ \pi_n\end{smallmatrix}\right]} N(n+1) \oplus N(n-1) \xrightarrow{[\pi_{n+1}, -\iota_{n-1}]} N(n) \to 0$$

is an Auslander-Reiten sequence in $\mathcal{N}^{\text{f.d.}}$. (We set $N(0) = 0$.)

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Part 3. **Modules of finite length**

This part is completely independent of Part 2, one only needs the terminology and the results from Part 1.

12. **Filtrations of modules**

12.1. **Schreier's Theorem.** Let V be a module and let $U_0, \dots, U_s$ be submodules of V such that

$$0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_s = V.$$

This is called a **filtration** of V with **factors** U_i/U_{i-1}. The **length** of this filtration is

$$|\{1 \leq i \leq s \mid U_i/U_{i-1} \neq 0\}|.$$

A filtration

$$0 = U_0' \subseteq U_1' \subseteq \cdots \subseteq U_t' = V$$

is a **refinement** of the filtration above if

$$\{U_i \mid 0 \leq i \leq s\} \subseteq \{U_j' \mid 0 \leq j \leq t\}.$$

Two filtrations $U_0 \subseteq U_1 \subseteq \cdots \subseteq U_s$ and $V_0 \subseteq V_1 \subseteq \cdots \subseteq V_t$ of V are called **isomorphic** if $s = t$ and there exists a bijection $\pi\colon [0,s] \to [0,t]$ (where for integers i and j we write $[i,j] = \{k \in \mathbb{Z} \mid i \leq k \leq j\}$) such that

$$U_i/U_{i-1} \cong V_{\pi(i)}/V_{\pi(i)-1}$$

for $1 \leq i \leq s$.

Theorem 12.1 (Schreier)**.** *Any two given filtrations of a module V have isomorphic refinements.*

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Before we prove this theorem, we need the following lemma:

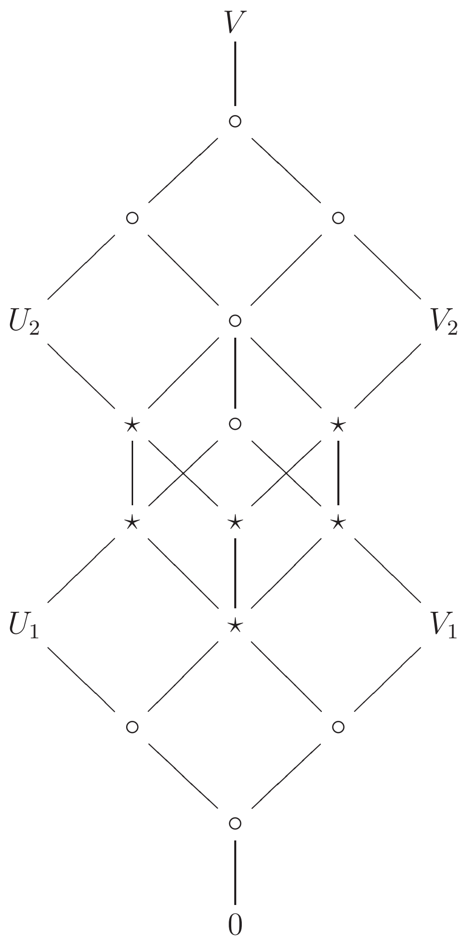
Lemma 12.2 (Butterfly Lemma)**.** *Let $U_1 \subseteq U_2$ and $V_1 \subseteq V_2$ be submodules of a module V. Then we have*

$$\begin{aligned}(U_1 + V_2 \cap U_2)/(U_1 + V_1 \cap U_2) &\cong (U_2 \cap V_2)/((U_1 \cap V_2) + (U_2 \cap V_1)) \\ &\cong (V_1 + U_2 \cap V_2)/(V_1 + U_1 \cap V_2).\end{aligned}$$

The name "butterfly" comes from the picture

$$\begin{tikzcd}[arrows=dash]
U_1 + V_2 \cap U_2 \arrow[d] \arrow[drr] & & & & V_1 + U_2 \cap V_2 \arrow[d] \arrow[dll] \\
U_1 + V_1 \cap U_2 \arrow[drr] & & U_2 \cap V_2 \arrow[d] & & V_1 + U_1 \cap V_2 \arrow[dll] \\
& & (U_1 \cap V_2) + (U_2 \cap V_1) & &
\end{tikzcd}$$

which occurs as the part marked with $\star$ of the picture



Note that $V_1 + U_2 \cap V_2 = (V_1 + U_2) \cap V_2 = V_1 + (U_2 \cap V_2)$, since $V_1 \subseteq V_2$. But we have $(V_2 + U_2) \cap V_1 = V_1$ and $V_2 + (U_2 \cap V_1) = V_2$. Thus the expression $V_2 + U_2 \cap V_1$ would not make any sense.

Proof of Lemma 12.2. Note that $U_1 \cap V_2 \subseteq U_2 \cap V_2$. Recall that for submodules U and U' of a module V we always have

$$U/(U \cap U') \cong (U + U')/U'.$$

$$\begin{tikzcd} & U+U' \arrow[dl, dash] \arrow[dr, dash] & \\ U \arrow[dr, dash] & & U' \arrow[dl, dash] \\ & U \cap U' & \end{tikzcd}$$

Since $U_1 \subseteq U_2$ and $V_1 \subseteq V_2$ we get

$$\begin{aligned}(U_1 + V_1 \cap U_2) + (U_2 \cap V_2) &= U_1 + (V_1 \cap U_2) + (U_2 \cap V_2)\\ &= U_1 + (U_2 \cap V_2)\\ &= U_1 + V_2 \cap U_2\end{aligned}$$

and

$$\begin{aligned}(U_1 + V_1 \cap U_2) \cap (U_2 \cap V_2) &= (U_1 + V_1) \cap U_2 \cap (U_2 \cap V_2)\\ &= (U_1 + V_1 \cap U_2) \cap V_2\\ &= ((V_1 \cap U_2) + U_1) \cap V_2\\ &= (V_1 \cap U_2) + (U_1 \cap V_2).\end{aligned}$$

The result follows. □

Proof of Theorem 12.1. Assume we have two filtrations

$$0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_s = V$$

and

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_t = V$$

of a module V. For $1 \leq i \leq s$ and $0 \leq j \leq t$ define

$$U_{ij} = U_{i-1} + V_j \cap U_i.$$

Thus we obtain

$$\begin{aligned}0 = U_{10} \subseteq U_{11} \subseteq \cdots \subseteq U_{1t} &= U_1\\ U_1 = U_{20} \subseteq U_{21} \subseteq \cdots \subseteq U_{2t} &= U_2\\ \vdots \qquad\qquad\qquad\qquad & \vdots\\ U_{s-1} = U_{s0} \subseteq U_{s1} \subseteq \cdots \subseteq U_{st} &= U_s = V.\end{aligned}$$

Similarly, set

$$V_{ji} = V_{j-1} + U_i \cap V_j.$$

This yields

$$\begin{aligned}0 = V_{10} \subseteq V_{11} \subseteq \cdots \subseteq V_{1s} &= V_1\\ V_1 = V_{20} \subseteq V_{21} \subseteq \cdots \subseteq V_{2s} &= V_2\\ \vdots \qquad\qquad\qquad\qquad & \vdots\\ V_{t-1} = V_{t0} \subseteq V_{t1} \subseteq \cdots \subseteq V_{ts} &= V_t = V.\end{aligned}$$

For $1 \leq i \leq s$ and $1 \leq j \leq t$ define

$$F_{ij} = U_{ij}/U_{i,j-1} \quad \text{and} \quad G_{ji} = V_{ji}/V_{j,i-1}.$$

The filtration $(U_{ij})_{ij}$ is a refinement of the filtration $(U_i)_i$ and its factors are the modules F_{ij}. Similarly, the filtration $(V_{ji})_{ji}$ is a refinement of $(V_j)_j$ and has factors

G_{ji}. Now the Butterfly Lemma 12.2 implies $F_{ij} \cong G_{ji}$, namely

$$\begin{aligned} F_{ij} &= U_{ij}/U_{i,j-1} \\ &= (U_{i-1} + V_j \cap U_i)/(U_{i-1} + V_{j-1} \cap U_i) \\ &\cong (V_{j-1} + U_i \cap V_j)/(V_{j-1} + U_{i-1} \cap V_j) \\ &= G_{ji}. \end{aligned}$$

This finishes the proof. □

A filtration

$$0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_s = V.$$

of a module V with all factors U_i/U_{i-1} ($1 \leq i \leq s$) being simple is called a **composition series** of V. In this case we call s (i.e. the number of simple factors) the **length** of the composition series. We call the U_i/U_{i-1} the **composition factors** of V.

12.2. **The Jordan-Hölder Theorem.** As an important corollary of Theorem 12.1 we obtain the following:

Corollary 12.3 (Jordan-Hölder Theorem)**.** *Assume that a module V has a composition series of length l. Then the following hold:*

- *Any filtration of V has length at most l and can be refined to a composition series;*
- *All composition series of V have length l.*

Proof. Let

$$0 = U_0 \subset U_1 \subset \cdots \subset U_l = V$$

be a composition series, and let

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_t = V$$

be a filtration. By Schreier's Theorem 12.1 there exist isomorphic refinements of these filtrations. Let $F_i = U_i/U_{i-1}$ be the factors of the filtration $(U_i)_i$. Thus F_i is simple. If $(U_i')_i$ is a refinement of $(U_i)_i$, then its factors are $F_1, \ldots, F_l$ together with some 0-modules. The corresponding refinement of $(V_j)_j$ has exactly $l+1$ submodules. Thus $(V_j)_j$ has at most l different non-zero factors. In particular, if $(V_j)_j$ is already a composition series, then $t = l$. □

If V has a composition series of length l, then we say V has **length** l, and we write $l(V) = l$. Otherwise, V has **infinite length** and we write $l(V) = \infty$.

Assume $l(V) < \infty$ and let S be a simple module. Then $[V : S]$ is the number of composition factors in a (and thus in all) composition series of V which are isomorphic to S. One calls $[V : S]$ the **Jordan-Hölder multiplicity** of S in V.

Let $l(V) < \infty$. Then $([V : S])_{S \in \mathcal{S}}$ is called the **dimension vector** of V, where $\mathcal{S}$ is a complete set of representatives of isomorphism classes of the simple modules. Note that only finitely many entries of the dimension vector are non-zero. We get

$$\sum_{S \in \mathcal{S}} [V : S] = l(V).$$

Example: If $J = \emptyset$, then a J-module is just given by a vector space V. In this case $l(V) = \dim V$ if V is finite-dimensional. It also follows that V is simple if and only if $\dim V = 1$. If V is infinite-dimensional, then $\dim V$ is a cardinality and we usually write $l(V) = \infty$.

For modules of finite length, the Jordan-Hölder multiplicities and the length are important invariants.

12.3. **Exercises. 1**: Let V be a module of finite length, and let $V_1, \ldots, V_t$ be submodules of V. Show: If

$$l\left(\sum_{i=1}^{t} V_i\right) = l(V),$$

then $V = \bigoplus_{i=1}^{t} V_i$.

2: Let V_1 and V_2 be modules, and let S be a factor of a filtration of $V_1 \oplus V_2$. Show: If S is simple, then there exists a filtration of V_1 or of V_2 which contains a factor isomorphic to S.

3: Construct indecomposable modules V_1 and V_2 with $l(V_1) = l(V_2) = 2$, and a filtration of $V_1 \oplus V_2$ containing a factor T of length 2 such that T is not isomorphic to V_1 or V_2.

4: Determine all composition series of the 2-module $V = (K^5, \phi, \psi)$ where

$$\phi = \begin{bmatrix} c_0 & & & & \\ & c_0 & & & \\ & & c_1 & & \\ & & & c_2 & \\ & & & & c_3 \end{bmatrix} \quad \text{and} \quad \psi = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ & 0 & 0 & 1 & 1 \\ & & 0 & & \\ & & & 0 & \\ & & & & 0 \end{bmatrix}$$

with pairwise different elements c_0, c_1, c_2, c_3 in K.

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13. **Digression: Local rings**

We need some basic notations from ring theory. This might seem a bit boring but will be of great use later on.

13.1. **Local rings.** Let R be a ring. Then $r \in R$ is **right-invertible** if there exists some $r' \in R$ such that $rr' = 1$, and r is **left-invertible** if there exists some $r'' \in R$ such that $r''r = 1$. We call r' a **right inverse** and r'' a **left inverse** of r. If r is both right- and left-invertible, then r is **invertible**.

Example: Let V be a vector space with basis $\{e_i \mid i \geq 1\}$. Define a linear map $f\colon V \to V$ by $f(e_i) = e_{i+1}$ for all i, and a linear map $g\colon V \to V$ by $g(e_1) = 0$ and $g(e_i) = e_{i-1}$ for all $i \geq 2$. Then we have $gf = 1_V$, thus f is left-invertible and g is right-invertible. Note also that $fg \neq 1_V$, since for example $fg(e_1) = 0$.

Lemma 13.1. *If r' is a right inverse and r'' a left inverse of an element r, then $r' = r''$. In particular, there is only one right inverse and only one left inverse.*

Proof. We have $r' = 1r' = r''rr' = r''1 = r''$. □

Lemma 13.2. *Assume that r is right-invertible. Then the following are equivalent:*

- *r is left-invertible;*
- *There exists only one right inverse of r.*

Proof. Assume that r is right-invertible, but not left-invertible. Then $rr' = 1$ and $r'r \neq 1$ for some r'. This implies

$$r(r' + r'r - 1) = rr' + rr'r - r = 1.$$

But $r' + r'r - 1 \neq r'$. □

An element r in a ring R is **nilpotent** if $r^n = 0$ for some $n \geq 1$.

Lemma 13.3. *Let r be a nilpotent element in a ring R, then $1 - r$ is invertible.*

Proof. We have $(1-r)(1+r+r^2+r^3+\cdots) = 1$. (Note that this sum is finite, since r is nilpotent.) One also easily checks that $(1+r+r^2+r^3+\cdots)(1-r) = 1$. Thus $(1-r)$ is right-invertible and left-invertible and therefore invertible. □

A ring R is **local** if the following hold:

- $1 \neq 0$;
- If $r \in R$, then r or $1 - r$ is invertible.

(Recall that the only ring with $1 = 0$ is the 0-ring, which contains just one element. Note that we do not exclude that for some elements $r \in R$ both r and $1 - r$ are invertible.)

Local rings occur in many different contexts. For example, they are important in Algebraic Geometry: One studies the local ring associated to a point x of a curve (or more generally of a variety or a scheme) and hopes to get a "local description", i.e. a description of the curve in a small neighbourhood of the point x.

Examples: $K[T]$ is not local (T is not invertible, and $1-T$ is also not invertible), $\mathbb{Z}$ is not local, every field is a local ring.

Let U be an additive subgroup of a ring R. Then U is a **right ideal** of R if for all $u \in U$ and all $r \in R$ we have $ur \in U$, and U is a **left ideal** if for all $u \in U$ and all $r \in R$ we have $ru \in U$. One calls U an **ideal** if it is a right and a left ideal.

If I and J are ideals of a ring R, then the **product** IJ is the additive subgroup of R generated by all (finite) sums of the form $\sum_s i_s j_s$ where $i_s \in I$ and $j_s \in J$. It is easy to check that IJ is again an ideal. For $n \geq 0$, define $I^0 = R$, $I^1 = I$ and $I^{n+2} = I(I^{n+1}) = (I^{n+1})I$.

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A left ideal U is a **maximal left ideal** if it is maximal in the set of all proper left ideals, i.e. if $U \subset R$ and for every left ideal U' with $U \subseteq U' \subset R$ we have $U = U'$. Similarly, define a **maximal right ideal**.

Recall that an element $e \in R$ is an **idempotent** if $e^2 = e$.

Lemma 13.4. *Let $e \in R$ be an idempotent. If e is left-invertible or right-invertible, then $e = 1$*

Proof. If e is left-invertible, then $re = 1$ for some $r \in R$. Also $e = 1e = (re)e = re = 1$. The other case is done similarly. □

Lemma 13.5. *Assume that R is a ring which has only 0 and 1 as idempotents. Then all left-invertible and all right-invertible elements are invertible.*

Proof. Let r be left-invertible, say $r'r = 1$. Then rr' is an idempotent, which by our assumption is either 0 or 1. If $rr' = 1$, then r is right-invertible and therefore invertible. If $rr' = 0$, then $1 = r'r = r'rr'r = 0$, thus $R = 0$. The only element $0 = 1$ in $R = 0$ is invertible. The other case is done similarly. □

Proposition 13.6. *The following properties of a ring R are equivalent:*

(i) *We have $0 \neq 1$, and if $r \in R$, then r or $1-r$ is invertible (i.e. R is a local ring);*
(ii) *There exist non-invertible elements in R, and the set of these elements is closed under $+$;*
(iii) *The set of non-invertible elements in R form an ideal;*
(iv) *R contains a proper left ideal, which contains all proper left ideals;*
(v) *R contains a proper right ideal which contains all proper right ideals.*

Remark: Property (iv) implies that R contains exactly one maximal left ideal. Using the Axiom of Choice, the converse is also true.

Proof. We first show that under the assumptions (i), (ii) and (iv) the elements 0 and 1 are the only idempotents in R, and therefore every left-invertible element and every right-invertible element will be invertible.

Let $e \in R$ be an idempotent. Then $1-e$ is also an idempotent. It is enough to show that e or $1-e$ are invertible: If e is invertible, then $e = 1$. If $1-e$ is invertible, then $1-e = 1$ and therefore $e = 0$.

Under (i) we assume that either e or $1-e$ are invertible, and we are done. Also under (ii) we know that e or $1-e$ is invertible: If e and $1-e$ are both non-invertible, then $1 = e + (1-e)$ is non-invertible, a contradiction. Finally, assume that under (iv) we have a proper left ideal I containing all proper left ideals, and assume that e and $1-e$ are both non-invertible. We claim that both elements and therefore also their sum have to belong to I, a contradiction, since 1 cannot be in I. Why does $e \in I$ hold? Since e is non-invertible, we know that e is not left-invertible. Therefore Re is a proper left ideal, which must be contained in I. Since $1-e$ is also non-invertible, we know that $1-e \in I$.

(i) $\implies$ (ii): $0 \neq 1$ implies that 0 is not invertible. Assume r_1, r_2 are not invertible. Assume also that $r_1 + r_2$ is invertible. Thus $x(r_1 + r_2) = 1$ for some $x \in R$. We get $xr_1 = 1 - xr_2$. Now (i) implies that xr_2 or $1 - xr_2$ is invertible. Without loss of generality let xr_1 be invertible. Thus there exists some y such that $1 = yxr_1$. This implies that r_1 is left-invertible and therefore invertible, a contradiction.

(ii) $\implies$ (i): The existence of non-invertible elements implies $R \neq 0$, and therefore we have $0 \neq 1$. Let $r \in R$. If r and $1-r$ are non-invertible, then by (ii) we get that $1 = r + (1-r)$ is non-invertible, a contradiction.

(ii) $\implies$ (iii): Let I be the set of non-invertible elements in R. Then by (ii) we know that I is a subgroup of $(R, +)$. Given $x \in I$ and $r \in R$ we have to show that $rx \in I$ and $xr \in I$. Assume rx is invertible. Then there is some y with $yrx = 1$, thus x is left-invertible and therefore x is invertible, a contradiction. Thus $rx \in I$. Similarly, we can show that $xr \in I$.

(iii) $\implies$ (iv): Let I be the set of non-invertible elements in R. By (iii) we get that I is an ideal and therefore a left ideal. Since $1 \notin I$ we get $I \subset R$. Let $U \subset R$ be a proper left ideal. Claim: $U \subseteq I$. Let $x \in U$, and assume $x \notin I$. Thus x is invertible. So there is some $y \in R$ such that $yx = 1$. Then for $r \in R$ we have $r = r1 = (ry)x \in U$. Thus $R \subseteq U$ which implies $U = R$, a contradiction. Similarly we prove (iii) $\implies$ (v).

(iv) $\implies$ (i): Let I' be a proper left ideal of R that contains all proper left ideals. We show that all elements in $R \setminus I'$ are invertible: Let $r \notin I'$. Then Rr is a left ideal of R which is not contained in I', thus we get $Rr = R$. So there is some $r' \in R$ such that $r'r = 1$, in other words, r is left-invertible and therefore invertible. Now let $r \in R$ be arbitrary. We claim that r or $1-r$ belong to $R \setminus I'$: If both elements belong to I', then so does $1 = r + (1-r)$, a contradiction. Thus either r or $1-r$ is invertible. Similarly we prove (v) $\implies$ (i). $\square$

If R is a local ring, then

$$I := \{r \in R \mid r \text{ non-invertible } \}$$

is called the **radical** (or **Jacobson radical**) of R.

Corollary 13.7. *The Jacobson radical I of a local ring R is the only maximal left ideal and also the only maximal right ideal of R. It contains all proper left and all proper right ideals of R.*

Proof. Part (iii) of the above proposition tells us that I is indeed an ideal in R. Assume now $I \subset I' \subseteq R$ with I' a left (resp. right) ideal. Take $r \in I' \setminus I$. Then r is invertible. Thus there exists some r' such that $r'r = rr' = 1$. This implies $I' = R$. So we proved that I' is a maximal left and also a maximal right ideal. Furthermore, the proof of (iii) $\implies$ (iv) in the above proposition shows that I contains all proper left ideals, and similarly one shows that I contains all proper right ideals of R. $\square$

If I is the Jacobson radical of a local ring R, then the **radical factor ring** R/I is a ring without left ideals different from 0 and R/I. It is easy to check that R/I is a skew field. (For $\overline{r} \in R/I$ with $\overline{r} \neq 0$ and $r \in R \setminus I$ there is some $s \in R$ such that $sr = 1 = rs$. In R/I we have $\overline{s} \cdot \overline{r} = \overline{sr} = \overline{1} = \overline{rs} = \overline{r} \cdot \overline{s}$.)

Example: For $c \in K$ set

$$R = \{f/g \mid f, g \in K[T], g(c) \neq 0\},$$
$$\mathfrak{m} = \{f/g \in R \mid f(c) = 0, g(c) \neq 0\}.$$

Then $\mathfrak{m}$ is an ideal in the ring R. In fact, $\mathfrak{m} = (T - c)R$: One inclusion is obtained since

$$(T - c)\frac{f}{g} = \frac{(T - c)f}{g},$$

and the other inclusion follows since $f(c) = 0$ implies $f = (T-c)h$ for some $h \in K[T]$ and therefore

$$\frac{f}{g} = (T - c)\frac{h}{g}.$$

If $r \in R \setminus \mathfrak{m}$, then $r = f/g$ with $f(c) \neq 0$ and $g(c) \neq 0$. Thus $r^{-1} = g/f \in R$ is an inverse of r.

If $r = f/g \in \mathfrak{m}$, then r is not invertible: For any $f'/g' \in R$, the product $f/g \cdot f'/g' = ff'/gg'$ always lies in $\mathfrak{m}$, since $(ff')(c) = f(c)f'(c) = 0$ and thus it cannot be the identity and therefore r is not invertible.

Thus we proved that $R \setminus \mathfrak{m}$ is exactly the set of invertible elements in R, and the set $\mathfrak{m}$ of non-invertible elements forms an ideal. So by the above theorem, R is a local ring.

13.2. **Exercises. 1**: A module V is called **local** if it contains a maximal submodule U, which contains all proper submodules of V. Show: If V is local, then V contains exactly one maximal submodule. Construct an example which shows that the converse is not true.

2: Show: Every module of finite length is a sum of local submodules.

3: Let V be a module of length n. Show: V is semisimple if and only if V cannot be written as a sum of $n-1$ local submodules.

4: Let $R = K[X, Y]$ be the polynomial ring in two (commuting) variables. Show:

- R is not local;
- 0 and 1 are the only idempotents in R;
- The Jacobson radical of R is 0.

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14. **Modules of finite length**

14.1. **Some length formulas for modules of finite length.**

Lemma 14.1. *Let U be a submodule of a module V and let $W = V/U$ be the corresponding factor module. Then V has finite length if and only if U and W have finite length. In this case, we get*

$$l(V) = l(U) + l(W)$$

and for every simple module S we have

$$[V : S] = [U : S] + [W : S].$$

Proof. Assume that V has length n. Thus every chain of submodules of V has length at most n. In particular this is true for all chains of submodules of U. This implies $l(U) \leq n$. The same holds for chains of submodules which all contain U. Such chains correspond under the projection homomorphism $V \to V/U$ to the chains of submodules of $V/U = W$. Thus we get $l(W) \leq n$. So if V has finite length, then so do U and W.

Vice versa, assume that U and $W = V/U$ have finite length. Let

$$0 = U_0 \subset U_1 \subset \cdots \subset U_s = U \quad \text{and} \quad 0 = W_0 \subset W_1 \subset \cdots \subset W_t = W$$

be composition series of U and W, respectively. We can write W_j in the form $W_j = V_j/U$ for some submodule $U \subseteq V_j \subseteq V$. We obtain a chain

$$0 = U_0 \subset U_1 \subset \cdots \subset U_s = U = V_0 \subset V_1 \subset \cdots \subset V_t = V$$

of submodules of V such that

$$V_j/V_{j-1} \cong W_j/W_{j-1}$$

for all $1 \leq j \leq t$. This chain is a composition series of V, since the factors U_i/U_{i-1} with $1 \leq i \leq s$, and the factors V_j/V_{j-1} with $1 \leq j \leq t$ are simple. If S is simple, then the number of composition factors of V which are isomorphic to S is equal to the number of indices i with $U_i/U_{i-1} \cong S$ plus the number of indices j with $V_j/V_{j-1} \cong S$. In other words, $[V : S] = [U : S] + [W : S]$. □

Corollary 14.2. *Let V be a module of finite length. If $0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_t = V$ is a filtration of V, then*

$$l(V) = \sum_{i=1}^{t} l(U_i/U_{i-1}).$$

Corollary 14.3. *Let U_1 and U_2 be modules of finite length. Then*

$$l(U_1 \oplus U_2) = l(U_1) + l(U_2).$$

Corollary 14.4. *Let V be a module of finite length, and let U_1 and U_2 be submodules of V. Then*

$$l(U_1) + l(U_2) = l(U_1 + U_2) + l(U_1 \cap U_2).$$

Proof. Set $U' := U_1 \cap U_2$ and $U'' := U_1 + U_2$.

$$\begin{tikzcd} & U'' \arrow[dl, dash] \arrow[dr, dash] & \\ U_1 \arrow[dr, dash] & & U_2 \arrow[dl, dash] \\ & U' & \end{tikzcd}$$

Then

$$U''/U' \cong (U_1/U') \oplus (U_2/U').$$

Thus

$$\begin{aligned} l(U_1) &= l(U') + l(U_1/U'), \\ l(U_2) &= l(U') + l(U_2/U'), \\ l(U'') &= l(U') + l(U_1/U') + l(U_2/U'). \end{aligned}$$

This yields the result. □

Corollary 14.5. *Let V and W be modules and let $f : V \to W$ be a homomorphism. If V has finite length, then*

$$l(V) = l(\operatorname{Ker}(f)) + l(\operatorname{Im}(f)).$$

If W has finite length, then

$$l(W) = l(\operatorname{Im}(f)) + l(\operatorname{Cok}(f)).$$

Proof. Use the isomorphisms $V/\operatorname{Ker}(f) \cong \operatorname{Im}(f)$ and $W/\operatorname{Im}(f) \cong \operatorname{Cok}(f)$. □

Recall that for every homomorphism $f\colon V \to W$ there are short exact sequences

$$0 \to \mathrm{Ker}(f) \to V \to \mathrm{Im}(f) \to 0$$

and

$$0 \to \mathrm{Im}(f) \to W \to \mathrm{Cok}(f) \to 0.$$

Corollary 14.6. *For every short exact sequence*

$$0 \to U \to V \to W \to 0$$

of modules with $l(V) < \infty$, we have $l(V) = l(U) + l(W)$.

Corollary 14.7. *Let V be a module of finite length, and let $f\colon V \to V$ be an endomorphism of V. Then the following statements are equivalent:*

(i) *f is injective;*
(ii) *f is surjective;*
(iii) *f is an isomorphism;*
(iv) *$l(\mathrm{Im}(f)) = l(V)$.*

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Lemma 14.8. *If V is a module of finite length, then V is a finite direct sum of indecomposable modules.*

Proof. This is proved by induction on $l(V)$. The statement is trivial if V is indecomposable. Otherwise, let $V = V_1 \oplus V_2$ with V_1 and V_2 two non-zero submodules. Then proceed by induction. □

Recall that in Section 6.2 we studied the radical $\mathrm{rad}(V)$ of a module V. The following lemma shows that $V/\mathrm{rad}(V)$ is well behaved if V is of finite length:

Lemma 14.9. *Let V be a module of finite length. Then $V/\mathrm{rad}(V)$ is semisimple.*

Proof. Assume that $l(V/\mathrm{rad}(V)) = n$. Inductively we look for maximal submodules $U_1, \ldots, U_n$ of V such that for $1 \le t \le n$ and $V_t := \bigcap_{i=1}^t U_i$ we have

$$V/V_t \cong \bigoplus_{i=1}^{t} V/U_i$$

and $l(V/V_t) = t$. Note that V/U_i is simple for all i.

For $t = 1$ there is nothing to show. If $U_1, \ldots, U_t$ are already constructed and if $t < n$, then $\mathrm{rad}(V) \subset V_t$. Thus there exists a maximal submodule U_{t+1} with $V_t \cap U_{t+1} \subset V_t$. Since $V_t \not\subseteq U_{t+1}$, we know that $U_{t+1} \subset V_t + U_{t+1}$. The maximality of U_{t+1} implies that $V_t + U_{t+1} = V$. Set $V_{t+1} := V_t \cap U_{t+1}$.

$$\begin{tikzcd}[column sep=small] & V \arrow[dl, no head] \arrow[dr, no head] & \\ V_t \arrow[dr, no head] & & U_{t+1} \arrow[dl, no head] \\ & V_{t+1} & \end{tikzcd}$$

Thus we obtain

$$\begin{aligned} V/V_{t+1} &= V_t/V_{t+1} \oplus U_{t+1}/V_{t+1} \\ &\cong V/U_{t+1} \oplus V/V_t \\ &\cong V/U_{t+1} \oplus \bigoplus_{i=1}^{t} V/U_i. \end{aligned}$$

The last of these isomorphisms comes from the induction assumption. □

14.2. **The Fitting Lemma.**

Lemma 14.10 (Fitting)**.** *Let V be a module of finite length, say $l(V) = n$, and let $f \in \mathrm{End}(V)$. Then we have*

$$V = \mathrm{Im}(f^n) \oplus \mathrm{Ker}(f^n).$$

In particular, if V is indecomposable, then $\mathrm{Im}(f^n) = 0$ or $\mathrm{Ker}(f^n) = 0$.

Proof. We have

$$0 = \mathrm{Ker}(f^0) \subseteq \mathrm{Ker}(f^1) \subseteq \mathrm{Ker}(f^2) \subseteq \cdots .$$

(For $x \in \mathrm{Ker}(f^i)$ we get $f^i(x) = 0$ and therefore $f^{i+1}(x) = 0$.)

Assume that $\mathrm{Ker}(f^{i-1}) = \mathrm{Ker}(f^i)$ for some i. It follows that $\mathrm{Ker}(f^i) = \mathrm{Ker}(f^{i+1})$. (Assume $f^{i+1}(x) = 0$. Then $f^i(f(x)) = 0$ and therefore $f(x) \in \mathrm{Ker}(f^i) = \mathrm{Ker}(f^{i-1})$. This implies $f^i(x) = f^{i-1}(f(x)) = 0$. Thus $\mathrm{Ker}(f^{i+1}) \subseteq \mathrm{Ker}(f^i)$.)

If

$$0 = \mathrm{Ker}(f^0) \subset \mathrm{Ker}(f^1) \subset \cdots \subset \mathrm{Ker}(f^i),$$

then $l(\mathrm{Ker}(f^i)) \geq i$. This implies $i \leq n$, and therefore $\mathrm{Ker}(f^m) = \mathrm{Ker}(f^n)$ for all $m \geq n$.

We have

$$\cdots \subseteq \mathrm{Im}(f^2) \subseteq \mathrm{Im}(f) \subseteq \mathrm{Im}(f^0) = V.$$

(For $x \in \mathrm{Im}(f^i)$ we get $x = f^i(y) = f^{i-1}(f(y))$ for some $y \in V$. Thus $x \in \mathrm{Im}(f^{i-1})$.)

Assume that $\mathrm{Im}(f^{i-1}) = \mathrm{Im}(f^i)$. Then $\mathrm{Im}(f^i) = \mathrm{Im}(f^{i+1})$. (For every $y \in V$ there exists some z with $f^{i-1}(y) = f^i(z)$. This implies $f^i(y) = f^{i+1}(z)$. Thus $\mathrm{Im}(f^i) \subseteq \mathrm{Im}(f^{i+1})$.)

If

$$\mathrm{Im}(f^i) \subset \cdots \subset \mathrm{Im}(f^1) \subset \mathrm{Im}(f^0) = V,$$

then $l(\mathrm{Im}(f^i)) \leq n - i$, which implies $i \leq n$. Thus $\mathrm{Im}(f^m) = \mathrm{Im}(f^n)$ for all $m \geq n$.

So we proved that

$$\mathrm{Ker}(f^n) = \mathrm{Ker}(f^{2n}) \quad \text{and} \quad \mathrm{Im}(f^n) = \mathrm{Im}(f^{2n}).$$

We claim that $\mathrm{Im}(f^n) \cap \mathrm{Ker}(f^n) = 0$: Let $x \in \mathrm{Im}(f^n) \cap \mathrm{Ker}(f^n)$. Then $x = f^n(y)$ for some y and also $f^n(x) = 0$, which implies $f^{2n}(y) = 0$. Thus we get $y \in \mathrm{Ker}(f^{2n}) = \mathrm{Ker}(f^n)$ and $x = f^n(y) = 0$.

Next, we show that $\mathrm{Im}(f^n) + \mathrm{Ker}(f^n) = V$: Let $v \in V$. Then there is some w with $f^n(v) = f^{2n}(w)$. This is equivalent to $f^n(v - f^n(w)) = 0$. Thus $v - f^n(w) \in \mathrm{Ker}(f^n)$. Now $v = f^n(w) + (v - f^n(w))$. This finishes the proof. □

Corollary 14.11. *Let V be an indecomposable module of finite length n, and let $f \in \mathrm{End}(V)$. Then either f is an isomorphism, or f is nilpotent (i.e. $f^n = 0$).*

Proof. If $\mathrm{Im}(f^n) = 0$, then $f^n = 0$, in particular f is nilpotent. Now assume that $\mathrm{Ker}(f^n) = 0$. Then f^n is injective, which implies that f is injective ($f(x) = 0 \implies f^n(x) = 0 \implies x = 0$). Thus f is an isomorphism. □

Combining the above with Lemma 13.3 we obtain the following important result:

Corollary 14.12. *Let V be an indecomposable module of finite length. Then $\mathrm{End}(V)$ is a local ring.*

Let V be a module, and let $R = \mathrm{End}(V)$ be the endomorphism ring of V. Assume that $V = V_1 \oplus V_2$ be a direct decomposition of V. Then the map $e\colon V \to V$ defined by $e(v_1, v_2) = (v_1, 0)$ is an idempotent in $\mathrm{End}(V)$. Now $e = 1$ if and only if $V_2 = 0$, and $e = 0$ if and only if $V_1 = 0$.

It follows that the endomorphism ring of any decomposable module contains idempotent which are not 0 or 1.

Example: The 1-module $V = (K[T], T\cdot)$ is indecomposable, but its endomorphism ring $\mathrm{End}(V) \cong K[T]$ is not local.

Lemma 14.13. *Let V be a module. If $\mathrm{End}(V)$ is a local ring, then V is indecomposable.*

Proof. If a ring R is local, then its only idempotents are 0 and 1. Then the result follows from the discussion above. □

14.3. **The Harada-Sai Lemma.**

Lemma 14.14 (Harada-Sai)**.** *Let V_i be indecomposable modules of length at most n where $1 \leq i \leq m = 2^n$, and let $f_i\colon V_i \to V_{i+1}$ where $1 \leq i < m$ be homomorphisms. If $f_{m-1} \cdots f_2 f_1 \neq 0$, then at least one of the homomorphisms f_i is an isomorphism.*

Proof. We show this by induction on a: Let $a \leq n$. Let V_i, $1 \leq i \leq m = 2^a$ be modules of length at most n, and $f_i\colon V_i \to V_{i+1}$, $1 \leq i < m$ homomorphisms. If

$$l(\mathrm{Im}(f_{m-1} \cdots f_2 f_1)) > n - a,$$

then at least one of the homomorphisms is an isomorphism.

If $a = 1$, then there is just one homomorphism, namely $f_1\colon V_1 \to V_2$. If $l(\mathrm{Im}(f_1)) > n - 1$, then f_1 is an isomorphism. Remember that by our assumption both modules V_1 and V_2 have length at most n.

Assume the statement holds for $a < n$. Define $m = 2^a$. Let V_i be indecomposable modules with $1 \leq i \leq 2m$, and for $1 \leq i < 2m$ let $f_i\colon V_i \to V_{i+1}$ be homomorphisms. Let $f = f_{m-1} \cdots f_1$, $g = f_m$ and $h = f_{2m-1} \cdots f_{m+1}$. Thus

$$V_1 \xrightarrow{f} V_m \xrightarrow{g} V_{m+1} \xrightarrow{h} V_{2m}.$$

Assume $l(\mathrm{Im}(hgf)) > n - (a + 1)$. We can assume that $l(\mathrm{Im}(f)) \leq n - a$ and $l(\mathrm{Im}(h)) \leq n - a$, otherwise we know by induction that one of the homomorphisms f_i is an isomorphism.

Since

$$l(\mathrm{Im}(f)) \geq l(\mathrm{Im}(gf)) \geq l(\mathrm{Im}(hgf)) > n - (a + 1)$$

and

$$l(\mathrm{Im}(h)) \geq l(\mathrm{Im}(hgf)),$$

it follows that $l(\mathrm{Im}(f)) = n - a = l(\mathrm{Im}(h))$ and therefore $l(\mathrm{Im}(hgf)) = n - a$.

Since $\mathrm{Im}(f)$ and $\mathrm{Im}(hgf)$ have the same length, we get $\mathrm{Im}(f) \cap \mathrm{Ker}(hg) = 0$. Now $\mathrm{Im}(f)$ has length $n - a$, and $\mathrm{Ker}(hg)$ has length $l(V_m) - l(\mathrm{Im}(hg))$. This implies $l(\mathrm{Im}(hg)) = n - a$, because

$$l(\mathrm{Im}(hgf)) \leq l(\mathrm{Im}(hg)) \leq l(\mathrm{Im}(h)).$$

So we see that $\mathrm{Im}(f) + \mathrm{Ker}(hg) = V_m$. In this way, we obtained a direct decomposition

$$V_m = \mathrm{Im}(f) \oplus \mathrm{Ker}(hg).$$

But V_m is indecomposable, and $\mathrm{Im}(f) \neq 0$. It follows that $\mathrm{Ker}(hg) = 0$. In other words, hg is injective, and so g is injective.

In a similar way, we can show that g is also surjective: Namely

$$V_{m+1} = \mathrm{Im}(gf) \oplus \mathrm{Ker}(h):$$

Since $\mathrm{Im}(gf)$ and $\mathrm{Im}(hgf)$ have the same length, we get

$$\mathrm{Im}(gf) \cap \mathrm{Ker}(h) = 0.$$

On the other hand, the length of $\mathrm{Ker}(h)$ is

$$l(V_{m+1}) - l(\mathrm{Im}(h)) = l(V_{m+1}) - (n - a).$$

Since V_{m+1} is indecomposable, $\mathrm{Im}(gf) \neq 0$ implies $V_{m+1} = \mathrm{Im}(gf)$. Thus gf is surjective, which yields that g is surjective as well.

Thus we have shown that $g = f_m$ is an isomorphism. □

Corollary 14.15. *If V is an indecomposable module of finite length n, and if I denotes the radical of* $\mathrm{End}(V)$*, then $I^n = 0$.*

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Proof. Let S be a subset of $\mathrm{End}(V)$, and let SV be the set of all (finite) sums of the form $\sum_i f_i(v_i)$ with $f_i \in S$ and $v_i \in V$. This is a submodule of V. (It follows from the definition, that SV is closed under addition. Since all f_i are linear maps, SV is also closed under scalar multiplication. Finally, for $V = (V, \phi_j)_j$ we have $\phi_j(SV) \subseteq SV$, since

$$\phi_j\left(\sum_i f_i(v_i)\right) = \sum_i f_i(\phi_j(v_i)),$$

because all the f_i are homomorphisms.)

For $i \geq 0$ we can look at the submodule I^iV of V. Thus

$$\cdots \subseteq I^2V \subseteq IV \subseteq I^0V = V.$$

If $I^{i-1}V = I^iV$, then $I^iV = I^{i+1}V$.

The Harada-Sai Lemma implies $I^m = 0$ for $m = 2^n - 1$, thus also $I^mV = 0$. Thus there exists some t with

$$0 = I^tV \subset \cdots \subset I^2V \subset IV \subset I^0V = V.$$

This is a filtration of the module V, and since V has length n, we conclude $t \leq n$. This implies $I^nV = 0$ and therefore $I^n = 0$. □

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14.4. **Exercises.** **1**: Find the original references for Schreier's Theorem, the Jordan-Hölder Theorem, the Fitting Lemma and the Harada-Sai Lemma.

2: Let V be a module with a simple submodule S, such that S is contained in every non-zero submodule of V. Assume that every endomorphism of S occurs as the restriction of an endomorphism of V. Show: The endomorphism ring of V is local, and its radical factor ring is isomorphic to the endomorphism ring of S.

In particular: The endomorphism ring of $N(\infty)$ is a local ring with radical factor ring isomorphic to the ground field K.

3: Let $V = (K[T], T\cdot)$. Show that V is indecomposable and that $\mathrm{End}(V)$ is not a local ring.

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15. **Direct summands of finite direct sums**

15.1. **The Exchange Theorem.** If the endomorphism ring of a module V is local, then V is indecomposable. In representation theory we are often interested in the indecomposable direct summands of a module. Then one can ask if these direct summands are in some sense uniquely determined (at least up to isomorphism).

Lemma 15.1. *For $i = 1, 2$ let $h_i \colon V \to Y_i$ be homomorphisms. Let $Y = Y_1 \oplus Y_2$ and*

$$f = {}^t[h_1, h_2] \colon V \to Y.$$

If h_1 is an isomorphism, then

$$Y = \mathrm{Im}(f) \oplus Y_2.$$

Proof. For $y \in Y$, write $y = y_1 + y_2$ with $y_1 \in Y_1$ and $y_2 \in Y_2$. Since h_1 is surjective, there is some $v \in V$ with $h_1(v) = y_1$. We get

$$y = y_1 + y_2 = h_1(v) + y_2 = h_1(v) + h_2(v) - h_2(v) + y_2 = f(v) + (-h_2(v) + y_2).$$

Now $f(v) \in \mathrm{Im}(f)$ and $-h_2(v) + y_2 \in Y_2$. So we proved that $\mathrm{Im}(f) + Y_2 = Y$.

For $y \in \mathrm{Im}(f) \cap Y_2$, there is some $v \in V$ with $y = f(v)$. Furthermore, $y = f(v) = h_1(v) + h_2(v)$. Since $y \in Y_2$, we get $h_1(v) = y - h_2(v) \in Y_1 \cap Y_2 = 0$. Since h_1 is injective, $h_1(v) = 0$ implies $v = 0$. Thus $y = f(v) = f(0) = 0$. $\square$

Theorem 15.2 (Exchange Theorem)**.** *Let $V, W_1, \ldots, W_m$ be modules, and define*

$$W = \bigoplus_{j=1}^{m} W_j.$$

Let $f \colon V \to W$ be a split monomorphism. If the endomorphism ring of V is local, then there exists some t with $1 \leq t \leq m$ and a direct decomposition of W_t of the form $W_t = V' \oplus W_t'$ such that

$$W = \mathrm{Im}(f) \oplus W_t' \oplus \bigoplus_{j \neq t} W_j \quad \textit{and} \quad V' \cong V.$$

If we know additionally that $W = \mathrm{Im}(f) \oplus W_1 \oplus C$ for some submodule C of W, then we can assume $2 \leq t \leq m$.

Proof. Since f is a split monomorphism, there is a homomorphism $g \colon W \to V$ with $gf = 1_V$. Write $f = {}^t[f_1, \ldots, f_m]$ and $g = [g_1, \ldots, g_m]$ with homomorphisms

$f_j : V \to W_j$ and $g_j : W_j \to V$. Thus we have

$$gf = \sum_{j=1}^{m} g_j f_j = 1_V.$$

Since $\mathrm{End}(V)$ is a local ring, there is some t with $1 \le t \le m$ such that $g_t f_t$ is invertible. Without loss of generality assume that $g_1 f_1$ is invertible.

Since $g_1 f_1$ is invertible, f_1 is a split monomorphism, thus $f_1 : V \to W_1$ is injective and $\mathrm{Im}(f_1) \oplus W_1' = W_1$ for some submodule W_1' of W_1. Let $h : V \to \mathrm{Im}(f_1)$ be defined by $h(v) = f_1(v)$ for all $v \in V$. Thus we can write

$$f_1 : V \to W_1 = \mathrm{Im}(f_1) \oplus W_1'$$

in the form $f_1 = {}^t[h, 0]$. Thus

$$f = {}^t[h, 0, f_2, \dots, f_m] : V \to \mathrm{Im}(f_1) \oplus W_1' \oplus W_2 \oplus \cdots \oplus W_m.$$

Since h is an isomorphism, the result follows from Lemma 15.1. (Choose $h_1 = h$, $Y_1 = \mathrm{Im}(f_1)$, $h_2 = [0, f_1, \dots, f_m]$ and $Y_2 = W_1' \oplus W_2 \oplus \cdots \oplus W_m$.)

Finally, we assume that $W = \mathrm{Im}(f) \oplus W_1 \oplus C$ for some submodule C of W. Let $g : W \to \mathrm{Im}(f)$ be the projection from W onto $\mathrm{Im}(f)$ with kernel $W_1 \oplus C$ followed by the isomorphism $f^{-1} : \mathrm{Im}(f) \to V$ defined by $f^{-1}(f(v)) = v$. It follows that $gf = 1_V$.

We can write $g = [g_1, \dots, g_m]$ with homomorphisms $g_j : W_j \to V$, thus g_j is just the restriction of g to W_j. By assumption $g_1 = 0$ since W_1 lies in the kernel of g. In the first part of the proof we have chosen some $1 \le t \le m$ such that $g_t f_t$ is invertible in $\mathrm{End}(V)$. Since $g_1 = 0$, we see that $t > 1$. □

15.2. **Consequences of the Exchange Theorem.**

Corollary 15.3. *Let $V, X, W_1, \dots, W_m$ be modules, and let*

$$V \oplus X \cong W := \bigoplus_{j=1}^{m} W_j.$$

If $\mathrm{End}(V)$ is a local ring, then there exists some t with $1 \le t \le m$ and a direct decomposition $W_t = V' \oplus W_t'$ with $V' \cong V$ and

$$X \cong W_t' \oplus \bigoplus_{j \ne t} W_j.$$

Proof. The composition of the inclusion $\iota : V \to V \oplus X$ and of an isomorphism $i : V \oplus X \to \bigoplus_{j=1}^{m} W_j$ is a split monomorphism

$$f : V \to \bigoplus_{j=1}^{m} W_j,$$

and the cokernel $\mathrm{Cok}(f)$ is isomorphic to X.

$$\begin{tikzcd}
0 \arrow[r] & V \arrow[r, "\iota"] \arrow[d, equal] & V \oplus X \arrow[r] \arrow[d, "i"] & X \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & V \arrow[r, "f"] & W \arrow[r] & \mathrm{Cok}(f) \arrow[r] & 0
\end{tikzcd}$$

The Exchange Theorem provides a t with $1 \leq t \leq m$ and a direct decomposition $W_t = V' \oplus W_t'$ with $V' \cong V$ such that

$$W = \mathrm{Im}(f) \oplus W_t' \oplus \bigoplus_{j \neq t} W_j.$$

This direct decomposition of W shows that the cokernel of f is also isomorphic to $Z = W_t' \oplus \bigoplus_{j \neq t} W_j$. This implies $X \cong Z$. In particular, we have $W_t = V' \oplus W_t' \cong V \oplus W_t'$. □

Corollary 15.4 (Cancellation Theorem)**.** *Let V, X_1, X_2 be modules with $V \oplus X_1 \cong V \oplus X_2$. If $\mathrm{End}(V)$ is a local ring, then $X_1 \cong X_2$.*

Proof. We apply Corollary 15.3 with $X = X_1$, $W_1 = V$ and $W_2 = X_2$. There are two cases: In the first case there is a direct decomposition $V = W_1 = V' \oplus W_1'$ with $V' \cong V$ and $X_1 \cong W_1' \oplus W_2$. Since V is indecomposable, $V \cong V' \oplus W_1'$ implies $W_1' = 0$. Therefore $X_1 \cong W_2 = X_2$. In the second case, there is a direct decomposition $X_2 = W_2 = V' \oplus W_2'$ with $V' \cong V$ and $X_1 \cong V \oplus W_2'$, thus $X_2 \cong V \oplus W_2' \cong X_1$. □

Corollary 15.5 (Krull-Remak-Schmidt Theorem)**.** *Let $V_1, \ldots, V_n$ be modules with local endomorphism rings, and let $W_1, \ldots, W_m$ be indecomposable modules. If*

$$\bigoplus_{i=1}^{n} V_i \cong \bigoplus_{j=1}^{m} W_j$$

then $n = m$ and there exists a permutation π such that $V_i \cong W_{\pi(i)}$ for all $1 \leq i \leq n$.

Proof. We proof this via induction on n: For $n = 0$ there is nothing to show. Thus let $n \geq 1$. Set $V = V_1$ and $X = \bigoplus_{i=2}^{n} V_i$. By Corollary 15.3 there is some $1 \leq t \leq m$ and a direct decomposition $W_t = V_1' \oplus W_t'$ with $V_1' \cong V_1$ and $X \cong W_t' \oplus \bigoplus_{j \neq t} W_j$. The indecomposability of W_t implies $W_t' = 0$. This implies

$$\bigoplus_{i=2}^{n} V_i \cong \bigoplus_{j \neq t} W_j.$$

By induction $n - 1 = m - 1$, and there exists a bijection

$$\pi\colon \{2, \ldots, n\} \to \{1, \ldots, m\} \setminus \{t\}$$

such that $V_i \cong W_{\pi(i)}$ for all $2 \leq i \leq n$. Now just set $\pi(1) = t$. □

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Rema(r)k: In the literature the Krull-Remak-Schmidt Theorem is often called Krull-Schmidt Theorem. But in fact Remak was the first to prove such a result in the context of finite groups, which Krull then generalized to modules. The result was part of Robert Remaks Doctoral Dissertation which he published in 1911. He was born in 1888 and murdered in Auschwitz in 1942.

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Corollary 15.6. *Let $V_1, \dots, V_n$ be modules with local endomorphism ring, and let U be a direct summand of $\bigoplus_{i=1}^n V_i$. Then there exists a subset $I \subseteq \{1, \dots, n\}$ such that*

$$U \cong \bigoplus_{i \in I} V_i.$$

Proof. We prove this via induction on n: For $n = 0$ there is nothing to show. Thus let $n \geq 1$. Set $V = V_1$, $X = \bigoplus_{i=2}^n V_i$ and $W_1 = U$. Let W_2 be a direct complement of U in $\bigoplus_{i=1}^n V_i$. Thus

$$V \oplus X = W_1 \oplus W_2.$$

There are two cases: In the first case there is a direct decomposition $W_1 = U = V' \oplus U'$ with $V' \cong V$ and $X \cong U' \oplus W_2$. Since U' is isomorphic to a direct summand of X, induction yields a subset $I' \subseteq \{2, \dots, n\}$ such that $U' \cong \bigoplus_{i \in I'} V_i$. Thus with $I := I' \cup \{1\}$ we get

$$U = V' \oplus U' \cong V_1 \oplus U' \cong \bigoplus_{i \in I} V_i.$$

In the second case there is a direct decomposition $W_2 = V' \oplus W_2'$ with $V' \cong V$ and $X \cong U \oplus W_2'$. Thus U is also isomorphic to a direct summand of X. Therefore there is a subset $I \subseteq \{2, \dots, n\}$ with $U \cong \bigoplus_{i \in I} V_i$. □

15.3. **Examples.** We present some examples which show what happens if we work with indecomposable direct summands, whose endomorphism ring is not local.

Assume $|J| = 2$, thus $M = (K[T_1, T_2], T_1 \cdot, T_2 \cdot)$ is a J-module. Let U_1 and U_2 be non-zero submodules of M. We claim that $U_1 \cap U_2 \neq 0$: Let $u_1 \in U_1$ and $u_2 \in U_2$ be non-zero elements. Then we get $u_1 u_2 \in U_1 \cap U_2$, and we have $u_1 u_2 \neq 0$.

In other words, the module M is uniform. (Recall that a module V is called **uniform** if for all non-zero submodules U_1 and U_2 of V we have $U_1 \cap U_2 \neq 0$.) This implies that every submodule of M is indecomposable.

The submodules U of M are the ideals of $K[T_1, T_2]$. If U is generated by elements $p_1, \dots, p_t$, we write $U = I(p_1, \dots, p_t)$. (One can show that every ideal in $K[T_1, T_2]$ is finitely generated, but we do not need this here.)

Now let U_1, U_2 be ideals with $U_1 + U_2 = K[T_1, T_2]$. This yields an exact sequence

$$0 \to U_1 \cap U_2 \xrightarrow{f} U_1 \oplus U_2 \xrightarrow{g} M \to 0,$$

where $f = {}^t[\iota, -\iota]$ and $g = [\iota, \iota]$. Here we denote all inclusion homomorphisms just by ι.

This sequence splits: Since g is surjective, there is some $u_1 \in U_1$ and $u_2 \in U_2$ with $g(u_1, u_2) = 1$. If we define

$$h \colon M \to U_1 \oplus U_2$$

by $h(p) = (pu_1, pu_2)$ for $p \in K[T_1, T_2]$, then this is a homomorphism, and we have $gh = 1_M$. This implies

$$M \oplus (U_1 \cap U_2) \cong U_1 \oplus U_2.$$

This setup allows us to construct some interesting examples:

Example 1: Let $U_1 = I(T_1, T_2)$ and $U_2 = I(T_1 - 1, T_2)$. We obtain

$$M \oplus (U_1 \cap U_2) \cong I(T_1, T_2) \oplus I(T_1 - 1, T_2).$$

Now M is a cyclic module, but $I(T_1, T_2)$ and $I(T_1 - 1, T_2)$ are not. Thus $I(T_1, T_2) \oplus I(T_1 - 1, T_2)$ contains a cyclic direct summand, but none of the indecomposable direct summands $I(T_1, T_2)$ and $I(T_1 - 1, T_2)$ is cyclic. (We have $U_1 \cap U_2 = I(T_1(T_1 - 1), T_2)$, but this fact is not used here.)

Example 2: Let $U_1 = I(T_1)$ and $U_2 = I(T_1^2 - 1, T_1T_2)$. We obtain $U_1 \cap U_2 = I(T_1^3 - T_1, T_1T_2)$ and

$$M \oplus I(T_1^3 - T_1, T_1T_2) \cong I(T_1) \oplus I(T_1^2 - 1, T_1T_2).$$

The map $f \mapsto T_1 f$ yields an isomorphism $M \to I(T_1)$, but the modules $I(T_1^3 - T_1, T_1T_2)$ and $I(T_1^2 - 1, T_1T_2)$ are not isomorphic. Thus in this situation there is no cancellation rule.

Example 3: Here is another (trivial) example for the failure of the cancellation rule: Let $J = \emptyset$, and let V be an infinite dimensional K-vector space. Then we have

$$V \oplus K \cong V \cong V \oplus 0.$$

Thus we cannot cancel V. On the other hand, in contrast to Example 2, V is not an indecomposable module.

15.4. **Exercises. 1**: Let $V = (K[T], T\cdot)$. Show:

(a): The direct summands of $V \oplus V$ are 0, $V \oplus V$ and all the submodules of the form

$$U_{f,g} := \{(hf, hg) \mid h \in K[T]\}$$

where f and g are polynomials with greatest common divisor 1.

(b): There exist direct summands U of $V \oplus V$ such that none of the modules 0, $V \oplus V$, $U_{1,0} = V \oplus 0$ and $U_{0,1} = 0 \oplus V$ are a direct complement of U in $V \oplus V$.

2: Let $M_1, \ldots, M_t$ be pairwise non-isomorphic modules of finite length, and let $m_i \geq 1$ for $1 \leq i \leq t$. Define

$$V = \bigoplus_{i=1}^{t} M_i^{m_i},$$

and let $R = \operatorname{End}(V)$ be the endomorphism ring of V. Show: There exists an idempotent e in R such that $e(V)$ is isomorphic to $\bigoplus_{i=1}^{t} M_i$, and we have $R = ReR$.

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Part 4. Modules II: A-Modules

16. Modules over algebras

16.1. **Representations of an algebra.** Let A and B be K-algebras. A map $\eta\colon A \to B$ is a K-**algebra homomorphism** if η is a ring homomorphism which is also K-linear. In other words, for all $a_1, a_2 \in A$ and all $\lambda \in K$ the map η satisfies the following:

$$\begin{aligned} \eta(a_1 + a_2) &= \eta(a_1) + \eta(a_2), \\ \eta(\lambda a) &= \lambda\eta(a), \\ \eta(a_1 a_2) &= \eta(a_1)\eta(a_2), \\ \eta(1_A) &= 1_B. \end{aligned}$$

An example of an algebra is the endomorphism ring $\mathrm{End}_K(V)$ of a K-vector space V. The underlying set of $\mathrm{End}_K(V)$ is the set of K-linear maps $f\colon V \to V$. Addition and scalar multiplication are defined pointwise, and the multiplication is given by the composition of maps. Thus we have

$$\begin{aligned} (f_1 + f_2)(v) &= f_1(v) + f_2(v), \\ (\lambda f)(v) &= \lambda(f(v)) = f(\lambda v), \\ (f_1 f_2)(v) &= f_1(f_2(v)) \end{aligned}$$

for all $f, f_1, f_2 \in \mathrm{End}_K(V)$, $\lambda \in K$ and $v \in V$.

Similarly, the set $M_n(K)$ of $n \times n$-matrices with entries in K forms naturally a K-algebra.

From the point of view of representation theory, these algebras are very boring (they are "semisimple"). We will meet more interesting algebras later on.

A **representation** of a K-algebra A is a K-algebra homomorphism

$$\eta\colon A \to \mathrm{End}_K(V)$$

where V is a K-vector space. We want to write down explicitely what this means: To every $a \in A$ we associate a map $\eta(a)\colon V \to V$ such that the following hold:

$$\begin{aligned} &(R_1) & \eta(a)(v_1 + v_2) &= \eta(a)(v_1) + \eta(a)(v_2), \\ &(R_2) & \eta(a)(\lambda v) &= \lambda(\eta(a)(v)), \\ &(R_3) & \eta(a_1 + a_2)(v) &= \eta(a_1)(v) + \eta(a_2)(v), \\ &(R_4) & \eta(\lambda a)(v) &= \lambda(\eta(a)(v)), \\ &(R_5) & \eta(a_1 a_2)(v) &= \eta(a_1)(\eta(a_2)(v)), \\ &(R_6) & \eta(1_A)(v) &= v \end{aligned}$$

for all $a, a_1, a_2 \in A$, $v, v_1, v_2 \in V$ and $\lambda \in K$. The conditions (R_1) and (R_2) just mean that for every $a \in A$ the map $\eta(a)\colon V \to V$ is K-linear. The other rules show that η is an algebra homomorphism: (R_3) and (R_4) say that η is K-linear, (R_5)

means that η is compatible with the multiplication, and (R_6) shows that the unit element of A is mapped to the unit element of $\mathrm{End}_K(V)$.

16.2. **Modules over an algebra.** An A-**module structure** on V (or more precisely, a **left** A-**module structure** on V) is a map

$$\sigma\colon A \times V \to V$$

(where we write $a \cdot v$ or av instead of $\sigma(a, v)$) such that for all $a, a_1, a_2 \in A$, $v, v_1, v_2 \in V$ and $\lambda \in K$ the following hold:

$$(M_1) \qquad a(v_1 + v_2) = av_1 + av_2,$$

$$(M_2) \qquad a(\lambda v) = \lambda(av),$$

$$(M_3) \qquad (a_1 + a_2)v = a_1 v + a_2 v,$$

$$(M_4) \qquad (\lambda a)v = \lambda(av),$$

$$(M_5) \qquad (a_1 a_2)v = a_1(a_2 v),$$

$$(M_6) \qquad 1_A v = v.$$

The conditions (M_1) and (M_2) are the K-linearity in the second variable, and (M_3) and (M_4) are the K-linearity in the first variable. Condition (M_5) gives the compatibility with the multiplication, and (M_6) ensures that 1_A acts as the identity on V. The map σ is sometimes called **scalar multiplication**. An A-**module** (**left** A-**module**) is a vector space V together with an A-module structure on V.

Thus an A-module V has two scalar multiplications: the one coming from V as a vector space over our ground field K, and the other one from the A-module structure. In the latter case, the scalars are elements of A. The scalar multiplication with elements of K is just a special case of the scalar multiplication with elements of the algebra, because $\lambda \cdot v = (\lambda \cdot 1_A) \cdot v$ for all $\lambda \in K$ and $v \in V$.

16.3. **Modules and representations.** Let $\mathrm{Abb}(V, V)$ be the set of all (set theoretic) maps $V \to V$. If we have any (set theoretic) map $\eta\colon A \to \mathrm{Abb}(V, V)$, then we can define a map $\overline{\eta}\colon A \times V \to V$ by

$$\overline{\eta}(a, v) = \eta(a)(v).$$

This defines a bijection between the set of all maps $A \to \mathrm{Abb}(V, V)$ and the set of maps $A \times V \to V$.

Lemma 16.1. *Let A be a K-algebra, and let V be a K-vector space. If $\eta\colon A \to \mathrm{End}_K(V)$ is a map, then η is a representation of A if and only if $\overline{\eta}\colon A \times V \to V$ is an A-module structure on V.*

Proof. If $\eta\colon A \to \mathrm{End}_K(V)$ is a representation, we obtain a map

$$\overline{\eta}\colon A \times V \to V$$

which is defined by $\overline{\eta}((a, v)) := \eta(a)(v)$. Then $\overline{\eta}$ defines an A-module structure on V.

Vice versa, let $\sigma\colon A \times V \to V$ be an A-module structure. Then the map $\overline{\sigma}\colon A \to \mathrm{End}_K(V)$ which is defined by $\overline{\sigma}(a)(v) := \sigma(a, v)$ is a representation of A.

Now it is easy to match the conditions (R_i) and (M_i) for $1 \le i \le 6$. □

Let V be an A-module. We often write $V = {}_AV$ and say "V is a left A-module". Often we are a bit sloppy and just say: "V is an A-module", "V is a module over A", "V is a module" (if it is clear which A is meant), or "V is a representation" without distinguishing between the two concepts of a "module" and a "representation".

16.4. **A-modules and $|A|$-modules.** Let $\eta\colon A \to \mathrm{End}_K(V)$ be a representation of A. Since we associated to every $a \in A$ an endomorphism $\eta(a)$ of a vector space V, we see immediately that each representation of A gives us an $|A|$-module, namely

$$(V, \eta(a))_{a \in |A|}.$$

By $|A|$ we just mean the underlying set of the algebra A. (Of course $|A|$ is just A itself, but we can forget about the extra structure (like multiplication etc.) which turns the set A into an algebra.) But note that the endomorphisms $\eta(a)$ are not just arbitrary and cannot be chosen independently of each other: They satisfy very strong extra conditions which are given by $(R_3), \dots, (R_6)$.

So we see that every A-module is an $|A|$-module.

This means that we can use the terminology and theory of modules which we developed in the previous chapters in the context of A-modules. (We just interpret them as $|A|$-modules.) The $|A|$-modules are the maps $A \times V \to V$ which satisfy the axioms (M_1) and (M_2), and the A-modules are exactly the $|A|$-modules which additionally satisfy $(M_3), \dots, (M_6)$.

If ${}_AV$ is an A-module, then every submodule and every factor module (in the sense of the general module definition) is again an A-module. If ${}_AV_i$, $i \in I$ are A-modules, then the direct sum $\bigoplus_{i \in I} {}_AV_i$ and the product $\prod_{i \in I} {}_AV_i$ are again A-modules.

As suggested in the considerations above, if ${}_AV$ and ${}_AW$ are A-modules, then a map $f\colon V \to W$ is an **A-module homomorphism** if it is a homomorphism of $|A|$-modules. In other words, for all $v, v_1, v_2 \in V$ and $a \in A$ we have

$$f(v_1 + v_2) = f(v_1) + f(v_2),$$
$$f(av) = af(v).$$

We write $\mathrm{Hom}_A(V, W)$ or just $\mathrm{Hom}(V, W)$ for the set of homomorphisms ${}_AV \to {}_AW$. Recall that $\mathrm{Hom}_A(V, W)$ is a K-vector space (with respect to addition and scalar multiplication). Similarly, let $\mathrm{End}_A(V)$ or $\mathrm{End}(V)$ be the endomorphism ring of V.

By $\mathrm{Mod}(A)$ we denote the K-linear category with all A-modules as objects, and with A-module homomorphisms as morphisms. We call $\mathrm{Mod}(A)$ the **category of (left) A-modules**. By $\mathrm{mod}(A)$ we denote the category of all finite-dimensional A-modules. This is a full subcategory of $\mathrm{Mod}(A)$.

16.5. **Free modules.** Let V be an A-module. A subset U of V is a submodule if and only if U is closed under addition and scalar multiplication with scalars from A.

If X is a subset of V, then the submodule $U(X)$ generated by X is the set of all (finite) linear combinations $\sum_{i=1}^n a_ix_i$ with $x_1,\dots,x_n \in X$ and $a_1,\dots,a_n \in A$: Clearly the elements of the form $\sum_{i=1}^n a_ix_i$ have to belong to $U(X)$. On the other hand, the set of all elements, which can be written in such a way, is closed under addition and scalar multiplication. Thus they form a submodule and this submodule contains X.

For $x \in V$ let $Ax = \{ax \mid a \in A\}$. Thus Ax is the submodule of V generated by x. Similarly, for all subsets $X \subseteq V$ we have

$$U(X) = \sum_{x\in X} Ax.$$

If A is an algebra, then the multiplication map $\mu\colon A \times A \to A$ satisfies all properties of an A-module structure, where $V = A$ as a vector space. Thus by our convention we denote this A-module by ${}_AA$. The corresponding representation

$$A \to \mathrm{End}_K(A)$$

with $a \mapsto \lambda_a$ is the **regular representation**. Here for $a \in A$ the map $\lambda_a\colon A \to A$ is defined by $\lambda_a(x) = ax$, thus λ_a is the left multiplication map with a.

A **free** A**-module** is by definition a module V which is isomorphic to a (possibly infinite) direct sum of copies of ${}_AA$.

If V is an A-module, then a subset X of V is a **free generating set** if the following two conditions are satisfied:

- X is a generating set of V, i.e. $V = \sum_{x\in X} Ax$;
- If $x_1,\dots,x_n$ are pairwise different elements in X and $a_1,\dots,a_n$ are arbitrary elements in A with
$$\sum_{i=1}^n a_ix_i = 0,$$
then $a_i = 0$ for all $1 \le i \le n$.

(Compare the definition of a free generating set with the definition of a basis of a vector space, and with the definition of a linearly independent set of vectors.)

Lemma 16.2. *An A-module is free if and only if it has a free generating set.*

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Proof. Let W be a direct sum of copies of ${}_AA$, say $W = \bigoplus_{i\in I} W_i$ with $W_i = {}_AA$ for all $i \in I$. By e_i we denote the 1-element of W_i. (In coordinate notation: All coefficients of e_i are 0, except the ith coefficient is the element $1_A \in {}_AA = W_i$.) Thus the set $\{e_i \mid i \in I\}$ is a free generating set of W.

If $f\colon W \to V$ is an isomorphism of A-modules, and if X is a free generating set of W, then $f(X)$ is a free generating set of V.

Vice versa, we want to show that every A-module V with a free generating set X is isomorphic to a free module. We take a direct sum of copies of ${}_AA$, which are indexed by the elements in X. Thus $W = \bigoplus_{x\in X} W_x$ where $W_x = {}_AA$ for all x. As before, let e_x be the 1-element of $W_x = {}_AA$. Then every element in W can be written as a (finite) sum $\sum_{x\in X} a_xe_x$ with $a_x \in A$ for all $x \in X$, and $a_x = 0$ for almost all (i.e. all but finitely many) $x \in X$. We define a map $f\colon W \to V$ by

$$f\left(\sum_{x\in X} a_xe_x\right) = \sum_{x\in X} a_xx.$$

It is easy to check that f is an A-module homomorphism which is surjective and injective, thus it is an isomorphism of A-modules. $\square$

If F is a free A-module with free generating set X, then the cardinality of X is called the **rank** of F. Thus F has finite rank, if X is a finite set.

Let F be a free A-module, and let W be an arbitrary A-module. If X is a free generating set of F, and if we choose for every $x \in X$ an element $w_x \in W$, then there exists exactly one A-module homomorphism $f\colon F \to W$ such that $f(x) = w_x$ for all $x \in X$. Namely,

$$f\left(\sum_{x\in X} a_xx\right) = \sum_{x\in X} a_xw_x$$

for all $x \in X$ and all $a_x \in A$. If the set $\{w_x \mid x \in X\}$ is a generating set of the A-module W, then the homomorphism f is surjective. Thus in this case W is isomorphic to a factor module of F. So we proved the following result:

Theorem 16.3. *Every A-module is isomorphic to a factor module of a free A-module.*

Inside the category of all $|A|$-modules, we can now characterize the A-modules as follows: They are exactly the modules which are isomorphic to some factor module of some free A-module. Thus up to isomorphism one obtains all A-modules by starting with ${}_AA$, taking direct sums of copies of ${}_AA$ and then taking all factor modules of these direct sums.

Every finitely generated A-module is isomorphic to a factor module of a free module of finite rank. In particular, each simple A-module is isomorphic to a factor module of a free module of rank one. Thus, we get the following:

Lemma 16.4. *Let A be a finite-dimensional K-algebra. For an A-module M the following are equivalent:*

(i) *M is finitely generated;*
(ii) *M is finite-dimensional as a K-vector space;*
(iii) *$l(M) < \infty$.*

16.6. **The opposite algebra.** If A is a K-algebra, then we denote the **opposite algebra** of A by $A^{\rm op}$. Here we work with the same underlying vector space, but the multiplication map is changed: To avoid confusion, we denote the multiplication of $A^{\rm op}$ by $\star$, which is defined as

$$a_1 \star a_2 = a_2 \cdot a_1 = a_2 a_1$$

for all $a_1, a_2 \in A$ (where $\cdot$ is the multiplication of A). This defines again an algebra. Of course we have $(A^{\rm op})^{\rm op} = A$.

Lemma 16.5. *If A is an algebra, then* $\mathrm{End}_A({}_AA) \cong A^{\rm op}$.

Proof. As before let $\lambda_a \colon A \to A$ be the left multiplication with $a \in A$, i.e. $\lambda_a(x) = ax$ for all $x \in A$. Similarly, let $\rho_a \colon A \to A$ be the right multiplication with $a \in A$, i.e. $\rho_a(x) = xa$ for all $x \in A$. It is straightforward to check that the map

$$\rho \colon A^{\rm op} \to \mathrm{End}_A({}_AA)$$

defined by $\rho(a) = \rho_a$ is an algebra homomorphism. In particular we have

$$\begin{aligned} \rho(a_1 \star a_2)(x) &= \rho(a_2 a_1)(x) \\ &= x(a_2 a_1) = (x a_2) a_1 \\ &= (\rho(a_2)(x)) \cdot a_1 \\ &= \rho(a_1)\rho(a_2)(x) \end{aligned}$$

for all $a_1, a_2, x \in A$. The map ρ is injective: If $a \in |A|$, then $\rho(a)(1) = 1 \cdot a = a$. Thus $\rho(a) = 0$ implies $a = \rho(a)(1) = 0$.

We know that

$$\lambda_a \rho_b = \rho_b \lambda_a$$

for all $a, b \in A$. (This follows directly from the associativity of the multiplication in A.) In other words, the vector space endomorphisms ρ_b are endomorphisms of the A-module ${}_AA$, and ρ yields an embedding of $A^{\rm op}$ into $\mathrm{End}_A({}_AA)$.

It remains to show that every endomorphism f of ${}_AA$ is a right multiplication: Let $f(1) = b$. We claim that $f = \rho_b$: For $a \in A$ we have

$$f(a) = f(a \cdot 1) = a \cdot f(1) = a \cdot b = \rho_b(a).$$

This finishes the proof. $\square$

16.7. **Right A-modules.** A **right A-module structure** on V is a map

$$\rho \colon V \times A \to V$$

(where we write $v \cdot a$ or va instead of $\rho(v, a)$) such that for all $a, a_1, a_2 \in A$, $v, v_1, v_2 \in V$ and $\lambda \in K$ the following hold:

(M_1') $$(v_1 + v_2)a = v_1 a + v_2 a,$$

(M_2') $$(\lambda v)a = \lambda(va),$$

(M_3') $$v(a_1 + a_2) = va_1 + va_2,$$

(M_4') $$v(\lambda a) = \lambda(va),$$

(M_5') $$v(a_1 a_2) = (va_1)a_2,$$

(M_6') $$v1_A = v.$$

A **right** A**-module** is a vector space V together with a right A-module structure on V. We often write $V = V_A$ and say "V is a right A-module".

16.8. **Examples.** For A-modules V and W the homomorphism space $\mathrm{Hom}_A(V, W)$ carries a (left) $\mathrm{End}_A(W)$-module structure defined by

$$\mathrm{End}_A(W) \times \mathrm{Hom}_A(V, W) \to \mathrm{Hom}_A(V, W), \quad (f, g) \mapsto fg,$$

and $\mathrm{Hom}_A(V, W)$ has a right $\mathrm{End}_A(V)$-module structure given by

$$\mathrm{Hom}_A(V, W) \times \mathrm{End}_A(V) \to \mathrm{Hom}_A(V, W), \quad (g, f) \mapsto gf.$$

One can also turn ${}_AV$ into a module over $\mathrm{End}_A(V)$ by

$$\mathrm{End}_A(V) \times V \to V, \quad (f, v) \mapsto f(v).$$

16.9. **Direct decompositions of the regular representation.** Let A be a K-algebra, and let

$${}_AA = \bigoplus_{i \in I} P_i$$

be a direct decomposition of the regular module ${}_AA$ with modules $P_i \neq 0$ for all $i \in I$. Thus every element $a \in A$ is of the form $a = \sum_{i \in I} a_i$ with $a_i \in P_i$. (Only finitely many of the a_i are allowed to be non-zero.) In particular let

$$1 = 1_A = \sum_{i \in I} e_i$$

with $e_i \in P_i$.

Lemma 16.6. *For all $i \in I$ we have $P_i = Ae_i$.*

Proof. Since P_j is a submodule of ${}_AA$ and $e_j \in P_j$, we know that $Ae_j \subseteq P_j$. Vice versa, let $x \in P_j$. We have

$$x = x \cdot 1 = \sum_{i \in I} xe_i.$$

Since x belongs to P_j, and since ${}_AA$ is the direct sum of the submodules P_i, we get $x = xe_j$ (and $xe_i = 0$ for all $i \neq j$). In particular, $x \in Ae_j$. $\square$

Lemma 16.7. *The index set I is finite.*

Proof. Only finitely many of the e_i are different from 0. If $e_i = 0$, then $P_i = Ae_i = 0$, a contradiction to our assumption. □

A set $\{f_i \mid i \in I\} \subseteq R$ of idempotents in a ring R is a **set of pairwise orthogonal idempotents** if $f_i f_j = 0$ for all $i \neq j$. Such a set of pairwise orthogonal idempotents is **complete** if $1 = \sum_{i \in I} f_i$.

Lemma 16.8. *The set $\{e_i \mid i \in I\}$ defined above is a complete set of pairwise orthogonal idempotents.*

Proof. We have
$$e_j = e_j \cdot 1 = \sum_{i \in I} e_j e_i.$$
As in the proof of Lemma 16.6, the unicity of the decomposition of an element in a direct sum yields that $e_j = e_j e_j$ and $e_j e_i = 0$ for all $i \neq j$. □

Warning: Given a direct decomposition ${}_AA = \bigoplus_{i \in I} P_i$. If we choose idempotents $e_i \in P_i$ with $P_i = Ae_i$, then these idempotents do not have to be orthogonal to each other. For example, let $A = M_2(K)$ be the algebra of 2×2-matrices with entries in K. Take
$$e_1 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
and define $P_i = Ae_i$. We obtain ${}_AA = P_1 \oplus P_2$. The elements e_1 and e_2 are idempotents, but they are not orthogonal.

Lemma 16.8 shows that any direct decomposition of ${}_AA$ yields a complete set of orthogonal idempotents in A. Vice versa, assume that $f_i, i \in I$ is a complete set of orthogonal idempotents in an algebra A, then
$${}_AA = \bigoplus Af_i$$
is a direct decomposition of ${}_AA$.

Example: Let B be an algebra, and let $A = M_n(B)$ be the algebra of $n \times n$-matrices with entries in B for some $n \in \mathbb{N}_1$. Let e_{ij} be the $n \times n$-matrix with entry 1 at the position (i, j) and all other entries 0. For brevity write $e_i = e_{ii}$. The diagonal matrices e_i, $1 \leq i \leq n$ form a complete set of orthogonal idempotents in A. Note that Ae_i contains exactly the matrices whose only non-zero entries are in the ith column. It follows immediately that
$${}_AA = \bigoplus_{i=1}^{n} Ae_i.$$
Note also that the modules Ae_i are isomorphic to each other: We get an isomorphism $Ae_i \to Ae_j$ via right multiplication with e_{ij}.

Instead of working with this isomorphism, we could also argue like this: Let $X = B^n$ be the vector space of n-tupels with coefficients in B. We interpret these n-tupels as $n \times 1$-matrices. So matrix multiplication yields an A-module structure on X. It is clear that X and Ae_i have to be isomorphic: X and Ae_i only differ by the fact that Ae_i contains some additional 0-columns.

Warning: The above direct decomposition of $M_n(B)$ is for $n \geq 2$ of course not the only possible decomposition. For example for $n = 2$ and any $x \in B$ the matrices

$$\begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & -x \\ 0 & 1 \end{pmatrix}$$

form also a complete set of orthogonal idempotents in $M_2(B)$. In this case

$$M_2(B) = M_2(B) \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix} \oplus M_2(B) \begin{pmatrix} 0 & -x \\ 0 & 1 \end{pmatrix},$$

where

$$M_2(B) \begin{pmatrix} 1 & x \\ 0 & 0 \end{pmatrix}$$

consists of the matrices of the form

$$\begin{pmatrix} b_1 & b_1x \\ b_2 & b_2x \end{pmatrix}$$

with $b_1, b_2 \in B$, and

$$M_2(B) \begin{pmatrix} 0 & -x \\ 0 & 1 \end{pmatrix}$$

consists of the matrices whose only non-zero entries are in the second column.

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16.10. **Modules over factor algebras.** Let A be a K-algebra, and let I be an ideal in A. Define $B = A/I$. If $M = {}_BM$ is a B-module, then we can turn M into an A-module by defining $a \cdot m := \overline{a}m$ for all $a \in A$ and $m \in M$. We write $\iota_B^A(M)$ or ${}_AM$ for this A-module. (But often we just write M.) For this A-module $M = \iota_B^A(M)$ we obviously have $I \cdot M = 0$. We say that M is **annihilated** by I.

Vice versa, if X is an A-module with $I \cdot X = 0$, then we can interpret X as a B-module: For $b \in B$ and $x \in X$, we write $b = a + I$ and then define $b \cdot x := ax$. This is well defined since $I \cdot X = 0$. It is easy to check that this turns X into an B-module.

These two constructions are inverse to each other. Thus we can identify the B-modules with the A-modules, which are annihilated by I.

The following is obviously also true: If M_1 and M_2 are A-modules, which are annihilated by I, then a map $M_1 \to M_2$ is A-linear if and only if it is B-linear. Thus we get $\mathrm{Hom}_A(M_1, M_2) = \mathrm{Hom}_B(M_1, M_2)$.

Proposition 16.9. *Let I be an ideal in a K-algebra A, and let $B = A/I$. If we associate to each B-module M the A-module $\iota_B^A(M)$, then we obtain an embedding of the category of B-modules into the category of A-modules. The image of the functor consists of all A-modules, which are annihilated by I.*

16.11. **Modules over products of algebras.** Let R and S be rings. Recall that the **product** $R \times S$ of R and S is again a ring with componentwise addition and multiplication. Thus $(r, s) + (r', s') = (r + r', s + s')$ and $(r, s) \cdot (r', s') = (rr', ss')$.

Similarly, if A and B are algebras, then $A \times B$ is again an algebra. In this case, define $e_A = (1, 0)$ and $e_B = (0, 1)$. These form a complete set of orthogonal idempotents. We have $(A \times B)e_A = A \times 0$ and $(A \times B)e_B = 0 \times B$. These are ideals in $A \times B$, and we can identify the factor algebra $(A \times B)/(A \times 0)$ with B, and $(A \times B)/(0 \times B)$ with A.

Let $C = A \times B$, and let M be a C-module. We get $M = e_AM \oplus e_BM$ as a direct sum of vector spaces, and the subspaces e_AM and e_BM are in fact submodules of M. The submodule e_AM is annihilated by $0 \times B = (A \times B)e_B$, thus e_AM can be seen as a module over $(A \times B)/(0 \times B)$ and therefore as a module over A: For $a \in A$ and $m \in M$ define $a \cdot e_Am = (a, 0)e_Am = (a, 0)m$. Similarly, e_BM is a B-module. Thus we wrote M as a direct sum of an A-module and a B-module.

Vice versa, if M_1 is an A-module and M_2 is a B-module, then the direct sum $M_1 \oplus M_2$ of vector spaces becomes an $(A \times B)$-module by defining $(a, b) \cdot (m_1, m_2) = (am_1, bm_2)$ for $a \in A$, $b \in B$, $m_1 \in M_1$ and $m_2 \in M_2$. In particular, we can interpret all A-modules and all B-modules as $(A \times B)$-modules: If M is an A-module, just define $(a, b)m = am$ for $a \in A$, $b \in B$ and $m \in M$. (This is the same as applying $\iota_A^{A \times B}$ to M.) We call an $(A \times B)$-modules, which is annihilated by $0 \times B$ just an A-module, and an $(A \times B)$-modules, which is annihilated by $A \times 0$ is just a B-module.

Thus we proved the following result:

Proposition 16.10. *Let A and B be algebras. Then each $(A \times B)$-module is the direct sum of an A-module and a B-module.*

In particular, indecomposable modules over $A \times B$ are either A-modules or B-modules.

Warning: If $A = B$, we have to be careful. If we say that an A-module M can be seen as a $(A \times B)$-module, we have to make clear which copy of A we mean, thus if we regard M as a module over $A \times 0$ or $0 \times A$.

16.12. **Bimodules.** Let A and B be K-algebras. An A-B-**bimodule** V is a K-vector space V together with two module structures

$$\mu_A \colon A \times V \to V \quad \text{and} \quad \mu_B \colon B \times V \to V$$

such that for all $a \in A$, $b \in B$ and $v \in V$ we have

$$\mu_A(a, \mu_B(b, v)) = \mu_B(b, \mu_A(a, v)).$$

Using our short notation av for $\mu_A(a, v)$ and bv instead of $\mu_B(b, v)$, we can write this as

$$a(bv) = b(av).$$

Note that for all $\lambda \in K$ and $v \in V$ we have

$$\mu_A(\lambda \cdot 1_A, v) = \lambda v = \mu_B(\lambda \cdot 1_B, v).$$

Warning: In many books, our A-B-bimodules are called A-B^{op}-bimodules.

Assume that M is an A-B-bimodule. We get a canonical map $c\colon B \to \mathrm{End}_A(M)$ which sends $b \in B$ to the scalar multiplication $b\cdot\colon M \to M$ which maps m to bm. It is easy to check that the image of c lies in $\mathrm{End}_A(M)$: We have

$$c(b)(am) = b(am) = a(bm) = a(c(b)(m))$$

for all $a \in A$, $b \in B$ and $m \in M$.

Example: Let M be an A-module, and let $B := \mathrm{End}_A(M)$ be its endomorphism algebra. Then M is an A-B-bimodule. Namely, M becomes a B-module by

$$\mu_B(f, m) = f(m)$$

for all $f \in \mathrm{End}_A(M)$ and $m \in M$. But we also have $f(am) = af(m)$.

The next result shows that bimodule structures allow us to see homomorphism spaces again as modules.

Lemma 16.11. *Let M be an A-B-bimodule, and let N an A-C-bimodule. Then $\mathrm{Hom}_A(M, N)$ is an B^{op}-C-bimodule via*

$$b(c(f(m))) = c(f(bm))$$

for all $b \in B$, $c \in C$, $f \in \mathrm{Hom}_A(M, N)$ and $m \in M$.

Proof. Let $\star$ be the multiplication in B^{op}, and set $H := \mathrm{Hom}_A(M, N)$. It is clear that the two maps $B^{\mathrm{op}} \times H \to H$, $(b, f) \mapsto (bf\colon m \mapsto f(bm))$ and $C \times H \to H$, $(c, f) \mapsto (cf\colon m \mapsto cf(m))$ are bilinear. We also have $1_B \cdot f = f$ and $1_C \cdot f = f$ for all $f \in H$.

For $b_1, b_2 \in B$ and $f \in H$ we have

$$\begin{aligned}((b_1 \star b_2)f)(m) &= f((b_1 \star b_2)m)\\ &= f((b_2 b_1)m)\\ &= f(b_2(b_1 m))\\ &= (b_2 f)(b_1 m)\\ &= (b_1(b_2 f))(m).\end{aligned}$$

This shows that

$$(b_1 \star b_2)f = b_1(b_2(f)).$$

Similarly,

$$\begin{aligned}((c_1c_2)f)(m) &= (c_1c_2)(f(m))\\ &= c_1(c_2(f(m)))\\ &= c_1((c_2f)(m))\\ &= (c_1(c_2f))(m).\end{aligned}$$

shows that $(c_1c_2)f = c_1(c_2f)$ for all $c_1, c_2 \in C$ and $f \in H$. □

Let M be an A-B-bimodule. This gives a covariant functor

$$\operatorname{Hom}_A(M, -): \operatorname{Mod}(A) \to \operatorname{Mod}(B^{\mathrm{op}}).$$

Similarly, if N is an A-C-bimodule we get a contravariant functor

$$\operatorname{Hom}_A(-, N): \operatorname{Mod}(A) \to \operatorname{Mod}(C).$$

16.13. **Modules over tensor products of algebras.** Let A and B be K-algebras. Then $A \otimes_K B$ is again a K-algebra with multiplication

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) := (a_1a_2 \otimes b_1b_2).$$

(One has to check that this is well defined and that one gets indeed a K-algebra.)

Proposition 16.12. *The category of A-B-bimodules is equivalent to the category of $A \otimes_K B$-modules.*

Sketch of proof. Let M be an A-B-bimodule. This becomes an $A \otimes_K B$-module via

$$(a \otimes b)m := abm$$

for all $a \in A$, $b \in B$ and $m \in M$. The same rule applied the other way round turns an $A \otimes_K B$-module into an A-B-bimodule. □

16.14. **Exercises. 1**: Let $A = K\langle X_1, \dots, X_n\rangle$ be the K-algebra of polynomials in n non-commuting variables $X_1, \dots, X_n$, and let $J = \{1, \dots, n\}$. Show: The category of J-modules is equivalent to $\operatorname{Mod}(A)$.

In particular, $\operatorname{Mod}(K[T])$ is equivalent to the category of 1-modules.

2: Let A be a K-algebra. Show that the category of left A-modules is equivalent to the category of right A^{op}-modules.

17. **Semisimple algebras**

17.1. **Semisimple algebras and their modules.**

Theorem 17.1. *Let A be an algebra. Then the following are equivalent:*

(i) *The module ${}_AA$ is semisimple;*
(ii) *Every A-module is semisimple;*
(iii) *There exist K-skew fields D_i and natural numbers n_i where $1 \leq i \leq s$ such that*
$$A \cong \prod_{i=1}^{s} M_{n_i}(D_i).$$

An algebra A is called **semisimple** if one of the equivalent conditions in the above theorem is satisfied.

The opposite algebra A^{op} of a semisimple algebra A is again semisimple. This follows directly from Condition (iii): If D is a skew field, then D^{op} is also a skew field. For an arbitrary ring R there is an isomorphism
$$M_n(R)^{\mathrm{op}} \to M_n(R^{\mathrm{op}})$$
which maps a matrix Φ to its transpose ${}^t\Phi$.

Proof of Theorem 17.1. The implication (ii) $\Longrightarrow$ (i) is trivial.

(i) $\Longrightarrow$ (ii): Let ${}_AA$ be a semisimple module. Since direct sums of semisimple modules are again semisimple, we know that all free A-modules are semisimple. But each A-modules is a factor module of a free module, and factor modules of semisimple modules are semisimple. Thus all A-modules are semisimple.

(i) $\Longrightarrow$ (iii): Since ${}_AA$ is semisimple, we know that ${}_AA$ is a direct sum of simple modules. By the results in Section 16.9 this direct sum has to be finite. Thus $\mathrm{End}_A({}_AA)$ is a finite product of matrix rings over K-skew fields. We know that $\mathrm{End}_A({}_AA) \cong A^{\mathrm{op}}$, thus A^{op} is a finite product of matrix rings over K-skew fields. Thus the same holds for A.

(iii) $\Longrightarrow$ (i): Let A be a product of s matrix rings over K-skew fields. We want to show that ${}_AA$ is semisimple. It is enough to study the case $s = 1$: If $A = B \times C$, then the modules ${}_BB$ and ${}_CC$ are semisimple, and therefore ${}_AA$ is also semisimple.

Let $A = M_n(D)$ for some K-skew field D and some $n \in \mathbb{N}_1$. Let $S = D^n$ be the set of column vectors of length n with entries in D. It is easy to show that S is a simple $M_n(D)$-module. (One only has to show that if $x \neq 0$ is some non-zero column vector in S, then $M_n(D)x = S$.) On the other hand, we can write ${}_AA$ as a direct sum of n copies of S. Thus ${}_AA$ is semisimple. □

Let $A = M_n(D)$ for some K-skew field D and some $n \in \mathbb{N}_1$. We have shown that ${}_AA$ is a direct sum of n copies of the simple module S consisting of column vectors of length n with entries in D. It follows that every A-module is a direct sum of copies of S. (Each free module is a direct sum of copies of S, and each module is a factor module of a free module. If a simple A-module T is isomorphic to a factor module of a free A-module, we obtain a non-zero homomorphism $S \to T$. Thus $T \cong S$ by Schur's Lemma.) If

$$A \cong \prod_{i=1}^{s} M_{n_i}(D_i),$$

then there are exactly s isomorphism classes of simple A-modules.

Proposition 17.2. *Let K be an algebraically closed field. If A is a finite-dimensional semisimple K-algebra, then*

$$A \cong \prod_{i=1}^{s} M_{n_i}(K)$$

for some natural numbers n_i, $1 \leq i \leq s$.

Proof. First, we look at the special case $A = D$, where D is a K-skew field: Let $d \in D$. Since D is finite-dimensional, the powers d^i with $i \in \mathbb{N}_0$ cannot be linearly independent. Thus there exists a non-zero polynomial p in $K[T]$ such that $p(d) = 0$. We can assume that p is monic. Since K is algebraically closed, we can write it as a product of linear factors, say $p = (T - c_1) \cdots (T - c_n)$ with $c_i \in K$. Thus in D we have $(d - c_1) \cdots (d - c_n) = 0$. Since D has no zero divisors, we get $d - c_i = 0$ for some i, and therefore $d = c_i \in K$.

Now we investigate the general case: We know that A is isomorphic to a product of matrix rings of the form $M_{n_i}(D_i)$ with K-skew fields D_i and $n_i \in \mathbb{N}_1$. Since A is finite-dimensional, every K-skew field D_i must be finite-dimensional over K. But since K is algebraically closed, and the D_i are finite-dimensional K-skew fields we get $D_i = K$. □

The **centre of a ring** R is by definition the set of elements $c \in R$ such that $cr = rc$ for all $r \in R$. We denote the centre of R by $C(R)$. If R and S are rings, then $C(R \times S) = C(R) \times C(S)$.

Lemma 17.3. *If $A \cong \prod_{i=1}^{s} M_{n_i}(K)$, then the centre of A is s-dimensional.*

Proof. It is easy to show that the centre of a matrix ring $M_n(K)$ is just the set of scalar matrices. Thus we get

$$C\left(\prod_{i=1}^{s} M_{n_i}(K)\right) = \prod_{i=1}^{s} C(M_{n_i}(K)) \cong \prod_{i=1}^{s} K.$$

□

End of Lecture 21

17.2. **Examples: Group algebras.** Let G be a group, and let $K[G]$ be a K-vector space with basis $\{e_g \mid g \in G\}$. Define

$$e_g e_h := e_{gh}.$$

Extending this linearly turns the vector space $K[G]$ into a K-algebra. One calls $K[G]$ the **group algebra** of G over K. Clearly, $K[G]$ is finite-dimensional if and only if G is a finite group.

A **representation** of G over K is a group homomorphism

$$\rho\colon G \to \mathrm{GL}(V)$$

where V is a K-vector space. In the obvious way one can define homomorphisms of representations. It turns out that the category of representations of G over K is equivalent to the category $\mathrm{Mod}(K[G])$ of modules over the group algebra $K[G]$. If V is a $K[G]$-module, then for $g \in G$ and $v \in V$ we often write gv instead of $e_g v$.

The representation theory of G depends very much on the field K, in particular, the characteristic of K plays an important role.

Theorem 17.4 (Maschke)**.** *Let G be a finite group, and let K be a field such that the characteristic of K does not divide the order of G. Then every $K[G]$-module is semisimple.*

Proof. It is enough to show that every finite-dimensional $K[G]$-module is semisimple. Let U be a submodule of a finite-dimensional $K[G]$-module V. Write

$$V = U \oplus W$$

with W a subspace of V. But note that W is not necessarily a submodule.

Let $\theta\colon V \to V$ be the projection onto U. So $\theta(u) = u$ and $\theta(w) = 0$ for all $u \in U$ and $w \in W$. Define $f\colon V \to V$ by

$$f(v) = \frac{1}{|G|} \sum_{g \in G} g^{-1}\theta(gv).$$

Here we use our assumption on the characteristic of K, otherwise we would divide by 0, which is forbidden in mathematics.

We claim that $f \in \mathrm{End}_{K[G]}(V)$: Clearly, f is a linear map. For $h \in G$ we have

$$f(hv) = \frac{1}{|G|} \sum_{g \in G} g^{-1}\theta(ghv).$$

Set $x_g := gh$. Thus $g^{-1} = hx_g^{-1}$. So we get

$$f(hv) = \frac{1}{|G|} \sum_{g \in G} hx_g^{-1}\theta(x_g v) = hf(v).$$

Thus f is an endomorphism of V.

We have $\mathrm{Im}(f) = U$: Namely, $\mathrm{Im}(f) \subseteq U$ since each term in the sum is in U. If $u \in U$, then

$$f(u) = \frac{1}{|G|}\sum_{g\in G} g^{-1}\theta(gu) = \frac{1}{|G|}\sum_{g\in G} g^{-1}gu = \frac{1}{|G|}\sum_{g\in G} u = u.$$

Clearly, this implies $U \cap \mathrm{Ker}(f) = 0$: Namely, if $0 \neq u \in U \cap \mathrm{Ker}(f)$, then $f(u) = u = 0$, a contradiction.

We have $\dim \mathrm{Ker}(f) + \dim \mathrm{Im}(f) = \dim V$. This implies

$$V = \mathrm{Im}(f) \oplus \mathrm{Ker}(f) = U \oplus \mathrm{Ker}(f),$$

and $\mathrm{Ker}(f)$ is a submodule of V. Now let U be a simple submodule of V. We get $V = U \oplus \mathrm{Ker}(f)$. By induction on $\dim V$, $\mathrm{Ker}(f)$ is semisimple, thus V is semisimple. □

17.3. **Remarks.** Let G be a finite group, and let K be a field. If $\mathrm{char}(K)$ does not divide the order of G, then $K[G]$ is semisimple. In this case, from our point of view, the representation theory of $K[G]$ is very boring. (But be careful: If you say this at the wrong place and wrong time, you will be crucified.)

More interesting is the **modular representation theory** of G, i.e. the study of representations of G over K where $\mathrm{char}(K)$ does divide $|G|$.

For example, if $G = S_n$ is the symmetric group of bijective maps $\{1,\dots,n\} \to \{1,\dots,n\}$, then one can parametrize the simple $K[G]$-modules by certain partitions. But in the modular case, it is not even known which dimensions these simple modules have.

Another interesting question is the following: Given two finite groups G and H, and let K be a field. When are the module categories $\mathrm{mod}(K[G])$ and $\mathrm{mod}(K[H])$ derived equivalent? (We will learn about derived categories and derived equivalences later on.)

Again let G be a finite group, let K be a field such that $\mathrm{char}(K)$ divides $|G|$, and let $S_1,\dots,S_n$ be a set of representatives of the isomorphism classes of simple $K[G]$-modules. We define a quiver Γ associated to $K[G]$ as follows: Its vertices are $S_1,\dots,S_n$. There is an arrow $S_i \to S_j$ if and only if there exists a non-split short exact sequence

$$0 \to S_j \to E \to S_i \to 0$$

of $K[G]$-modules. (In fact, one should work with multiple arrows here, namely the number of arrows $S_i \to S_j$ should be equal to $\dim \mathrm{Ext}^1_{K[G]}(S_i, S_j)$, but we did not introduce Ext-groups yet...)

The connected components of Γ parametrize the "blocks of $K[G]$": A K-algebra A is **connected** if it cannot be written as a product $A = A_1 \times A_2$ of non-zero K-algebras A_1 and A_2. Now write $K[G]$ as a product

$$K[G] = B_1 \times \cdots \times B_t$$

of connected algebras B_i. It turns out that the B_i are uniquely determined up to isomorphism and reordering. They are called the **blocks** of $K[G]$. The simple representations of a block B_i correspond to the vertices of a connected component Γ_i of Γ. To understand the representation theory of $K[G]$ is now the same as understanding the representation theory of each of the blocks. Such blocks are in general not any longer group algebras. Thus to understand group algebras, one is forced to study larger classes of finite-dimensional algebras. Each block is a "selfinjective algebra".

17.4. **Exercises.** Let $G = \mathbb{Z}_2$ be the group with two elements, and let K be a field.

1: Assume $\mathrm{char}(K) \neq 2$. Show: Up to isomorphism there are exactly two simple $K[G]$-modules.

2: Assume $\mathrm{char}(K) = 2$. Show: Up to isomorphism there are exactly two indecomposable $K[G]$-modules, and one of them is not simple.

3: Assume that K is an infinite field with $\mathrm{char}(K) = 2$. Construct an infinite number of 2-dimensional pairwise non-isomorphic representations of $K[G \times G]$.

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18. **Modules defined by idempotents**

In this section let A be a K-algebra.

Lemma 18.1. *Let e be an idempotent in A. The endomorphism ring $\mathrm{End}_A(Ae)$ of the A-module Ae is isomorphic to $(eAe)^{\mathrm{op}}$. In particular, $\mathrm{End}_A({}_AA)$ is isomorphic to A^{op}. We obtain an isomorphism*

$$\eta\colon \mathrm{End}_A(Ae) \to (eAe)^{\mathrm{op}}$$

which maps $f \in \mathrm{End}_A(Ae)$ to $f(e)$. Vice versa, for each $a \in A$, the inverse $\eta^{-1}(eae)$ is the right multiplication with eae.

Proof. Let $f \in \mathrm{End}_A(Ae)$, and let $a = f(e) \in Ae$. Then $a = ae$ because a belongs to Ae. Since f is a homomorphism, and e is an idempotent we have $a = f(e) = f(e^2) = ef(e) = ea$. Thus $a = eae \in eAe$. Clearly, the map defined by $\eta(f) = f(e)$ is K-linear.

Let $f_1, f_2 \in \mathrm{End}_A(Ae)$, and let $\eta(f_i) = f_i(e) = a_i$ for $i = 1, 2$. We get

$$\eta(f_1f_2) = (f_1f_2)(e) = f_1(f_2(e)) = f_1(a_2) = f_1(a_2e) = a_2f_1(e) = a_2a_1.$$

Thus η yields an algebra homomorphism $\eta\colon \mathrm{End}_A(Ae) \to (eAe)^{\mathrm{op}}$. (Note that the unit element of $(eAe)^{\mathrm{op}}$ is e.)

The algebra homomorphism η is injective: If $\eta(f) = 0$, then $f(e) = 0$ and therefore $f(ae) = af(e) = 0$ for all $a \in A$. Thus $f = 0$.

The map η is also surjective: For every $a \in A$ let $\rho_{eae}\colon Ae \to Ae$ be the right multiplication with eae defined by $\rho_{eae}(be) = beae$ where $b \in A$. This map is obviously an endomorphism of the A-module Ae, and we have $\eta(\rho_{eae}) = \rho_{eae}(e) = eae$.

Thus we have shown that η is bijective. In particular, the inverse $\eta^{-1}(eae)$ is the right multiplication with eae. □

Lemma 18.2. *If X is an A-module, then $\mathrm{Hom}_A(Ae, X) \cong eX$ as vector spaces.*

Proof. Let $\eta\colon \mathrm{Hom}_A(Ae, X) \to eX$ be the map defined by $\eta(f) = f(e)$. Since $f(e) = f(e^2) = ef(e)$, we have $f(e) \in eX$, thus this is well defined. It is also clear that η is K-linear.

If $f_1, f_2 \in \mathrm{Hom}_A(Ae, X)$, then $\eta(f_1) = \eta(f_2)$ implies $f_1(e) = f_2(e)$, and therefore

$$f_1(ae) = af_1(e) = af_2(e) = f_2(ae)$$

for all $a \in A$. So $f_1 = f_2$. This proves that η is injective.

Next, let $ex \in eX$. Define $f_x\colon Ae \to X$ by $f_x(ae) = aex$. It follows that $f_x(a_1a_2e) = a_1f_x(a_2e)$ for all $a_1, a_2 \in A$. Thus $f_x \in \mathrm{Hom}_A(Ae, X)$ and $\eta(f_x) = f_x(e) = ex$. So η is surjective. □

An idempotent $e \neq 0$ in a ring R is called **primitive** if e is the only non-zero idempotent in eRe.

Lemma 18.3. *A non-zero idempotent e in a ring R is primitive if and only if the following hold: Let $e = e_1 + e_2$ with e_1 and e_2 orthogonal idempotents, then $e_1 = 0$ or $e_2 = 0$.*

Proof. Let e_1 and e_2 be orthogonal idempotents with $e = e_1 + e_2$. Then $ee_1e = e_1$ and $ee_2e = e_2$. Thus e_1 and e_2 belong to eRe.

Vice versa, if e' is an idempotent in eRe, then e' and $e - e'$ is a pair of orthogonal idempotents with sum equal to e. □

Lemma 18.4. *Let e, e' be idempotents in A. Then the following are equivalent:*

(i) *The modules Ae and Ae' are isomorphic;*
(ii) *There exist some $x \in eAe'$ and $y \in e'Ae$ such that $xy = e$ and $yx = e'$.*

Proof. We can identify $\mathrm{Hom}_A(Ae, M)$ with eM: We just map $f\colon Ae \to M$ to $f(e)$. Since $e = e^2$ we get $f(e) = f(e^2) = ef(e)$, thus $f(e) \in eM$. Thus the homomorphisms $f \in \mathrm{Hom}_A(Ae, Ae')$ correspond to the elements in eAe'.

(i) $\Longrightarrow$ (ii): If Ae and Ae' are isomorphic, there exist homomorphisms $f\colon Ae \to Ae'$ and $g\colon Ae' \to Ae$ such that $gf = 1_{Ae}$. Set $x = f(e)$ and $y = g(e')$. Thus $x \in eAe'$, $y \in e'Ae$, $xy = e$ and $yx = e'$.

(ii) $\Longrightarrow$ (i): Assume there exist elements $x \in eAe'$ and $y \in e'Ae$ with $xy = e$ and $yx = e'$. Let $f\colon Ae \to Ae'$ be the right multiplication with x, and let $g\colon Ae' \to Ae$ be the right multiplication with y. Then f and g are A-module homomorphisms, and we have $gf = 1_{Ae}$ and $fg = 1_{Ae'}$. Thus the A-modules Ae and Ae' are isomorphic. □

The statement (ii) in the above lemma is left-right symmetric. Thus (i) and (ii) are also equivalent to

(iii) The A^{op}-modules eA and $e'A$ are isomorphic.

We want to compare A-modules and eAe-modules. If M is an A-module, then eM is an eAe-module.

Lemma 18.5. *Let e be an idempotent in A. If S is a simple A-module with $eS \neq 0$, then eS is a simple eAe-module.*

Proof. We have to show that every element $x \neq 0$ in eS generates the eAe-module eS. Since $x \in S$, we get $Ax = S$. Thus $eAex = eAx = eS$. Here we used that $x = ex$ for every element $x \in eS$. □

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19. **Quivers and path algebras**

Path algebras are an extremely important class of algebras. In fact, one of our main aims is to obtain a better understanding of their beautiful representation theory and also of the numerous links between representation theory of path algebras and other areas of mathematics.

Several parts of this section are taken from Crawley-Boevey's excellent lecture notes on representation theory of quivers.

19.1. **Quivers and path algebras.** Recall: A **quiver** is a quadruple

$$Q = (Q_0, Q_1, s, t)$$

where Q_0 and Q_1 are finite sets, and $s, t\colon Q_1 \to Q_0$ are maps. The elements in Q_0 are the **vertices** of Q, and the elements in Q_1 the **arrows**. For an arrow $a \in Q_1$ we call $s(a)$ the **starting vertex** and $t(a)$ the **terminal vertex** of a.

Thus we can think of Q as a finite directed graph. But note that multiple arrows and loops (a **loop** is an arrow a with $s(a) = t(a)$) are allowed.

Let $Q = (Q_0, Q_1, s, t)$ be a quiver. A sequence

$$a = (a_1, a_2, \dots, a_m)$$

of arrows $a_i \in Q_1$ is a **path** in Q if $s(a_i) = t(a_{i+1})$ for all $1 \leq i \leq m-1$. Such a path has **length** m, we write $l(a) = m$. Furthermore set $s(a) = s(a_m)$ and $t(a) = t(a_1)$. Instead of $(a_1, a_2, \ldots, a_m)$ we often just write $a_1 a_2 \cdots a_m$.

Additionally there is a path e_i of length 0 for each vertex $i \in Q_0$, and we set $s(e_i) = t(e_i) = i$.

The **path algebra** KQ of Q over K is the K-algebra with basis the set of all paths in Q. The multiplication of paths a and b is defined as follows:

If $a = e_i$ is of length 0, then

$$ab := a \cdot b := \begin{cases} b & \text{if } t(b) = i, \\ 0 & \text{otherwise.} \end{cases}$$

If $b = e_i$, then

$$ab := a \cdot b := \begin{cases} a & \text{if } s(a) = i, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, assume that $a = (a_1, \ldots, a_l)$ and $b = (b_1, \ldots, b_m)$ are paths of length $l, m \geq 1$. Then

$$ab := a \cdot b := \begin{cases} (a_1, \ldots, a_l, b_1, \ldots, b_m) & \text{if } s(a_l) = t(b_1), \\ 0 & \text{else.} \end{cases}$$

19.2. **Examples.** **1**: Let Q be the following quiver:

$$\begin{tikzcd} 2 \arrow[d, shift right, "c"'] \arrow[d, shift left, "d"] & 4 \arrow[l, "a"'] \arrow[r, "b"] \arrow[d, "e"] & 5 \\ 1 \arrow[r, "f"] & 3 & \end{tikzcd}$$

Then the paths in Q are

$$e_1, e_2, e_3, e_4, e_5, a, b, c, d, e, f, ca, da, fc, fd, fda, fca.$$

Thus, KQ is a 17-dimensional K-algebra. Here are some examples of multiplications of paths:

$$\begin{aligned} e_3 \cdot e_4 &= 0, \\ fc \cdot a &= fca, \\ a \cdot fc &= 0, \\ b \cdot e_4 &= b, \\ e_4 \cdot b &= 0, \\ e_5 \cdot b &= b. \end{aligned}$$

The algebra KQ has a unit element, namely $1 := e_1 + e_2 + e_3 + e_4 + e_5$.

2: Let Q be the quiver

$$Q: \quad \begin{tikzcd} 1 \arrow[loop left] \end{tikzcd}$$

Then KQ is isomorphic to the polynomial ring $K[T]$ in one variable T.

3: Let Q be the quiver

$$Q: \quad \circlearrowleft 1 \circlearrowright$$

Then KQ is isomorphic to the *free algebra* $K\langle X, Y\rangle$ in two non-commuting variables X and Y.

19.3. **Idempotents in path algebras.** Let $A = KQ$ for some quiver Q, and assume that $Q_0 = \{1, \ldots, n\}$.

Then the e_i are orthogonal idempotents, in other words $e_i^2 = e_i$ and $e_i e_j = 0$ for all $i \neq j$. Clearly,

$$1 = \sum_{i=1}^{n} e_i$$

is the identity of A. The vector spaces Ae_i and e_jA have as bases the set of paths starting in i and the set of paths ending in j, respectively. Furthermore, e_jAe_i has as a basis the set of paths starting in i and ending in j. We have

$$A = \bigoplus_{i=1}^{n} Ae_i.$$

Clearly, each Ae_i is a left A-module. So this is a direct decomposition of the regular representation ${}_AA$.

Lemma 19.1. *If $0 \neq x \in Ae_i$ and $0 \neq y \in e_iA$, then $xy \neq 0$.*

Proof. Look at the longest paths p and q involved in x and y, respectively. In the product xy the coefficient of pq cannot be zero. $\square$

Lemma 19.2. *The e_i are primitive idempotents.*

Proof. If $\mathrm{End}_A(Ae_i) \cong (e_iAe_i)^{\mathrm{op}}$ contains an idempotent f, then $f^2 = f = fe_i$. This implies $f(e_i - f) = 0$. Now use Lemma 19.1. $\square$

Corollary 19.3. *The A-modules Ae_i are indecomposable.*

Proof. The only idempotents in $\mathrm{End}_A(Ae_i)$ are 0 and 1. $\square$

Lemma 19.4. *If $e_i \in Ae_jA$, then $i = j$.*

Proof. The vector space Ae_jA has as a basis the paths passing through the vertex j. $\square$

Lemma 19.5. *If $i \neq j$, then $Ae_i \not\cong Ae_j$.*

Proof. Assume $i \neq j$ and that there exists an isomorphism $f\colon Ae_i \to Ae_j$. Set $y = f(e_i)$. It follows from Lemma 18.2 that $y \in e_iAe_j$. Let $g = f^{-1}$, and let $x = g(e_j)$. This implies

$$(gf)(e_i) = g(y) = g(ye_j) = yg(e_j) = yx = e_i.$$

A similar calculation shows that $xy = e_j$. But $y \in e_iAe_j$ and $x \in Ae_i$. Thus $y = e_iye_j$ and $x = xe_i$. This implies $e_j = xy = xe_iye_j \in Ae_iA$, a contradiction to Lemma 19.4. □

End of Lecture 22

19.4. **Representations of quivers.** A **representation** (or more precisely a K-**representation**) of a quiver $Q = (Q_0, Q_1, s, t)$ is given by a K-vector space V_i for each vertex $i \in Q_0$ and a linear map

$$V_a\colon V_{s(a)} \to V_{t(a)}$$

for each arrow $a \in Q_1$. Such a representation is called **finite-dimensional** if V_i is finite-dimensional for all i. In this case,

$$\dim V := \sum_{i \in Q_0} \dim V_i$$

is the **dimension** of the representation V.

For a path $p = (a_1, \ldots, a_m)$ of length $m \geq 1$ in Q, define

$$V_p := V_{a_1} \circ V_{a_2} \circ \cdots \circ V_{a_m} : V_{s(p)} \to V_{t(p)}.$$

A **morphism**

$$\theta\colon V \to W$$

between representations $V = (V_i, V_a)_{i,a}$ and $W = (W_i, W_a)_{i,a}$ is given by linear maps $\theta_i\colon V_i \to W_i$, $i \in Q_0$ such that the diagram

$$\begin{tikzcd} V_{s(a)} \arrow[r, "\theta_{s(a)}"] \arrow[d, "V_a"'] & W_{s(a)} \arrow[d, "W_a"] \\ V_{t(a)} \arrow[r, "\theta_{t(a)}"] & W_{t(a)} \end{tikzcd}$$

commutes for each $a \in Q_1$. The vector space of homomorphisms from V to W is denoted by $\operatorname{Hom}(V, W)$, or more precisely by $\operatorname{Hom}_Q(V, W)$.

A morphism $\theta = (\theta_i)_i\colon V \to W$ is an **isomorphism** if each θ_i is an isomorphism. In this case, we write $V \cong W$.

The **composition** $\psi \circ \theta$ of two morphisms $\theta\colon V \to W$ and $\psi\colon W \to X$ is given by $(\psi \circ \theta)_i = \psi_i \circ \theta_i$.

The K-representations form a K-category denoted by $\operatorname{Rep}(Q) = \operatorname{Rep}_K(Q)$. The full subcategory of finite-dimensional representations is denoted by $\operatorname{rep}(Q) = \operatorname{rep}_K(Q)$.

A **subrepresentation** of a representation $(V_i, V_a)_{i,a}$ is given by a tuple $(U_i)_i$ of subspaces U_i of V_i such that

$$V_a(U_{s(a)}) \subseteq U_{t(a)}$$

for all $a \in Q_1$. In this case, we obtain a representation $(U_i, U_a)_{i,a}$ where $U_a\colon U_{s(a)} \to U_{t(a)}$ is defined by $U_a(u) = V_a(u)$ for all $u \in U_{s(a)}$.

The **direct sum** of representations $V = (V_i, V_a)_{i,a}$ and $W = (W_i, W_a)_{i,a}$ is defined in the obvious way, just take $V \oplus W := (V_i \oplus W_i, V_a \oplus W_a)_{i,a}$.

Now we can speak about **simple representations** and **indecomposable representations**. (As for modules, it is part of the definition of a simple and of an indecomposable representation V that $V \neq 0$.)

19.5. **Examples.** **1**: For $i \in Q_0$ let S_i be the representation with

$$(S_i)_j = \begin{cases} K & \text{if } i = j, \\ 0 & \text{else.} \end{cases}$$

for all $j \in Q_0$, and set $(S_i)_a = 0$ for all $a \in Q_1$. Obviously, S_i is a simple representation.

2: For $\lambda \in K$ let V_λ be the representation

$$K \xrightarrow{\lambda} K$$

of the quiver $1 \longrightarrow 2$. Then $V_\lambda \cong V_\mu$ if and only if $\lambda = 0 = \mu$ or $\lambda \neq 0 \neq \mu$. We have

$$\dim \mathrm{Hom}(V_\lambda, V_\mu) = \begin{cases} 0 & \text{if } \lambda \neq 0 \text{ and } \mu = 0, \\ 1 & \text{if } \mu \neq 0, \\ 2 & \text{if } \lambda = 0 = \mu. \end{cases}$$

3: For λ_1, λ_2 let V_{λ_1,λ_2} be the representation

$$\begin{tikzcd} K \arrow[r, shift left, "\lambda_1"] \arrow[r, shift right, "\lambda_2"'] & K \end{tikzcd}$$

of the quiver $1 \rightrightarrows 2$. Then $V_{\lambda_1,\lambda_2} \cong V_{\mu_1,\mu_2}$ if and only if there exists some $c \neq 0$ with $c(\lambda_1, \lambda_2) = (\mu_1, \mu_2)$: Assume there exists an isomorphism

$$\theta = (\theta_1, \theta_2) \colon V_{\lambda_1,\lambda_2} \to V_{\mu_1,\mu_2}.$$

Thus $\theta = (a, b)$ for some $a, b \in K^*$. We obtain a diagram

$$\begin{tikzcd} K \arrow[r, "a"] \arrow[d, shift right, "\lambda_1"'] \arrow[d, shift left, "\lambda_2"] & K \arrow[d, shift right, "\mu_1"'] \arrow[d, shift left, "\mu_2"] \\ K \arrow[r, "b"] & K \end{tikzcd}$$

satisfying $b\lambda_1 = \mu_1 a$ and $b\lambda_2 = \mu_2 a$. Set $c = a^{-1}b$. It follows that $c(\lambda_1, \lambda_2) = (\mu_1, \mu_2)$.

4: For $\lambda \in K$ let V_λ be the representation

$$\begin{tikzcd} K \arrow[loop left, "\lambda"] \end{tikzcd}$$

of the *1-loop quiver*. Then $V_\lambda \cong V_\mu$ if and only if $\lambda = \mu$.

5: Let V be the representation

$$\begin{tikzcd} K \arrow[r, shift left, "{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]}"] \arrow[r, shift right, "{\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]}"'] & K^2 \end{tikzcd}$$

of the quiver $1 \rightrightarrows 2$. The subrepresentations of V are (K, K^2) and $(0, U)$ where U runs through all subspaces of K^2. It is easy to check that none of these subrepresentations is a direct summand of V. Thus V is an indecomposable representation.

19.6. **Representations of quivers and modules over path algebras.** Let $V = (V_i, V_a)_{i,a}$ be a representation. Let

$$\eta\colon KQ \times \bigoplus_{i \in Q_0} V_i \to \bigoplus_{i \in Q_0} V_i$$

be the map defined by

$$\eta(e_j, v_i) = \begin{cases} v_i & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \eta(p, v_i) = \begin{cases} V_p(v_i) & \text{if } i = s(p), \\ 0 & \text{otherwise} \end{cases}$$

where the e_j are the paths of length 0, p runs through the set of paths of length at least one and $v_i \in V_i$. Then we extend these rules linearly.

Vice versa, let V be a KQ-module, i.e. there is a KQ-module structure

$$\eta\colon KQ \times V \to V$$

on the K-vector space V. For each path e_i of length 0 define $V_i := e_i V$, which is clearly a K-vectorspace. It follows that

$$V = \bigoplus_{i \in Q_0} V_i.$$

For each arrow $a \in Q_1$ define a linear map

$$V_a\colon V_{s(a)} \to V_{t(a)}$$

by $V_a(v) := \eta(a, v)$ for all $v \in V_{s(a)}$. Then $(V_i, V_a)_{i,a}$ is obviously a representation of Q.

We leave it as an exercise to show that these constructions yield equivalences of K-categories between $\operatorname{Rep}_K(Q)$ and $\operatorname{Mod}(KQ)$.

So from now on we can use all the terminology and the results we obtained for modules over algebras also for representations of quivers. In particular, we get a Jordan-Hölder and a Krull-Remak-Schmidt Theorem, we can ask for Auslander-Reiten sequences of quiver representations, etc. We will often not distinguish any longer between a representation of Q and a module over KQ.

If $V = (V_i, V_a)_{i,a}$ is a representation of Q let

$$d = \underline{\dim}(V) = (\dim V_i)_{i \in Q_0}$$

be its **dimension vector**. If V is a finite-dimensional indecomposable representation, then $\underline{\dim}(V) \in \mathbb{N}^{Q_0}$ is called a **root** of Q. A root d is a **Schur root** if there exists a representation V with $\mathrm{End}_Q(V) \cong K$ and $\underline{\dim}(V) = d$. Assume that d is a root. If there exists a unique (up to isomorphism) indecomposable representation V with $\dim(V) = d$, then d is called a **real root**. Otherwise, d is an **imaginary root**.

A representation V of Q is **rigid** (or **exceptional**) if each short exact sequence

$$0 \to V \to W \to V \to 0$$

splits, i.e. if $W \cong V \oplus V$.

Here are some typical problems appearing in representation theory of quivers:

(i) Classify all indecomposable representations of Q. (This is usually very hard and can only be achieved for very few quivers.)
(ii) Determine all roots of Q. Determine the real roots and the Schur roots of Q.
(iii) Classify all rigid representations of Q.
(iv) Compute the Auslander-Reiten quiver of $\mathrm{mod}(KQ)$, or at least try to describe the shape of its connected components.
(v) How does the representation theory of a quiver Q change, if we change the orientation of an arrow of Q?

19.7. **Exercises.** **1**: Let Q be a quiver. Show that KQ is finite-dimensional if and only if Q has no oriented cycles.

2: Let $V = (K \xleftarrow{1} K \xrightarrow{1} K)$ and $W = (K \xleftarrow{1} K \to 0)$ be representations of the quiver $1 \leftarrow 2 \to 3$. Show that $\mathrm{Hom}_Q(V, W)$ is one-dimensional, and that $\mathrm{Hom}_Q(W, V) = 0$.

3: Let Q be the quiver

$$1 \to 2 \to \cdots \to n.$$

Show that KQ is isomorphic to the subalgebra

$$A := \{M \in M_n(K) \mid m_{ij} = 0 \text{ if there is no path from } j \text{ to } i\}$$

of $M_n(K)$.

4: Let Q be any quiver. Determine the centre of KQ. (Reminder: The **centre** $C(A)$ of an algebra A is defined as $C(A) = \{a \in A \mid ab = ba \text{ for all } b \in A\}$.)

5: Let Q be a quiver with n vertices. Show that there are n isomorphism classes of simple KQ-modules if and only if Q has no oriented cycles.

6: Let Q be a quiver. Show that the categories $\mathrm{Rep}_K(Q)$ and $\mathrm{Mod}(KQ)$ are equivalent.

7: Construct an indecomposable representation of the quiver

$$\begin{tikzcd} & & \circ \arrow[d] & & \\ \circ \arrow[r] & \circ \arrow[r] & \circ & \circ \arrow[l] & \circ \arrow[l] \end{tikzcd}$$

with dimension vector

$$\begin{matrix} & & 1 & & \\ 1 & 2 & 3 & 2 & 1 \end{matrix}$$

8: Show: If $V = (V_i, V_a)_{i \in Q_0, a \in Q_1}$ is an indecomposable representation of the quiver

$$Q\colon \qquad \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ \longrightarrow \circ$$

then $\dim V_i \leq 1$ for all $i \in Q_0$.

Construct the Auslander-Reiten quiver of Q.

9: Let Q be the following quiver:

$$\begin{tikzcd} & & \circ \arrow[d] & & & \\ \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \arrow[r] & \circ \end{tikzcd}$$

Let $A = KQ$. Write ${}_AA$ as a direct sum of indecomposable representations and compute the dimension of the indecomposable direct summands.

10: Let

$$A = \begin{bmatrix} K[T]/(T^2) & 0 \\ K[T]/(T^2) & K \end{bmatrix}.$$

This gives a K-algebra via the usual matrix multiplication. (The elements of A are of the form

$$\begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$$

where $a, b \in K[T]/(T^2)$ and $c \in K$.) Show that A is isomorphic to KQ/I where Q is the quiver

$$\begin{tikzcd} \circ \arrow[loop left, "\alpha"] \arrow[r] & \circ \end{tikzcd}$$

and I is the ideal in KQ generated by the path $\alpha^2 := (\alpha, \alpha)$.

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20. **Digression: Classification problems in Linear Algebra**

Many problems in Linear Algebra can be reformulated using quivers. In this section, we give some examples of this kind.

20.1. **Classification of endomorphisms.**

$$Q: \quad \begin{tikzcd} 1 \arrow[loop left] \end{tikzcd}$$

Let K be an algebraically closed field, and let V be a finite-dimensional K-vector space of dimension n. By $\operatorname{End}_K(V)$ we denote the set of K-linear maps $V \to V$, and by $G = \mathrm{GL}(V)$ the set of invertible K-linear maps $V \to V$. For $f \in \mathrm{End}_K(V)$ let

$$Gf = \{g^{-1}fg \mid g \in G\} \subseteq \mathrm{End}_K(V)$$

be the G-*orbit* of f. One easily checks that for $f_1, f_2 \in \mathrm{End}_K(V)$ we have either $Gf_1 = Gf_2$ or $Gf_1 \cap Gf_2 = \emptyset$.

Question 20.1. *Can we classify all G-orbits?*

Answer: Of course we can, since we paid attention in our Linear Algebra lectures.

For $n = 0$ everything is trivial, there is just one orbit containing only the zero map. Thus assume $n \geq 1$. Fix a basis B of V. Now each map $f \in \mathrm{End}_K(V)$ is (with respect to B) given by a particular matrix, which (via conjugation) can be transformed to a Jordan normal form. It follows that each orbit Gf is uniquely determined by a set

$$\{(n_1, \lambda_1), (n_2, \lambda_2), \dots, (n_t, \lambda_t)\}$$

where the n_i are positive integers with $n_1 + \cdots + n_t = n$, and the λ_i are elements in K. Here (n_i, λ_i) stands for a Jordan block of size n_i with Eigenvalue λ_i.

20.2. **Classification of homomorphisms.**

$$Q: \qquad 1 \longrightarrow 2$$

Let K be any field, and let V_1 and V_2 be finite-dimensional K-vector spaces of dimension n_1 and n_2, respectively. By $\mathrm{Hom}_K(V_1, V_2)$ we denote the set of K-linear maps $V_1 \to V_2$, and let $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$. For $f \in \mathrm{Hom}_K(V_1, V_2)$ let

$$Gf = \{h^{-1}fg \mid (g, h) \in G\} \subseteq \mathrm{Hom}_K(V_1, V_2)$$

be the G-*orbit* of f.

Question 20.2. *Can we classify all G-orbits?*

Answer: Of course we can. This is even easier than the previous problem: Fix bases B_1 and B_2 of V_1 and V_2, respectively. Then each $f \in \mathrm{Hom}_K(V_1, V_2)$ is given by a matrix with respect to B_1 and B_2. Now using row and column transformations (which can be expressed in terms of matrix multiplication from the left and right) we can transform the matrix of f to a matrix of the form

$$\begin{pmatrix} E_r & 0 \\ 0 & 0 \end{pmatrix}$$

where E_r is the $r \times r$-unit matrix and the zeros are matrices with only zero entries. Here r is the rank of the matrix of f.

It turns out that there are $1+\min\{n_1, n_2\}$ different G-orbits where n_i is the dimension of V_i.

20.3. **The Kronecker problem.**

$$Q: \qquad 1 \rightrightarrows 2$$

Let K be an algebraically closed field, and let V_1 and V_2 be finite-dimensional K-vector spaces. Let $G = \mathrm{GL}(V_1) \times \mathrm{GL}(V_2)$. For $(f_1, f_2) \in \mathrm{Hom}_K(V_1, V_2) \times \mathrm{Hom}_K(V_1, V_2)$ let

$$G(f_1, f_2) = \{(h^{-1}f_1g, h^{-1}f_2g) \mid (g, h) \in G\} \subseteq \mathrm{Hom}_K(V_1, V_2) \times \mathrm{Hom}_K(V_1, V_2)$$

be the G-*orbit* of (f_1, f_2).

Question 20.3. *Can we classify all G-orbits?*

Answer: Yes, we can do that, by we will need a bit of theory here. As you can see, the problem became more complicated, because we simultaneously transform the matrices of f_1 and f_2 with respect to some fixed bases of V_1 and V_2.

Example: The orbits $G(K \xrightarrow{1} K, K \xrightarrow{\lambda} K)$ and $G(K \xrightarrow{1} K, K \xrightarrow{\mu} K)$ are equal if and only if $\lambda = \mu$.

20.4. **The n-subspace problem.**

$$\begin{tikzcd} Q: & 1 \arrow[dr] & 2 \arrow[d] & \cdots & n \arrow[dll] \\ & & 0 & & \end{tikzcd}$$

An n-*subspace configuration* is just an $n+1$-tuple $(V, V_1, \ldots, V_n)$ where V is a vector space and the V_i are subspaces of V. We call

$$\underline{\dim}(V, V_1, \ldots, V_n) = (\dim V, \dim V_1, \ldots, \dim V_n)$$

the *dimension vector* of the n-subspace configuration $(V, V_1, \ldots, V_n)$.

We say that two n-subspace configurations $(V, V_1, \ldots, V_n)$ and $(W, W_1, \ldots, W_n)$ are *isomorphic* if there exists an isomorphism (= bijective linear map) $f: V \to W$ such that the following hold:

- $f(V_i) \subseteq W_i$;
- The linear maps $f_i: V_i \to W_i$ defined by $f_i(v_i) = f(v_i)$ where $1 \le i \le n$ and $v_i \in V_i$ are isomorphisms.

In particular, two isomorphic n-subspace configurations have the same dimension vector.

Problem 20.4. *Classify all n-subspace configurations up to isomorphism.*

We can reformulate this problem as follows: Let $V, V_1, \ldots, V_n$ be vector spaces such that $\dim V_i \le \dim V$ for all i. Set

$$Z = \mathrm{Inj}(V_1, V) \times \cdots \times \mathrm{Inj}(V_n, V)$$

where $\mathrm{Inj}(V_i, V)$ denotes the set of injective linear maps from $V_i \to V$. Let $G = \mathrm{GL}(V) \times \mathrm{GL}(V_1) \times \cdots \times \mathrm{GL}(V_n)$. Each element $(f_1, \ldots, f_n)$ can be thought of as an n-subspace configuration given by $(V, \mathrm{Im}(f_1), \ldots, \mathrm{Im}(f_n))$.

Then G acts on Z as follows: For $(f_1, \ldots, f_n)$ and $g = (g_0, g_1, \ldots, g_n) \in G$ define

$$g \cdot (f_1, \ldots, f_n) = (g_0^{-1} f_1 g_1, \ldots, g_0^{-1} f_n g_n)$$

and let

$$G(f_1, \ldots, f_n) = \{g \cdot (f_1, \ldots, f_n) \mid g \in G\}$$

be the G-orbit of $(f_1, \ldots, f_n)$. Classifying all n-subspace configurations with dimension vector $(\dim V, \dim V_1, \ldots, \dim V_n)$ up to isomorphism corresponds to classifying the G-orbits in Z.

It turns out that Problem 20.4 is much too hard for large n. But for small n one can solve it.

Given two n-subspace configurations $(V, V_1, \ldots, V_n)$ and $(W, W_1, \ldots, W_n)$, we define their *direct sum* by

$$(V, V_1, \ldots, V_n) \oplus (W, W_1, \ldots, W_n) = (V \oplus W, V_1 \oplus W_1, \ldots, V_n \oplus W_n).$$

It follows that $(V, V_1, \ldots, V_n) \oplus (W, W_1, \ldots, W_n)$ is again an n-subspace configuration.

An n-subspace configuration $(V, V_1, \ldots, V_n)$ is **indecomposable** if it is not isomorphic to the direct sum of two non-zero n-subspace configurations. (We say that an n-subspace configuration $(V, V_1, \ldots, V_n)$ is *zero*, if $V = 0$.)

One can prove that any n-subspace configuration can be written (in a "unique way") as a direct sum of indecomposable n-subspace configurations. Thus to classify all n-subspace configurations, it is enough to classify the indecomposable ones.

We will see for which n there are only finitely many indecomposable n-subspace configurations.

Instead of asking for the classification of all n-subspace configurations, we might ask the following easier question:

Problem 20.5. *Classify the dimension vectors of the indecomposable n-subspace configurations.*

It turns out that there is a complete answer to Problem 20.5.

20.5. **Exercises. 1**: Classify all indecomposable 3-subspace configurations. Does the result depend on the field K?

2: Solve the Kronecker problem as described above for $V_1 = V_2 = K^2$ where K is an algebraically closed field.

3: Find the publication of Kronecker where he solves the Kronecker problem.

21. **Large and small submodules**

21.1. **Large and small submodules.** Let V be a module, and let U be a submodule of V. The module U is called **large** in V if $U \cap U' \neq 0$ for all non-zero submodules U' of V. The module U is **small** in V if $U + U' \subset V$ for all proper submodules U' of V.

Lemma 21.1. *Let U_1 and U_2 be submodules of a module V. If U_1 and U_2 are large in V, then $U_1 \cap U_2$ is large in V. If U_1 and U_2 are small in V, then $U_1 + U_2$ is small in V.*

Proof. Let U_1 and U_2 be large submodules of V. If U is an arbitrary non-zero submodule of V, then $U_2 \cap U \neq 0$, since U_2 is large. But we also get $U_1 \cap (U_2 \cap U) = U_1 \cap U_2 \cap U \neq 0$, since U_1 is large. This implies $(U_1 \cap U_2) \cap U \neq 0$. Thus $U_1 \cap U_2$ is large as well.

If U_1 and U_2 are small submodule of V, and if U is an arbitrary submodule of V with $U_1 + U_2 + U = V$, then $U_2 + U = V$, since U_1 is small. But this implies $U = V$, since U_2 is small as well. □

Lemma 21.2. *For $1 \leq i \leq n$ let U_i be a submodule of a module V_i. Set $U = U_1 \oplus \cdots \oplus U_n$ and $V = V_1 \oplus \cdots \oplus V_n$. Then the following hold:*

- *U is large in V if and only if U_i is large in V_i for all i;*
- *U is small in V if and only if U_i is small in V_i for all i.*

Proof. Let U be large in V. For some j let $W_j \neq 0$ be a submodule of V_j. Now we consider W_j as a submodule of V. Since U is large in V, we get that $W_j \cap U \neq 0$. But we have

$$W_j \cap U = (W_j \cap V_j) \cap U = W_j \cap (V_j \cap U) = W_j \cap U_j.$$

This shows that $W_j \cap U_j \neq 0$. So we get that U_j is large in W_j.

To show the converse, it is enough to consider the case $n = 2$. Let U_i be large in V_i for $i = 1, 2$. Set $V = V_1 \oplus V_2$. We first show that $U_1 \oplus V_2$ is large in V: Let $W \neq 0$ be a submodule of V. If $W \subseteq V_2$, then $0 \neq W \subseteq U_1 \oplus V_2$. If $W \not\subseteq V_2$, then $V_2 \subset W + V_2$ and therefore $V_1 \cap (W + V_2) \neq 0$. This is a submodule of V_1, thus $U_1 \cap V_1 \cap (W + V_2) \neq 0$ because U_1 is large in V_1. Since $U_1 \cap (W + V_2) \neq 0$, there exists a non-zero element $u_1 \in U_1$ with $u_1 = w + v_2$ where $w \in W$ and $v_2 \in V_2$. This implies $w = u_1 - v_2 \in W \cap (U_1 \oplus V_2)$. Since $0 \neq u_1 \in V_1$ and $v_2 \in V_2$ we get $w \neq 0$. Thus we have shown that $U_1 \oplus V_2$ is large in V. In the same way one shows that $V_1 \oplus U_2$ is large in V. The intersection of these two modules is $U_1 \oplus U_2$. But the intersection of two large modules is again large. Thus $U_1 \oplus U_2$ is large in V.

Next, assume that U is small in V. For some j let W_j be a submodule of V_j with $U_j + W_j = V_j$. Set

$$W := W_j \oplus \bigoplus_{i \neq j} V_i.$$

This is a submodule of V with $U + W = V$. Since U is small in V, we get $W = V$, and therefore $W_j = V_j$.

To show the converse, it is enough to consider the case $n = 2$. For $i = 1, 2$ let U_i be small in V_i, and set $V = V_1 \oplus V_2$. We show that $U_1 = U_1 \oplus 0$ is small in V: Let W be a submodule of V with $U_1 + W = V$. Since $U_1 \subseteq V_1$ we get

$$U_1 + (W \cap V_1) = (U_1 + W) \cap V_1 = V \cap V_1 = V_1.$$

Now U_1 is small in V_1, which implies $W \cap V_1 = V_1$. Therefore $V_1 \subseteq W$. In particular, $U_1 \subseteq W$ and $W = U_1 + W = V$. In the same way one shows that $U_2 = 0 \oplus U_2$ is small in V. Since the sum of two small modules is again small, we conclude that $U_1 \oplus U_2$ is small in V. □

End of Lecture 23

Let V be a module, and let U be a submodule of V. A submodule U' of V is called a **maximal complement** of U in V if the following hold:

- $U \cap U' = 0$;
- If U'' is a submodule with $U' \subset U''$, then $U \cap U'' \neq 0$.

If U' is a maximal complement of U, then $U + U' = U \oplus U'$.

Lemma 21.3. *Let V be a module. Every submodule U of V has a maximal complement. If U' is a maximal complement of U, then $U \oplus U'$ is large in V.*

Proof. We show the existence by using Zorn's Lemma: Let U be a submodule of V, and let $\mathcal{W}$ be the set of all submodules W of V with $U \cap W = 0$. Clearly, this set is non-empty, and if W_i, $i \in I$ form a chain in $\mathcal{W}$, then also

$$\bigcup_{i \in I} W_i$$

is in $\mathcal{W}$. Thus $\mathcal{W}$ contains maximal elements. But if U' is maximal in $\mathcal{W}$, then U' is a maximal complement of U.

If U' is a maximal complement of U in V, and if W is a submodule of V with $(U \oplus U') \cap W = 0$, then $U + U' + W = U \oplus U' \oplus W$. Thus $U \cap (U' \oplus W) = 0$. The maximality of U' yields $U' \oplus W = U'$. This implies $W = 0$. It follows that $U \oplus U'$ is large in V. □

Recall that a module V is called **uniform** if $U_1 \cap U_2 \neq 0$ for all non-zero submodules U_1 and U_2 of V. It is easy to show that V is uniform if and only if all non-zero submodules of V are large. It follows that a module V is uniform and has a simple socle if and only if V contains a large simple submodule.

Lemma 21.4. *Let $U \neq 0$ be a cyclic submodule of a module V, and let W be a submodule of V with $U \not\subseteq W$. Then there exists a submodule W' of V with $W \subseteq W'$ such that $U \not\subseteq W'$ and W' is maximal with these properties. Furthermore, for each such W', the module V/W' is uniform and has a simple socle, and we have*

$$\mathrm{soc}(V/W') = (U + W')/W'.$$

Proof. Assume U is generated by x. Let $\mathcal{V}$ be the set of all submodules V' of V with $W \subseteq V'$ and $x \notin V'$.

Since W belongs to $\mathcal{V}$, we known that $\mathcal{V}$ is non-empty. If V_i, $i \in I$ is a chain of submodules in $\mathcal{V}$, then $\bigcup_{i \in I} V_i$ also belongs to $\mathcal{V}$. (For each $y \in \bigcup_{i \in I} V_i$ we have $y \in V_i$ for some i.) Now Zorn's Lemma yields a maximal element in $\mathcal{V}$.

Let W' be maximal in $\mathcal{V}$. Thus we have $W \subseteq W'$, $x \notin W'$ and $U \not\subseteq W'$. If now W'' is a submodule of V with $W' \subset W''$, then W'' does not belong to $\mathcal{V}$. Therefore $x \in W''$ and also $U \subseteq W''$.

Since $W' \subset U + W'$, we know that $(U + W')/W' \neq 0$. Every non-zero submodule of V/W' is of the form W''/W' for some submodule W'' of V with $W' \subset W''$. It follows that $U \subseteq W''$ and that $(U + W')/W' \subseteq W''/W'$. This shows that $(U + W')/W'$ is simple. We also get that $(U + W')/W'$ is large in V/W'. This implies $\mathrm{soc}(V/W') = (U + W')/W'$. □

Corollary 21.5. *Let $U \neq 0$ be a cyclic submodule of a module V, and let W be a submodule of V with $U \not\subseteq W$. If $U + W = V$, then there exists a maximal submodule W' of V with $W \subseteq W'$.*

Proof. Let W' be a submodule of V with $W \subseteq W'$ and $U \not\subseteq W'$ such that W' is maximal with these properties. Assume $U + W = V$. This implies $U + W' = V$. By Lemma 21.4 we know that

$$V/W' = (U + W')/W' = \mathrm{soc}(V/W')$$

is simple. Thus W' is a maximal submodule of V. □

Corollary 21.6. *For a finitely generated module V the following hold:*

(i) $\mathrm{rad}(V)$ *is small in V;*
(ii) *If $V \neq 0$, then* $\mathrm{rad}(V) \subset V$*;*
(iii) *If $V \neq 0$, then V has maximal submodules.*

Proof. Clearly, (i) implies (ii) and (iii). Let us prove (i): Assume V is a finitely generated module, and let $x_1, \dots, x_n$ be a generating set of V. Furthermore, let W be a proper submodule of V. We show that $W + \mathrm{rad}(V)$ is a proper submodule: For $0 \leq t \leq n$ let W_t be the submodule of V which is generated by W and the elements $x_1, \dots, x_t$. Thus we obtain a chain of submodules

$$W = W_0 \subseteq W_1 \subseteq \cdots \subseteq W_n = V.$$

Since $W \subset V$, there exists some t with $W_{t-1} \subset W_t = V$. Let U be the (cyclic) submodule generated by x_t. We get

$$U + W_{t-1} = W_t = V,$$

and $U \not\subseteq W_{t-1}$. By Corollary 21.5 this implies that there exists a maximal submodule W' of V with $W_{t-1} \subseteq W'$. Since W' is a maximal submodule of V, we get $\mathrm{rad}(V) \subseteq W'$. Thus

$$W + \mathrm{rad}(V) \subseteq W + W' = W' \subset V.$$

This shows that $\mathrm{rad}(V)$ is small in V. □

Corollary 21.7. *Every proper submodule of a finitely generated module V is contained in a maximal submodule of V.*

Proof. This follows from the proof of Corollary 21.6. □

Proposition 21.8. *Let V be a module. The intersection of all large submodules of V is equal to* $\mathrm{soc}(V)$.

Proof. Let U_0 be the intersection of all large submodules of V. We want to show that $\mathrm{soc}(V)$ is contained in every large submodule of V. This implies then $\mathrm{soc}(V) \subseteq U_0$.

Let U be a large submodule of V. Assume $\mathrm{soc}(V)$ is not contained in U. Then $U \cap \mathrm{soc}(V)$ is a proper submodule of $\mathrm{soc}(V)$. Since $\mathrm{soc}(V)$ is generated by simple submodules, there exists a simple submodule S of V which is not contained in U. Now S is simple and therefore $U \cap S = 0$. Since $S \neq 0$, this is a contradiction. This implies $\mathrm{soc}(V) \subseteq U_0$.

Vice versa, we claim that U_0 is semisimple: Let W be a submodule of U_0. We have to show that W is a direct summand of U_0. Let W' be a maximal complement of W in V. Since $W \cap W' = 0$, we get $W \cap (W' \cap U_0) = 0$. It follows that $W + (W' \cap U_0) = U_0$: Since $W + W'$ is large in V, we have $U_0 \subseteq W + W'$. Thus

$$W + (W' \cap U_0) = (W + W') \cap U_0 = U_0.$$

Here we used modularity. Summarizing, we see that $W' \cap U_0$ is a direct complement of W in U_0. Thus W is a direct summand of U_0. This shows that U_0 is semisimple, which implies $U_0 \subseteq \mathrm{soc}(V)$. □

Proposition 21.9. *Let V be a module. The sum of all small submodules of V is equal to* $\mathrm{rad}(V)$. *A cyclic submodule U of V is small in V if and only if $U \subseteq$* $\mathrm{rad}(V)$.

Proof. Let W be a maximal submodule of V. If U is a small submodule of V, we get $U \subseteq W$. (Otherwise $W \subset U + W = V$ by the maximality of W, and therefore $W = V$ since U is small in V.) Thus every small submodule of V is contained in $\mathrm{rad}(V)$. The same is true, if there are no maximal submodules in V, since in this case we have $\mathrm{rad}(V) = V$.

Let U be a cyclic submodule contained in $\mathrm{rad}(V)$. We want to show that U is small in V. Let U' be a proper submodule of V. Assume that $U + U' = V$. Since U' is a proper submodule, U cannot be a submodule of U'. Thus there exists a maximal submodule W' with $U' \subseteq W'$. Since $U + U' = V$, we obtain $U + W' = V$. In particular, U is not contained in W'. But U lies in the radical of V, and is therefore a submodule of any maximal submodule of V, a contradiction. This proves that $U + U' \subset V$, thus U is small in V.

Let U_0 be the sum of all small submodule of V. We have shown already that $U_0 \subseteq \mathrm{rad}(V)$. Vive versa, we show that $\mathrm{rad}(V) \subseteq U_0$: Let $x \in \mathrm{rad}(V)$. The cyclic submodule $U(x)$ generated by x is small, thus it is contained in U_0. In particular, $x \in U_0$. Thus we proved that $U_0 = \mathrm{rad}(V)$. □

21.2. **Local modules defined by idempotents.** Recall that a module V is called **local** if it contains a maximal submodule U, which contains all proper submodules of V.

Lemma 21.10. *Let e be an idempotent in A. Then Ae is a local module if and only if eAe is a local ring.*

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Proof. Let Ae be a local module, and let M be the maximal submodule of Ae. For every element $x \in Ae$ we have $x = xe$, thus $M = Me$. We have $eM = eAe \cap M$. (Clearly, $eM \subseteq eAe \cap M$. The other inclusion follows from the fact that e is an idempotent: If $a \in A$ and $eae \in M$, then $eae = e(eae) \in eM$.)

In particular we have $eM = eMe \subseteq eAe$. We have $e \in eAe$, but e does not belong to M or eM. Thus $eMe \subset eAe$.

We claim that eMe is an ideal in eAe: Clearly, $eAe \cdot eMe \subseteq eMe$. Since the right multiplications with the elements from eAe are the endomorphisms of Ae, we have $Me \cdot eAe \subseteq Me$. ($Me$ is the radical of the module Ae.) Thus $eMe \cdot eAe \subseteq eMe$.

If $x \in eAe \setminus eMe$, then $x \in Ae$ and $x \notin M$. (Note that $exe = x$.) Thus x generates the local module Ae. It follows that there exists some $y \in A$ with $yx = e$. Because $x = ex$, we have

$$eye \cdot x = eyx = e^2 = e.$$

Thus x is left-invertible in eAe, and eye is right-invertible in eAe.

The element eye does not belong to eM, since eM is closed under right multiplication with elements from eAe, and $e \notin eM$. So we get $eye \in eAe \setminus eMe$.

Thus also the element eye has a left inverse in eAe. This proves that eye is invertible in eAe. It follows that exe is invertible in eAe: Namely, we have

$$(eye)^{-1} \cdot eye \cdot x = (eye)^{-1}e.$$

Multiplying both sides of this equation from the right with eye yields $x \cdot eye = e$.

We have shown that all elements in $eAe \setminus eMe$ are invertible, thus eAe is a local ring.

Vice versa, assume that eAe is a local ring. Then Ae is a non-zero cyclic module, thus it has maximal submodules. Let M_1 be a maximal submodule, and let M_2 be any proper submodule of Ae. Suppose M_2 is not contained in M_1. This implies $Ae = M_1 + M_2$, thus $e = x_1 + x_2$ with $x_i \in M_i$. We have $e = ee = ex_1 + ex_2$. Since eAe is a local ring, one of the elements $ex_i = ex_ie$, $i = 1, 2$ is invertible in eAe. If $e = yex_i$, then e belongs to $Ax_i \subseteq M_i$, thus $Ae = M_i$. By assumption both modules M_1 and M_2 are proper submodules of Ae. This contradiction shows that M_1 contains all proper submodules of Ae, thus Ae is a local module. □

21.3. **Exercises.** **1**: Are small submodules really "small", and are large submodules really "large"?

2: Classify the small submodules of $(K[T], T\cdot)$ and $N(\infty)$.

Are $(K[T], T\cdot)$ and $N(\infty)$ uniform modules?

3: Find an example of a module V and a submodule U of V such that U is large and small in V.

4: When is 0 large (resp. small) in a module V?

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22. **The Jacobson radical of an algebra**

In this section let A be a K-algebra.

22.1. **The radical of an algebra.** The **radical** of A is defined as

$$J(A) := \mathrm{rad}({}_AA).$$

In other words, $J(A)$ is the intersection of all maximal left ideals of A. Often one calls $J(A)$ the **Jacobson radical** of A.

Since the A-module ${}_AA$ is finitely generated (it is cyclic), we know that ${}_AA$ contains maximal submodules, provided $A \neq 0$. In particular, $J(A) = A$ if and only if $A = 0$.

Lemma 22.1. *The radical $J(A)$ is a two-sided ideal.*

Proof. As an intersection of left ideals, $J(A)$ is a left ideal. It remains to show that $J(A)$ is closed with respect to right multiplication. Let $a \in A$, then the right multiplication with a is a homomorphism ${}_AA \to {}_AA$, and it maps $\mathrm{rad}({}_AA)$ to $\mathrm{rad}({}_AA)$. □

Lemma 22.2. *If A is semisimple, then $J(A) = 0$.*

Proof. Obvious. (Why?) □

Lemma 22.3. *Let $x \in A$. The following statements are equivalent:*

(i) *$x \in J(A)$;*
(ii) *For all $a_1, a_2 \in A$, the element $1 + a_1xa_2$ has an inverse;*
(iii) *For all $a \in A$, the element $1 + ax$ has a left inverse;*
(iv) *For all $a \in A$, the element $1 + xa$ has a right inverse.*

Proof. (i) $\implies$ (ii): Let $x \in J(A)$. We have to show that $1 + x$ is invertible. Since $x \in J(A)$, we know that x belongs to all maximal left ideals. This implies that $1+x$ does not belong to any maximal left ideal (because 1 is not contained in any proper ideal).

We claim that $A(1 + x) = A$: The module ${}_AA$ is finitely generated. Assume that $A(1 + x)$ is a proper submodule of ${}_AA$. Then Corollary 21.7 implies that $A(1 + x)$ must be contained in a maximal submodule of ${}_AA$, a contradiction.

Therefore there exists some $a \in A$ with $a(1 + x) = 1$. Let $y = a - 1$. We have $a = 1 + y$, thus $(1 + y)(1 + x) = 1$, which implies $y + x + yx = 0$. This implies $y = (-1 - y)x \in Ax \subseteq J(A)$. Thus also $1 + y$ has a left inverse. We see that $1 + y$ is left invertible and also right invertible. Thus its right inverse $1 + x$ is also its left inverse. Since $J(A)$ is an ideal, also a_1xa_2 belongs to $J(A)$ for all $a_1, a_2 \in A$. Thus all elements of the form $1 + a_1xa_2$ are invertible.

(ii) $\implies$ (iii): Obvious.

(iii) $\implies$ (i): If $x \notin J(A)$, then there exists a maximal left ideal M, which does not contain x. This implies $A = M + Ax$, thus $1 = y - ax$ for some $y \in M$ and $a \in A$. We get $1 + ax = y$, and since y belongs to the maximal left ideal M, y cannot have a left inverse.

(iii) $\iff$ (iv): Condition (ii) is left-right symmetric. □

Corollary 22.4. *The radical $J(A)$ of A is the intersection of all maximal right ideals.*

Proof. Condition (ii) in Lemma 22.3 is left-right symmetric. □

Lemma 22.5. *If I is a left ideal or a right ideal of A, which consists only of nilpotent elements, then I is contained in $J(A)$.*

Proof. Let I be a left ideal of A, and assume all elements in I are nilpotent. It is enough to show that for all $x \in I$ the element $1+x$ is left-invertible. (If $a \in A$, then $ax \in I$.) Since x is nilpotent, we can define

$$z = \sum_{i \geq 0} (-1)^i x^i = 1 - x + x^2 - x^3 + \cdots .$$

We get $(1 + x)z = 1 = z(1 + x)$. The left-right symmetry shows that every right ideal, which consists only of nilpotent elements is contained in the radical. □

Warning: Nilpotent elements do not have to belong to the radical, as the following example shows: Let $A = M_2(K)$. Then A is a semisimple algebra, thus $J(A) = 0$.

But of course A contains many nilpotent elements, for example

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

But observe that there are elements y in Ax which are not nilpotent. In other words $1 + y$ is not invertible. For example

$$e = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$$

is in Ax and $1 + e$ is not invertible. We can also construct maximal left ideals of A which do not contain x.

Proposition 22.6. *Let $a \in A$. Then $a \in J(A)$ if and only if $aS = 0$ for all simple A-modules S.*

Proof. Let T be a simple A-module, and let x be a non-zero element in T. The map $f \colon {}_AA \to T$ defined by $f(b) = bx$ is an A-module homomorphism. Since $x \neq 0$, we have $f \neq 0$. Since T is simple, f is surjective and the kernel of f is a maximal left ideal. It follows from the definition of $J(A)$ that $J(A)$ is contained in the kernel of f. Thus $J(A)x = 0$, and therefore $J(A)T = 0$.

Vice versa, assume $aS = 0$ for all simple A-modules S. We assume that a does not belong to $J(A)$. Since $J(A)$ is the intersection of all maximal left ideals, there exists a maximal left ideal I with $a \notin I$. We know that $S_I := {}_AA/I$ is a simple A-module. For $b \in A$ set $\overline{b} = b + I$. It follows that $\overline{a} \neq 0$. Since

$$a \cdot \overline{1} = \overline{a \cdot 1} = \overline{a} \neq 0$$

we have $aS_I \neq 0$. This contradiction shows that $a \in J(A)$. □

In other words, the radical $J(A)$ is the intersection of the annihilators of the simple A-modules. (Given an A-module M, the **annihilator** of M in A is the set of all $a \in A$ such that $aM = 0$.)

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Corollary 22.7. *For every A-module M we have $J(A)M \subseteq \mathrm{rad}(M)$.*

Proof. If M' is a maximal submodule of M, then M/M' is a simple A-module, thus $J(A)(M/M') = 0$. This implies $J(A)M \subseteq M'$. Since $J(A)M$ is contained in all maximal submodules of M, it is also contained in the intersection of all maximal submodules of M. □

Warning: In general, we do not have an equality $J(A)M = \mathrm{rad}(M)$: Let $A = K[T]$. Then $J(K[T]) = 0$, and therefore $J(K[T])M = 0$ for all $K[T]$-modules M. But for the $K[T]$-module $N(2)$ we have $\mathrm{rad}(N(2)) \cong N(1) \neq 0$.

Corollary 22.8. *If M is a finitely generated A-module, M' is a submodule of M and $M' + J(A)M = M$, then $M' = M$.*

Proof. Assume M is finitely generated and M' is a submodule of M with $M' + J(A)M = M$. By Corollary 22.7 we know that $J(A)M \subseteq \operatorname{rad}(M)$. Thus $M' + \operatorname{rad}(M) = M$. Since M is finitely generated Corollary 21.6 implies that $\operatorname{rad}(M)$ is small in M. Thus we get $M' = M$. □

Corollary 22.9 (Nakayama Lemma)**.** *If M is a finitely generated A-module such that $J(A)M = M$, then $M = 0$.*

Proof. In Corollary 22.8 take $M' = 0$. □

Lemma 22.10. *The algebra A is a local ring if and only if $A/J(A)$ is a skew field.*

Proof. If A is a local ring, then $J(A)$ is a maximal left ideal. Thus $A/J(A)$ is a ring which contains only one proper left ideal, namely the zero ideal. Thus $A/J(A)$ is a skew field.

Vice versa, if $A/J(A)$ is a skew field, then $J(A)$ is a maximal left ideal. We have to show that $J(A)$ contains every proper left ideal: Let L be a left ideal, which is not contained in $J(A)$. Thus $J(A) + L = A$. Now $J(A) = \operatorname{rad}({}_AA)$ is a small submodule of ${}_AA$, since ${}_AA$ is finitely generated. Thus $L = A$. □

Theorem 22.11. *If $A/J(A)$ is semisimple, then for all A-modules M we have*

$$J(A)M = \operatorname{rad}(M).$$

Proof. We have seen that $J(A)M \subseteq \operatorname{rad}(M)$. On the other hand, $M/J(A)M$ is annihilated by $J(A)$, thus it is an $A/J(A)$-module. Since $A/J(A)$ is semisimple, each $A/J(A)$-module is a semisimple $A/J(A)$-module, thus also a semisimple A-module. But if $M/J(A)M$ is semisimple, then $\operatorname{rad}(M)$ has to be contained in $J(A)M$. □

Examples: If $A = K[T]$, then $A/J(A) = A/0 = A$. So $A/J(A)$ is not semisimple. If A is an algebra with $l({}_AA) < \infty$ (for example if A is finite-dimensional), then $A/J(A) = {}_AA/\operatorname{rad}({}_AA)$ is semisimple.

Lemma 22.12. *If e is an idempotent in A, then $J(eAe) = eJ(A)e = J(A) \cap eAe$.*

Proof. We have $J(A) \cap eAe \subseteq eJ(A)e$, since $x \in eAe$ implies $x = exe$. Thus, if additionally $x \in J(A)$, then $x = exe$ belongs to $eJ(A)e$.

Next we show that $eJ(A)e \subseteq J(eAe)$: Let $x \in J(A)$. If $a \in A$, then $1 + eae \cdot x \cdot e$ is invertible, thus there exists some $y \in A$ with $y(1 + eaexe) = 1$. This implies $eye(e + eaexe) = ey(1 + eaexe)e = e$. Thus all elements in $e + eAe(exe)$ are left-invertible. This shows that exe belongs to $J(eAe)$.

Finally, we show that $J(eAe) \subseteq J(A) \cap eAe$: Clearly, $J(eAe) \subseteq eAe$, thus we have to show $J(eAe) \subseteq J(A)$. Let S be a simple A-module. Then $eS = 0$, or eS is a simple eAe-module. Thus $J(eAe)eS = 0$, and therefore $J(eAe)S = 0$, which implies $J(eAe) \subseteq J(A)$. □

22.2. **Exercises.** **1**: Let Q be a quiver. Show that the radical $J(KQ)$ has as a basis the set of all paths from i to j such that there is no path from j to i, where i and j run through the set of vertices of Q.

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Part 5. Projective modules

23. Projective modules

In this section let A be a K-algebra. As before, let $\mathrm{Mod}(A)$ be the category of left A-modules, and let $\mathrm{mod}(A)$ be the full subcategory of finitely generated A-modules.

23.1. Definition of a projective module.

An A-module P is called **projective** if for any epimorphism $g\colon M \to N$ and any homomorphism $h\colon P \to N$ there exists a homomorphism $h'\colon P \to M$ such that $g \circ h' = h$. This is called the "lifting property".

$$\begin{tikzcd} & M \arrow[d, "g"] \\ P \arrow[ur, dashed, "h'"] \arrow[r, "h"] & N \end{tikzcd}$$

An A-module P is projective if and only if for every epimorphism $g\colon M \to N$ of A-modules the induced map

$$\mathrm{Hom}_A(P,g)\colon\ \mathrm{Hom}_A(P,M) \to \mathrm{Hom}_A(P,N)$$

is surjective. For every A-module X, the functor $\mathrm{Hom}_A(X,-)$ is left exact. Thus a module P is projective if and only if $\mathrm{Hom}_A(P,-)$ is exact.

(A functor $F\colon \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ is **exact** if for every short exact sequence $0 \to U \to V \to W \to 0$ in $\mathrm{Mod}(A)$ the sequence $0 \to F(U) \to F(V) \to F(W) \to 0$ is exact in $\mathrm{Mod}(B)$.)

Recall that an A-module F is **free** if F is isomorphic to a direct sum of copies of the regular representation ${}_AA$.

Lemma 23.1. *Free modules are projective.*

Proof. Let F be a free A-module with free generating set X. Let $g\colon M \to N$ be an epimorphism of A-modules, and let $h\colon F \to N$ be any homomorphism of A-modules. For every $x \in X$ we look at the image $h(x)$. Since g is surjective there exists some $m_x \in M$ with $g(m_x) = h(x)$. Define a homomorphism $h'\colon F \to M$ by $h'(x) = m_x$. Since X is a free generating set of X, there exists exactly one such homomorphism h'. For every $x \in X$ we have $(g \circ h')(x) = h(x)$, and this implies $gh' = h$, since X is a generating set of F. □

The map h' constructed in the proof of the above lemma is in general not uniquely determined. There can be many different maps h' with the property $gh' = h$:

For example, for $F = {}_AA$ the set $\{1_A\}$ is a free generating set. Let M be an A-module and let U be a submodule of M. By $g\colon M \to M/U$ we denote the corresponding projection map. This is a typical epimorphism. For $x \in M$ let $\overline{x} = x+U$ be the corresponding residue class in M/U. Let $h\colon {}_AA \to M/U$ be an arbitrary homomorphism. Then $h(1_A) = \overline{x}$ for some $x \in M$. Now the homomorphisms

$h' \colon {}_AA \to M$ such that $gh' = h$ correspond to the elements in $\overline{x} = x + U = \{x + u \mid u \in U\}$, namely we can take any $x + u \in x + U$ and then define $h'(1_A) = x + u$. Thus if $U \neq 0$, then there are many such homomorphisms h'.

Lemma 23.2. *A direct sum of modules is projective if and only if each direct summand is projective.*

Proof. Let $g \colon M \to N$ be an epimorphism. First let $P = \bigoplus_{i \in I} P_i$ with P_i projective for all I. For a homomorphism $h \colon P \to N$ let $h_i \colon P_i \to N$ be its restriction to P_i. For every h_i there exists a lifting, i.e. there exists a homomorphism $h_i' \colon P_i \to M$ with $gh_i' = h_i$. Define $h' \colon \bigoplus_{i \in I} P_i \to M$ such that the restriction of h' to P_i is just h_i'. This implies $gh' = h$.

Vice versa, let P be a projective module, and let $P = P_1 \oplus P_2$ be a direct decomposition of P. For a homomorphism $h_1 \colon P_1 \to N$ let $h = [h_1, 0] \colon P_1 \oplus P_2 \to N$. Since P is projective, there exists a homomorphism $h' \colon P \to M$ with $gh' = h$. We can write $h' = [h_1', h_2']$ with $h_i' \colon P_i \to M$. It follows $gh_1' = h_1$. □

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Lemma 23.3. *For a module P the following are equivalent:*

(i) *P is projective;*
(ii) *Every epimorphism $M \to P$ splits;*
(iii) *P is isomorphic to a direct summand of a free module.*

Furthermore, a projective module P is a direct summand of a free module of rank c if and only if P has a generating set of cardinality c.

Proof. (i) $\implies$ (ii): Let M be an A-module, and let $g \colon M \to P$ be an epimorphism. For the identity map $1_P \colon P \to P$ the lifting property gives a homomorphism $h' \colon P \to M$ with $g \circ h' = 1_P$. Thus g is a split epimorphism.

(ii) $\implies$ (iii): There exists an epimorphism $f \colon F \to P$ where F is a free A-module. Since f splits, P is isomorphic to a direct summand of a free module.

(iii) $\implies$ (i): The class of projective modules contains all free modules and is closed under direct summands.

Now we proof the last statement of the Lemma: If P has a generating set X of cardinality c, then let F be a free module of rank c. Thus F has a generating set of cardinality c. We get an epimorphism $F \to P$ which has to split.

Vice versa, if P is a direct summand of a free module F of rank c, then P has a generating set of cardinality c: We choose an epimorphism $f \colon F \to P$, and if X is a generating set of F, then $f(X)$ is a generating set of P. □

Thus an A-module P is finitely generated and projective if and only if P is a direct summand of a free module of finite rank.

Let $\mathrm{Proj}(A)$ be the full subcategory of Mod(A) of all projective A-modules, and set $\mathrm{proj}(A) := \mathrm{Proj}(A) \cap \mathrm{mod}(A)$.

Warning: There exist algebras A such that ${}_AA$ is isomorphic to ${}_AA \oplus {}_AA$. Thus the free module ${}_AA$ has rank n for any positive integer n. For example, take as A the endomorphism algebra of an infinite dimensional vector space.

Corollary 23.4. *If P and Q are indecomposable projective modules, and if $p\colon P \to Q$ an epimorphism, then p is an isomorphism.*

Proof. Since Q is projective, p is a split epimorphism. But P is indecomposable and $Q \neq 0$. Thus p has to be an isomorphism. $\square$

Recall that a submodule U of a module M is **small** in M if $U + U' \subset M$ for all proper submodules U' of M.

As before, by $J(A)$ we denote the **radical** of an algebra A.

Lemma 23.5. *If P is a projective A-module, then*

$$J(\mathrm{End}_A(P)) = \{f \in \mathrm{End}_A(P) \mid \mathrm{Im}(f) \text{ is small in } P\}.$$

Proof. Let $J := J(\mathrm{End}_A(P))$. Let $f\colon P \to P$ be an endomorphism such that the image $\mathrm{Im}(f)$ is small in P. If $g \in \mathrm{End}_A(P)$ is an arbitrary endomorphism, then $\mathrm{Im}(fg) \subseteq \mathrm{Im}(f)$, thus $\mathrm{Im}(fg)$ is also small in P. Clearly, we have

$$P = \mathrm{Im}(1_P) = \mathrm{Im}(1_P + fg) + \mathrm{Im}(fg).$$

Since $\mathrm{Im}(fg)$ is small, we get that $1_P + fg$ is surjective. But P is projective, therefore $1_P + fg$ is a split epimorphism. Thus there exists some $h \in \mathrm{End}_A(P)$ with $(1_P + fg)h = 1_P$. We have shown that the element $1_P + fg$ has a right inverse for all $g \in \mathrm{End}_A(P)$. Thus f belongs to J.

Vice versa, assume $f \in J$. Let U be a submodule of P with $P = \mathrm{Im}(f) + U$, and let $p\colon P \to P/U$ be the projection. Since $P = \mathrm{Im}(f) + U$, we know that pf is surjective. But P is projective, therefore there exists some $p'\colon P \to P$ with $p = pfp'$. Now $1_P - fp'$ is invertible, because $f \in J$. Since $p(1_P - fp') = 0$ we get $p = 0$. This implies $U = P$. It follows that $\mathrm{Im}(f)$ is small in P. $\square$

Corollary 23.6. *Let P be a projective A-module. If $\mathrm{rad}(P)$ is small in P, then*

$$J(\mathrm{End}_A(P)) = \{f \in \mathrm{End}_A(P) \mid \mathrm{Im}(f) \subseteq \mathrm{rad}(P)\}.$$

Proof. Each small submodule of a module M is contained in $\mathrm{rad}(M)$. If $\mathrm{rad}(M)$ is small in M, then the small submodules of M are exactly the submodules of $\mathrm{rad}(M)$. $\square$

23.2. **The radical of a projective module.**

Lemma 23.7. *If P is a projective A-module, then* $\mathrm{rad}(P) = J(A)P$.

Proof. By definition $J(A) = \mathrm{rad}({}_AA)$ and $J(A)A = J(A)$. This shows that the statement is true for $P = {}_AA$. Now let M_i, $i \in I$ be a family of modules. We have

$$J(A)\left(\bigoplus_{i\in I} M_i\right) = \bigoplus_{i\in I} J(A)M_i \quad \text{and} \quad \mathrm{rad}\left(\bigoplus_{i\in I} M_i\right) = \bigoplus_{i\in I} \mathrm{rad}(M_i).$$

We know that $J(A)M \subseteq \mathrm{rad}(M)$ for all modules M. Thus we get

$$J(A)\left(\bigoplus_{i\in I} M_i\right) = \bigoplus_{i\in I} J(A)M_i \subseteq \bigoplus_{i\in I} \mathrm{rad}(M_i) = \mathrm{rad}\left(\bigoplus_{i\in I} M_i\right).$$

This is a proper inclusion only if there exists some i with $J(A)M_i \subset \mathrm{rad}(M_i)$. Thus, if $J(A)M_i = \mathrm{rad}(M_i)$ for all i, then $J(A)\left(\bigoplus_{i\in I} M_i\right) = \mathrm{rad}\left(\bigoplus_{i\in I} M_i\right)$. This shows that the statement is true for free modules.

Vice versa, if $J(A)\left(\bigoplus_{i\in I} M_i\right) = \mathrm{rad}\left(\bigoplus_{i\in I} M_i\right)$, then $J(A)M_i = \mathrm{rad}(M_i)$ for all i.

Since projective modules are direct summands of free modules, and since we proved the statement already for free modules, we obtain it for all projective modules. □

23.3. **Cyclic projective modules.**

Lemma 23.8. *Let P be an A-module. Then the following are equivalent:*

(i) *P is cyclic and projective;*
(ii) *P is isomorphic to a direct summand of ${}_AA$;*
(iii) *P is isomorphic to a module of the form Ae for some idempotent $e \in A$.*

Proof. We have shown before that a submodule U of ${}_AA$ is a direct summand of ${}_AA$ if and only if there exists an idempotent $e \in A$ such that $U = Ae$.

If e is any idempotent in A, then ${}_AA = Ae \oplus A(1-e)$ is a direct decomposition. Thus Ae is a direct summand of ${}_AA$, and Ae is projective and cyclic.

Vice versa, let U be a direct summand of ${}_AA$, say ${}_AA = U \oplus U'$. Write $1 = e + e'$ with $e \in U$ and $e' \in U'$. This implies $U = Ae$ and $U' = Ae'$, and one checks easily that e, e' form a complete set of orthogonal idempotents. □

Lemma 23.9. *Let P be a projective A-module. If P is local, then* $\mathrm{End}_A(P)$ *is a local ring.*

Proof. If P is local, then P is obviously cyclic. A cyclic projective module is of the form Ae for some idempotent $e \in A$, and its endomorphism ring is $(eAe)^{\mathrm{op}}$. We have seen that Ae is a local module if and only if eAe is a local ring. Furthermore, we know that eAe is a local ring if and only if $(eAe)^{\mathrm{op}}$ is a local ring. □

The converse of Lemma 23.9 is also true. We will not use this result, so we skip the proof.

End of Lecture 27

23.4. **Submodules of projective modules.** Let A be an algebra, and let

$$0 \to M' \to P \to M \to 0$$

be a short exact sequence of A-modules. If M is an arbitrary module, then such a sequence exists, because every module is factor module of a projective module. If we fix M and look at all possible sequence of the above form, then the modules M' are similar to each other, in fact they are "stably equivalent".

Every module can be written as a factor module of a projective module, but the submodules of projective modules are in general a very special class of modules.

For example there are algebras, where submodules of projective modules are always projective.

An A-module M is called **torsion free** if M is isomorphic to a submodule of a projective module.

Lemma 23.10 (Schanuel)**.** *Let P and Q be projective modules, let U be a submodule of P and V a submodule of Q. If $P/U \cong Q/V$, then $U \oplus Q \cong V \oplus P$.*

Proof. Let $M := P/U$, and let $p\colon P \to M$ be the projection map. Similarly, let $q\colon Q \to M$ be the epimorphism with kernel V (since Q/V is isomorphic to M such a q exists).

We construct the pullback of (p, q) and obtain a commutative diagram

$$\begin{tikzcd}
E \arrow[r, "p'"] \arrow[d, "q'"'] & Q \arrow[d, "q"] \\
P \arrow[r, "p"] & M
\end{tikzcd}$$

where p' is an epimorphism with kernel isomorphic to $U = \operatorname{Ker}(p)$, and q' is an epimorphism with kernel isomorphic to $V = \operatorname{Ker}(q)$: Set $U' := \operatorname{Ker}(p')$ and $V' := \operatorname{Ker}(q')$. We get a diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & & V' \arrow[r, dashed, "i_V"] \arrow[d] & V \arrow[d] & \\
0 \arrow[r] & U' \arrow[r] \arrow[d, dashed, "i_U"'] & E \arrow[r, "p'"] \arrow[d, "q'"'] & Q \arrow[r] \arrow[d, "q"] & 0 \\
0 \arrow[r] & U \arrow[r] & P \arrow[r, "p"] \arrow[d] & M \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

We can assume $E = \{(v, w) \in P \oplus Q \mid p(v) = q(w)\}$, $q'(v, w) = v$ and $p'(v, w) = w$ for all $(v, w) \in E$. Now it is easy to define homomorphisms i_U and i_V such that everything commutes, and then ones shows that i_U and i_V are in fact isomorphisms.

Since Q is projective, p' is a split epimorphism, which implies $E \cong U \oplus Q$. Since P is projective as well, q' is a split epimorphism, thus $E \cong V \oplus P$. $\square$

Lemma 23.11. *Let*

$$0 \to U \xrightarrow{u} P \xrightarrow{p} W \to 0$$

be a short exact sequence of A-modules with P a projective module. Then this sequence induces every short exact sequence which ends in W.

Proof. Let

$$0 \to U' \xrightarrow{f} V' \xrightarrow{g} W \to 0$$

be an arbitrary short exact sequence of A-modules. Since g is an epimorphism, the lifting property of the projective module P yields a homomorphism $p' \colon P \to V'$ such that $gp' = p$. This implies $gp'u = pu = 0$. Thus $p'u$ can be factorized through the kernel of g. So there exists some $h \colon U \to U'$ such that $p'u = fh$. Thus we obtain the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & U \arrow[r, "u"] \arrow[d, "h"'] & P \arrow[r, "p"] \arrow[d, "p'"'] & W \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & U' \arrow[r, "f"] & V' \arrow[r, "g"] & W \arrow[r] & 0
\end{tikzcd}$$

This shows that (f, g) is the short exact sequence induced by h. $\square$

Lemma 23.12. *Let U be a submodule of a projective module P such that for every endomorphism f of P we have $f(U) \subseteq U$. Define*

$$f_* \colon P/U \to P/U$$

by $f_*(x + U) := f(x) + U$. Then the following hold:

(i) *f_* is an endomorphism of P/U;*
(ii) *The map $f \mapsto f_*$ defines a surjective algebra homomorphism*

$$\mathrm{End}_A(P) \to \mathrm{End}_A(P/U)$$

with kernel $\{f \in \mathrm{End}_A(P) \mid \mathrm{Im}(f) \subseteq U\}$.

Proof. Let $p \colon P \to P/U$ be the projection. Thus f_* is defined via $p \circ f = f_* \circ p$. It is easy to show that this is really an A-module homomorphism, and that $f \mapsto f_*$ defines an algebra homomorphism. The description of the kernel is also obvious.

It remains to show the surjectivity: Here we use that P is projective. If g is an endomorphism of P/U, there exists a lifting of $g \circ p \colon P \to P/U$. In other words there exists a homomorphism $g' \colon P \to P$ such that $p \circ g' = g \circ p$.

$$\begin{tikzcd} & & P \arrow[d, "p"] \\ P \arrow[rru, dashed, "g'"] \arrow[r, "p"'] & P/U \arrow[r, "g"'] & P/U \end{tikzcd}$$

Thus we get $g'_* = g$. □

Let M be an A-module. For all $f \in \mathrm{End}_A(M)$ we have $f(\mathrm{rad}(M)) \subseteq \mathrm{rad}(M)$. Thus the above lemma implies that for any projective module P there is a surjective algebra homomorphism $\mathrm{End}_A(P) \to \mathrm{End}_A(P/\mathrm{rad}(P))$, and the kernel is the set of all endomorphisms of P whose image is contained in $\mathrm{rad}(P)$.

We have shown already: If $\mathrm{rad}(P)$ is a small submodule of P, then the set of all endomorphisms of P whose image is contained in $\mathrm{rad}(P)$ is exactly the radical of $\mathrm{End}_A(P)$. Thus, we proved the following:

Corollary 23.13. *Let P be a projective A-module. If $\mathrm{rad}(P)$ is small in P, then*

$$\mathrm{End}_A(P)/J(\mathrm{End}_A(P)) \cong \mathrm{End}_A(P/\mathrm{rad}(P)).$$

23.5. **Projective covers.** Let M be an A-module. A homomorphism $p \colon P \to M$ is a **projective cover** of M if the following hold:

- P is projective;
- p is an epimorphism;
- $\mathrm{Ker}(p)$ is a small submodule of P.

In this situation one often calls the module P itself a projective cover of M and writes $P = P(M)$.

Lemma 23.14. *Let P be a finitely generated projective module. Then the projection map $P \to P/\mathrm{rad}(P)$ is a projective cover.*

Proof. The projection map is surjective and its kernel is $\mathrm{rad}(P)$. By assumption P is projective. For every finitely generated module M the radical $\mathrm{rad}(M)$ is a small submodule of M. Thus $\mathrm{rad}(P)$ is small in P. □

Warning: If P is an arbitrary projective A-module, then $\mathrm{rad}(P)$ is not necessarily small in P: For example, let A be the subring of $K(T)$ consisting of all fractions of the form f/g such that g is not divisible by T. This is a local ring. Now let P be a free A-module of countable rank, for example the module of all sequences $(a_0, a_1, \ldots)$ with $a_i \in A$ for all i such that only finitely many of the a_i are non-zero. The radical $U = \mathrm{rad}(P)$ consists of all such sequences with a_i divisible by T for all i. We define a homomorphism $f \colon P \to {}_AK(T)$ by

$$f(a_0, a_1, \ldots) = \sum_{i \geq 0} T^{-i} a_i = a_0 + \frac{a_1}{T} + \frac{a_2}{T^2} + \cdots.$$

Let W be the kernel of f. Since $f \neq 0$, W is a proper submodule of P. On the other hand we will show that $U + W = P$. Thus $U = \mathrm{rad}(P)$ is not small in P. Let $a = (a_0, a_1, \dots)$ be a sequence in P and choose n such that $a_j = 0$ for all $j > n$. Define $b = (b_0, b_1, \dots)$ by

$$b_{n+1} = \sum_{i=0}^{n} T^{n-i+1} a_i = a_0 T^{n+1} + a_1 T^n + \cdots + a_n T$$

and $b_j = 0$ for all $j \neq n+1$. Since b belongs to TA, we know that b is in U. On the other hand $f(b - a) = 0$, thus $b - a \in W$. We see that $a = b - (b - a)$ belongs to $U + W$.

Given two projective covers $p_i \colon P_i \to M_i$, $i = 1, 2$, then the direct sum

$$p_1 \oplus p_2 \colon P_1 \oplus P_2 \to M_1 \oplus M_2$$

is a projective cover. Here

$$p_1 \oplus p_2 = \begin{pmatrix} p_1 & 0 \\ 0 & p_2 \end{pmatrix}.$$

The map $p_1 \oplus p_2$ is obviously an epimorphism and its kernel is $\mathrm{Ker}(p_1) \oplus \mathrm{Ker}(p_2)$. By assumption $\mathrm{Ker}(p_i)$ is small in P_i, thus $\mathrm{Ker}(p_1) \oplus \mathrm{Ker}(p_2)$ is small in $P_1 \oplus P_2$.

Warning: Given infinitely many projective modules P_i with small submodules U_i, then $\bigoplus_{i \in I} U_i$ is not necessarily small in $\bigoplus_{i \in I} P_i$.

Lemma 23.15 (Projective covers are unique)**.** *Let $p_1 \colon P_1 \to M$ be a projective cover, and let $p_2 \colon P_2 \to M$ be an epimorphism with P_2 projective. Then the following hold:*

- *There exists a homomorphism $f \colon P_2 \to P_1$ such that $p_1 \circ f = p_2$;*
- *Each homomorphism $f \colon P_2 \to P_1$ with $p_1 \circ f = p_2$ is a split epimorphism;*
- *If p_2 is also a projective cover, then every homomorphism $f \colon P_2 \to P_1$ with $p_1 \circ f = p_2$ is an isomorphism.*

Proof. Since p_1 is an epimorphism, and since P_2 is projective, there exists a homomorphism f with $p_1 f = p_2$.

$$\begin{tikzcd} & P_1 \arrow[d, "p_1"] \\ P_2 \arrow[ur, dashed, "f"] \arrow[r, "p_2"'] & M \end{tikzcd}$$

We have to show that each such f is a split epimorphism: We show that

$$\mathrm{Ker}(p_1) + \mathrm{Im}(f) = P_1.$$

For $x \in P_1$ we have $p_1(x) \in M$. Since p_2 is surjective, there exists some $x' \in P_2$ such that $p_1(x) = p_2(x') = (p_1 f)(x')$. Thus $p_1(x - f(x')) = 0$. We see that $x - f(x')$ belongs to $\mathrm{Ker}(p_1)$, thus $x = (x - f(x')) + f(x')$ lies in $\mathrm{Ker}(p_1) + \mathrm{Im}(f)$. Now $\mathrm{Ker}(p_1)$ is small in P_1, which implies $\mathrm{Im}(f) = P_1$. We have shown that f is surjective. But each epimorphism to a projective module is a split epimorphism.

Now we assume that p_2 is also a projective cover. Again let $f\colon P_2 \to P_1$ be a homomorphism with $p_1 f = p_2$. Since f is a split epimorphism, there exists a submodule U of P_2 with $\mathrm{Ker}(f) \oplus U = P_2$. We show that

$$\mathrm{Ker}(p_2) + U = P_2.$$

If $y \in P_2$, then there exists some $y' \in P_1$ with $p_2(y) = p_1(y')$. We know that $f\colon P_2 \to P_1$ is surjective, thus y' has a preimage in P_2. Since $P_2 = \mathrm{Ker}(f) \oplus U$, we can find this preimage in U. Thus there is some $u \in U$ with $f(u) = y'$. Summarizing, we get $p_2(y) = p_1(y') = (p_1 f)(u) = p_2(u)$. We see that $y - u$ belongs to $\mathrm{Ker}(p_2)$, thus $y = (y-u) + u \in \mathrm{Ker}(p_2) + U$. Since $\mathrm{Ker}(p_2)$ is small in P_2 we get $U = P_2$ and therefore $\mathrm{Ker}(f) = 0$. Thus f is also injective.

So projective covers are (up to isomorphism) uniquely determined. Also, if $p\colon P \to M$ is a projective cover, and $f\colon P \to P$ is a homomorphism with $p \circ f = p$, then f is an isomorphism. □

Corollary 23.16. *Let P and Q be finitely generated projective modules. Then $P \cong Q$ if and only if $P/\mathrm{rad}(P) \cong Q/\mathrm{rad}(Q)$.*

Proof. Since P and Q are finitely generated projective modules, the projections $p\colon P \to P/\mathrm{rad}(P)$ and $q\colon Q \to Q/\mathrm{rad}(Q)$ are projective covers. If $f\colon P/\mathrm{rad}(P) \to Q/\mathrm{rad}(Q)$ is an isomorphism, then $f \circ p\colon P \to Q/\mathrm{rad}(Q)$ is a projective cover. The uniqueness of projective covers yields $P \cong Q$. The other direction is obvious. □

Corollary 23.17. *Let P be a direct sum of local projective modules. If U is a submodule of P which is not contained in $\mathrm{rad}(P)$, then there exists an indecomposable direct summand P' of P which is contained in U.*

Proof. Let $P = \bigoplus_{i \in I} P_i$ with local projective modules P_i. Let U be a submodule of P which is not contained in $\mathrm{rad}(P)$. We have $\mathrm{rad}(P) = \bigoplus_{i \in I} \mathrm{rad}(P_i)$ and therefore

$$P/\mathrm{rad}(P) = \bigoplus_{i \in I} P_i/\mathrm{rad}(P_i).$$

Let $u\colon U \to P$ be the inclusion map, and let $p\colon P \to P/\mathrm{rad}(P)$ be the projection. Finally, for every $i \in I$ let $\pi_i\colon P/\mathrm{rad}(P) \to P_i/\mathrm{rad}(P_i)$ also be the projection. The composition pu is not the zero map. Thus there exists some $i \in I$ with $\pi_i pu \neq 0$. Since $P_i/\mathrm{rad}(P_i)$ is a simple module, $\pi_i pu$ is surjective. Let $p_i\colon P_i \to P_i/\mathrm{rad}(P_i)$ be the projection. Since P_i is a local projective module, p_i is a projective cover. By the surjectivity of $\pi_i pu$ the lifting property of P_i yields an $f\colon P_i \to U$ such that $\pi_i puf = p_i$. Now we use that p_i is an epimorphism: The lifting property of P gives us a homomorphism $g\colon P \to P_i$ with $p_i g = \pi_i p$.

$$\begin{tikzcd}
& U \arrow[d, "u"] \\
& P \arrow[d, "p"] \arrow[dddl, dashed, "g"'] \\
& P/\operatorname{rad}(P) \arrow[d, "\pi_i"] \\
P_i \arrow[uuur, dashed, "f"] \arrow[r, "p_i"] & P_i/\operatorname{rad}(P_i)
\end{tikzcd}$$

Thus we have

$$p_i g u f = \pi_i p u f = p_i.$$

Since p_i is a projective cover, guf must be an isomorphism. Thus we see that uf is a split monomorphism whose image $P' := \operatorname{Im}(uf)$ is a direct summand of P which is isomorphic to P_i. Clearly, P' as the image of uf is contained in $U = \operatorname{Im}(f)$. □

Lemma 23.18. *Let P be a finitely generated projective A-module, and let M be a finitely generated module. For a homomorphism $p\colon P \to M$ the following are equivalent:*

(i) *p is a projective cover;*
(ii) *p is surjective and* $\operatorname{Ker}(p) \subseteq \operatorname{rad}(P)$*;*
(iii) *p induces an isomorphism $P/\operatorname{rad}(P) \to M/\operatorname{rad}(M)$.*

Proof. (i) $\Longrightarrow$ (ii): Small submodules of a module are always contained in the radical.

(ii) $\Longrightarrow$ (iii): Since $\operatorname{Ker}(p) \subseteq \operatorname{rad}(P)$ we have $\operatorname{rad}(P/\operatorname{Ker}(p)) = \operatorname{rad}(P)/\operatorname{Ker}(p)$. Now p induces an isomorphism $P/\operatorname{Ker}(p) \to M$ which maps $\operatorname{rad}(P/\operatorname{Ker}(p))$ onto $\operatorname{rad}(M)$ and induces an isomorphism $P/\operatorname{rad}(P) \to M/\operatorname{rad}(M)$.

(iii) $\Longrightarrow$ (i): We assume that $p\colon P \to M$ induces an isomorphism $p_*\colon P/\operatorname{rad}(P) \to M/\operatorname{rad}(M)$. This implies $\operatorname{rad}(M) + \operatorname{Im}(p) = M$. Since M is a finitely generated module, its radical is a small submodule. Thus $\operatorname{Im}(p) = M$. We see that p is an epimorphism. Since p_* is injective, the kernel of p must be contained in $\operatorname{rad}(P)$. The radical $\operatorname{rad}(P)$ is small in P because P is finitely generated. Now $\operatorname{Ker}(p) \subseteq \operatorname{rad}(P)$ implies that $\operatorname{Ker}(p)$ is small in P. □

24. **Digression: The stable module category**

Let $\mathcal{C}$ be a K-linear category. An **ideal** $\mathcal{I}$ in $\mathcal{C}$ is defined as follows: To each pair (C, C') of objects $C, C' \in \mathcal{C}$ there is a subspace $\mathcal{I}(C, C')$ of $\operatorname{Hom}_{\mathcal{C}}(C, C')$ such that for arbitrary morphisms $f\colon D \to C$, $h\colon C' \to D'$ and $g \in \mathcal{I}(C, C')$ we have $h \circ g \circ f \in \mathcal{I}(D, D')$.

If $\mathcal{I}$ is an ideal in $\mathcal{C}$ we can define the **factor category** $\mathcal{C}/\mathcal{I}$ as follows: The objects are the same as in $\mathcal{C}$ and

$$\mathrm{Hom}_{\mathcal{C}/\mathcal{I}}(C, C') := \mathrm{Hom}_{\mathcal{C}}(C, C')/\mathcal{I}(C, C').$$

The composition of morphisms is defined in the obvious way.

If $\mathcal{X}$ is a class of objects in $\mathcal{C}$ which is closed under finite direct sums, then we say that $f\colon C \to C'$ **factors through** $\mathcal{X}$, if $f = f_2 \circ f_1$ with $f_1\colon C \to X$, $f_2\colon X \to C'$ and $X \in \mathcal{X}$. Let $\mathcal{I}(\mathcal{X})(C, C')$ be the set of morphisms $C \to C'$ which factor through $\mathcal{X}$. In this way we obtain an ideal $\mathcal{I}(\mathcal{X})$ in $\mathcal{C}$.

Now let A be an arbitrary K-algebra, and as before let $\mathrm{Proj}(A)$ be the full subcategory of projective A-modules. The **stable module category** of $\mathrm{Mod}(A)$ is defined as

$$\underline{\mathrm{Mod}}(A) = \mathrm{Mod}(A)/\mathcal{I}(\mathrm{Proj}(A)).$$

Define

$$\underline{\mathrm{Hom}}(M, N) := \mathrm{Hom}_{\mathrm{Mod}(A)/\mathrm{Proj}(A)}(M, N) = \mathrm{Hom}_A(M, N)/\mathcal{I}(\mathrm{Proj}(A))(M, N).$$

Similarly, we define $\underline{\mathrm{mod}}(A) = \mathrm{mod}(A)/\mathcal{I}(\mathrm{proj}(A))$.

Thus the objects of $\underline{\mathrm{Mod}}(A)$ are the same ones as in $\mathrm{Mod}(A)$, namely just the A-modules. But it follows that all projective A-modules become zero objects in $\underline{\mathrm{Mod}}(A)$: If P is a projective A-module, then 1_P lies in $\mathcal{I}(\mathrm{Proj}(A))(P, P)$. Thus 1_P becomes zero in $\underline{\mathrm{Mod}}(A)$. Vice versa, if a module M is a zero object in $\underline{\mathrm{Mod}}(A)$, then M is a projective A-module: If 1_M factors through a projective A-module, then M is a direct summand of a projective module and therefore also projective.

Now Schanuel's Lemma implies the following: If M is an arbitrary module, and if $p\colon P \to M$ and $p'\colon P' \to M$ are epimorphisms with P and P' projective, then the kernels $\mathrm{Ker}(p)$ and $\mathrm{Ker}(p')$ are isomorphic in the category $\underline{\mathrm{Mod}}(A)$.

If we now choose for every module M an epimorphism $p\colon P \to M$ with P projective, then $M \mapsto \mathrm{Ker}(p)$ yields a functor $\mathrm{Mod}(A) \to \underline{\mathrm{Mod}}(A)$. If we change the choice of P and p, then the isomorphism class of $\mathrm{Ker}(p)$ in $\underline{\mathrm{Mod}}(A)$ does not change.

So it makes sense to work with an explicit construction of a projective module P and an epimorphism $p\colon P \to M$. Let M be a module, and let $F(M)$ be the free module with free generating set $|M|$ (the underlying set of the vector space M). Define

$$p(M)\colon F(M) \to M$$

by $m \mapsto m$. In this way we obtain a functor $F\colon \mathrm{Mod}(A) \to \mathrm{Mod}(A)$

Let $\Omega(M)$ be the kernel of $p(M)$, and let

$$u(M)\colon \Omega(M) \to F(M)$$

be the corresponding inclusion. Then $\Omega\colon \mathrm{Mod}(A) \to \mathrm{Mod}(A)$ is a functor and $u\colon \Omega \to F$ is a natural transformation. We obtain a short exact sequence

$$0 \to \Omega(M) \to F(M) \to M \to 0$$

with $F(M)$ a free (and thus projective) module. One calls Ω the **loop functor** or **syzygy functor**. This is a functor but it is not at all additive.

For example, if $M = 0$, then $F(M) = {}_AA$ and $\Omega(M) = {}_AA$.

Future: Let A be a finite-dimensional K-algebra. We will meet stable homomorphism spaces in Auslander-Reiten theory, for example the Auslander-Reiten formula reads

$$\mathrm{Ext}^1_A(N, \tau(M)) \cong \mathrm{D}\underline{\mathrm{Hom}}_A(M, N),$$

for all finite-dimensional A-modules M and N. If A has finite global dimension, we have

$$\underline{\mathrm{mod}}(\widehat{A}) \cong D^b(A)$$

where $\widehat{A}$ is the repetitive algebra of A and $D^b(A)$ is the derived category of bounded complexes of A-modules.

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25. **Projective modules over finite-dimensional algebras**

There is a beautiful general theory on projective modules, however one can cut this short and concentrate on finite-dimensional projective modules over finite-dimensional algebras. The results in this section can be generalized considerably. The general theory is developed in Sections 27 and 28.

Theorem 25.1 (Special case of Theorem 28.1)**.** *Let A be a finite-dimensional algebra. Then $A/J(A)$ is semisimple.*

Proof. The module ${}_AA$ is a finite direct sum of local modules, thus ${}_AA/\operatorname{rad}({}_AA)$ is a finite sum of simple modules and therefore semisimple. □

Theorem 25.2 (Special case of Theorem 28.2)**.** *Let A be a finite-dimensional algebra. If*

$${}_AA = P_1 \oplus \cdots \oplus P_n$$

is a direct decomposition of the regular representation into indecomposable modules P_i, then each finite-dimensional indecomposable projective A-module is isomorphic to one of the P_i.

Proof. For each finite-dimensional indecomposable projective A-module P there exists an epimorphism $F \to P$ with F a free A-module of finite rank. In particular F is finite-dimensional. Since P is projective, this epimorphism splits. Then we use the Krull-Remak-Schmidt Theorem. □

Theorem 25.3 (Special case of Theorem 28.3)**.** *Let A be a finite-dimensional algebra. The map $P \mapsto P/\operatorname{rad}(P)$ yields a bijection between the isomorphism classes of finite-dimensional indecomposable projective A-modules and the isomorphism classes of simple A-modules.*

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Proof. If P and Q are isomorphic modules, then $P/\operatorname{rad}(P)$ and $Q/\operatorname{rad}(Q)$ are also isomorphic. Therefore $P \mapsto P/\operatorname{rad}(P)$ yields a well defined map.

The map is surjective: Let S be a simple module. We write ${}_AA = \bigoplus_{i=1}^n P_i$ with indecomposable modules P_i. Since P_i is of finite length and indecomposable, we know that $\operatorname{End}_A(P_i)$ is a local ring. Furthermore, $P_i = Ae_i$ for some idempotent $e_i \in A$. Since $\operatorname{End}_A(P_i) \cong (e_iAe_i)^{\mathrm{op}}$ we know that e_iAe_i is also a local ring. Therefore, Lemma 21.10 implies that Ae_i is a local module.

There exists a non-zero homomorphism ${}_AA \to S$, and therefore for at least one index i we get a non-zero map $f\colon P_i \to S$. Since S is simple, we know that f is surjective. Furthermore, the kernel of f is $\operatorname{rad}(P_i)$ because P_i is local. Thus S is isomorphic to $P_i/\operatorname{rad}(P_i)$.

The map is injective: Let P and Q be finite-dimensional indecomposable projective modules such that $P/\operatorname{rad}(P) \cong Q/\operatorname{rad}(Q)$. Then Corollary 23.16 implies that $P \cong Q$. □

If P is a local projective module, then $S := P/\operatorname{rad}(P)$ is a simple module and $P(S) := P$ is the projective cover of S.

Theorem 25.4 (Special case of Theorem 28.4)**.** *Let A be a finite-dimensional algebra, and let P be a finite-dimensional indecomposable projective A-module. Set $S := P/\operatorname{rad}(P)$. Then the following hold:*

(i) *P is local;*
(i) $\operatorname{End}_A(P)$ *is a local ring;*
(iii) $J(\operatorname{End}_A(P)) = \{f \in \operatorname{End}_A(P) \mid \operatorname{Im}(f) \subseteq \operatorname{rad}(P)\}$*;*
(iv) *Each endomorphism of P induces an endomorphism of S, and we obtain an algebra isomorphism* $\operatorname{End}_A(P)/J(\operatorname{End}_A(P)) \to \operatorname{End}_A(S)$*;*
(v) *The multiplicity of P in a direct sum decomposition ${}_AA = \bigoplus_{i=1}^m P_i$ with indecomposable modules P_i is exactly the dimension of S as an* $\operatorname{End}_A(S)$*-module.*

Proof. We have shown already that each finite-dimensional indecomposable module is local and has a local endomorphism ring. Since P is finitely generated, $\operatorname{rad}(P)$ is small in P. Now (iii) and (iv) follow from Lemma 23.12 and Corollary 23.13. It remains to prove (v): We write ${}_AA = \bigoplus_{i=1}^m P_i$ with indecomposable modules P_i.

Then

$$J(A) = \operatorname{rad}({}_AA) = \bigoplus_{i=1}^{m} \operatorname{rad}(P_i) \quad \text{and} \quad {}_AA/J(A) = \bigoplus_{i=1}^{m} P_i/\operatorname{rad}(P_i).$$

The multiplicity of P in this decomposition (in other words, the number of direct summands P_i which are isomorphic to P) is equal to the multiplicity of S in the direct decomposition ${}_AA/J(A) = \bigoplus_{i=1}^{m} P_i/\operatorname{rad}(P_i)$. But this multiplicity of S is the dimension of S as an $\operatorname{End}_A(S)$-module. (Recall that $A/J(A)$ is a semisimple algebra, and that $\operatorname{End}_A(S)$ is a skew field.) □

Theorem 25.5. *Let A be a finite-dimensional algebra. Then every finitely generated module has a projective cover.*

Proof. Let M be a finitely generated A-module. There exists a finitely generated projective module P and an epimorphism $p\colon P \to M$. We write $P = \bigoplus_{i=1}^{n} P_i$ with indecomposable modules P_i. We can assume that P is chosen such that n is minimal. We want to show that $\operatorname{Ker}(p) \subseteq \operatorname{rad}(P)$:

Assume $\operatorname{Ker}(p)$ is not a submodule of $\operatorname{rad}(P)$. Then there exists an indecomposable direct summand P' of P which is contained in $\operatorname{Ker}(p)$, see Corollary 23.17. But then we can factorize p through P/P' and obtain an epimorphism $P/P' \to M$. Since P' is an indecomposable direct summand of P, the Krull-Remak-Schmidt Theorem implies that P/P' is a direct sum of $n-1$ indecomposable modules, which is a contradiction to the minimality of n. Thus we have shown that $\operatorname{Ker}(p) \subseteq \operatorname{rad}(P)$.

Since M is finitely generated, $\operatorname{rad}(M)$ is small in M, and therefore every submodule $U \subseteq \operatorname{rad}(M)$ is small in M. □

Let A be a finite-dimensional algebra, and let M be a finitely generated A-module. How do we "construct" a projective cover of M?

Let $\varepsilon\colon M \to M/\operatorname{rad}(M)$ be the canonical projection. The module $M/\operatorname{rad}(M)$ is a finitely generated $A/J(A)$-module. Since $A/J(A)$ is semisimple, also $M/\operatorname{rad}(M)$ is semisimple. So $M/\operatorname{rad}(M)$ can be written as a direct sum of finitely many simple modules S_i, say

$$M/\operatorname{rad}(M) = \bigoplus_{i=1}^{n} S_i.$$

For each module S_i we choose a projective cover $q_i\colon P(S_i) \to S_i$. Set $P = \bigoplus_{i=1}^{n} P(S_i)$ and

$$q = \bigoplus_{i=1}^{n} q_i\colon P \to M/\operatorname{rad}(M).$$

Since P is projective there exists a lifting $p\colon P \to M$ of q, i.e. p is a homomorphism with $\varepsilon \circ p = q$. Thus we get a commutative diagram

$$\begin{tikzcd} & M \arrow[d, "\varepsilon"] \\ P \arrow[ur, dashed, "p"] \arrow[r, "q"'] & M/\operatorname{rad}(M) \end{tikzcd}$$

Since P and M are finitely generated we see that p is a projective cover.

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26. **Projective modules over basic algebras**

A K-algebra A is a **basic algebra** if the following hold:

- A is finite-dimensional;
- $A/J(A) \cong \underbrace{K \times K \times \cdots \times K}_{n \text{ times}}$.

In this case, there are n isomorphism classes of simple A-modules, and each simple A-module is 1-dimensional.

Let $Q = (Q_0, Q_1, s, t)$ be a quiver. By KQ^+ we denote the subspace of KQ generated by all paths of length at least one. Clearly, KQ^+ is an ideal in KQ.

An ideal $\mathcal{I}$ in KQ is an **admissible ideal** if there exists some $m \geq 2$ such that

$$(KQ^+)^m \subseteq \mathcal{I} \subseteq (KQ^+)^2.$$

It follows that $A := KQ/\mathcal{I}$ is a finite-dimensional K-algebra.

Theorem 26.1 (Gabriel). *A K-algebra A is basic if and only if $A \cong KQ/\mathcal{I}$ where Q is a quiver and $\mathcal{I}$ is an admissible ideal.*

Proof. Later. □

Theorem 26.2 (Gabriel). *Let A be a finite-dimensional K-algebra with K algebraically closed. Then there exists a uniquely determined quiver Q and an admissible ideal $\mathcal{I}$ in KQ such that the categories* $\operatorname{mod}(A)$ *and* $\operatorname{mod}(KQ/\mathcal{I})$ *are equivalent.*

Proof. Later. □

We will actually not use Theorems 26.1 and 26.2 very often. But of course these results are still of central importance, because they tell us that path algebras and their quotients by admissible ideals are not at all exotic. They are hidden behind every finite-dimensional algebra over an algebraically closed field.

Assume now that $\mathcal{I}$ is an admissible ideal in a path algebra KQ and set $A := KQ/\mathcal{I}$.

For each $i \in Q_0$ let S_i be a 1-dimensional K-vector space, and let

$$\eta\colon A \times S_i \to S_i$$

be the A-module structure defined by

$$\eta(\overline{a}, s) = \begin{cases} s & \text{if } a = e_i, \\ 0 & \text{otherwise} \end{cases}$$

for all $s \in S_i$ and all paths a in Q. It is clear that the modules S_i, $i \in Q_0$ are pairwise non-isomorphic 1-dimensional (and therefore simple) A-modules.

Define $A^+ = KQ^+/\mathcal{I}$. This is an ideal in A. The algebra A is (as a vector space) generated by the residue classes $\overline{p} = p + I$ of all paths p in Q.

Lemma 26.3. *A^+ is an ideal in KQ, and all elements in A^+ are nilpotent.*

Proof. Clearly, A^+ is (as a vector space) generated by the residue classes $\overline{p} = p + I$ of all paths p in Q with $l(p) \geq 1$. Now it is obvious that A^+ is an ideal.

Since A is finite-dimensional, we get that every element in A^+ is nilpotent. (When we take powers of a linear combination of residue classes of paths of length at least one, we get linear combinations of residue classes of strictly longer paths, which eventually have to be zero for dimension reasons.) □

Corollary 26.4. $A^+ \subseteq J(A)$.

Proof. By Lemma 22.5 an ideal consisting just of nilpotent elements is contained in the radical $J(A)$. □

Lemma 26.5. *$\{e_i + \mathcal{I} \mid i \in Q_0\}$ is a linearly independent subset of A.*

Proof. This follows because $\mathcal{I} \subseteq (KQ^+)^2 \subseteq KQ^+$. □

Corollary 26.6. $\dim A^+ = \dim KQ - |Q_0|$.

By abuse of notation we denote the residue class $e_i + \mathcal{I}$ also just by e_i.

Lemma 26.7. $\dim J(A) \leq \dim KQ - |Q_0|$.

Proof. We know that $J(A)$ consists of all elements $x \in A$ such that $xS = 0$ for all simple A-modules S, see Proposition 22.6. By definition $e_iS_i \neq 0$ for all $i \in Q_0$. Thus none of the e_i belongs to $J(A)$. □

Thus for dimension reasons, we obtain the following result:

Lemma 26.8. *We have* $A^+ = J(A)$ *and* $\dim J(A) = \dim KQ - |Q_0|$.

Lemma 26.9. *e_iAe_i is a local ring for all $i \in Q_0$.*

Proof. As a vector space, e_iAe_i is generated by all residue classes of paths p in Q with $s(p) = t(p) = i$. By Lemma 22.12 we know that $J(e_iAe_i) = e_iJ(A)e_i$. We proved already that $A^+ = J(A)$. It follows that $J(e_iAe_i)$ is (as a vector space) generated by all paths p with $s(p) = t(p) = i$ and $l(p) \geq 1$. Thus

$$\dim e_iAe_i/J(e_iAe_i) = 1.$$

Therefore e_iAe_i is a local ring. □

Theorem 26.10. *Let $A = KQ/\mathcal{I}$ where $\mathcal{I}$ is an admissible ideal in a path algebra KQ. Then the following hold:*

(i) ${}_AA = \bigoplus_{i \in Q_0} Ae_i$ *is a direct decomposition of the regular representation into indecomposables;*
(ii) *Each finite-dimensional indecomposable projective A-module is isomorphic to one of the Ae_i;*
(iii) Ae_i *is a local module with* $\operatorname{top}(Ae_i) := Ae_i/\operatorname{rad}(Ae_i) \cong S_i$*;*
(iv) *The S_i are the only simple A-modules;*
(v) $A/J(A) \cong \bigoplus_{i \in Q_0} S_i$*;*
(vi) *A is a basic algebra.*

Proof. There exists a non-zero homomorphism $\pi_i \colon Ae_i \to S_i$ defined by $\pi_i(ae_i) = ae_i \cdot 1$. (Recall that the underlying vector space of S_i is just our field K.) It follows that π_i is an epimorphism.

Since e_iAe_i is a local ring, we know that the modules Ae_i are local (and indecomposable). This implies

$$Ae_i/\operatorname{rad}(Ae_i) \cong S_i.$$

The rest of the theorem follows from results we proved before for arbitrary finite-dimensional algebras. □

End of Lecture 30

End of Semester 1 of this series of lecture courses

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27. Direct summands of infinite direct sums

27.1. The General Exchange Theorem.

Theorem 27.1 (General Exchange Theorem)**.** *Let M be a module with direct decompositions*

$$M = U \oplus \bigoplus_{i=1}^{m} M_i = U \oplus N \oplus V.$$

We assume that $N = \bigoplus_{j=1}^{n} N_j$ such that the endomorphism rings of the N_j are local. Then for $1 \leq i \leq m$ there exist direct decompositions $M_i = M_i' \oplus M_i''$ such that

$$M = U \oplus N \oplus \bigoplus_{i=1}^{m} M_i' \quad \textit{and} \quad N \cong \bigoplus_{i=1}^{m} M_i''.$$

Proof. We prove the theorem by induction on n. For $n = 0$ there is nothing to show: We can choose $M_i' = M_i$ for all i.

Let

$$M = U \oplus \bigoplus_{i=1}^{m} M_i = U \oplus \bigoplus_{j=1}^{n} N_j \oplus V$$

be direct decompositions of M, and assume that the endomorphism rings of the modules N_j are local. Take

$$M = U \oplus N' \oplus (N_n \oplus V)$$

where $N' = \bigoplus_{j=1}^{n-1} N_j$. By the induction assumption there are direct decompositions $M_i = X_i \oplus Y_i$ such that

$$M = U \oplus N' \oplus \bigoplus_{i=1}^{m} X_i \quad \text{and} \quad N' \cong \bigoplus_{i=1}^{m} Y_i.$$

Now we look at the direct decomposition

$$M = (U \oplus N') \oplus \bigoplus_{i=1}^{m} X_i$$

and the inclusion homomorphism from N_n into M. Then we apply the Exchange Theorem (see Skript 1) to this situation: We use that $N_n \oplus (U \oplus N')$ is a direct summand of M. For $1 \leq i \leq m$ we obtain a direct decomposition $X_i = M_i' \oplus X_i'$ such that

$$M = (U \oplus N') \oplus N_n \oplus \bigoplus_{i=1}^{m} M_i'$$

with $\bigoplus_{i=1}^{m} X_i' \cong N_n$. Note that only one of the modules X_i' is non-zero. Define $M_i'' = X_i' \oplus Y_i$. This implies

$$M_i = X_i \oplus Y_i = M_i' \oplus X_i' \oplus Y_i = M_i' \oplus M_i''$$

and

$$\bigoplus_{i=1}^{m} M_i'' = \bigoplus_{i=1}^{m} X_i' \oplus \bigoplus_{i=1}^{m} Y_i \cong N' \oplus N_n = N.$$

This finishes the proof. □

If $M = \bigoplus_{i \in I} M_i$ is a direct sum of modules M_i, and J is a subset of the index set I, we define

$$M_J := \bigoplus_{i \in J} M_i.$$

We want to study modules which can be written as direct sums of modules with local endomorphism ring. The key result in this situation is the following:

Theorem 27.2. *Let $M = \bigoplus_{i\in I} M_i$ be a direct sum of modules M_i with local endomorphism rings, and let U be a direct summand of M. Then the following hold:*

(a) *For every element $u \in U$ there exists a direct decomposition $U = U' \oplus U''$ and a finite subset $J \subseteq I$ such that $u \in U'$ and $U' \cong M_J$;*
(b) *If M/U is indecomposable, then there exists some $i \in I$ with $M = U \oplus M_i$.*

Proof. For $u \in U$ there exists a finite subset I' of I such that $u \in M_{I'}$. Since U is a direct summand of M, we can choose a direct decomposition $M = U \oplus C$. By the General Exchange Theorem 27.1 there exist submodules $U'' \subseteq U$ and $C'' \subseteq C$ such that $M = M_{I'} \oplus U'' \oplus C''$. Define

$$U' = (M_{I'} \oplus C'') \cap U \quad \text{and} \quad C' = (M_{I'} \oplus U'') \cap C.$$

We claim that

$$U = U' \oplus U'' \quad \text{and} \quad C = C' \oplus C''.$$

It is enough to show the first equality: Of course we have $U' \cap U'' = 0$, since $(M_{I'} \oplus C'') \cap U'' = 0$. Since $U'' \subset U$, we get by modularity

$$\begin{aligned} U = M \cap U &= (U'' \oplus M_{I'} \oplus C'') \cap U \\ &= U'' + ((M_{I'} \oplus C'') \cap U) \\ &= U'' + U'. \end{aligned}$$

We see that

$$U' \oplus U'' \oplus C' \oplus C'' = U \oplus C = M = M_{I'} \oplus U'' \oplus C'' \tag{1}$$

and therefore

$$U' \oplus C' \cong M/(U'' \oplus C'') \cong M_{I'}.$$

By the Krull-Remak-Schmidt Theorem there exists a subset $J \subseteq I'$ with $U' \cong M_J$. Of course u belongs to $U' = (M_{I'} \oplus C'') \cap U$. Thus we constructed a direct decomposition $U = U' \oplus U''$ with $u \in U'$ and $U' \cong M_J$ with J a finite subset of I. This proves part (a) of the theorem.

We started with an arbitrary direct decomposition $M = U \oplus C$, and now we want to prove (b) for the direct summand C (and not for U). Thus we assume that M/C is indecomposable. Since $U \cong M/C$, we know that U is indecomposable. Let u be a non-zero element in U. As before we obtain a direct decomposition $U = U' \oplus U''$ with $u \in U'$. We see that $U'' = 0$. Thus Equation 1 reduces to

$$U' \oplus C' \oplus C'' = M = \bigoplus_{i\in I'} M_i \oplus C''.$$

Now C' is isomorphic to a direct summand of $M_{I'}$, thus by the Krull-Remak-Schmidt Theorem it is a finite direct sum of modules with local endomorphism rings. Thus we can apply the General Exchange Theorem 27.1 and obtain direct decompositions $M_i = M_i' \oplus M_i''$ for $i \in I'$ such that

$$M = C' \oplus \bigoplus_{i \in I'} M_i' \oplus C'' = C \oplus \bigoplus_{i \in I'} M_i'.$$

Since M/C is indecomposable, we know that exactly one of the modules M_i', say M_{i_0}', is non-zero. On the other hand, $M_{i_0} = M_{i_0}' \oplus M_{i_0}''$ is indecomposable, and therefore $M_{i_0}' = M_{i_0}$. Thus $M = C \oplus M_{i_0}$. This proves part (b) for the direct summand C of M. □

Corollary 27.3. *Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of modules M_i with local endomorphism rings. Then every non-zero direct summand of M has a direct summand which is isomorphic to one of the M_i's.*

Proof. If U is a non-zero direct summand of M, then choose some $0 \neq u \in U$. Then part (a) of Theorem 27.2 yields a direct decomposition $U = U' \oplus U''$ with $u \in U'$ and $U' \cong M_J$ for some finite index set $J \subseteq I$. Since $0 \neq u \in U'$, we know that J is non-empty. If $i \in J$, then U has a direct summand isomorphic to M_i. □

Corollary 27.4. *Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of modules M_i with local endomorphism rings. If U is an indecomposable direct summand of M, then $U \cong M_i$ for some $i \in I$.*

Proof. Choose $0 \neq u \in U$. We get a direct decomposition $U = U' \oplus U''$ and a finite non-empty index set $J \subseteq I$ with $u \in U'$ and $U' \cong M_J$. Since U is indecomposable, $U = U' \cong M_i$ with $i \in J$. □

27.2. **The Krull-Remak-Schmidt-Azumaya Theorem.**

Theorem 27.5 (Azumaya)**.** *Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of modules M_i with local endomorphism rings. Let $U = \bigoplus_{j \in J} U_j$ be a direct summand of M. For every indecomposable module N let $I(N)$ be the set of indices $i \in I$ with $M_i \cong N$, and let $J(N)$ be the set of indices $j \in J$ with $U_j \cong N$. Then we have*

$$|J(N)| \leq |I(N)|.$$

Proof. First, let $J(N)$ be finite and non-empty, and let $j_0 \in J(N)$. Corollary 27.4 yields that there exists some $i_0 \in I$ with $M_{i_0} \cong U_{j_0}$. The Cancellation Theorem implies that

$$\bigoplus_{i \in I \setminus \{i_0\}} M_i \cong \bigoplus_{j \in J \setminus \{j_0\}} U_j.$$

By induction we obtain $|J(N)| \leq |I(N)|$.

Next, assume that $J(N)$ is infinite. For $t \in J$ define $U_t' = \bigoplus_{j \neq t} U_j$. Let $i \in I(N)$, and let J_i be the set of all $t \in J$ with $M = M_i \oplus U_t'$. Obviously, J_i is a subset of $J(N)$, because $M_i \oplus U_t' = U_t \oplus U_t'$ implies $U_t \cong M_i \cong N$.

On the other hand, if $t \in J(N)$, then U'_t is a maximal direct summand of M. Thus there exists some $i \in I$ with $U'_t \oplus M_i = M$, and we see that $t \in J_i$. We proved that

$$\bigcup_{i \in I(N)} J_i = J(N).$$

We claim that every set of the form J_i is finite: Let $x \neq 0$ be an element in M_i. There exists a finite subset $J(x) \subseteq J$ such that $x \in \bigoplus_{j \in J(x)} U_j$. If $t \notin J(x)$, then $\bigoplus_{j \in J(x)} U_j \subseteq U'_t$, and therefore $x \in M_i \cap U'_t$ which implies $t \notin J_i$. We see that J_i is a subset of the finite set $J(x)$.

Since $J(N)$ is infinite, $I(N)$ has to be infinite as well. The cardinality of $\bigcup_{i \in I(N)} J_i$ is at most $|I(N)|$, thus $|J(N)| \leq |I(N)|$. □

Corollary 27.6. *Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of modules M_i with local endomorphism rings. Let U be a direct summand of M such that $U = \bigoplus_{j \in J} U_j$ with U_j indecomposable for all $j \in J$. Then there exists an injective map $\sigma\colon J \to I$ such that $U_j \cong M_{\sigma(j)}$ for all $j \in J$. In particular, U is isomorphic to $M_{I'}$ for some subset I' of I.*

Proof. Let $U = \bigoplus_{j \in J} U_j$ with indecomposable module U_j. We choose a direct complement C of U, thus

$$M = U \oplus C = \bigoplus_{j \in J} U_j \oplus C.$$

To this decomposition we apply the above Theorem 27.5. Thus for any indecomposable module N we have $|J(N)| \leq |I(N)|$. Thus there is an injective map $\sigma\colon J \to I$ such that $U_j \cong M_{\sigma(j)}$ for all j. We can identify J with a subset I' of I, and we obtain $U \cong M_{I'}$. □

Corollary 27.7 (Krull-Remak-Schmidt-Azumaya). *Assume that $M = \bigoplus_{i \in I} M_i$ is a direct sum of modules M_i with local endomorphism rings. Let $M = \bigoplus_{j \in J} U_j$ with indecomposable modules U_j. Then there exists a bijection $\sigma\colon I \to J$ such that $M_i \cong U_{\sigma(i)}$.*

Proof. By Corollary 27.6 there is an injective map $\sigma\colon I \to J$ with $M_i \cong U_{\sigma(i)}$ for all i. By Corollaries 27.4 and 27.6 we know that the modules U_j have local endomorphism rings. Thus for every indecomposable module N we have not only $|J(N)| \leq |I(N)|$, but also the reverse $|I(N)| \leq |J(N)|$, which implies $|J(N)| = |I(N)|$. Thus we can construct a bijection $\sigma\colon I \to J$ with $M_i \cong U_{\sigma(i)}$ for all i. □

27.3. **The Crawley-Jønsson-Warfield Theorem.**

Theorem 27.8 (Crawley-Jønsson-Warfield). *Let $M = \bigoplus_{i \in I} M_i$ be a direct sum of modules M_i with local endomorphism rings. If U is a countably generated direct summand of M, then there exists a subset J of I with*

$$U \cong M_J := \bigoplus_{j \in J} M_j.$$

Proof. Let U be a countably generated direct summand of M, and let $u_1, u_2, \dots$ be a countable generating set of U. Inductively we construct submodules U_t and V_t of U with

$$U = U_1 \oplus \cdots \oplus U_t \oplus V_t$$

such that $u_1, \dots, u_t \in \bigoplus_{i=1}^t U_i$, and such that each U_t is a direct sum of indecomposable modules.

As a start of our induction we take $t = 0$, and there is nothing to show. Now assume that we constructed alread $U_1, \dots, U_t, V_t$ with the mentioned properties.

Let $u_{t+1} = x_{t+1} + y_{t+1}$ with $x_{t+1} \in \bigoplus_{i=1}^t U_i$ and $y_{t+1} \in V_t$. Now V_t is a direct summand of M and $v_{t+1} \in V_t$. Thus by Theorem 27.2, (a) there exists a direct decomposition $V_t = U_{t+1} \oplus V_{t+1}$ with $y_{t+1} \in U_{t+1}$ such that U_{t+1} is a direct sum of modules of the form M_i. Since $y_{t+1} \in U_{t+1}$, we know that $u_{t+1} = x_{t+1} + y_{t+1}$ belongs to $\bigoplus_{i=1}^{t+1} U_i$.

We obtain the direct decomposition

$$U_1 \oplus \cdots \oplus U_t \oplus U_{t+1} \oplus V_{t+1},$$

and the modules U_i with $1 \leq i \leq t+1$ have the desired form.

By construction, the submodules U_i with $i \in \mathbb{N}_1$ form a direct sum. This direct sum is a submodule of U, and it also contains all elements u_i with $i \in \mathbb{N}_1$. Since these elements form a generating set for U, we get $U = \bigoplus_i U_i$. Thus we wrote U as a direct sum of indecomposable modules. Corollary 27.6 shows now that U is of the desired form. □

27.4. **Kaplansky's Theorem.** Let U be a submodule of a module M, and let $M = M_1 \oplus M_2$ be a direct decomposition of M. We call U **compatible with the direct decomposition** $M = M_1 \oplus M_2$ if $U = (U \cap M_1) + (U \cap M_2)$.

By $\aleph_0$ we denote the first infinite cardinal number. (For example $\mathbb{Q}$ is of cardinality $\aleph_0$.) Let c be a cardinal number. A module M is called c-**generated** if M has a generating set of cardinality at most c, and M is **countably generated** if M is $\aleph_0$-generated.

Theorem 27.9 (Kaplansky)**.** *Let $c \geq \aleph_0$ be a cardinal number. The class of modules, which are direct sums of c-generated modules, is closed under direct summands.*

Proof. For each $i \in I$ let M_i be a c-generated module, and let $M = \bigoplus_{i \in I} M_i$. We can assume $M_i \neq 0$ for all i. Let $M = X \oplus Y$. We want to show that X is a direct sum of c-generated submodules. Set $e = e(X, Y)$. If U is a submodule of M, define $\sigma(U) = eU + (1-e)U$. We call a subset $J \subseteq I$ *compatible* if M_J is compatible with the decomposition $X \oplus Y$. A set with cardinality at most c is called a *c-set*.

We start with some preliminary considerations:

(1) If $J \subseteq I$ is a c-set, then M_J is c-generated.

Proof: For every $i \in I$ choose a c-set X_i which generates M_i. Then set $X_J = \bigcup_{i \in J} X_i$ which is a generating set of M_J. (Here we assume that $M_J = \bigoplus_{i \in J} M_i$ is an inner direct sum, so the M_i are indeed submodules of M_J.) Since J and also all the X_i are c-sets, the cardinality of X_J is at most c^2. Since c is an infinite cardinal, we have $c^2 = c$.

(2) If U is a c-generated submodule of M, then eU, $(1-e)U$ and $\sigma(U)$ are c-generated.

Proof: If X is a generating set of U, then $eX = \{ex \mid x \in X\}$ is a generating set of eU. Similarly, $(1-e)X$ is a generating set of $(1-e)U$.

(3) For every c-generated submodule U of M there exists a c-set $J \subseteq I$ such that $U \subseteq M_J$.

Proof: Let X be a generating set of U. For every $x \in X$ there exists a finite subset $J(x) \subseteq I$ with $x \in M_{J(x)}$. Define $J = \bigcup_{x \in X} J(x)$. Now all sets $J(x)$ are finite, X is a c-set and c is an infinite cardinal number, thus we conclude that J is a c-set. By construction X is contained in the submodule M_J of M. Since U is (as a module) generated by X, we know that U is a submodule of M_J.

(4) For every c-generated submodule U of M there exists a compatible c-set $J \subseteq I$ such that $U \subseteq M_J$.

Proof: Let U be a c-generated submodule of M. By (3) we can find a c-set $J(1) \subseteq I$ such that $U \subseteq M_{J(1)}$. We can form $\sigma(M_{J(1)})$.

Inductively we construct c-sets $J(1) \subseteq J(2) \subseteq \cdots \subseteq I$ such that $\sigma(M_{J(t)}) \subseteq \sigma(M_{J(t+1)})$ for all $t \geq 1$. (Here we use (1), (2) and (3).) Define $J = \bigcup_{t \geq 1} J(t)$. We have $M_J = \bigcup_{t \geq 1} M_{J(t)}$. Since $J(t)$ is a c-set, and since c is an infinite cardinal number, the set J is also a c-set. It remains to show that M_J is compatible with the decomposition $M = X \oplus Y$, in other words, we have to show that for every $x \in M_J$ also ex belongs to M_J: Since $x \in M_J$ we have $x \in M_{J(t)}$ for some t. Therefore $ex \in eM_{J(t)} \subseteq \sigma(M_{J(t)}) \subseteq M_{J(t+1)} \subseteq M_J$.

(5) If $I(j)$ is a compatible subset of I, then $eM_{I(j)} \subseteq M_{I(j)}$. Set $J = \bigcup I(j)$. If $eM_{I(j)} \subseteq M_{I(j)}$ for every j, then $eM_J \subseteq M_J$.

Now we can start with the proof of the theorem:

Let $I(\alpha)$ be an ordered chain of compatible subsets of I with the following properties:

(i) The cardinality of $I(\alpha+1) \setminus I(\alpha)$ is at most c;
(ii) If λ is a limit number, then $I(\lambda) = \bigcup_{\alpha < \lambda} I(\alpha)$;
(iii) We have $\bigcup_\alpha I(\alpha) = I$.

Here $I(\alpha)$ is defined inductively: Let $I(0) = 0$. If α is an ordinal number with $I(\alpha) \subset I$, choose some $x \in M_I \setminus M_{I(\alpha)}$. Let U_x be the submodule generated by x. By (4) there exists a compatible subset $J(x)$ of I with cardinality at most c such that U_x is contained in $M_{J(x)}$. Define $I(\alpha+1) = I(\alpha) \cup J(x)$. By (5) this is again a compatible set. For a limit number λ define $I(\lambda)$ as in (ii). It follows from (5) that $I(\lambda)$ is compatible.

Since $I(\alpha)$ is compatible, we get a decomposition

$$M_{I(\alpha)} = X(\alpha) \oplus Y(\alpha)$$

with $X(\alpha) \subseteq X$ and $Y(\alpha) \subseteq Y$. Let us stress that the submodules $X(\alpha)$ and $Y(\alpha)$ are direct summands of M. We have $X(\alpha) \subseteq X(\alpha+1)$, and $X(\alpha)$ is a direct summand of $X(\alpha+1)$, say $X(\alpha+1) = X(\alpha) \oplus U(\alpha+1)$. If λ is a limit ordinal number, then $X(\alpha) = \bigcup_{\alpha<\lambda} X(\alpha)$. We get

$$X = \bigoplus_{\alpha} U(\alpha)$$

and

$$U(\alpha+1) = X(\alpha+1)/X(\alpha) = eM_{I(\alpha+1)}/eM_{I(\alpha)} \cong eM_{I(\alpha+1)\setminus I(\alpha)}.$$

By (i), (1) and (2) we obtain that $U(\alpha)$ is c-generated. □

Corollary 27.10. *Let $M = \bigoplus_{i\in I} M_i$ be a direct sum of countably generated modules M_i with local endomorphism rings. If U is a direct summand of M, then there exists a subset $J \subseteq I$ such that $U \cong M_J$.*

Proof. By Theorem 27.9 every direct summand U of M is a direct sum of countably generated modules U_j. On the other hand, we know that every countably generated direct summand U_j of M is a direct sum of indecomposable modules, thus U ist a direct sum of indecomposable modules. Finally, we use Corollary 27.6. □

Question 27.11. *Is the class of modules, which are direct sums of modules with local enomorphism rings, closed under direct summands?*

The following is a direct consequence of Theorem 27.9:

Theorem 27.12 (Kaplansky)**.** *Every projective module is a direct sum of countably generated projective modules.*

Proof. Each projective A-module is a direct summand of a direct sum of modules of the form ${}_AA$. The module ${}_AA$ is cyclic, in particular it is countably generated. Thus we can apply Theorem 27.9, where we set $c = \aleph_0$. □

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28. **Projective modules over semiperfect algebras**

In this section we generalize the results from Section 25.

An algebra A is **semiperfect** if the following equivalent conditions are satisfied:

(i) ${}_AA$ is a (finite) direct sum of local modules;

(ii) ${}_AA$ is a (finite) direct sum of modules with local endomorphism ring;
(iii) The identity 1_A is a sum of pairwise orthogonal idempotents e_i such that the rings e_iAe_i are local.

If A is semiperfect, then A^{op} is semiperfect as well. This follows since condition (iii) is left-right symmetric.

Examples:

(a) Finite-dimensional algebras are semiperfect.
(b) Let $M_1, \dots, M_n$ be indecomposable A-modules with local endomorphism rings, and let $M = M_1 \oplus \cdots \oplus M_n$. Then $\mathrm{End}_A(M)$ is semiperfect, since condition (iii) is obviously satisfied.

Theorem 28.1. *Let A be a semiperfect algebra. Then $A/J(A)$ is semisimple.*

Proof. The module ${}_AA$ is a finite direct sum of local modules, thus ${}_AA/\operatorname{rad}({}_AA)$ is a finite sum of simple modules and therefore semisimple. □

Warning: The converse of the above theorem does not hold: By $K(T)$ we denote the field of rational functions in one variable T. Thus $K(T)$ consists of fractions f/g of polynomials $f, g \in K[T]$ where $g \neq 0$. Now let A be the subring of $K(T)$ consisting of all rational functions f/g such that neither T nor $T-1$ divide g. The radical $J(A)$ of A is the ideal generated by $T(T-1)$, and the corresponding factor ring $A/J(A)$ is isomorphic to $K \times K$, in particular it is semisimple. Note that A has no zero divisors, but $A/J(A)$ contains the two orthogonal idempotents $\overline{-T+1}$ and $\overline{T}$. For example

$$(-T+1)^2 = T^2 - 2T + 1 = (T^2 - T) - T + 1,$$

and modulo $T^2 - T$ this is equal to $-T+1$.

Theorem 28.2. *Let A be a semiperfect algebra. Then the following hold:*

- *Each projective A-module is a direct sum of indecomposable modules;*
- *Each indecomposable projective module is local and has a local endomophism ring;*
- *If ${}_AA = P_1 \oplus \cdots \oplus P_n$ is a direct decomposition of the regular representation into indecomposable modules P_i, then each indecomposable projective A-module is isomorphic to one of the P_i.*

Proof. Since A is a semiperfect algebra, the module ${}_AA$ is a direct sum of local modules. Thus let

$${}_AA = \bigoplus_{i=1}^{m} Q_i$$

with local modules Q_i. As a direct summand of ${}_AA$ each Q_i is of the form Ae_i for some idempotent e_i. In particular Q_i is cyclic. The endomorphism ring of an A-module of the form Ae (where e is an idempotent) is $(eAe)^{\mathrm{op}}$, and if Ae is local, then so is eAe. Thus also $(eAe)^{\mathrm{op}}$ is a local ring.

Let P be a projective A-module. Thus P is a direct summand of a free A-module F. We know that F is a direct sum of modules with local endomorphism ring, namely of copies of the Q_i. Kaplansky's Theorem implies that

$$P = \bigoplus_{j \in J} P_j$$

is a direct sum of countably generated modules P_j. By the Crawley-Jønsson-Warfield Theorem each P_j (and therefore also P) is a direct sum of modules of the form Q_i.

So each projective module is a direct sum of indecomposable modules, and each indecomposable projective module is of the form Q_i, in particular it is local.

If ${}_AA = \bigoplus_{k=1}^n P_k$ is another direct decomposition with indecomposable modules P_k, then by the Krull-Remak-Schmidt Theorem we get $m = n$, and each P_k is isomorphic to some Q_i. □

Theorem 28.3. *Let A be a semiperfect algebra. The map $P \mapsto P/\operatorname{rad}(P)$ yields a bijection between the isomorphism classes of indecomposable projective A-modules and the isomorphism classes of simple A-modules.*

Proof. By Theorem 28.2 we know that each indecomposable projective A-module is local and isomorphic to a direct summand of ${}_AA$. Now we can continue just as in the proof of Theorem 25.3. □

Theorem 28.4. *Let A be a semiperfect algebra, and let P be an indecomposable projective A-module. Set $S := P/\operatorname{rad}(P)$. Then the following hold:*

(i) *P is local;*
(ii) *$\operatorname{End}_A(P)$ is a local ring;*
(iii) *$J(\operatorname{End}_A(P)) = \{f \in \operatorname{End}_A(P) \mid \operatorname{Im}(f) \subseteq \operatorname{rad}(P)\}$;*
(iv) *Each endomorphism of P induces an endomorphism of S, and we obtain an algebra isomorphism $\operatorname{End}_A(P)/J(\operatorname{End}_A(P)) \to \operatorname{End}_A(S)$;*
(v) *The multiplicity of P in a direct sum decomposition ${}_AA = \bigoplus_{i=1}^m P_i$ with indecomposable modules P_i is exactly the dimension of S as an $\operatorname{End}_A(S)$-module.*

Proof. We have shown already that each indecomposable projective A-module P is local and isomorphic to a direct summand of ${}_AA$. Therefore $\operatorname{End}_A(P)$ is local. In particular, $\operatorname{rad}(P)$ is small in P. Now copy the proof of Theorem 25.4. □

End of Lecture 31

29. **Digression: Projective modules in other areas of mathematics**

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