# EXCEPTIONAL OBJECTS IN HEREDITARY CATEGORIES

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ABSTRACT. Let $k$ be a field and $\mathcal{A}$ a finite dimensional $k$-category which is a hereditary length category. We are going to show that the support algebra of any object of $\mathcal{A}$ without self-extension is a finite dimensional $k$-algebra. An object in $\mathcal{A}$ is said to be exceptional provided it is indecomposable and has no self-extensions. For an algebraically closed field $k$, Schofield has exhibited an algorithm for obtaining all exceptional objects starting from the simple ones. We will present a proof which works for arbitrary fields $k$.

Let $\mathcal{A}$ be an abelian category. The category $\mathcal{A}$ is said to be *hereditary* provided $\mathrm{Ext}^2$ vanishes everywhere. Also, we recall that $\mathcal{A}$ is said to be a *length category* provided every object in $\mathcal{A}$ has finite length.

Let $k$ be a field. We say that $\mathcal{A}$ is a *k-category* provided $k$ operates centrally on all Hom-sets and such that the composition of maps is bilinear. Such a $k$-category is said to be *finite dimensional* provided the vector spaces $\mathrm{Hom}(X, Y)$ are finite dimensional, for all objects $X, Y$ in $\mathcal{A}$.

Exceptional objects have been studied in various contexts. The terminology 'exceptional' was first used by Rudakov and his school [Ru] when dealing with vector bundles. The relevance of exceptional objects in the representation theory of finite dimensional hereditary $k$-algebras is well accepted; these exceptional modules are just the indecomposable partial tilting modules, they have also been called stones by Kerner [K1] and Schur modules by Unger [U]. We may refer to a recent survey of Kerner [K2] dealing with objects in hereditary length categories, or at least with representations of wild quivers.

The aim of this report is to focus attention to some interesting developments in the representation theory of finite dimensional hereditary algebras. This theory has an apparent combinatorial flavour; one of the reasons is the role the exceptional modules play. The existence of non-trivial finite dimensional modules without self-extensions should be considered as a feature which is peculiar to non-commutative representation theory. As we want to show, the existence of such modules seems also to be a kind of finiteness condition. We will present a proof of a very useful theorem of Schofield [S2], for an arbitrary base field $k$.

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This result describes certain types of filtrations of an exceptional module $X$ using as factors exceptional modules again; it is a kind of Jordan-Hölder theorem, but the classical unicity assertion is replaced by the assertion that there are precisely $s(X)-1$ essentially different kinds of filtrations, where $s(X)$ is the number of isomorphism classes of composition factors of $X$. Our presentation should be considered as a variation of the considerations by Crawley-Boevey [CB] dealing with a braid group operation on exceptional sequences, see also [R]. Along the way, we will focus attention to the so-called Bongartz complement of a sincere exceptional module.

## 1. Subfactors of objects without self-extensions.

Let $\mathcal{A}$ be an abelian category and $A$ an object in $\mathcal{A}$. Let $A'' \subseteq A' \subseteq A$ be a chain of subobjects. Then $A'/A''$ is said to be a *subfactor* of $A$. If $\mathcal{U}$ is a subcategory of $\mathcal{A}$, we denote by $\mathcal{I}(\mathcal{U})$ the class of all subfactors of objects in $\mathcal{U}$.

Recall that for any object $A$ of $\mathcal{A}$, one denotes by $\operatorname{add} A$ the full subcategory given by all direct summands of finite direct sums of copies of $A$.

**Lemma 1.1.** *Let $\mathcal{A}$ be a hereditary abelian category, and $\mathcal{U}$ a subcategory which is closed under extensions. Then $\mathcal{I}(\mathcal{U})$ is closed under extensions.*

Proof. Let $A, B$ be objects in $\mathcal{U}$. Let $A'' \subseteq A' \subseteq A$ and $B'' \subseteq B' \subseteq B$ be chains of subobjects. Thus, $A'/A''$ and $B'/B''$ are subfactors of objects in $\mathcal{U}$, and we consider an extension: assume that there is given an exact sequence

$$0 \to A'/A'' \to C \to B'/B'' \to 0.$$

We construct stepwise the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A'/A'' & \longrightarrow & C & \longrightarrow & B'/B'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & A/A'' & \longrightarrow & D & \longrightarrow & B'/B'' & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & B'/B'' & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \longrightarrow & F & \longrightarrow & B/B'' & \longrightarrow & 0 \\
 & & \| & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & A & \longrightarrow & G & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

First, we form the induced exact sequence with respect to the inclusion $A'/A'' \to A/A''$ and obtain $D$ with an inclusion map $C \to D$. The exact sequence with middle term $D$ is induced from a third exact sequence with respect to the canonical epimorphism $A \to A/A'$; here we use that $\mathcal{A}$ is hereditary. In the diagram above, this third exact sequence has middle term $E$ and there is an epimorphism $E \to D$. Again using that $\mathcal{A}$ is hereditary, there exists an exact sequence with middle term $F$ and a monomorphism $E \to F$ which induces the third exact sequence with respect to the inclusion map $B'/B'' \to B/B''$. Finally, we form the induced sequence with respect to the canonical epimorphism $B \to B/B''$ and obtain an object $G$ and an epimorphism $G \to F$. The maps

$$C \to D \leftarrow E \to F \leftarrow G$$

show that $C$ is a subfactor of $G$. Since $A, B$ belong to $\mathcal{U}$, and $\mathcal{U}$ is closed under extensions, the object $G$ belongs to $\mathcal{U}$. This completes the proof.

**Remark 1.2.** The conclusion of Lemma 1.1 may be reformulated as follows: *$\mathcal{I}(\mathcal{U})$ is a Serre subcategory of $\mathcal{A}$.* Recall that a subcategory $\mathcal{B}$ of $\mathcal{A}$ is said to be a *Serre subcategory* provided for any exact sequence $0 \to A_1 \to A_2 \to A_3 \to 0$ in $\mathcal{A}$, the object $A_2$ belongs to $\mathcal{B}$ if and only if both $A_1, A_3$ belong to $\mathcal{B}$.

## 2. The support of an object without self-extension

Let $\mathcal{A}$ be a length category. For any object $A$ in $\mathcal{A}$, we denote its isomorphism class by $[A]$. We denote by $\mathcal{S}(\mathcal{A})$ the 'set' of isomorphism classes of simple objects in $\mathcal{A}$ (it may not be a set, for set-theoretical reasons, thus we have used the quotation marks). Given two simple objects $S$, $S'$ in $\mathcal{A}$, we draw an arrow $[S] \to [S']$ provided $\operatorname{Ext}^1(S, S') \neq 0$. In this way, $\mathcal{S}(\mathcal{A})$ becomes a 'quiver' (again, we use quotation marks, for set-theoretical reasons).

The support $\operatorname{supp} A$ of an object $A$ in $\mathcal{A}$ is the set of isomorphism classes of composition factors of $A$, this is a 'subset' of $\mathcal{S}(\mathcal{A})$ (but since $\operatorname{supp} A$ is finite, we now deal with a 'subset' which really is a set). We consider $\operatorname{supp} A$ as a full 'subquiver' of $\mathcal{S}(\mathcal{A})$. If $\mathcal{S}'$ is a 'subset' of $\mathcal{S}$, we denote by $\mathcal{E}(\mathcal{S}')$ the class of objects of $\mathcal{A}$ with all composition factors belonging to $\mathcal{S}'$.

**Proposition 2.1.** *Let $\mathcal{A}$ be a hereditary length category. Let $A$ be an object in $\mathcal{A}$ with $\operatorname{Ext}^1(A, A) = 0$. Then the support $\operatorname{supp} A$ of $A$ is a directed quiver, and $\mathcal{E}(\operatorname{supp} A) = \mathcal{I}(\operatorname{add} A)$. If $\mathcal{A}$ is, in addition, a finite dimensional $k$-category, then the $k$-space $\operatorname{Ext}^1(S, S')$ is finite dimensional, for every pair $S, S'$ of composition factors of $A$.*

Proof: Since $\operatorname{Ext}^1(A, A) = 0$, the subcategory $\operatorname{add} A$ of $\mathcal{A}$ is closed under extensions. According to Lemma 1.1, the class $\mathcal{I}(\operatorname{add} A)$ is closed under extensions. The composition factors of $A$ belong to $\mathcal{I}(\operatorname{add} A)$, thus any object in

$\mathcal{E}(\operatorname{supp} A)$ belongs to $\mathcal{I}(\operatorname{add} A)$. Of course, conversely, the composition factors of subfactors of objects in $\operatorname{add} A$ belong to $\operatorname{supp} A$.

Let $t$ be the Loewy length of $A$, this is the minimal length $t$ of a filtration

$$0 = B_0 \subset B_1 \subset \cdots \subset B_t = A$$

of $A$ with semisimple factors $B_i/B_{i-1}$. Note that any object in $\operatorname{add} A$, and therefore also any any subfactor of such an object has Loewy length at most $t$.

Now assume that there is an oriented cycle in the quiver $\operatorname{supp} A$, say $[S_0] \to [S_1] \to \cdots \to [S_s] = [S_0]$, with simple objects $S_i$. Since $\mathcal{A}$ is hereditary, one may construct serial objects $U_n$ in $\mathcal{A}$ of arbitarily large finite length $n$, such that the composition factors of $U_n$ are of the form $S_0, \ldots, S_{s-1}$. In particular, $U_n$ belongs to $\mathcal{I}(\operatorname{add} A)$. But the Loewy length of $U_n$ is equal to its length $n$, since $U_n$ is serial. It follows that $n \le t$. This contradiction shows that there cannot be any oriented cycle in $\operatorname{supp} A$.

Let us assume now that $\mathcal{A}$ is, in addition, a finite dimensional $k$-category. Let us start with a composition series

$$0 = A_0 \subset A_1 \subset \cdots \subset A_n = A$$

of $A$, and let $S_i = A_i/A_{i-1}$. Let $b$ be the $k$-dimension of the endomorphism ring of $\bigoplus_{i=1}^n S_i$. Clearly, $b$ is a common bound for the $k$-dimension of $\operatorname{Hom}(A', A'')$, where $A', A''$ are subfactors of $A$. Let us fix $1 \le i, j \le n$. The embedding $S_i \subset A/A_{i-1}$ yields a surjection

$$\operatorname{Ext}^1(A/A_{i-1}, S_j) \to \operatorname{Ext}^1(S_i, S_j).$$

The exact sequence

$$0 \to A_{i-1} \to A \to A/A_{i-1} \to 0$$

yields an exact sequence

$$\operatorname{Hom}(A_{i-1}, S_j) \to \operatorname{Ext}^1(A/A_{i-1}, S_j) \to \operatorname{Ext}^1(A, S_j).$$

The epimorphism $A_j \to S_j$ yields a surjection

$$\operatorname{Ext}^1(A, A_j) \to \operatorname{Ext}^1(A, S_j),$$

and finally we consider the exact sequence

$$0 \to A_j \to A \to A/A_j \to 0;$$

it yields an exact sequence

$$\operatorname{Hom}(A, A/A_j) \to \operatorname{Ext}^1(A, A_j) \to \operatorname{Ext}^1(A, A).$$

Note that the last term is zero. Altogether we see that

$$\begin{aligned}
\dim \operatorname{Ext}^1(S_i, S_j) &\leq \dim \operatorname{Ext}^1(A/A_{i-1}, S_j) \\
&\leq \dim \operatorname{Hom}(A_{i-1}, S_j) + \dim \operatorname{Ext}^1(A, S_j) \\
&\leq \dim \operatorname{Hom}(A_{i-1}, S_j) + \dim \operatorname{Ext}^1(A, A_j) \\
&\leq \dim \operatorname{Hom}(A_{i-1}, S_j) + \dim \operatorname{Hom}(A, A/A_j) \\
&\leq 2\, b.
\end{aligned}$$

This completes the proof.

**Corollary 2.2.** *Let $\mathcal{A}$ be a finite dimensional hereditary length $k$-category. Let $A$ be an object without self-extensions with support* $\operatorname{supp} A$. *Then there exists a finite dimensional hereditary $k$-algebra $\Lambda$ such that the category $\mathcal{E}(\operatorname{supp} A)$ is equivalent to the category of all $\Lambda$-modules of finite length. Under such an equivalence, $A$ corresponds to a faithful $\Lambda$-module.*

If $\Lambda$ is any $k$-algebra, a $\Lambda$-module $X$ is said to be *sincere* provided every simple $\Lambda$-module occurs as a composition factor of $X$.

**Corollary 2.3.** *Let $\Lambda$ be a hereditary $k$-algebra and $X$ a finite dimensional $\Lambda$-module. Assume that $X$ is sincere and has no self-extensions. Then $\Lambda$ is finite dimensional and $X$ is faithful.*

Proof: Apply the previous considerations to the category $\mathcal{A}$ of all finite dimensional $\Lambda$-modules.

It seems that finite dimensional modules without self-extensions have been considered before mainly for $k$-algebras $\Lambda$ which are finite dimensional. For $\Lambda$ hereditary, the corollary asserts that in essence one obtains in this way all such modules. The fact that for a finite dimensional hereditary $k$-algebra, a sincere module without self-extension is faithful, is well-known, see for example Kerner [K2], Lemma 8.3.

**Remark 2.4.** Let $\mathcal{A}$ be a hereditary length category with an exceptional object $A$ whose support is $\mathcal{S}(\mathcal{A})$. If $\mathcal{A}$ is a finite dimensional $k$-category, then Corollary 2.2 shows that $\mathcal{A}$ has enough projective objects and enough injective objects. In general, this may not be the case: consider a field extension $k \subset K$ of infinite degree, let $\Lambda = \begin{bmatrix} k & 0 \\ K & K \end{bmatrix}$, and let $\mathcal{A}$ be the category of all (left) $\Lambda$-modules of finite length. The indecomposable projective $\Lambda$-module $P$ of length 2 is exceptional and satisfies $\operatorname{supp} P = \mathcal{S}(\mathcal{A})$. The category $\mathcal{A}$ has enough projective objects, but not enough injective objects.

**3. Schofield's Theorem.**

Let $\mathcal{A}$ be a finite dimensional $k$-category which is a hereditary length category. We are going to present a theorem of Schofield which yields an inductive way for constructing all exceptional objects $A$ in $\mathcal{A}$. The theorem asserts that any exceptional object $A$ is obtained as the middle term of a suitable exact sequence

$$(*) \qquad 0 \to U^u \to A \to V^v \to 0$$

where $U, V$ are again exceptional objects and $u, v$ are positive integers. More precisely, there is such an exact sequence where $U, V$ are exceptional objects and where the objects $U, V$ satisfy in addition the following conditions:

$$(**) \qquad \mathrm{Hom}(U,V) = \mathrm{Hom}(V,U) = \mathrm{Ext}^1(U,V) = 0.$$

A pair $(V, U)$ of exceptional objects satisfying these conditions $(**)$ is called an *orthogonal exceptional pair* (the general notion of an exceptional pair will be recalled below). Given an orthogonal exceptional pair $(V, U)$, we want to consider the full subcategory $\mathcal{E}(U, V)$ of all objects of $\mathcal{A}$ which have a filtration with factors of the form $U$ and $V$. Note that for any object $A$ in $\mathcal{E}(U, V)$, there exists an exact sequence of the form $(*)$ with non-negative integers $u, v$.

The reduction problem to be considered is the following: Given an exceptional object $A$, we want to find orthogonal exceptional pairs $(V, U)$ such that $A$ belongs to $\mathcal{E}(U, V)$, but $A$ is not one of the two simple objects of $\mathcal{E}(U, V)$. One may ask for all possible pairs of this kind, and it is amazing that there exists an intrinsic characterization of the number of such pairs.

**Theorem 3.1** (Schofield). *Let $\mathcal{A}$ be a finite dimensional $k$-category which is a hereditary length category. Let $A$ be an exceptional object in $\mathcal{A}$. Then there are precisely $s(A)-1$ orthogonal exceptional pairs $(V_i, U_i)$ such that $A$ belongs to $\mathcal{E}(U_i, V_i)$ and is not a simple object in $\mathcal{E}(U_i, V_i)$.*

Proof: We want to find exact sequences of the form $(*)$. Note that the objects $U, V$ have to belong to $\mathcal{E}(\operatorname{supp} A)$, thus we may assume that $\mathcal{A}$ is equal to $\mathcal{E}(\operatorname{supp} A)$. This means that we may assume that $\mathcal{A}$ is the category of all finite length $\Lambda$-modules, where $\Lambda$ is a finite dimensional hereditary $k$-algebra and that we consider a faithful exceptional $\Lambda$-module.

Thus, let $\Lambda$ be a finite dimensional hereditary $k$-algebra and $X$ a faithful exceptional $\Lambda$-module.

We will need some preliminary considerations. A pair $(B, A)$ of exceptional objects in a hereditary abelian category $\mathcal{A}$ is said to be an *exceptional pair* provided we have $\mathrm{Hom}(A, B) = \mathrm{Ext}^1(A, B) = 0$.

Let $(Y, X)$ be an exceptional pair of $\Lambda$-modules. We define $\mathcal{C}(X, Y)$ to be the closure of the full subcategory with objects $X, Y$ under kernels, images, cokernels and extensions; of course, in case $(Y, X)$ is in addition orthogonal, then $\mathcal{C}(X, Y) = \mathcal{E}(X, Y)$. Let us recall the following facts: This subcategory $\mathcal{C}(X, Y)$ is an exact abelian subcategory, it is the smallest exact abelian subcategory of the category of all $\Lambda$-modules containing $X, Y$ and being closed under extensions. It is of importance that $\mathcal{C}(X, Y)$ is equivalent to the category of all finite length modules over a finite dimensional hereditary $k$-algebra $\Theta$ with precisely 2 simple modules $S, T$; these modules $S, T$ have no self-extensions and they satisfy $\mathrm{Ext}^1(S, T) = 0$. Under such an equivalence, the pair $(Y, X)$ corresponds to an exceptional pair of $\Theta$-modules. The proofs rely on the use of perpendicular categories as considered by Geigle-Lenzing [GL] and Schofield [S1], see Crawley-Boevey [CB] (the latter paper assumes that $k$ is an algebraically closed field, but the relevant proofs needed here are valid in our more general setting).

The (finite dimensional) exceptional $\Theta$-modules are well-known: they are just the preprojective and the preinjective $\Theta$-modules. Also, the exceptional pairs of $\Theta$-modules are easy to describe: For any exceptional $\Theta$-module $X$, there is (up to isomorphism) a unique module $Y$ such that $(Y, X)$ is an exceptional pair of $\Theta$-modules. Finally, if $(Y, X)$ is an exceptional pair of $\Theta$-modules, and $\mathrm{Hom}(Y, X) \neq 0$, then we must have $\mathrm{Ext}^1(Y, X) = 0$ (so that $X \oplus Y$ is a module without self-extensions). Note that the last assertion remains valid for arbitrary exceptional pairs of $\Lambda$-modules.

From now on, we fix a faithful exceptional $\Lambda$-module $X$. Let us stress the following conclusion: *the orthogonal exceptional pairs $(V, U)$ with $X$ in $\mathcal{E}(U, V)$ and $X$ not simple in $\mathcal{E}(U, V)$ correspond bijectivly to the exceptional pairs $(Y, X)$ such that $X$ is not simple in $\mathcal{C}(X, Y)$*; at least if the pairs in question are considered as pairs of isomorphism classes, not as pairs of modules. Namely, if the pair $(V, U)$ is given, then there is (up to isomorphism) a unique $\Lambda$-module $Y$ in $\mathcal{E}(U, V)$ such that $(Y, X)$ is an exceptional pair in $\mathcal{E}(U, V)$ and therefore in $\mathcal{A}$. Also, we have $\mathcal{E}(U, V) = \mathcal{C}(X, Y)$. Conversely, if $(Y, X)$ is an exceptional pair in $\mathcal{A}$, then $\mathcal{C}(X, Y)$ is a hereditary length category with precisely two simple objects, say $U, V$, and such that $(V, U)$ is an (even orthogonal) exceptional sequence. Again we have $\mathcal{E}(U, V) = \mathcal{C}(X, Y)$. If we assume that $X$ is not simple in $\mathcal{E}(U, V)$, then the pair $(U, V)$ cannot be exceptional, thus not only the set $\{V, U\}$, but the pair $(V, U)$ is uniquely determined by the pair $(Y, X)$.

Thus, our aim is to classify all exceptional pairs $(Y, X)$ such that $X$ is not simple in $\mathcal{C}(X, Y)$. It will turn out that there is a constructive way of obtaining these pairs.

**Lemma 3.2.** *Let $(Y, X)$ be an exceptional pair. Then the following assertions are equivalent:*
(i) *$X$ is not simple in $\mathcal{C}(X, Y)$.*
(ii) *$Y$ is not injective in $\mathcal{C}(X, Y)$.*

(iii) *$Y$ is cogenerated by $X$.*

Proof: This follows from an easy inspection of all the exceptional sequences of $\Theta$-modules. In order to see the implication (ii) $\implies$ (iii), just use the almost split sequence in $\mathcal{C}(X,Y)$ starting with $Y$, its left hand map is a monomorphism of the form $Y \to X^n$, for some $n$.

Consider now an exceptional module $X$. In case $X$ is projective, we denote by $X'$ the direct sum of the remaining indecomposable projective modules, one from each isomorphism class and call it the *Bongartz complement* for $X$. Otherwise, let $X'$ be the universal extension of ${}_\Lambda\Lambda$ by copies of $X$, thus there is an exact sequence

$$0 \to {}_\Lambda\Lambda \to X' \to X^m \to 0$$

for some $m$, we have $\operatorname{Ext}^1(X,X') = 0$, and $m$ is chosen minimal. Note that $m > 0$, since $X$ is not projective (since $\Lambda$ is hereditary, a minimal projective resolution of $X$ is of the form $0 \to P_1 \to P_0 \to X \to 0$, and this shows that $\operatorname{Ext}^1(X,P_0) \neq 0$, thus $\operatorname{Ext}^1(X,P) \neq 0$ for some indecomposable projective module, thus $\operatorname{Ext}^1(X,{}_\Lambda\Lambda) \neq 0$). Also note that the minimality of $m$ is equivalent to the requirement that $X$ does not occur as a direct summand of $X'$. This module $X'$ is called the *Bongartz complement* for $X$.

Consider now again the general case of an exceptional module $X$ and let $X'$ be its Bongratz complement. It is well-known (and easy to see) that $X \oplus X'$ is a tilting module. The Bongartz complement of a module without self-extension has been used before in many different situations, and a lot is known about its properties. For the convenience of the reader, we will include proofs of all the facts which are relevant for our consideration.

**Lemma 3.3.** *Let $X$ be a faithful exceptional module. The Bongartz complement $X'$ of $X$ is cogenerated by $X$ and therefore* $\operatorname{Hom}(X,X') = 0$.

Proof: Let us first show that $X'$ is cogenerated by $X$. First, consider the case of $X$ being projective, say the projective cover of the simple module $E$. Since $X$ is faithful, we have $\operatorname{Hom}(P,X) \neq 0$, for any indecomposable projective module $P$. But any non-zero map $P \to X$ is a monomorphism, since $\Lambda$ is hereditary. This shows that the Bongartz complement $X'$ of $X$ is cogenerated by $X$.

Now, assume that $X$ is not projective and take the defining exact sequence

$$0 \to {}_\Lambda\Lambda \to X' \to X^m \to 0.$$

Since $X$ is faithful, there is a monomorphism $\alpha\colon {}_\Lambda\Lambda \to X^s$ for some $s$. We obtain a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & {}_\Lambda\Lambda & \longrightarrow & X' & \longrightarrow & X^m & \longrightarrow & 0 \\
 & & {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\alpha'}\big\downarrow & & \big\| & & \\
0 & \longrightarrow & X^s & \longrightarrow & X'' & \longrightarrow & X^m & \longrightarrow & 0.
\end{array}$$

The lower sequence splits, since $X$ is exceptional, thus $X''$ is isomorphic to $X^{s+m}$. With $\alpha$ also $\alpha'$ is injective, thus $X'$ is cogenerated by $X$, also in this case.

Let us assume that there exists a non-zero homomorphism $\beta\colon X \to X'$. Since $X'$ is cogenerated by $X$, we find $\beta'\colon X' \to X$ such that $\beta'\beta \neq 0$. This is a non-zero endomorphism of the exceptional module $X$, thus invertible. But this implies that $\beta$ is a split monomorphism, impossible.

As a consequence, we obtain the following characterization of the indecomposable direct summands of the Bongartz complement of a faithful exceptional module:

**Lemma 3.4.** *Let $X$ be a faithful exceptional module, let $Y$ be indecomposable. The following assertions are equivalent:*
(i) *$Y$ is a direct summand of the Bongartz complement of $X$.*
(ii) *$(Y, X)$ is an exceptional pair and $Y$ is cogenerated by $X$.*

Proof: Let $X'$ be the Bongartz complement of $X$. First, let us assume that $Y$ is a direct summand of $X'$. In particular, we have $\mathrm{Ext}^1(X,Y) = 0$. According to the previous Lemma, $Y$ is cogenerated by $X$ and $\mathrm{Hom}(X,Y) = 0$. But this also means that $(Y,X)$ is an exceptional pair.

For the proof of the converse, we first note the following: Let $Z$ be a module cogenerated by $X$, say with a monomorphism $\gamma\colon Z \to X^t$, and let $Z'$ be a module with $\mathrm{Ext}^1(X,Z') = 0$. The long exact sequence for $\mathrm{Hom}(-,Z')$ yields an epimorphism $\mathrm{Ext}^1(X^t,Z') \to \mathrm{Ext}^1(Z,Z')$, thus we see that $\mathrm{Ext}^1(Z,Z') = 0$.

Now, let us assume that $(Y,X)$ is an exceptional pair and that $Y$ is cogenerated by $X$. Since $\mathrm{Hom}(Y,X) \neq 0$, we have $\mathrm{Ext}^1(Y,X) = 0$. The previous considerations yield $\mathrm{Ext}^1(X',Y) = 0$ and $\mathrm{Ext}^1(Y,X') = 0$, since both modules $X', Y$ are cogenerated by $X$ and since they satisfy $\mathrm{Ext}^1(X,X') = 0$ and $\mathrm{Ext}^1(X,Y) = 0$. It follows that $X \oplus X' \oplus Y$ is a tilting module. As a consequence, $Y$ is isomorphic to a direct summand of $X \oplus X'$. Since $\mathrm{Hom}(X,Y) = 0$, we see that $Y$ is isomorphic to a direct summand of $X'$.

Proof of Schofield's Theorem: Since $X$ is a faithful $\Lambda$-module, $s = s(X)$ is the number of simple $\Lambda$-modules. Let $Y_1, \dots, Y_{s-1}$ be pairwise non-isomorphic direct summands of the Bongartz complement $X'$ of $X$ (recall that a tilting module has precisely $s$ isomorphism classes of indecomposable direct summands). Then, the pairs $(Y_i, X)$ are exceptional with $Y_i$ being cogenerated by $X$, thus $X$ is not simple in the subcategory $\mathcal{C}(Y_i, X)$.

On the other hand, consider an exceptional pair $(Y,X)$ with $X$ not simple in $\mathcal{C}(X,Y)$. Then $Y$ is cogenerated by $X$, thus $Y$ is isomorphic to a direct summand of $X'$, thus to one of the modules $Y_i$. This completes the proof.

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