The Hall Algebra Approach to Quantum Groups

Claus Michael Ringel

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Introduction

Given any Dynkin diagram Δ of type $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$, we may endow its edges with an orientation; we obtain in this way a *quiver* (an oriented graph) $\vec{\Delta}$, and the corresponding path algebra $k\vec{\Delta}$, where k is a field. We may consider the representations of $\vec{\Delta}$ over k, or, equivalently, the $k\vec{\Delta}$-modules. In case k is a finite field, one may define a multiplication on the free abelian group with basis the isomorphism classes of $k\vec{\Delta}$-modules by counting filtrations of modules; the ring obtained in this way is called the Hall algebra $\mathcal{H}(k\vec{\Delta})$.

We denote by $\mathbb{Z}[v]$ the polynomial ring in one variable v, and we set $q = v^2$. Also, let $A = \mathbb{Z}[v, v^{-1}]$.

The free $\mathbb{Z}[q]$-module $\mathcal{H}(\vec{\Delta})$ with basis the isomorphism classes of $k\vec{\Delta}$-modules can be endowed with a multiplication so that $\mathcal{H}(\vec{\Delta})/(q - |k|) \simeq \mathcal{H}(k\vec{\Delta})$, for any finite field k of cardinality $|k|$, thus $\mathcal{H}(\vec{\Delta})$ may be called the *generic* Hall algebra. The generic Hall algebra satisfies relations which are very similar to the ones used by Jimbo and Drinfeld in order to define a q-deformation $U_q(\mathbf{n}_+(\Delta))$ of the Kostant $\mathbb{Z}$-form $U(\mathbf{n}_+(\Delta))$. Here, $\mathbf{g}(\Delta) = \mathbf{n}_-(\Delta) \oplus \mathbf{h}(\Delta) \oplus \mathbf{n}_+(\Delta)$ is a triangular decomposition of the complex simple Lie algebra $\mathbf{g}(\Delta)$ of type Δ. Note that $U_q(\mathbf{n}_+(\Delta))$ is an A-algebra, and we can modify the multiplication of $\mathcal{H}(\Delta) \otimes_{\mathbb{Z}[q]} A$ using the Euler characteristic on the Grothendieck group $K_0(k\vec{\Delta})$ in order to obtain the *twisted* Hall algebra $\mathcal{H}_*(\vec{\Delta})$ with

$$\boxed{U_q(\mathbf{n}_+(\Delta)) \simeq \mathcal{H}_*(\vec{\Delta})}$$

What is the advantage of the Hall algebra approach? Assume we have identified $U_q(\mathbf{n}_+(\Delta))$ with $\mathcal{H}_*(\vec{\Delta})$. Note that the ring $U_q(\mathbf{n}_+(\Delta))$ is defined by generators and relations, whereas $\mathcal{H}_*(\vec{\Delta})$ is a free A-module with a prescribed basis.

The presentation of $U_q(\mathbf{n}_+(\Delta))$ gives us a presentation for the (twisted) Hall algebra, and this may be interpreted as follows: the Jimbo-Drinfeld relations are the universal relations for comparing the numbers of composition series of modules over algebras with a prescribed quiver.

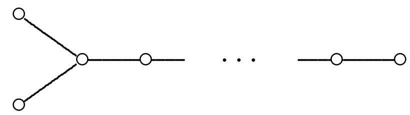
On the other hand, in $\mathcal{H}_*(\vec{\Delta})$, there is the prescribed basis given by the $k\vec{\Delta}$-modules, and we obtain in this way a basis for $U_q(\mathbf{n}_+(\Delta))$, thus normal forms for its elements, and this makes calculations in $U_q(\mathbf{n}_+(\Delta))$ easier. Also, the basis elements themselves gain more importance, more flavour. Since they may be interpreted as modules, one can discuss about their module theoretical, homological or geometrical properties: whether they are indecomposable, or multiplicityfree and so on.

The basis of $U_q(\mathbf{n}_+(\Delta))$ obtained in this way depends on the chosen orientation of Δ, and Lusztig has proposed a base change which leads to a basis which is independent of such a choice and which he calls the *canonical* basis. This basis also was constructed by Kashiwara and called the *crystal* basis of $U_q(\mathbf{n}_+(\Delta))$.

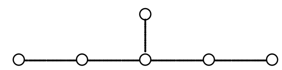
Here is the list of the Dynkin diagrams $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$:

A_n

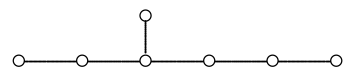
D_n



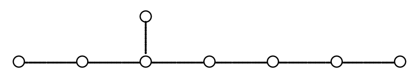
E_6



E_7



E_8



0. Preliminaries

Consider the following polynomials in a variable T, where $n, m \in \mathbb{N}_0$ and $m \leq n$

$$F^n(T) := \frac{(T^n - 1)(T^{n-1} - 1) \cdots (T - 1)}{(T - 1)^n},$$

$$G^n_m(T) := \frac{(T^n - 1)(T^{n-1} - 1) \cdots (T^{n-m+1} - 1)}{(T^m - 1)(T^{m-1} - 1) \cdots (T - 1)}.$$

Note that the degree of the polynomial $F^n(T)$ is $\binom{n}{2}$, the degree of $G^n_m(T)$ is $m(n - m)$.

Let k be a finite field, denote its cardinality by $q_k = |k|$. The cardinality of the set of complete flags in k^n is just $F^n(q_k)$, and for $0 \leq m \leq n$, the number of m-dimensional subspaces of k^n is $G^n_m(q_k)$.

Let $A' = \mathbb{Q}(v)$ be the rational function field over $\mathbb{Q}$ in one variable v, and let us consider its subring $A = \mathbb{Z}[v, v^{-1}]$. We denote by $\overline{}\colon A' \to A'$ the field automorphism with $\overline{v} = v^{-1}$; it has order 2, and it sends A onto itself.

We set $q = v^2$; we will have to deal with $F^n(q)$ and $G^n_m(q)$. We define

$$[n] := \frac{v^n - v^{-n}}{v - v^{-1}} = v^{n-1} + v^{n-3} + \cdots + v^{-n+1},$$

thus $[0] = 0,\ [1] = 1,\ [2] = v + v^{-1},\ [3] = v^2 + 1 + v^{-2}$, and so on. Let

$$[n]! := \prod_{m=1}^{n} [m],$$

$$\begin{bmatrix} n \\ m \end{bmatrix} := \frac{[n]!}{[m]![n-m]!} \quad \text{where} \quad 0 \leq m \leq n.$$

There are the following identities:

$$[n] = v^{-n+1} \frac{q^n - 1}{q - 1}$$

$$[n]! = v^{-\binom{n}{2}} F^n(q)$$

$$\begin{bmatrix} n \\ m \end{bmatrix} = v^{-m(n-m)} G^n_m(q).$$

1. The definition of $U_q(\mathbf{n}_+(\Delta))$

Let $\Delta = (a_{ij})_{ij}$ be a symmetric $(n \times n)$-matrix with diagonal entries $a_{ii} = 2$, and with off-diagonal entries 0 and -1. (Such a matrix is called a *simply-laced generalized Cartan matrix.*)

Note that Δ defines a graph with n vertices labelled $1, 2, \ldots, n$ with edges $\{i, j\}$ provided $a_{ij} = -1$. Often we will not need the labels of the vertices, then we will present the vertices by small dots $\circ$. Of particular interest will be the Dynkin diagrams $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$.

Given Δ, we define $U'_q(\mathbf{n}_+(\Delta))$ as the A'-algebra with generators $E_1, \ldots, E_n$ and relations

$$\begin{aligned} E_iE_j - E_jE_i &= 0 \quad &\text{if} \quad a_{ij} = 0, \\ E_i^2E_j - (v + v^{-1})E_iE_jE_i + E_jE_i^2 &= 0 \quad &\text{if} \quad a_{ij} = -1. \end{aligned}$$

We denote

$$E_i^{(m)} := \frac{1}{[\,m\,]!}E_i^m.$$

Let $U_q(\mathbf{n}_+(\Delta))$ be the A-subalgebra of $U'_q(\mathbf{n}_+(\Delta))$ generated by the elements $E_i^{(m)}$ with $1 \leq i \leq n$ and $m \geq 0$.

We denote by $^-\colon U'_q(\mathbf{n}_+(\Delta)) \to U'_q(\mathbf{n}_+(\Delta))$ the automorphism with $\overline{v} = v^{-1}$ and $\overline{E_i} = E_i$ for all i.

We denote by $\mathbb{Z}^n$ the free abelian group of rank n with basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$. Given an element $\mathbf{d} \in \mathbb{Z}^n$, say $\mathbf{d} = \sum d_i\mathbf{e}_i$, let $|\mathbf{d}| = \sum d_i$.

Note that the rings $U_q(\mathbf{n}_+(\Delta))$ and $U'_q(\mathbf{n}_+(\Delta))$ are $\mathbb{Z}^n$-graded, where we assign to E_i the degree $\mathbf{e}_i$. Given $\mathbf{d} \in \mathbb{Z}^n$, we denote by $U_q(\mathbf{n}_+(\Delta))_{\mathbf{d}}$ the set of homogeneous elements of degree $\mathbf{d}$, thus

$$U_q(\mathbf{n}_+(\Delta)) = \bigoplus_{\mathbf{d}} U_q(\mathbf{n}_+(\Delta))_{\mathbf{d}}.$$

Let Δ be of the form $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, or $\mathbb{E}_8$. We denote by $\Phi = \Phi(\Delta)$ the corresponding root system. We choose a basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of the root system, and denote by Φ^+ the set of positive roots (with respect to this choice). The choice of the basis yields a fixed embedding of Φ into $\mathbb{Z}^n$.

We will have to deal with maps $\alpha\colon \Phi^+ \to \mathbb{N}_0$. Given such a map α, we set

$$\mathbf{dim}\,\alpha := \sum_{\mathbf{a}} \alpha(\mathbf{a})\mathbf{a} \in \mathbb{Z}^n$$

and call it its *dimension vector.* We denote by $u(\mathbf{d})$ for $\mathbf{d} \in \mathbb{Z}^n$ the number of maps $\alpha\colon \Phi^+ \to \mathbb{N}_0$ with $\mathbf{dim}\,\alpha = \mathbf{d}$.

Consider the $\mathbb{Q}$-Lie-algebra $\mathbf{n}_+(\Delta)$ generated by $E_1, \dots, E_n$ with relations

$$\begin{aligned} [E_i, E_j] &= 0 \quad \text{if} \quad a_{ij} = 0, \\ [E_i, [E_i, E_j]] &= 0 \quad \text{if} \quad a_{ij} = -1. \end{aligned}$$

(Usually, one deals with the corresponding $\mathbb{C}$-algebra $\mathbf{n}_+(\Delta) \otimes_{\mathbb{Q}} \mathbb{C}$; here, it will be more convenient to consider the mentioned $\mathbb{Q}$-form.)

The universal enveloping algebra $U(\mathbf{n}_+(\Delta))$ is the $\mathbb{Q}$-algebra generated by the elements $E_1, \dots, E_n$ with relations

$$\begin{aligned} [E_i, E_j] &= 0 \quad \text{if} \quad a_{ij} = 0, \\ [E_i, [E_i, E_j]] &= 0 \quad \text{if} \quad a_{ij} = -1, \end{aligned}$$

thus we see:

Proposition. *We have*

$$U(\mathbf{n}_+(\Delta)) = U_q(\mathbf{n}_+(\Delta)) \otimes_A \mathbb{Q}[v, v^{-1}]/(v-1).$$

Of course, $\mathbf{n}_+(\Delta)$ and $U(\mathbf{n}_+(\Delta))$ both are $\mathbb{Z}^n$-graded, where again we assign to E_i the degree $\mathbf{e}_i$. For any non-zero homogeneous element L of $\mathbf{n}_+(\Delta)$, we denote by $\mathbf{dim}\,L$ its degree. It is well-known that $\mathbf{n}_+(\Delta)$ has a basis $E_{\mathbf{a}}$ indexed by the positive roots, such that $\mathbf{dim}\,E_{\mathbf{a}} = \mathbf{a}$. As a consequence, we obtain the following consequence:

Proposition. *The $\mathbb{Q}$-dimension of $U(\mathbf{n}_+(\Delta))_{\mathbf{d}}$ is $u(\mathbf{d})$.*

Proof: Use the theorem of Poincaré-Birkhoff-Witt.

2. Rings and modules, path algebras of quivers

Rings and modules.

Given a ring R, the R-modules which we will consider will be finitely generated right R-modules. The category of finitely generated right R-modules will be denoted by $\operatorname{mod} R$.

Let R be a ring. The direct sum of two R-modules M_1, M_2 will be denoted by $M_1 \oplus M_2$, the direct sum of t copies of M will be denoted by tM. The zero module will be denoted by 0_R or just by 0.

We write $M \simeq M'$, in case the modules M, M' are isomorphic, the isomorphism class of M will be denoted by $[M]$. For any module M, we denote by $s(M)$ the number of isomorphism classes of indecomposable direct summands of M.

Let R be a finite dimensional algebra over some field k. Let $n = s(R_R)$, thus n is the rank of the Grothendieck group $K_0(R)$ of all finite length modules modulo split exact sequences. Given such a module M, we denote its equivalence class in $K_0(R)$ by $\mathbf{dim}\, M$. There are precisely n isomorphism classes of simple R-modules $S_1, \dots, S_n$, and the elements $\mathbf{e}_i = \mathbf{dim}\, S_i$ form a basis of $K_0(R)$. If we denote the Jordan-Hölder multiplicity of S_i in M by $[M : S_i]$, then $\mathbf{dim}\, M = \sum_i [M : S_i]\mathbf{e}_i$.

We denote by $\operatorname{supp} M$ the support of M, it is the set of simple modules S with $[M : S] \neq 0$. (In case the simple modules are indexed by the vertices of a quiver, we also will consider $\operatorname{supp} M$ as a subset of the set of vertices of this quiver).

Path algebras of quivers

A *quiver* $\vec{\Delta} = (\vec{\Delta}_0, \vec{\Delta}_1, s, t)$ is given by two sets $\vec{\Delta}_0, \vec{\Delta}_1$, and two maps $s, t\colon \vec{\Delta}_1 \to \vec{\Delta}_0$. The elements of $\vec{\Delta}_0$ are called *vertices*, the elements of $\vec{\Delta}_1$ are called *arrows*; given $f \in \vec{\Delta}_1$, then we say that f starts in $s(f)$ and ends in $t(f)$, and we write $f\colon s(f) \to t(f)$. An arrow f with $s(f) = t(f)$ is called a *loop*, we always will assume that $\vec{\Delta}$ has no loops.

We denote by $k\vec{\Delta}$ the path algebra of the quiver $\vec{\Delta}$ over the field k. We will not distinguish between representations of $\vec{\Delta}$ over k and (right) $k\vec{\Delta}$-modules. Recall that a representation M of $\vec{\Delta}$ over k attaches to each vertex x of $\vec{\Delta}$ a vector space M_x over k, and to each arrow $f\colon s(f) \to t(f)$ a k-linear map $M_{s(f)} \to M_{t(f)}$. For any vertex x of $\vec{\Delta}$, we can define a simple $k\vec{\Delta}$-module $S(x)$ by attaching the one-dimensional k-space k to the vertex x, the zero space to the remaining vertices, and the zero map to all arrows. We stress that the number of arrows $x \to y$ is equal to $\dim_k \operatorname{Ext}^1(S(x), S(y))$. In case there is precisely one arrow starting in x and ending in y, there exists up to isomorphism a unique indecomposable representation of length 2 with top $S(x)$ and socle $S(y)$, we denote it by $\begin{smallmatrix} S(x) \\ S(y) \end{smallmatrix}$.

In case $\vec{\Delta}$ has as vertex set the set $\{1,2,\dots,n\}$, we define a corresponding $(n\times n)$-matrix $\Delta=(a_{ij})_{ij}$ as follows: let $a_{ii}=2$, for all i, and let a_{ij} be the number of arrows between i and j (take the arrows $i\to j$ as well as the arrows $j\to i$). In case there is at most one arrow between i and j we obtain a matrix as considered in section 1, and then we will call Δ the underlying graph of $\vec{\Delta}$.

Let us assume that $\vec{\Delta}$ is a finite quiver with n vertices, and let $\Lambda=k\vec{\Delta}$. We assume in addition that $\vec{\Delta}$ has no cyclic paths (a cyclic path is a path of length at least 1 starting and ending in the same vertex). As a consequence, $\vec{\Delta}$ is finite-dimensional, and there are precisely n simple Λ-modules, namely the modules $S(x)$, with x a vertex. Of course, if M is a representation of $\vec{\Delta}$, then $\mathbf{dim}\,M=\sum_x(\dim_k M_x)\,\mathbf{dim}\,S(x)$ in the Grothendieck group $K_0(\Lambda)$.

It is easy to see that Λ is hereditary, thus we can define on $K_0(\Lambda)$ a bilinear form via

$$\langle \mathbf{dim}\,X,\mathbf{dim}\,Y\rangle=\dim_k \operatorname{Hom}(X,Y)-\dim_k \operatorname{Ext}^1(X,Y)$$

where X,Y are Λ-modules of finite length. The corresponding quadratic form will be denoted by χ; thus for $\mathbf{d}\in K_0(\Lambda)$, we have $\chi(\mathbf{d})=\langle\mathbf{d},\mathbf{d}\rangle$. Of course, we have the following formula for all i,j

$$a_{ij}=\langle \mathbf{dim}\,S_i,\mathbf{dim}\,S_j\rangle+\langle \mathbf{dim}\,S_j,\mathbf{dim}\,S_i\rangle$$

Dynkin quivers

A quiver $\vec{\Delta}$ whose underlying graph is of the form $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ will be called a *Dynkin quiver.* We recall some well-known results:

Gabriel's Theorem. *Let $\vec{\Delta}$ be a Dynkin quiver. The map* **dim** *yields a bijection between the isomorphism classes of the indecomposable $k\vec{\Delta}$-modules and the positive roots for Δ.*

Let $\vec{\Delta}$ be a Dynkin quiver, and let $\Lambda=k\vec{\Delta}$. Given a positive root $\mathbf{a}$ for Δ, we denote by $M(\mathbf{a})$ or $M_\Lambda(\mathbf{a})$ the corresponding Λ-module; thus $M(\mathbf{a})=M_\Lambda(\mathbf{a})$ is an indecomposable Λ-module with $\mathbf{dim}\,M(\mathbf{a})=\mathbf{a}$. Similarly, given a map $\alpha\colon\Phi^+\to\mathbb{N}_0$, we denote by $M(\alpha)$ we denote the Λ-module

$$M(\alpha)=M_\Lambda(\alpha)=\bigoplus_{\mathbf{a}}\alpha(\mathbf{a})M(\mathbf{a}).$$

We obtain in this way a bijection between the maps $\Phi^+\to\mathbb{N}_0$ and the isomorphism classes of Λ-modules of finite length (according to the Krull-Schmidt theorem).

A finite dimensional k-algebra R is called *representation directed* provided there is only a finite number of (isomorphism classes of) indecomposable R-modules, say $M_1,\dots,M_m$, and they can be indexed in such a way that $\operatorname{Hom}(M_i,M_j)=0$ for $i>j$.

Proposition. *Let $\vec{\Delta}$ be a Dynkin quiver. Then $k\vec{\Delta}$ is representation directed.*

Let $\Phi^+ = \{ \mathbf{a}_1, \dots, \mathbf{a}_m \}$, we will assume that the ordering is chosen so that

$$\operatorname{Hom}(M_\Lambda(\mathbf{a}_i), M_\Lambda(\mathbf{a}_j)) \neq 0 \quad \text{implies} \quad i \leq j.$$

The subcategories $\mathcal{C}$, $\mathcal{D}$ of mod Λ are said to be *linearly separated* provided for modules C in $\operatorname{add}\mathcal{C}$, and D in $\operatorname{add}\mathcal{D}$ with $\mathbf{dim}\, C = \mathbf{dim}\, D$, we have $C = 0 = D$.

Lemma. *The subcategories* $\operatorname{add}\{M(\mathbf{a}_1), \dots, M(\mathbf{a}_{s-1})\}$ *and* $\operatorname{add}\{M(\mathbf{a}_s), \dots, M(\mathbf{a}_m)\}$ *are linearly separated.*

3. The Hall algebra of a finitary ring.

Given a ring R, we will be interested in the finite R-modules; here a module M will be said to be *finite* provided the cardinality of its underlying set is finite (not just that M is of finite length). Of course, for many rings the only finite R-module will be the zero-module, but for finite rings, in particular for finite-dimensional algebras over finite fields, all finite length modules are finite modules. A ring R will be said to be *finitary* provided the group $\operatorname{Ext}^1(S_1, S_2)$ is finite, for all finite simple R-modules S_1, S_2. (For a discussion of finitary rings, see [R1]).

We assume that R is a finitary ring. We mainly will consider path algebras of finite quivers over finite fields; of course, such a ring is finitary.

Given finite R-modules $N_1, N_2, \dots, N_t$ and M, let $\mathcal{F}^M_{N_1,\dots,N_t}$ be the set of filtrations

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_t = 0$$

such that M_{i-1}/M_i is isomorphic to N_i, for all $1 \leq i \leq t$. The cardinality of $\mathcal{F}^M_{N_1,\dots,N_t}$ will be denoted by $F^M_{N_1,\dots,N_t}$ or also by $\langle N_1 N_2 \dots N_t \diamondsuit M \rangle$. (These cardinalities are finite, since we assume that R is finitary.)

Let $\mathcal{H}(R)$ be the free abelian group with basis the set of isomorphism classes $[X]$ of finite R-modules, with a multiplication which we denote by the diamond sign $\diamond$

$$[N_1] \diamond [N_2] := \sum_{[M]} F^M_{N_1 N_2} [M] = \sum_{[M]} \langle N_1 N_2 \diamondsuit M \rangle [M].$$

Given an element $x \in \mathcal{H}(R)$, we denote its t^{th} power with respect to the diamond product by $x^{\diamond t}$.

Proposition. *$\mathcal{H}(R)$ is an associative ring with 1.*

Proof: The associativity follows from the fact that

$$([N_1] \diamond [N_2]) \diamond [N_3] = \sum_{[M]} F^M_{N_1 N_2 N_3}[M] = [N_1] \diamond ([N_2] \diamond [N_3]),$$

The unit element is just $[0_R]$, with 0_R the zero module.

In case R is a finite-dimensional algebra over some finite field, we assign to the isomorphism class $[M]$ the degree $\mathbf{dim}\, M \in \mathbb{Z}^n$. Let $\mathcal{H}(R)_{\mathbf{d}}$ be the free abelian group with basis the set of isomorphism classes $[M]$ of finite R-modules with $\mathbf{dim}\, M = \mathbf{d}$.

Proposition. *$\mathcal{H}(R) = \bigoplus_{\mathbf{d}} \mathcal{H}(R)_{\mathbf{d}}$ is a $\mathbb{Z}^n$-graded ring.*

Proof: We only have to observe that for $F^M_{N_1 N_2} \neq 0$, we have $\mathbf{dim}\, M = \mathbf{dim}\, N_1 + \mathbf{dim}\, N_2$.

From now on, let $\vec{\Delta}$ be a Dynkin quiver, let k be a field. We consider $\Lambda = k\vec{\Delta}$. Let $\{1, 2, \dots, n\}$ be the vertices of $\vec{\Delta}$, ordered in such a way that

$$\operatorname{Ext}^1(S_i, S_j) \neq 0 \quad \textit{implies} \quad i < j.$$

Let $\Phi^+ = \{\mathbf{a}_1, \dots, \mathbf{a}_m\}$, and we will assume that the ordering is chosen so that

$$\operatorname{Hom}(M_\Lambda(\mathbf{a}_i), M_\Lambda(\mathbf{a}_j)) \neq 0 \quad \textit{implies} \quad i \leq j.$$

Hall polynomials

Proposition. *Let $\alpha, \beta, \gamma \colon \Phi^+ \to \mathbb{N}_0$. There exists a polynomial $\phi^\beta_{\alpha,\gamma}(q) \in \mathbb{Z}[q]$ such that for any finite field k of cardinality q_k*

$$F^{M_\Lambda(\beta)}_{M_\Lambda(\alpha) M_\Lambda(\gamma)} = \phi^\beta_{\alpha,\gamma}(q_k).$$

For a proof, see [R1], Theorem 1, p.439.

The polynomials which arise in this way are called *Hall polynomials.*

Let $\vec{\Delta}$ be a Dynkin quiver. Let $\mathcal{H}(\vec{\Delta})$ be the free $\mathbb{Z}[q]$-module with basis the set of maps $\Phi^+ \to \mathbb{N}_0$. On $\mathcal{H}(\vec{\Delta})$, we define a multiplication by

$$\alpha_1 \diamond \alpha_2 := \sum_\beta \phi^\beta_{\alpha_1 \alpha_2}(q) \cdot \beta$$

Proposition. *$\mathcal{H}(\vec{\Delta})$ is an associative ring with 1, it is $\mathbb{Z}^n$-graded (the degree of $\alpha \colon \Phi^+ \to \mathbb{N}_0$ being $\mathbf{dim}\, \alpha$), and for any finite field k of cardinality q_k, the map $\alpha \mapsto [M_{k\vec{\Delta}}(\alpha)]$ yields an isomorphism*

$$\mathcal{H}(\vec{\Delta})/(q - q_k) \simeq \mathcal{H}(k\vec{\Delta}).$$

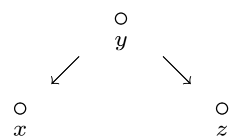
4. Loewy series.

For $\mathbf{d} = (d_1, \dots, d_n) \in \mathbb{N}_0^n$, let

$$w_\diamond(\mathbf{d}) := [d_1 S_1] \diamond \cdots \diamond [d_n S_n].$$

Note that the element $w_\diamond(\mathbf{d})$ only depends on the semisimple module $\bigoplus d_i S_i$ and not on the particular chosen ordering of the vertices of $\vec{\Delta}$, since $[d_i S_i] \diamond [d_j S_j] = [d_j S_j] \diamond [d_i S_i]$ in case $\operatorname{Ext}^1(S_i, S_j) = 0 = \operatorname{Ext}^1(S_j, S_i)$.

Remark: Recall that the vertices $\{1, 2, \dots, n\}$ of $\vec{\Delta}$ are ordered in such a way that $\operatorname{Ext}^1(S_i, S_j) \neq 0$ implies that $i < j$. Usually, there will be several possible orderings, for example in the case of $\mathbb{A}_3$ with orientation



we have to take $1 = y$ but we may take $2 = x,\ 3 = z$ or else $2 = z,\ 3 = x$. All possible orderings are obtained from each other by a finite sequence of transpositions $(i, i+1)$ in case $\operatorname{Ext}^1(S_i, S_{i+1}) = 0 = \operatorname{Ext}^1(S_{i+1}, S_i)$.

Lemma. *We have* $\langle w_\diamond(\mathbf{d}) \mathbin{\diamond\!\!\!\!\mid} M\rangle \neq 0$ *if and only if* $\mathbf{dim}\, M = \mathbf{d}$; *and, in this case,* $\langle w_\diamond(\mathbf{d}) \mathbin{\diamond\!\!\!\!\mid} M\rangle = 1$.

The proof is obvious.

Given a map $\alpha\colon \Phi^+ \to \mathbb{N}_0$, let

$$w_\diamond(\alpha) := w_\diamond(\alpha(\mathbf{a}_1)\mathbf{a}_1) \diamond \cdots \diamond w_\diamond(\alpha(\mathbf{a}_m)\mathbf{a}_m).$$

The element $w_\diamond(\alpha)$ does not depend on the chosen ordering of the positive roots.

Example. Consider the case $\mathbb{A}_2$. Thus, there is given a hereditary k-algebra Λ with two simple modules S_1, S_2 such that $\operatorname{Ext}^1(S_1, S_2) = k$. There is a unique indecomposable module of length 2, and we denote it by I. There are three positive roots $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, where $\mathbf{a}_2 = \mathbf{a}_1 + \mathbf{a}_3$. If we assume that $S_2 = M_\Lambda(\mathbf{a}_1)$ and $S_1 = M_\Lambda(\mathbf{a}_3)$, then the ordering is as desired. For $\alpha\colon \Phi^+ \to \mathbb{N}_0$, we obtain the following element

$$\begin{aligned} w_\diamond(\alpha) &= w_\diamond(\alpha(\mathbf{a}_1)\mathbf{a}_1) \diamond w_\diamond(\alpha(\mathbf{a}_2)\mathbf{a}_2) \diamond w_\diamond(\alpha(\mathbf{a}_3)\mathbf{a}_3) \\ &= [\alpha(\mathbf{a}_1)S_2] \diamond [\alpha(\mathbf{a}_2)S_1] \diamond [\alpha(\mathbf{a}_2)S_2] \diamond [\alpha(\mathbf{a}_3)S_1], \end{aligned}$$

the corresponding Λ-module is $M_\Lambda(\alpha) = \alpha(\mathbf{a}_1)S_2 \oplus \alpha(\mathbf{a}_2)I \oplus \alpha(\mathbf{a}_3)S_1$.

Lemma. *We have $\langle w_\diamond(\alpha) \diamondsuit M\rangle \neq 0$ if and only if there exists a filtration*

$$M = M_0 \supseteq M_1 \supseteq \cdots \supseteq M_m = 0$$

such that $\mathbf{dim}\, M_{i-1}/M_i = \alpha(\mathbf{a}_i)\mathbf{a}_i$.

The proof is obvious.

The set of maps $\Phi^+ \to \mathbb{N}_0$ will be ordered using the opposite of the lexicographical ordering: Given $\alpha, \beta \colon \Phi^+ \to \mathbb{N}_0$, we have $\beta < \alpha$ if and only if there exists some $1 \le j \le m$ such that $\beta(\mathbf{a}_i) = \alpha(\mathbf{a}_i)$ for all $i < j$, whereas $\beta(\mathbf{a}_j) > \alpha(\mathbf{a}_j)$.

Theorem 1. *Let $\alpha \colon \Phi^+ \to \mathbb{N}_0$. Then $\langle w_\diamond(\alpha) \diamondsuit M(\alpha)\rangle = 1$. On the other hand, given a module M with $\langle w_\diamond(\alpha) \diamondsuit M\rangle \neq 0$, then $M \simeq M(\beta)$ for some $\beta \le \alpha$.*

Before we present the proof, we need some preliminary considerations. Given $\alpha \colon \Phi^+ \to \mathbb{N}_0$, let us define for $0 \le t \le m$, the submodule $M_t(\alpha) = \bigoplus_{i>t} \alpha(\mathbf{a}_i) M_\Lambda(\mathbf{a}_i)$ of $M(\alpha)$. Thus we obtain a sequence of submodules

$$M(\alpha) = M_0(\alpha) \supseteq M_1(\alpha) \supseteq \cdots \supseteq M_m(\alpha) = 0.$$

Lemma. *Let U be a submodule of $M' = M_{t-1}(\beta)$ such that $\mathbf{dim}\, M'/U = u \cdot \mathbf{a}_j$ for some j. Then we have $j \ge t$. If $j = t$, then $U \supseteq M_t$ (and therefore $u \le \beta(\mathbf{a}_t)$), there is an isomorphism $M'/U \simeq uM(\mathbf{a}_j)$, and $U = M_t(\beta) \oplus U'$, with $U' \simeq (\beta(\mathbf{a}_t) - u)M(\mathbf{a}_t)$.*

Proof. We can assume $u > 0$.

Let us first assume that $M'/U \simeq u \cdot M(a_j)$. First of all, we show that $j \ge t$. For $j < t$, we have $\mathrm{Hom}(M(\mathbf{a}_i), M(\mathbf{a}_j) = 0$ for all $i \ge t$, thus $\mathrm{Hom}(M', M(\mathbf{a}_j)) = 0$, whereas there is given a non-zero map $M' \to M'/U \simeq u \cdot M(\mathbf{a}_j)$. Now assume $j = t$. Using the same argument, we see that the composition of the inclusion map $M_t(\beta) \to M'$ and the projection map $M' \to M'/U$ has to be zero, since $\mathrm{Hom}(M(\mathbf{a}_i), M(\mathbf{a}_t)) = 0$ for $i > t$. This shows that $U \supseteq M_t$, and consequently $u \le \beta(\mathbf{a}_t)$. The canonical projection $\beta(\mathbf{a}_t)M(\mathbf{a}_t) \simeq M'/M_t(\beta) \to M'/U$ splits, thus $U/M_t(\beta) \simeq (\beta(\mathbf{a}_t) - u)M(\mathbf{a}_t)$. But then also the projection $U \to U/M_t(\beta)$ splits (since $\mathrm{Ext}^1(M(\mathbf{a}_t), M(\mathbf{a}_i)) = 0$ for all $i > t$). This shows the existence of a direct complement U' in U to $M_t(\beta)$, and we have $U' \simeq U/M_t(\beta) \simeq (\beta(\mathbf{a}_t) - u)M(\mathbf{a}_t)$.

In general, we can write $M'/U = M(\gamma)$ for some $\gamma \colon \Phi^+ \to \mathbb{N}_0$. Choose s minimal with $\gamma(\mathbf{a}_s) > 0$. Let $U \subseteq V \subset M_{t-1}(\beta)$ such that $M_{t-1}(\beta)/V = M_s(\gamma)$. Then $M_{t-1}(\beta)/V \simeq M(\gamma)/M_s(\gamma) \simeq \gamma(a_s)M(\mathbf{a}_s)$, and we can apply the previous considerations. We see that $s \ge t$, and if $s = t$, then $\gamma(\mathbf{a}_t) \le \beta(\mathbf{a}_t)$. By assumption, $u \cdot \mathbf{a}_j = \mathbf{dim}\, M'/U = \sum_i \gamma(\mathbf{a}_i)\mathbf{a}_i = \sum_{i \ge s} \gamma(\mathbf{a}_i)\mathbf{a}_i$, with non-negative coefficients u and $\gamma(\mathbf{a}_i)$. We cannot have $j < s$, since $\{M(\mathbf{a}_1), \ldots, M(\mathbf{a}_{s-1})\}$ and $\{M(\mathbf{a}_s), \ldots, M(\mathbf{a}_m)\}$ are linearly separated. This shows that $j \ge s \ge t$. Now assume $j = t$, thus we have $j = s = t$. We must have $\gamma(\mathbf{a}_t) \le u$, and thus we can write $(u - \gamma(\mathbf{a}_t)) \cdot \mathbf{a}_t = \sum_{i>t} \gamma(\mathbf{a}_i)\mathbf{a}_i$ with non-negative coefficients $(u - \gamma(\mathbf{a}_t))$ and $\gamma(\mathbf{a}_i)$. Now, we use that $\{M(\mathbf{a}_1), \ldots, M(\mathbf{a}_t)\}$ and $\{M(\mathbf{a}_{t+1}), \ldots, M(\mathbf{a}_m)\}$ are linearly separated in order to conclude that $u-\gamma(\mathbf{a}_t)=0$ and $\gamma(\mathbf{a}_i)=0$ for all $i>t$. This shows that $M'/U\simeq\gamma(\mathbf{a}_t)M(\mathbf{a}_t)$ and therefore our first considerations do apply.

Proof of Theorem 1: First of all, the sequence

$$M(\alpha)=M_0(\alpha)\supseteq M_1(\alpha)\supseteq\cdots\supseteq M_m(\alpha)=0$$

shows that $\langle w_\diamond(\alpha)\ \phi\ M(\alpha)\rangle\neq 0$.

Let as assume that $\langle w_\diamond(\alpha)\ \phi\ M(\beta)\rangle\neq 0$ for some $\alpha,\beta\colon\Phi^+\to\mathbb{N}_0$. Thus, we know that there exists a sequence

$$M(\beta)=M_0\supseteq M_1\supseteq\cdots\supseteq M_m=0$$

such that $\mathbf{dim}\, M_{i-1}/M_i=\alpha(\mathbf{a}_i)\mathbf{a}_i$. Let $\beta(\mathbf{a}_i)=\alpha(\mathbf{a}_i)$ for all $i<j$. By induction, we claim that $M_i=M_i(\beta)$ for $i<j$. Assume, we know that $M_{i-1}=M_{i-1}(\beta)$. According to Lemma, the only submodule $M_i(=U)$ of $M_{i-1}(\beta)$ with $\mathbf{dim}\, M_{i-1}(\beta)/M_i=\beta(\mathbf{a}_i)$ is $M_i=M_i(\beta)$. In particular, we have $M_{j-1}=M_{j-1}(\beta)$. Again, using Lemma, we see that we must have $\alpha(\mathbf{a}_j)\leq\beta(\mathbf{a}_j)$, this shows that $\alpha\geq\beta$. Also, we see that for $\alpha=\beta$, we have $M_i=M_i(\beta)$ for all i, thus $\langle w_\diamond(\alpha)\ \phi\ M(\alpha)\rangle=1$. This completes the proof.

5. The fundamental relations

Lemma. *Let S_i,S_j be simple R-modules with*

$$\operatorname{Ext}^1(S_i,S_j)=0,\quad \operatorname{Ext}^1(S_j,S_i)=0.$$

Then we have

$$[S_i]\diamond[S_j]=[S_j]\diamond[S_i].$$

The proof is obvious.

Lemma. *Let k be a finite field of cardinality q_k. Let R be a k-algebra. Let S_i,S_j be simple R-modules such that*

$$\operatorname{Ext}^1(S_i,S_i)=0,\ \operatorname{Ext}^1(S_j,S_j)=0,\ \operatorname{Ext}^1(S_i,S_j)=k,\ \operatorname{Ext}^1(S_j,S_i)=0.$$

Then

$$[S_i]^{\diamond 2}\diamond[S_j]-(q_k+1)[S_i]\diamond[S_j]\diamond[S_i]+q_k[S_j]\diamond[S_i]^{\diamond 2}=0,$$
$$[S_i]\diamond[S_j]^{\diamond 2}-(q_k+1)[S_i]\diamond[S_j]\diamond[S_i]+q_k[S_j]^{\diamond 2}\diamond[S_i]=0.$$

Proof: Since $\mathrm{Ext}^1(S_i,S_j)=k$, there exists an indecomposable module M of length 2 with top S_i and socle S_j. Taking into account the assumptions $\mathrm{Ext}^1(S_i,S_i)=0=\mathrm{Ext}^1(S_j,S_i)$, we see that there are just two isomorphism classes of modules of length three with 2 composition factors of the form S_i and one of the form S_j, namely $X=M\oplus S_i$ and $Y=2S_i\oplus S_j$. It is easy to check that

$$\begin{aligned}
[S_i]^{\diamond 2}\diamond[S_j] &= (q_k+1)[X]+(q_k+1)[Y],\\
[S_i]\diamond[S_j]\diamond[S_i] &= \qquad\quad [X]+(q_k+1)[Y],\\
[S_j]\diamond[S_i]^{\diamond 2} &= \qquad\qquad\qquad (q_k+1)[Y].
\end{aligned}$$

This yields the first equality. Similarly, there are the two isomorphism classes of modules with 2 composition factors of the form S_j and one of the form S_i, namely $X'=M\oplus S_j$ and $Y'=S_i\oplus 2S_j$. It is easy to check that

$$\begin{aligned}
[S_i]\diamond[S_j]^{\diamond 2} &= (q_k+1)[X']+(q_k+1)[Y'],\\
[S_i]\diamond[S_j]\diamond[S_i] &= \qquad\quad [X']+(q_k+1)[Y'],\\
[S_j]^{\diamond 2}\diamond[S_i] &= \qquad\qquad\qquad (q_k+1)[Y'].
\end{aligned}$$

This yields the second equality.

As an immediate consequence we obtain:

Proposition. *The elements $[S_i]$ of $\mathcal{H}(\vec{\Delta})$ satisfy the following relations: Let $i<j$. If there is no arrow from i to j, then*

$$[S_i]\diamond[S_j]-[S_j]\diamond[S_i]=0,$$

if there is an arrow $i\to j$, then

$$\begin{aligned}
[S_i]^{\diamond 2}\diamond[S_j]-(q+1)[S_i]\diamond[S_j]\diamond[S_i]+q[S_j]\diamond[S_i]^{\diamond 2}&=0,\\
[S_i]\diamond[S_j]^{\diamond 2}-(q+1)[S_i]\diamond[S_j]\diamond[S_i]+q[S_j]^{\diamond 2}\diamond[S_i]&=0.
\end{aligned}$$

More general, if we start with simple modules S_i,S_j satisfying

$$\begin{aligned}
\mathrm{Ext}^1(S_i,S_i)&=0, &\mathrm{Ext}^1(S_j,S_j)&=0,\\
\mathrm{Ext}^1(S_i,S_j)&=k^t, &\mathrm{Ext}^1(S_j,S_i)&=0.
\end{aligned}$$

for some t, then we obtain relations which are similar to the Jimbo-Drinfeld relations which are used to define quantum groups for arbitrary symmetrizable generalized Cartan matrices. See [R3].

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6. The twisted Hall algebra

In the ring $A = \mathbb{Z}[v, v^{-1}]$, we denote the element v^2 by q. In this way, we have fixed an embedding of $\mathbb{Z}[q]$ into A. Consider the A-module

$$\mathcal{H}_*(\vec{\Delta}) = \mathcal{H}(\vec{\Delta}) \otimes_{\mathbb{Z}[q]} A.$$

In $\mathcal{H}_*(\vec{\Delta})$, we introduce a new multiplication $*$ by

$$\begin{aligned}[N_1] * [N_2] &:= v^{\dim_k \operatorname{Hom}(N_1,N_2) - \dim_k \operatorname{Ext}^1(N_1,N_2)} [N_1] \diamond [N_2] \\ &= v^{\langle \mathbf{dim}\, N_1, \mathbf{dim}\, N_2\rangle} [N_1] \diamond [N_2]\end{aligned}$$

where N_1, N_2 are Λ-modules.

The following assertion is rather obvious:

Proposition. *The free A-module $\mathcal{H}_*(\vec{\Delta})$ with the multiplication $*$ is an associative algebra with 1, and $\mathbb{Z}^n$-graded.*

We call $\mathcal{H}_*(\vec{\Delta})$ (with this multiplication) the *twisted Hall algebra* of $\vec{\Delta}$. For any element x, we denote its t^{th} power with respect to the $*$ multiplication by $x^{*(t)}$.

Using induction, one shows that

$$[N_1] * [N_2] * \cdots * [N_m] = v^{\sum_{i<j} \langle \mathbf{dim}\, N_i, \mathbf{dim}\, N_j\rangle} [N_1] \diamond [N_2] \diamond \ldots \diamond [N_m].$$

Example. Assume there is an arrow $i \to j$. Then

$$\begin{aligned}&[S_i] * [S_j] = v^{-1}[S_i] \diamond [S_j] = v^{-1}\Big(\begin{bmatrix} S_i \\ S_j \end{bmatrix} + [S_i \oplus S_j]\Big), \\ &[S_j] * [S_i] = [S_i \oplus S_j],\end{aligned}$$

thus

$$\begin{bmatrix} S_i \\ S_j \end{bmatrix} = v\,[S_i] * [S_j] - [S_i \oplus S_j] = v\,[S_i] * [S_j] - [S_j] * [S_i].$$

Proposition. *The elements $[S_i]$ of $\mathcal{H}_*(\vec{\Delta})$ satisfy the following relations:*

$$\begin{aligned}[S_i] * [S_j] - [S_j] * [S_i] &= 0 \qquad \textit{if} \quad a_{ij} = 0, \\ [S_i]^{*(2)} * [S_j] - (v + v^{-1})[S_i] * [S_j] * [S_i] + [S_j] * [S_i]^{*(2)} &= 0 \qquad \textit{if} \quad a_{ij} = -1.\end{aligned}$$

Proof: In case $a_{ij} = 0$, we must have $\operatorname{Ext}^1(S_i, S_j) = 0 = \operatorname{Ext}^1(S_j, S_i)$, and therefore we have $\langle \mathbf{dim}\, S_i, \mathbf{dim}\, S_j \rangle = 0 = \langle \mathbf{dim}\, S_j, \mathbf{dim}\, S_i \rangle$.

Now assume $a_{ij} = -1$. First, consider the case when $i < j$, thus $\dim_k \operatorname{Ext}^1(S_i, S_j) = 1$, and $\operatorname{Ext}^1(S_j, S_i) = 0$. In this case, we have

$$\langle \mathbf{dim}\, S_i, \mathbf{dim}\, S_j \rangle = -1, \quad \text{and} \quad \langle \mathbf{dim}\, S_j, \mathbf{dim}\, S_i \rangle = 0.$$

Also, $\langle \mathbf{dim}\, S_i, \mathbf{dim}\, S_i \rangle = 1$, thus

$$\begin{aligned} [S_i]^{*(2)} * [S_j] &= v^{-1}[S_i]^{\diamond 2} \diamond [S_j], \\ [S_i] * [S_j] * [S_i] &= [S_i] \diamond [S_j] \diamond [S_i], \\ [S_j] * [S_i]^{*(2)} &= v[S_j] \diamond [S_i]^{\diamond 2}, \end{aligned}$$

thus

$$\begin{aligned} &[S_i]^{*(2)} * [S_j] - (v{+}v^{-1})[S_i] * [S_j] * [S_i] + [S_j] * [S_i]^{*(2)} \\ &\qquad = v^{-1}[S_i]^{\diamond 2} \diamond [S_j] - (v + v^{-1})[S_i] \diamond [S_j] \diamond [S_i] + v[S_j] \diamond [S_i]^{\diamond 2} \\ &\qquad = v^{-1}\Big([S_i]^{\diamond 2} \diamond [S_j] - (q+1)[S_i] \diamond [S_j] \diamond [S_i] + q[S_j] \diamond [S_i]^{\diamond 2}\Big) \\ &\qquad = 0. \end{aligned}$$

Similarly, if $j < i$, so that $\dim_k \operatorname{Ext}^1(S_j, S_i) = 1$, and $\operatorname{Ext}^1(S_i, S_j) = 0$, then

$$\begin{aligned} &[S_i]^{*(2)} * [S_j] - (v{+}v^{-1})[S_i] * [S_j] * [S_i] + [S_j] * [S_i]^{*(2)} \\ &\qquad = v[S_i]^{\diamond 2} \diamond [S_j] - (v + v^{-1})[S_i] \diamond [S_j] \diamond [S_i] + v^{-1}[S_j] \diamond [S_i]^{\diamond 2} \\ &\qquad = v^{-1}\Big(q[S_i]^{\diamond 2} \diamond [S_j] - (q+1)[S_i] \diamond [S_j] \diamond [S_i] + [S_j] \diamond [S_i]^{\diamond 2}\Big) \\ &\qquad = 0. \end{aligned}$$

Also in general, the fundamental relations in $\mathcal{H}(\vec{\Delta})$ give rise to the Jimbo-Drinfeld relations in $\mathcal{H}_*(\vec{\Delta})$, see [R6].

Divided powers. Given an indecomposable module X, let

$$[X]^{*(t)} := \frac{1}{[\,t\,]!}[X]^{*(t)},$$

we claim that this is an element of $\mathcal{H}_*(\Lambda)$. Namely:

$$\begin{aligned} [X]^{*(t)} &= v^{\binom{t}{2}}[X]^{\diamond(t)} \\ &= v^{\binom{t}{2}}F^t(q)[tX] \\ &= v^{t(t-1)}[\,t\,]![tX], \end{aligned}$$

(since $F^t(q) = v^{\binom{t}{2}}[\,t\,]!$). Thus

$$\boxed{[X]^{(*(t))} = \frac{1}{[\,t\,]!}[X]^{*(t)} = v^{t(t-1)}[tX]}$$

.

Using divided powers, we can rewrite the fundamental relations

$$[S_i]^{*(2)} * [S_j] - (v+v^{-1})[S_i] * [S_j] * [S_i] + [S_j] * [S_i]^{*(2)} = 0$$

as follows:

$$\boxed{[S_i]^{(*2)} * [S_j] - [S_i] * [S_j] * [S_i] + [S_j] * [S_i]^{(*2)} = 0}$$

Recall that the vertices $\{\,1,2,\ldots,n\,\}$ of Λ are ordered in such a way that

$$\mathrm{Ext}^1(S_i,S_j) \neq 0 \quad \text{implies} \quad i<j.$$

In case M is semisimple, say $M = \bigoplus d_i S(i)$, we have

$$[M] = [d_nS_n] \diamond \cdots \diamond [d_1S_1] = [d_nS_n] * \cdots * [d_1S_1],$$

since for $i>j$ we have $\mathrm{Hom}(d_iS_i,d_jS_j) = 0 = \mathrm{Ext}^1(d_iS_i,d_jS_j)$. Also, recall that

$$[tS_i] = v^{-t(t-1)}[S_i]^{(*t)},$$

thus

$$[M] = [d_nS_n] * \cdots * [d_1S_1] = v^{-\sum d_i(d_i-1)}[S_n]^{(*d_n)} * \cdots * [S_1]^{(*d_1)}.$$

The words $w_*(\mathbf{d})$, $w_*(\alpha)$.

Recall that $\{\,1,2,\ldots,n\,\}$ is the set of vertices vertices of Λ, ordered in such a way that

$$\mathrm{Ext}^1(S_i,S_j) \neq 0 \quad \text{implies} \quad i<j.$$

Also, recall that $\Phi^+ = \{\,\mathbf{a}_1,\ldots,\mathbf{a}_m\,\}$ is the set of positive roots and we assume that the ordering is chosen so that

$$\mathrm{Hom}(M_\Lambda(\mathbf{a}_i),M_\Lambda(\mathbf{a}_j)) \neq 0 \quad \text{implies} \quad i \leq j.$$

Using the multiplication $*$, we define for $\mathbf{d} \in \mathbb{N}_0^n$ and $\alpha\colon \Phi^+ \to \mathbb{N}_0$

$$\begin{aligned} w_*(\mathbf{d}) &:= [S_1]^{*(d_1)} * \cdots * [S_n]^{*(d_n)}, \\ w_*(\alpha) &:= w_*(\alpha(\mathbf{a}_1)\mathbf{a}_1) * \cdots * w_*(\alpha(\mathbf{a}_m)\mathbf{a}_m). \end{aligned}$$

Lemma. *We have*

$$\boxed{w_*(\alpha) = v^{r(\alpha)} w_\diamond(\alpha), \quad \textit{with} \quad r(\alpha) := -\dim_k M_\Lambda(\alpha) + \dim_k \operatorname{End}(M_\Lambda(\alpha))}$$

.

Proof: We have for $\mathbf{d} = \sum d_i \mathbf{e}_i$

$$\begin{aligned} w_*(\mathbf{d}) &= [S_1]^{(*d_1)} * \cdots * [S_n]^{(*d_n)} \\ &= v^{\sum d_i^2 - \sum d_i} [d_1 S_1] * \cdots * [d_n S_n] \\ &= v^{\sum d_i^2 - \sum d_i - \sum_{i \to j} d_i d_j} [d_1 S_1] \diamond \cdots \diamond [d_n S_n] \\ &= v^{\chi(\mathbf{d}) - |\mathbf{d}|} [d_1 S_1] \diamond \cdots \diamond [d_n S_n], \\ &= v^{\chi(\mathbf{d}) - |\mathbf{d}|} w_\diamond(d) \end{aligned}$$

where we have used that $\operatorname{Hom}(d_i S_i, d_j S_j) = 0$ for $i > j$ and $\dim_k \operatorname{Ext}^1(d_i S_i, d_j S_j) = d_i d_j$ for $i \to j$. We apply this for $\mathbf{d} = \alpha(\mathbf{a}_i)\mathbf{a}_i$. We note that

$$\chi(\alpha(\mathbf{a}_i)\mathbf{a}_i) = \dim_k \operatorname{End}(M_\Lambda(\alpha(\mathbf{a}_i)\mathbf{a}_i)),$$

and

$$|\alpha(\mathbf{a}_i)\mathbf{a}_i| = \dim_k M_\Lambda(\alpha(\mathbf{a}_i)\mathbf{a}_i),$$

therefore

$$w_*(\alpha(\mathbf{a}_i)\mathbf{a}_i) = v^{\dim_k \operatorname{End}(M_\Lambda(\alpha(\mathbf{a}_i)\mathbf{a}_i)) - \dim_k M_\Lambda(\alpha(\mathbf{a}_i)\mathbf{a}_i)} w_\diamond(\alpha(\mathbf{a}_i)\mathbf{a}_i).$$

On the other hand,

$$\begin{aligned} w_*(\alpha) &= w_*(\alpha(\mathbf{a}_1)\mathbf{a}_1) * \cdots * w_*(\alpha(\mathbf{a}_m)\mathbf{a}_m) \\ &= v^{r'} w_*(\alpha(\mathbf{a}_1)\mathbf{a}_1) \diamond \cdots \diamond w_*(\alpha(\mathbf{a}_m)\mathbf{a}_m) \end{aligned}$$

with

$$\begin{aligned} r' &= \sum_{i<j} \langle \alpha(\mathbf{a}_i)\mathbf{a}_i, \alpha(\mathbf{a}_j)\mathbf{a}_j \rangle \\ &= \sum_{i<j} \dim_k \operatorname{Hom}(\alpha(\mathbf{a}_i) M_\Lambda(\mathbf{a}_i), \alpha(\mathbf{a}_j) M_\Lambda(\mathbf{a}_j)) \\ &= \dim_k \operatorname{rad} \operatorname{End}(M_\Lambda(\alpha)), \end{aligned}$$

here we use that for $i < j$, we have $\operatorname{Ext}^1(\alpha(\mathbf{a}_i)M_\Lambda(\mathbf{a}_i), \alpha(\mathbf{a}_j)M_\Lambda(\mathbf{a}_j)) = 0$, and that for $i > j$, we have $\operatorname{Hom}(\alpha(\mathbf{a}_i)M_\Lambda(\mathbf{a}_i), \alpha(\mathbf{a}_j)M_\Lambda(\mathbf{a}_j)) = 0$. Altogether, we see that

$$w_*(\alpha) = v^{\dim_k \operatorname{rad} \operatorname{End}(M_\Lambda(\alpha))} w_*(\alpha(\mathbf{a}_1)\mathbf{a}_1) \diamond \cdots \diamond w_*(\alpha(\mathbf{a}_m)\mathbf{a}_m) = v^r w_\diamond(\alpha),$$

with

$$\begin{aligned} r &= r' + \sum_i \dim_k \operatorname{End}(M_\Lambda(\alpha(\mathbf{a}_i)\mathbf{a}_i)) - \sum_i \dim_k M_\Lambda(\alpha(\mathbf{a}_i)\mathbf{a}_i) \\ &= \dim_k \operatorname{End}(M_\Lambda(\alpha)) - \dim_k M_\Lambda(\alpha). \end{aligned}$$

This completes the proof.

By definition, $\mathcal{H}_*(\Lambda)$ is the free A-module with basis elements the isomorphism classes $[M]$ of the finite Λ-modules. It seems to be worthwhile to consider besides these elements $[M]$ also their multiples

$$\langle M \rangle := v^{-\dim_k M + \dim_k \operatorname{End}(M)}[M].$$

Example.

$$\left\langle \begin{matrix} S_i \\ S_j \end{matrix} \right\rangle = v^{-2+1} \begin{bmatrix} S_i \\ S_j \end{bmatrix} = v^{-1}(v\,[S_i] * [S_j] - [S_j] * [S_i]) = [S_i] * [S_j] - v^{-1}[S_j] * [S_i].$$

Theorem 1′.

$$\boxed{w_*(\alpha) = \langle M_\Lambda(\alpha) \rangle + \sum_{\beta < \alpha} g_{\alpha\beta} \langle M_\Lambda(\beta) \rangle \quad \text{with} \quad g_{\alpha\beta} \in A}$$

Proof: This is a direct consequence of Theorem 1.

Lemma.

$$\boxed{\langle M_\Lambda(\alpha) \rangle = \langle \alpha(\mathbf{a}_1)M_\Lambda(\mathbf{a}_1) \rangle * \cdots * \langle \alpha(\mathbf{a}_m)M_\Lambda(\mathbf{a}_m) \rangle}$$

Proof:

$$\begin{aligned} &\langle \alpha(\mathbf{a}_1)M_\Lambda(\mathbf{a}_1) \rangle * \cdots * \langle \alpha(\mathbf{a}_m)M_\Lambda(\mathbf{a}_m) \rangle \\ &\quad = v^{-\sum |\alpha(\mathbf{a}_i)\mathbf{a}_i| + \sum \alpha(\mathbf{a}_i)^2} [\alpha(\mathbf{a}_1)M_\Lambda(\mathbf{a}_1)] * \cdots * [\alpha(\mathbf{a}_m)M_\Lambda(\mathbf{a}_m)] \\ &\quad = v^{-\sum |\alpha(\mathbf{a}_i)\mathbf{a}_i| + \sum \alpha(\mathbf{a}_i)^2} v^{\dim_k \operatorname{rad} \operatorname{End}(M(\alpha))} [\alpha(\mathbf{a}_1)M_\Lambda(\mathbf{a}_1)] \diamond \cdots \diamond [\alpha(\mathbf{a}_m)M_\Lambda(\mathbf{a}_m)] \\ &\quad = v^{-\dim_k M(\alpha) + \dim_k \operatorname{End}(M(\alpha))} [M_\Lambda(\alpha)] \\ &\quad = \langle M_\Lambda(\alpha) \rangle \end{aligned}$$

Example. Let us consider the explicit expression for $w_*(\mathbf{d})$, where $\mathbf{d} \in \mathbb{N}_0^n$.

$$w_*(\mathbf{d}) = \sum_{\mathbf{dim}\,\beta=\mathbf{d}} v^{-\delta(\beta)} \langle M_\Lambda(\beta)\rangle \quad \text{with} \quad \delta(\beta) := \dim_k \operatorname{Ext}^1(M_\Lambda(\beta), M_\Lambda(\beta)).$$

Proof: We have

$$w_*(\mathbf{d}) = v^{\chi(\mathbf{d})-|\mathbf{d}|} w_\diamond(\mathbf{d}) = v^{\chi(\mathbf{d})-|\mathbf{d}|} \sum_{\mathbf{dim}\,\beta=\mathbf{d}} [M_\Lambda(\beta)],$$

since any module $M_\Lambda(\beta)$ with $\mathbf{dim}\,\beta = \mathbf{d}$ has a unique filtration of type $w_\diamond(\mathbf{d})$. But

$$\begin{aligned} \chi(\mathbf{d}) - |\mathbf{d}| &= \dim_k \operatorname{End}(M_\Lambda(\beta)) - \dim_k \operatorname{Ext}^1(M_\Lambda(\beta), M_\Lambda(\beta)) - |\mathbf{d}| \\ &= -\delta(\beta) + r(\beta). \end{aligned}$$

Thus,

$$\begin{aligned} w_*(\mathbf{d}) &= v^{\chi(\mathbf{d})-|\mathbf{d}|} \sum_{\mathbf{dim}\,\beta=\mathbf{d}} [M_\Lambda(\beta)] \\ &= \sum_{\mathbf{dim}\,\beta=\mathbf{d}} v^{-\delta(\beta)} v^{r(\beta)} [M_\Lambda(\beta)] \\ &= \sum_{\mathbf{dim}\,\beta=\mathbf{d}} v^{-\delta(\beta)} \langle M_\Lambda(\beta)\rangle. \end{aligned}$$

More generally, given $\alpha, \beta \colon \Phi^+ \to \mathbb{N}_0$, we have to consider

$$\begin{aligned} \delta(\beta;\alpha) &= \dim_k \operatorname{Ext}^1(M(\beta), M(\beta)) - \dim_k \operatorname{Ext}^1(M(\alpha), M(\alpha)) \\ &= \dim_k \operatorname{End}(M(\alpha)) - \dim_k \operatorname{End}(M(\beta)), \end{aligned}$$

of course, we have $\delta(\beta) = \delta(\beta; 0)$.

7. The isomorphism between $U_q(\mathbf{n}_+(\Delta))$ and $\mathcal{H}_*(k\vec{\Delta})$ for $\vec{\Delta}$ a Dynkin quiver

Proposition. *The elements $[S_i]^{*(t)}$ with $1 \le i \le n$ and $t \ge 1$ generate $\mathcal{H}_*(\vec{\Delta})$ as a A-algebra.*

Proof: Let $\mathcal{H}'$ be the A-algebra generated by the elements $[S_i]^{*(t)}$ with $1 \le i \le n$ and $t \ge 1$. By induction on $\mathbf{dim}\,\alpha$, we show that $\langle M_\Lambda(\alpha)\rangle$ belongs to $\mathcal{H}'$.

If the support of α contains more than one element, then we use the formula

$$\langle M_\Lambda(\alpha)\rangle = \langle \alpha(\mathbf{a}_1)M_\Lambda(\mathbf{a}_1)\rangle * \cdots * \langle \alpha(\mathbf{a}_m)M_\Lambda(\mathbf{a}_m)\rangle.$$

By induction, all the elements $\langle \alpha(\mathbf{a}_i)M_\Lambda(\mathbf{a}_i)\rangle$ belong to $\mathcal{H}'$, thus also $\langle M_\Lambda(\alpha)\rangle$, and thefore $[M_\Lambda(\alpha)]$ belong to $\mathcal{H}'$.

In case the support of α consists of the unique element $\mathbf{a}_i$, let $\mathbf{d} = \alpha(\mathbf{a}_i)\mathbf{a}_i$, thus $M_\Lambda(\alpha) = M_\Lambda(\mathbf{d})$, and we know that

$$w_*(\mathbf{d}) = \langle M_\Lambda(\alpha)\rangle + \sum_{\substack{\mathbf{dim}\,\beta=\mathbf{d}\\ \beta\neq\alpha}} v^{-\delta(\beta)}\langle M_\Lambda(\beta)\rangle.$$

The support of any β with $\mathbf{dim}\,\beta = \mathbf{d}$ and $\beta \ne \alpha$ contains more than one element; as we have seen, this implies that the corresponding elements $\langle M_\Lambda(\beta)\rangle$ belong to $\mathcal{H}'$. Since also $w_*(\mathbf{d})$ is in $\mathcal{H}'$, we conclude that $\langle M_\Lambda(\alpha)\rangle$ belongs to $\mathcal{H}'$.

Of course, with $\langle M_\Lambda(\alpha)\rangle$ also $[M_\Lambda(\alpha)]$ belongs to $\mathcal{H}'$. This completes the proof.

The fundamental relations show that we may define a ring homomorphism

$$\eta\colon U_q(\mathbf{n}_+(\Delta)) \to \mathcal{H}_*(\vec{\Delta})$$

by $\eta(E_i) = [S_i]$. The Lemma above shows that this map is surjective.

Theorem. *The map $\eta\colon U_q(\mathbf{n}_+(\Delta)) \to \mathcal{H}_*(\vec{\Delta})$ is an isomorphism.*

We have to show that η is also injective. Let $A'' = \mathbb{Q}[v, v^{-1}]$, and $U'' = U''_q(\mathbf{n}_+(\Delta))$ the A''-subalgebra of $U'_q(\mathbf{n}_+(\Delta))$ generated by the elements $E_i^{(t)}$ with $1 \le i \le n$ and $t \ge 0$.

Also, let $\mathcal{H}''_*(\vec{\Delta}) = \mathcal{H}_*(\vec{\Delta}) \otimes_A A''$. Of course, the map η extends in a unique way to a map $\eta''\colon U'' \to \mathcal{H}''_*(\vec{\Delta})$ (thus $\eta''|U_q(\mathbf{n}_+(\Delta)) = \eta$). It remains to be seen that η'' is injective. Both U'' and $\mathcal{H}''_*(\vec{\Delta})$ are $\mathbb{Z}^n$-graded, and η'' respects this graduation, thus, for $\mathbf{d} \in \mathbb{Z}^n$, there is the corresponding map $\eta''_{\mathbf{d}}\colon U''_{\mathbf{d}} \to \mathcal{H}''_*(\vec{\Delta})_{\mathbf{d}}$, and we show that all these maps $\eta''_{\mathbf{d}}$ are injective.

The A''-module $U''_{\mathbf{d}}$ is torsionfree (since it is a submodule of $U'_q(\mathbf{n}_+(\Delta))$) and finitely generated. Since A'' is a principal ideal domain, we see that $U''_{\mathbf{d}}$ is a free A''-module. In order to calculate its rank, we consider the factor module $U''_{\mathbf{d}}/(v-1)$. As we have seen in section 1, we can identify $U''_{\mathbf{d}}/(v-1)$ with $U(\mathbf{n}_+(\Delta))_{\mathbf{d}}$, thus it has $\mathbb{Q}$-dimension $u(\mathbf{d})$. It follows that $U''_{\mathbf{d}}$ is a free A''-module of rank $u(\mathbf{d})$. On the other hand, $\mathcal{H}''_*(\vec{\Delta})_{\mathbf{d}}$ is the free A''-module with basis the set of maps $\alpha\colon \Phi^+ \to \mathbb{N}_0$ satisfying $\mathbf{dim}\,\alpha = \mathbf{d}$, thus it also is a free A''-module of rank $u(\mathbf{d})$. But any surjective map between free A''-modules of equal rank has to be an isomorphism. This completes the proof.

In our further considerations, it sometimes will be useful to identify $U_q(\mathbf{n}_+(\Delta))$ *and* $\mathcal{H}_*(\vec{\Delta})$ *via the map* η. *Under this identification, the generator* E_i *corresponds to the isomorphism class* $[S_i]$.

8. The canonical basis

For any pair $\beta < \alpha$ of maps $\Phi^+ \to \mathbb{N}_0$, Theorem $1'$ gives an element $g_{\alpha\beta} \in A$. Let $g_{\alpha\alpha} = 1$, and $g_{\alpha\beta} = 0$ in the remaining cases. We may consider $g = (g_{\alpha\beta})_{\alpha\beta}$ as a matrix using some total ordering of the indices; it is the base change matrix between the basis given by the elements $\langle M_\Lambda(\alpha)\rangle$ and the basis given by the elements $w_*(\alpha)$. Note that we may assume that g is a unipotent lower triangular matrix. Let $\overline{g}$ be obtained from g by applying the automorphism $\overline{}$, and g' the inverse of $\overline{g}$. Since $\overline{w_*(\alpha)} = w_*(\alpha)$, we see that

$$w_*(\alpha) = \overline{w_*(\alpha)} = \sum_\beta \overline{g_{\alpha\beta}}\,\overline{\langle M_\Lambda(\beta)\rangle},$$

thus

$$\overline{\langle M_\Lambda(\alpha)\rangle} = \sum_\beta g'_{\alpha\beta} w_*(\beta) = \sum_\beta \sum_\gamma g'_{\alpha\beta} g_{\beta\gamma} \langle M_\Lambda(\gamma)\rangle.$$

Let us denote by $h = g'g$ the matrix product, then h is again a unipotent lower triangular matrix, and $\overline{h} = h^{-1}$.

There exists a unique unipotent lower triangular matrix $u = (u_{\alpha\beta})_{\alpha,\beta}$ with off-diagonal entries in $\mathbb{Z}[v^{-1}]$ without constant term, such that $u = \overline{u}h$ (see [L6], 7.10, or also [D], 1.2).

The desired basis is

$$\boxed{C(\alpha) := \langle M_\Lambda(\alpha)\rangle + \sum_{\beta \prec \alpha} u_{\alpha\beta} \langle M_\Lambda(\beta)\rangle \quad \text{with} \quad u_{\alpha\beta} \in v^{-1}\mathbb{Z}[v^{-1}]}$$

.

this is called the *canonical basis* of $\mathcal{H}_*(\vec{\Delta})$ or also of $U_q(\mathbf{n}_+(\Delta))$.

Note that by construction the elements of the canonical basis are invariant under the automorphism $\overline{}$, since

$$\begin{aligned}\overline{C(\alpha)} &= \sum_\beta \overline{u_{\alpha\beta}}\,\overline{\langle M_\Lambda(\beta)\rangle} \\ &= \sum_{\beta,\gamma} \overline{u_{\alpha\beta}} h_{\beta\gamma} \langle M_\Lambda(\gamma)\rangle \\ &= \sum_\beta u_{\alpha\beta} \langle M_\Lambda(\beta)\rangle = C(\alpha).\end{aligned}$$

In fact, the element $C(\alpha)$ is characterized by the two properties

$$\boxed{C(\alpha) := \langle M_\Lambda(\alpha)\rangle + \sum_{\beta\prec\alpha} u_{\alpha\beta}\langle M_\Lambda(\beta)\rangle \quad \text{with} \quad u_{\alpha\beta} \in v^{-1}\mathbb{Z}[v^{-1}],}$$

and

$$\boxed{\overline{C(\alpha)} = C(\alpha)}$$

.

In particular, any monomial will satisfy the second property, thus in order to show that a monomial belongs to the canonical basis, we only have to verify the first property.

9. The case $\mathbb{A}_2$

We consider the quiver

$$1 \longrightarrow 2.$$

There are three positive roots $\mathbf{a}_1 = (1,0)$, $\mathbf{a}_2 = (1,1)$, $\mathbf{a}_3 = (0,1)$, with corresponding indecomposable modules $S_1 = M(1,0)$, $M(1,1)$, $S_2 = M(0,1)$. (For simplicity, we sometimes will denote the isomorphism class $[S_1]$ by 1, the isomorphism class $[S_2]$ by 2.)

The Auslander-Reiten quiver is of the form

$$\begin{array}{ccccc} & & M(1,1) & & \\ & \nearrow & & \searrow & \\ M(0,1) & & \cdots & & M(1,0) \end{array}$$

Let

$$M(c,r,s) = cM(0,1) \oplus rM(1,1) \oplus sM(1,0),$$

note that $M(c,r,s)$ has dimension vector $(c+r, r+s)$, it is given by a linear map

$$M(c,r,s)_1 = k^{s+r} \longrightarrow k^{r+c} = M(c,r,s)_2$$

of rank r (thus, s is the dimension of its kernel, c the dimension of its cokernel). We may visualize $M(c,r,s)$ as follows:

$$\begin{array}{cccc} 1 & & \circ\cdots\circ & \circ\cdots\circ \\ & & \downarrow\cdots\downarrow & \\ 2 & \circ\cdots\circ & \circ\cdots\circ & \\ & \underbrace{\qquad}_{c} & \underbrace{\qquad}_{r} & \underbrace{\qquad}_{s} \end{array}$$

Let $\epsilon(c,r,s) = \dim_k \operatorname{End} M(c,r,s)$, thus

$$\epsilon(c,r,s) = c^2 + r^2 + s^2 + cr + rs,$$

and for $0 \le i \le r$,

$$\epsilon(c+i, r-i, s+i) - \epsilon(c,r,s) = i(i+c+s).$$

Claim:

$$\langle [cS_2] \diamond [(r+s)S_1] \diamond [rS_2] \between M(c+i, r-i, s+i) \rangle = G_i^{c+i}.$$

Proof: We take an r-dimensional subspace U of the $(c+r)$-dimensional space $M(c,r,s)_2$ such that U contains a fixed $(r-i)$-dimensional subspace V (the image of the given map $M(c+i, r-i, s+i)_1 \to M(c+i, r-i, s+i)_2$), thus in the $(c+i)$-dimensional space $M(c+i, r-i, s+i)_2/V$, we choose an arbitrary i-dimensional subspace.

Similarly:

$$\langle [rS_1] \diamond [(c+r)S_2] \diamond [sS_1] \between M(c+i, r-i, s+i) \rangle = G_s^{s+i}.$$

Proof: Here, we take an s-dimensional subspace in the $(s+i)$-dimensional kernel of the map $M(c+i, r-i, s+i)_1 \to M(c+i, r-i, s+i)_2$, and the number of such subspaces is G_s^{s+i}.

It follows that

$$\boxed{2^{(*c)} * 1^{(*(r+s))} * 2^{(*r)} = \sum_{i=0}^{r} v^{-i(i+c+s)} G_i^{c+i} \langle M(c+i, r-i, s+i) \rangle}$$

and

$$\boxed{1^{(*r)} * 2^{(*(c+r))} * 1^{(*s)} = \sum_{i=0}^{r} v^{-i(i+c+s)} G_s^{s+i} \langle M(c+i, r-i, s+i) \rangle}$$

Note that in both expressions, the coefficient of $\langle M(c,r,s) \rangle$ itself is 1. Consider the coefficients of the summands with index $i > 0$. Since G_i^{c+i} has degree ic, we see that for $c \le s$, the coefficient $v^{-i(i+c+s)} G_i^{c+i}$ belongs to $v^{-1}\mathbb{Z}[v^{-1}]$, similarly, for $c \ge s$, the coefficient $v^{-i(i+c+s)} G_s^{s+i}$ belongs to $v^{-1}\mathbb{Z}[v^{-1}]$.

Let us consider the formulae in case $c = s$. In this case, the right hand sides coincide, since $G_i^{s+i} = G_s^{s+i}$. Thus, we see:

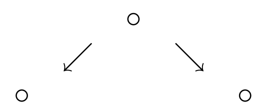
$$\boxed{2^{(*s)} * 1^{(*(r+s))} * 2^{(*r)} = 1^{(*r)} * 2^{(*(s+r))} * 1^{(*s)}}$$

This shows the following:

Proposition. *The canonical basis of* $U_q(\mathbf{n}_+(\mathbb{A}_2))$ *consists of the following elements: take the monomials* $2^{(*c)}*1^{(*(r+s))}*2^{(*r)}$ *with* $c \le s$ *and the monomials* $1^{(*r)}*2^{(*(c+r))}*1^{(*s)}$ *with* $c > s$.

10. The case $\mathbb{A}_3$.

Consider the following quiver

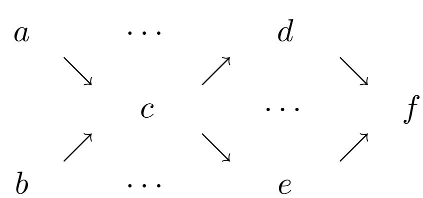


denote the source by 2, the sinks by 1 and 3, respectively.

The indecomposable representations have the following dimension vectors

$$\begin{aligned} a &= (100), \\ b &= (001), \\ c &= (111), \\ d &= (011), \\ e &= (110), \\ f &= (010). \end{aligned}$$

The Auslander-Reiten quiver is of the form



Consider the dimension vector (xyz), with positive integers x, y, z. Let $\alpha\colon \Phi \to \mathbb{N}_0$ with

$$M(\alpha) = M(c) \oplus (x-1)M(a) \oplus (y-1)M(f) \oplus (z-1)M(b).$$

We want to determine $C(\alpha)$.

Let $\beta, \beta', \gamma\colon \Phi \to \mathbb{N}_0$ with

$$\begin{aligned} M(\beta) &= M(d) \oplus xM(a) \oplus (y-1)M(f) \oplus (z-1)M(b), \\ M(\beta') &= M(e) \oplus (x-1)M(a) \oplus (y-1)M(f) \oplus zM(b), \\ M(\gamma) &= xM(a) \oplus yM(f) \oplus zM(b). \end{aligned}$$

We have

$$\begin{aligned}\epsilon(\alpha) &= x^2 - x + y^2 - y + z^2 - z + 1,\\ \epsilon(\beta) &= x^2 + y^2 - y + z^2 - z + 1,\\ \epsilon(\beta') &= x^2 - x + y^2 - y + z^2 + 1,\\ \epsilon(\gamma) &= x^2 + y^2 + z^2.\end{aligned}$$

Thus, we see that

$$\begin{aligned}\epsilon(\beta) - \epsilon(\alpha) &= x,\\ \epsilon(\beta') - \epsilon(\alpha) &= z,\\ \epsilon(\gamma) - \epsilon(\alpha) &= x + y + z - 1.\end{aligned}$$

On the other hand,

$$\begin{aligned}\langle [S_2] \diamond [xS_1] \diamond [zS_3] \diamond [(y-1)S_2] \between \langle M(\beta)\rangle\rangle &= 1\\ \langle [S_2] \diamond [xS_1] \diamond [zS_3] \diamond [(y-1)S_2] \between \langle M(\beta')\rangle\rangle &= 1\\ \langle [S_2] \diamond [xS_1] \diamond [zS_3] \diamond [(y-1)S_2] \between \langle M(\gamma)\rangle\rangle &= G^y_{y-1}.\end{aligned}$$

It follows that

$$2 * 1^{(*x)} * 3^{(*z)} * 2^{(*(y-1))} = \langle M(\alpha)\rangle + v^{-x}\langle M(\beta)\rangle + v^{-z}\langle M(\beta')\rangle + v^{-(x+z)}[y]\langle M(\gamma)\rangle.$$

The two coefficients v^{-x}, v^{-z} belong to $v^{-1}\mathbb{Z}[v^{-1}]$. In case $x+z \geq y$, also the last coefficient $v^{-(x+z)}[y]$ belongs to $v^{-1}\mathbb{Z}[v^{-1}]$. Thus we see:

If $x + z \geq y$, then $C(\alpha) = 2 * 1^{(*x)} * 3^{(*z)} * 2^{(*(y-1))}$

In case $x + z < y$, we use the following equality

$$v^{-(x+z)}[y] = [y - x - z] + v^{-y}[x + z],$$

in order to see that

$$\begin{aligned}2 * 1^{(*x)} * 3^{(*z)} * 2^{(*(y-1))} &- [y - x - z]1^{(*x)} * 3^{(*z)} * 2^{(*y)}\\ &= \langle M(\alpha)\rangle + v^{-x}\langle M(\beta)\rangle + v^{-z}\langle M(\beta')\rangle + v^{-y}[x + z]\langle M(\gamma)\rangle.\end{aligned}$$

Note that the last coefficient $v^{-y}[x + z]$ belongs to $v^{-1}\mathbb{Z}[v^{-1}]$.

For $x + z < y$, $C(\alpha) = 2 * 1^{(*x)} * 3^{(*z)} * 2^{(*(y-1))} - [y - x - z]1^{(*x)} * 3^{(*z)} * 2^{(*y)}$

Lemma. *If $c \geq a+d,\ c \geq b+e$, then $1^{*(a)} * 3^{*(b)} * 2^{*(c)} * 1^{*(d)} * 3^{*(e)}$ belongs to the canonical basis.*

Proof: Let $w_* = 1^{*(a)} * 3^{*(b)} * 2^{*(c)} * 1^{*(d)} * 3^{*(e)}$, and $w_\diamond = 1^{\diamond(a)} \diamond 3^{\diamond(b)} \diamond 2^{\diamond(c)} \diamond 1^{\diamond(d)} \diamond 3^{\diamond(e)}$ Let $M = M(d,c,e)$ be the generic module with dimension vector (d,c,e), let $S = aS_1 \oplus bS_3$. Since $d \leq a+d \leq c$, we see that $\operatorname{Hom}(M, S_1) = 0$. Similarly, Since $e \leq b+e \leq c$, we see that $\operatorname{Hom}(M, S_2) = 0$. Thus $\operatorname{Hom}(M, S) = 0$. Let $N = S \oplus M$. It follows that $\langle w_\diamond \,\phi\, N\rangle = 1$.

Now, consider any module N' with $\langle w \,\phi\, N'\rangle \neq 0$. It follows that N' maps surjectively to S, and, since S is projective, S is a direct summand of N'. Let i, j be maximal so that $S' = (a+i)S_1 \oplus (b+j)S_3$ is a direct summand of N', say $N' = S' \oplus M'$. Note that we have $\operatorname{Hom}(M', S') = 0$. Let M'' be the generic module with dimension vector equal to the dimension vector of M'. Let $\epsilon, \epsilon', \epsilon''$ be the dimension of the endomorphism rings of N, N', and $N'' = S' \oplus M''$ respectively. Then $\epsilon' \geq \epsilon''$.

Note that

$$\begin{aligned}\epsilon'' &= \dim_k \operatorname{End}(S') + \dim_k \operatorname{End}(M'') + \dim_k \operatorname{Hom}(S', M'')\\ &= q(S') + q(M'') + \langle S', M''\rangle\\ &= q(S' \oplus M'') - \langle M'', S'\rangle\\ &= q(a+d, c, b+e) + \dim \operatorname{Ext}^1(M'', S')\end{aligned}$$

where we first have used that $\operatorname{Hom}(M'', S') = 0$, then that $\operatorname{Ext}^1(S', M'') = 0$, and finally again that $\operatorname{Hom}(M'', S') = 0$.

Let us show that $\dim_k \operatorname{Ext}^1(M'', S') = (a+i)(c-d+i) + (b+j)(c-e+j)$. Note that M'' has no direct summand of the form S_1 or S_3, thus the number of indecomposable direct summands in any direct decomposition is just $\dim_k M_2 = c$, whereas the number of indecomposable direct summands with dimension vector (111) or (110) is $\dim_k M_1 = d-i$. Thus, the number of indecomposable direct summands with dimension vector (011) or (010) is $c-d+i$. It follows that $\dim_k \operatorname{Ext}^1(M'', S_1) = c-d+i$. Similarly, $\dim_k \operatorname{Ext}^1(M'', S_2) = c-e+i$.

As a consequence,

$$\begin{aligned}\epsilon'' &= q(a+d, c, b+e) + \dim_k \operatorname{Ext}^1(M'', S')\\ &= q(a+d, c, b+e) + (a+i)(c-d+i) + (b+j)(c-e+j).\end{aligned}$$

In particular, we also see that

$$\epsilon = q(a+d, c, b+e) + a(c-d) + b(c-e).$$

Therefore,

$$\begin{aligned}\epsilon' - \epsilon \geq \epsilon'' - \epsilon &= (a+i)(c-d+i) + (b+j)(c-e+j) - a(c-d) - b(c-e)\\ &= i(a+c-d+i) + j(b+c-d+i) \geq i(2a+i) + j(2b+j),\end{aligned}$$

since we assume that $c \geq a+d$, and $c \geq b+d$. In particular, in case $(i,j) \neq (0,0)$, we see that

$$\epsilon' - \epsilon > 2(ai+bj).$$

On the other hand, we clearly have

$$\langle w \diamond N' \rangle = G_a^{a+i} G_b^{b+j},$$

and this is a polynomial of degree $2(ai + bj)$. The coefficient of $w_* = 1^{*(a)} * 3^{*(b)} * 2^{*(c)} * 1^{*(d)} * 3^{*(e)}$ at $\langle N' \rangle$ is $v^{-\epsilon' + \epsilon} G_a^{a+i} G_b^{b+j}$, thus it belongs to $v^{-1}\mathbb{Z}[v^{-1}]$. This completes the proof.

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