Categorification of the Fibonacci Numbers Using Representations of Quivers

Philipp Fahr
Brehmstr. 51
D-40239 Düsseldorf, Germany
philfahr@gmx.net

Claus Michael Ringel
Department of Mathematics, Shanghai Jiao Tong University
Shanghai 200240, P. R. China, and
King Abdulaziz University, P O Box 80200
Jeddah, Saudi Arabia
ringel@math.uni-bielefeld.de

Abstract

In a previous paper we have presented a partition formula for the even-index Fibonacci numbers using the preprojective representations of the 3-Kronecker quiver and its universal cover, the 3-regular tree. Now we deal in a similar way with the odd-index Fibonacci numbers. The Fibonacci modules introduced here provide a convenient categorification of the Fibonacci numbers.

1 Introduction

We consider the Fibonacci numbers with $f_0 = 0$, $f_1 = 1$ and the recursion rule $f_{i+1} = f_i + f_{i-1}$ for $i \geq 1$. As suggested by many authors, one may use this recursion rule also for $i \leq 0$ and one obtains in this way Fibonacci numbers f_i for all integral indices i (Knuth [11] calls this extended set the negaFibonacci numbers); the sequence f_t with $-10 \leq t \leq 10$ looks as follows:

$$-55, 34, -21, 13, -8, 5, -3, 2, -1, 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55;$$

in general, $f_{-t} = (-1)^{t+1} f_t$. This rule, but also many other observations, show that Fibonacci numbers have quite different properties depending whether the index is even or odd, and the present report will strongly support this evidence.

Our aim is to outline a categorification of the Fibonacci numbers, or better of Fibonacci pairs, using the representation theory of quivers. By definition, *Fibonacci pairs* are the pairs of the form $[f_t, f_{t+2}]$ or $[f_t, f_{t-2}]$; such a pair will be called *even* or *odd* provided the index t of f_t is even or odd, respectively.

It is well-known that the dimension vectors of some indecomposable representations of the 3-Kronecker quiver Q are Fibonacci pairs. In a previous note [6], preprojective and preinjective representations were used in order to derive a partition formula for the even-index Fibonacci numbers f_{2n}. In the present paper we will exhibit a corresponding partition formula for the odd-index Fibonacci numbers f_{2n+1}. Now regular representations of Q play a role (see also Chen [3] and Ringel [14]). Actually, we will be dealing directly with the universal cover of Q, thus with graded representations of Q. The universal cover of Q is the quiver (T, E, Ω), where (T, E) is the 3-regular tree and Ω is a bipartite orientation. In section 4 we will introduce some representations $P_t(x)$ and $R_t(x, y)$ of the quiver (T, E, Ω) which we label Fibonacci modules. The Fibonacci modules will be used in section 4 and 6 in order to categorify the Fibonacci pairs (and hence the Fibonacci numbers). The dimension vectors of the Fibonacci modules are called Fibonacci vectors, they are studied already in secton 2 and 3 and provide the partition formula.

Categorification of a set of numbers means to consider instead of these numbers suitable objects in a category (here representations of quivers), so that the numbers in question occur as invariants of the objects (see for example Crane and Yetter [4]). Equality of numbers may be visualized by isomorphisms of objects, functional relations by functorial ties. We will see that certain addition formulas for Fibonacci numbers can be interpreted very well by displaying filtrations of Fibonocci modules.

The partition formula for the odd-index Fibonacci numbers presented here is due to Fahr [5], the remaining considerations are based on discussions of the authors during the time when Fahr was a PhD student at Bielefeld, the final version was written by Ringel. In a forthcoming paper [7] we will show that the numbers which are used in the partition formulae can be arranged quite well in two triangles which we call the Fibonacci partition triangles. They turn out to be quite similar to the Pascal triangle for the binomial coefficients and are strongly interrelated.

2 Fibonacci Vectors

Let us start with the 3-regular tree (T, E) with vertex set T and edge set E. We consider paths in this tree, thus sequences $x_0, x_1, \ldots, x_t$ of elements x_i of T such that all the sets $\{x_{i-1}, x_i\}$ belong to E, where $1 \leq i \leq t$, and $x_{i+1} \neq x_{i-1}$ for $0 < i < t$; if such a sequence exists, then we say that x_0 and x_t have distance t and write $d(x_0, x_t) = t$. As usual, vertices with distance 1 are said to be neighbors.

Let $K_0(T)$ be the free abelian group generated by the vertex set T, or better by elements $s(x)$ corresponding bijectively to the vertices $x \in T$. The elements of $K_0(T)$ can be written as finite sums $a = \sum_{x \in T} a_x s(x)$ with $a_x \in \mathbb{Z}$ (and almost all $a_x = 0$), or just as $a = (a_x)_{x \in T}$; the set of vertices x with $a_x \neq 0$ will be called the support of a.

We are going to single out suitable elements of $K_0(T)$ which we will call Fibonacci vectors: For every pair (x, y) with $\{x, y\} \in E$, let $r(x, y) = s(x) + s(y)$. The Fibonacci vectors will be obtained from the elements $s(x)$ and $r(x, y)$ by applying sequences of reflections. For $y \in T$ the reflection $\sigma^y : K_0(T) \to K_0(T)$ is defined as follows: For $a \in K_0(T)$, its image $\sigma^y a$ is given by $(\sigma^y a)_x = a_x$ for $x \neq y$, and $(\sigma^y a)_y = -a_y + \sum_{\{x,y\} \in E} a_x$. Observe that the reflections $\sigma^y, \sigma^{y'}$ commute provided y, y' are not neighbors.

We denote by Σ^x the composition of all the reflections σ^y with $d(x, y)$ being even (note that this composition is defined: first of all, since any element $a \in K_0(T)$ has finite support, the number of vertices y with $\sigma^y a \neq a$ is finite; second, given two different vertices y, y' such that the distance between x and y, as well as between x and y' is even, then y, y' cannot be neighbors, thus $\sigma^y, \sigma^{y'}$ commute). Similarly, we denote by $\underline{\Sigma}^x$ the composition of all the reflections σ^y with $d(x, y)$ being odd (again, this is well-defined).

We define $s_t(x)$ for all $t \geq 0$, using induction: First of all, $s_0(x) = s(x)$. If $s_t(x)$ is already defined for some $t \geq 0$, let $s_{t+1}(x) = \underline{\Sigma}^x s_t(x)$ in case t is even, and $s_{t+1}(x) = \Sigma^x s_t(x)$ in case t is odd, thus

$$\begin{aligned} s_0(x) &= s(x), \\ s_1(x) &= \underline{\Sigma}^x s(x), \\ s_2(x) &= \Sigma^x \underline{\Sigma}^x s(x), \\ s_3(x) &= \underline{\Sigma}^x \Sigma^x \underline{\Sigma}^x s(x), \\ &\dots \end{aligned}$$

Similarly, we define $r_t(x, y)$ for all $t \geq 0$: First of all, $r_0(x, y) = r(x, y)$. If $r_t(x, y)$ is already defined for some $t \geq 0$, let $r_{t+1}(x, y) = \underline{\Sigma}^x r_t(x, y)$ in case t is even, and $r_{t+1}(x, y) = \Sigma^x r_t(x, y)$ in case t is odd. The elements $s_t(x)$ with $t \geq 0$ as well as the elements $r_t(x, y)$ with $t \in \mathbb{Z}$ will be called the *Fibonacci vectors.*

Define

$$s_t(x)_- = \sum_{d(x,z) \not\equiv t} s_t(x)_z, \quad \text{and} \quad s_t(x)_+ = \sum_{d(x,z) \equiv t} s_t(x)_z,$$

where $\equiv$ means equivalence modulo 2, and similarly,

$$r_t(x, y)_- = \sum_{d(x,z) \not\equiv t} r_t(x, y)_z, \quad \text{and} \quad r_t(x, y)_+ = \sum_{d(x,z) \equiv t} r_t(x, y)_z.$$

Proposition 2.1. *The Fibonacci vectors have non-negative coordinates and*

$$s_t(x)_- = f_{2t}, \; s_t(x)_+ = f_{2t+2}, \; r_t(x, y)_- = f_{2t-1}, \; r_t(x, y)_+ = f_{2t+1}.$$

Proof. The proof for s_t has been given in a previous paper [6]. In the same way, one deals with $r_t(x, y)$. But note that in the previous paper [6] we have fixed some base point x_0, and we wrote Φ_0, Φ_1 instead of Σ^{x_0}, $\underline{\Sigma}^{x_0}$ respectively. Also, there, we have denoted the function $s_{2t}(x_0)$ by a_t, and the functions $s_{2t+1}(x_0)$ by $\Phi_1 a_t$. □

The functions $s_t(x_0)$ have been exhibited in the previous paper [6], for $0 \leq t \leq 5$. At the end of the present paper we present the functions $r_t(x, y)$ for $0 \leq t \leq 5$.

3 The Partition Formula

In the previous paper [6] the functions $s_t(x)$ were used in order to provide a partition formula for the even-index Fibonacci numbers (see also Prodinger [13] and Hirschhorn [9]). We now consider the functions $r_t(x, y)$ in order to obtain a corresponding partition formula for the odd-index Fibonacci numbers. In order to get all odd-index Fibonacci numbers, it is sufficient to look at the functions $r_{2t}(x, y)$, since

$$f_{4t-1} = r_{2t}(x, y)_-, \quad f_{4t+1} = r_{2t}(x, y)_+.$$

For any t, the numbers $s_t(x)_z$ only depend on the distance $d(x, z)$. In contrast, for $r_t(x, y)$ and $d(x, z) > 0$, there are now **two** values which occur as $r_t(x, y)$ depending on whether the path connecting x and z runs through y or not. Let us denote by $T_s(x, y)$ the set of vertices z of T with $d(x, z) = s$ and such that the path connecting x, z does not involve y, and $T'_s(x, y)$ shall denote the set of vertices z of T with $d(x, z) = s$, such that the path connecting x, z does involve y. Clearly, we have for $s \geq 1$

$$|T_s(x, y)| = 2^s, \qquad |T'_s(x, y)| = 2^{s-1}. \tag{1}$$

The proof of (1) is by induction: Of course, $T'_1(x, y) = \{y\}$ whereas $T_1(x, y)$ consists of the two remaining neighbors of x. For any vertex z with $d(x, z) = s > 0$, there are precisely two neighbors z', z'' such that $d(x, z') = d(x, z'') = s+1$, therefore $|T_{s+1}(x, y)| = 2|T_s(x, y)|$ and $|T'_{s+1}(x, y)| = 2|T'_s(x, y)|$.

Thus, let us look at the function $u_t : \mathbb{Z} \to \mathbb{N}$ defined for $s \geq 0$ by $u_t[s] = r_{2t}(x, y)_z$ with $z \in T_s(x, y)$, and for $s \leq -1$ by $u_t[s] = r_{2t}(x, y)_z$ with

$z \in T'_{-s}(x,y)$. Using the equality (1), we see:

$$
\begin{aligned}
f_{4t-1} &= r_{2t}(x,y)_- = \sum_{d(x,z)\text{ odd}} r_{2t}(x,y)_z \\
&= \sum_{s\geq 0\text{ odd}} |T_s(x,y)| u_t[s] + \sum_{s>0\text{ odd}} |T'_s(x,y)| u_t[-s] \\
&= \sum_{s\geq 0\text{ odd}} 2^s u_t[s] + \sum_{s>0\text{ odd}} 2^{s-1} u_t[-s] \\
f_{4t+1} &= r_{2t}(x,y)_+ = \sum_{d(x,z)\text{ even}} r_{2t}(x,y)_z \\
&= \sum_{s\geq 0\text{ even}} |T_s(x,y)| u_t[s] + \sum_{s>0\text{ even}} |T'_s(x,y)| u_t[-s] \\
&= \sum_{s\geq 0\text{ even}} 2^s u_t[s] + \sum_{s>0\text{ even}} 2^{s-1} u_t[-s]
\end{aligned}
$$

Without reference to r_{2t}, we can define the functions u_t directly as follows: We start with $u_0[0] = u_0[-1] = 1$ and $u_0[s] = 0$ for all other s. If u_t is already defined for some $t \geq 0$, then we define first $u_{t+1}[s]$ for odd integers s by the rule

$$
u_{t+1}[s] = \begin{cases} 2u_t[s-1] - u_t[s] + u_t[s+1] & \text{for} \quad s < 0 \\ u_t[s-1] - u_t[s] + 2u_t[s+1] & \text{for} \quad s > 0 \end{cases}
$$

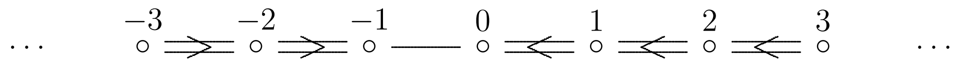
and in a second step for even integers s by

$$
u_{t+1}[s] = \begin{cases} 2u_{t+1}[s-1] - u_t[s] + u_{t+1}[s+1] & \text{for} \quad s < 0 \\ u_{t+1}[s-1] - u_t[s] + 2u_{t+1}[s+1] & \text{for} \quad s \geq 0 \end{cases}
$$

Here is the table of the numbers $u_t[s]$, for $t \leq 4$; in addition, we list the corresponding Fibonacci numbers $f_{-1}, f_1, f_3, \ldots, f_{17}$.

$t \backslash s$	$\cdots -6$	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	$8 \cdots$	f_{4t-1}	f_{4t+1}
0						1	1									1	1
1							1	1	1							2	5
2					1	1	4	2	3	1	1					13	34
3			1	1	6	5	17	8	13	4	5	1	1			89	233
4	1	1	8	7	32	24	77	35	60	19	26	6	7	1	1	610	1597

Remark. We can reformulate the recursion rules using the generalized Cartan matrix $A = (a_{ij})_{ij}$ (indexed over the integers) with Dynkin diagram



One pair of vertices is simply laced, for the remaining laced pairs, one of the numbers a_{ij}, a_{ji} is equal to -1, the other to -2 (as in the case of the Cartan matrix of type $\mathbb{B}_2$). In order to obtain u_{t+1} from u_t, apply first the reflections σ_i with i odd, then the reflections σ_i with i even.

4 Fibonacci Modules

We now endow (T, E) with a bipartite orientation Ω; in this way, we deal with a quiver (T, E, Ω) and we consider its representations. Note that there are just two bipartite orientations for (T, E). Given a vertex x of T, let us denote by Ω_t^x the bipartite orientation such that x is a sink in case t is even, and a source, in case t is odd. Let $P_t(x)$ be the indecomposable representation of (T, E, Ω_t^x) with dimension vector $s_t(x)$. If there is an edge between x and y, let $R_t(x, y)$ be the indecomposable representation of (T, E, Ω_t^x) with dimension vector $r_t(x, y)$; it is well-known that these modules exist and are unique up to isomorphism: for $t = 0$ this assertion is trivial, and the Bernstein-Gelfand-Ponomarev reflection functors [2] can be used in order to construct $P_{t+1}(x)$ from $P_t(x)$, as well as $R_{t+1}(x, y)$ from $R_t(x, y)$.

We call the modules $P_t(x)$ and $R_t(x, y)$ *Fibonacci modules.*

Proposition 4.1. *Let x be a vertex of T with neighbors y, y', y'' and $t \geq 1$ an integer. Assume that for even t, the vertex x is a sink, and if t is odd, x is a source. Then there are exact sequences*

$$0 \to P_{t-1}(y) \to P_t(x) \to R_t(x, y) \to 0,$$

$$0 \to P_{t-1}(y') \to R_t(x, y) \to R_{t-1}(y'', x) \to 0,$$

and they are unique up to ismomorphism.

Proof. This is clear for $t = 1$ and follows by induction using the Bernstein-Gelfand-Ponomarev reflection functors. $\square$

Corollary 4.2. *Let $x_0, x_1, \dots, x_t$ be a path with x_0 a sink. Then $P_t(x_t)$ has a filtration*

$$P_0(x_0) \subset P_1(x_1) \subset \cdots \subset P_t(x_t)$$

with factors

$$P_i(x_i)/P_{i-1}(x_{i-1}) = R_i(x_i, x_{i-1})$$

for $1 \leq i \leq t$.

Proof. Since x_0 is a sink, we see that x_i is a sink for i even and a source for i odd. Thus, the proposition provides exact sequences

$$0 \to P_{i-1}(x_{i-1}) \to P_i(x_i) \to R_i(x_i, x_{i-1}) \to 0,$$

for $1 \leq i \leq t$. □

Corollary 4.3. *Let $x_{-1}, x_0, x_1, \dots, x_t, x_{t+1}$ be a path with x_0 a source. For $0 \leq i \leq t$, let z_i be the neighbor of x_i different from x_{i-1} and x_{i+1}. Then there is an exact sequence*

$$0 \to P_0(z_0) \oplus \cdots \oplus P_t(z_t) \to R_{t+1}(x_t, x_{t+1}) \to R_0(x_{-1}, x_0) \to 0.$$

Proof. Here, x_0 is a source, thus we see that x_i is a source for i even and a sink for i odd. The proof is by induction on t, the case $t = 0$ should be clear. Assume that $t \geq 1$. We consider the vertices $x_0, \dots, x_{t+1}, z_1, \dots z_t$, but deal with the opposite orientation Ω' (so that now x_1 is a source); the corresponding respresentations will be distinguished by a dash, thus $P_i'(z_i)$ is the indecomposable representation of (T, E, Ω') with dimension vector $s_i(z_i)$, and so on. By induction, there is an exact sequence

$$0 \to P_0'(z_1) \oplus \cdots \oplus P_{t-1}'(z_t) \to R_t'(x_t, x_{t+1}) \to R_0'(x_0, x_1) \to 0.$$

Applying reflection functors (at all sinks), we obtain the exact sequence

$$0 \to P_1(z_1) \oplus \cdots \oplus P_t(z_t) \to R_{t+1}(x_t, x_{t+1}) \to R_1(x_0, x_1) \to 0.$$

Note that $P_0(z_0)$ is a subrepresentation of $R_1(x_1, x_0)$, thus we obtain the induced exact sequence

$$0 \to P_1(z_1) \oplus \cdots \oplus P_t(z_t) \to W \to P_0(z_0) \to 0,$$

where W is a subrepresentation of $R_{t+1}(x_t, x_{t+1})$ with $R_{t+1}(x_t, x_{t+1})/W$ being isomorphic to $R_1(x_1, x_0)/P_0(z_0) = R_0(x_{-1}, x_0)$.

On the other hand, the induced sequence has to split, since $P_0(z_0)$ is projective, thus W is the direct sum of the representations $P_i(z_i)$ with $0 \leq i \leq t$. This completes the proof. □

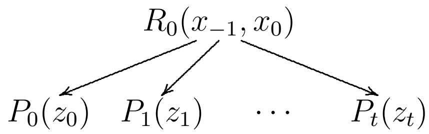
Of course, proposition 4.1 should be seen as a categorification of the defining equation

$$f_{t+1} = f_{t-1} + f_t$$

for the Fibonacci numbers. Corollaries 4.2 and 4.3 are categorifications of the equalities

$$f_{2t} = \sum_{i=1}^{t} f_{2i-1}, \qquad f_{2t+1} = 1 + \sum_{i=1}^{t} f_{2i}.$$

Remark. The representations $R_0(x_{-1}, x_0), P_0(z_0), \dots, P_t(z_t)$ are pairwise orthogonal bricks, the corresponding Ext-quiver is



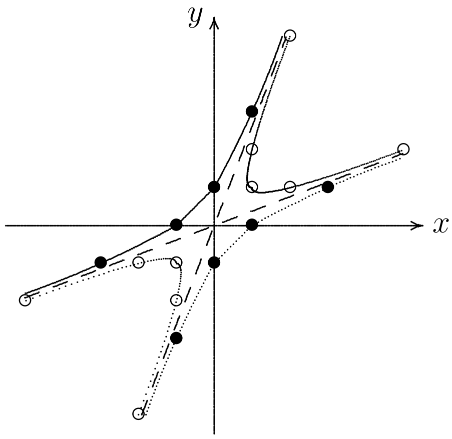
Let $\mathcal{F}$ denote the full subcategory of of all representations of (T, E, Ω) with a filtration with factors of the form $R_0(x_{-1}, x_0), P_0(z_0), \dots, P_t(z_t)$, then corollary 4.3 shows that $R_{t+1}(x_t, x_{t+1})$ is indecomposable projective in $\mathcal{F}$.

5 Fibonacci Pairs

The Fibonacci pairs are related to the integral quadratic form $q(x, y) = x^2 + y^2 - 3xy$ as follows:

Proposition 5.1. *A pair $[x, y] \in \mathbb{Z}^2$ is a Fibonacci pair if and only if $|q(x, y)| = 1$.*

The pairs $[x, y]$ with $|q(x, y)| = 1$ form two hyperbolas. The hyperbola of all pairs $[x, y]$ with $q(x, y) = 1$ yields the even Fibonacci pairs, these are the black dots on the following picture. The hyperbola $q(x, y) = -1$ yields the odd Fibonacci pairs, they are indicated by small circles.



The quadratic form q on $\mathbb{Z}^2$ defines the Kac-Moody root system [10] for the Cartan matrix $\begin{bmatrix} 2 & -3 \\ -3 & 2 \end{bmatrix}$ (see [6], section 6). The pairs $[x, y]$ with $q(x, y) = 1$ are the real roots, those with $q(x, y) \leq 0$ the imaginary roots. We identify $\mathbb{Z}^2$ with the Grothendieck group of the 3-Kronecker modules, and q is the Euler form. Given a 3-Kronecker module M, the corresponding element in the Grothendieck group is its dimension vector $\mathbf{dim}\, M$.

6 Recursion formula for Fibonacci numbers with index of the same parity.

This is the recursion formula:

$$f_{t+2} = 3f_t - f_{t-2}. \qquad (1)$$

Proof. The term $f_{t+2} - 3f_t + f_{t-2}$ is just the sum of $f_{t+2} - f_{t+1} - f_t$, of $f_{t+1} - f_t - f_{t-1}$ and of $-(f_t - f_{t-1} - f_{t-2})$, thus equal to zero. □

Note that the recursion formula can also be rewritten as

$$f_{t-2} = 3f_t - f_{t+2}, \qquad (2)$$

thus the same recurrence formula works both for going up and for going down. Also, we can write it in the form

$$f_{t-2} + f_{t+2} = 3f_t. \qquad (3)$$

The formulae (1) and (2) are categorified by looking at the reflection functors σ_+ and σ_-: Given a 3-Kronecker module $M = (M_1, M_2, \alpha, \beta, \gamma)$, we form $\sigma_+ M$ by taking $(\sigma_+ M)_2 = M_1$ and $(\sigma_+ M)_1$ as the kernel of the map $(\alpha, \beta, \gamma) : M_1^3 \to M_2$; note that if M is indecomposable and not the simple projective module P_0, then this kernel has dimension $3 \dim M_1 - \dim M_2$. Similarly, we take $(\sigma_- M)_1 = M_2$ and $(\sigma_+ M)_2$ is by definition the cokernel of the map $(\alpha, \beta, \gamma)^t : M_1 \to M_2^3$; this cokernel has dimension $3 \dim M_2 - \dim M_1$ provided M is indecomposable and not the simple injective module Q_0.

Let us now look at the formula (3). Let $P_n = (\sigma_-)^n P_0$ and $Q_n = (\sigma_+)^n Q_0$, with $n \in \mathbb{N}$. Also note that the indecomposable 3-Kronecker modules with dimension vector $[1, 1]$ are indexed by the elements λ of the projective plane $\mathbb{P}_2(k)$, such a module will be denoted by $R_{0,\lambda}$ and we form $R_{n,\lambda} = (\sigma_+)^n R_{0,\lambda}$ and $R_{-n,\lambda} = (\sigma_-)^n R_{0,\lambda}$, again with $n \in \mathbb{N}$. Of course, the dimension vectors of these modules are just Fibonacci pairs:

$$\mathbf{dim}\, P_n = [f_{2n}, f_{2n+2}],\ \mathbf{dim}\, Q_n = [f_{2n+2}, f_{2n}],\ \mathbf{dim}\, R_{m,\lambda} = [f_{2m+1}, f_{2m-1}],$$

for all $n \in \mathbb{N}, m \in \mathbb{Z}$. Let us consider the corresponding Auslander-Reiten sequences (see for example [1]), they are of the form

$$0 \to P_{n-1} \to P_n^3 \to P_{n+1} \to 0,$$

$$0 \to Q_{n+1} \to Q_n^3 \to Q_{n-1} \to 0,$$

$$0 \to R_{m+1,\lambda} \to E_{m,\lambda} \to R_{m-1,\lambda} \to 0,$$

with suitable indecomposable modules $E_{m,\lambda}$ having dimension vector equal to $3\,\mathbf{dim}\, R_{m,\lambda}$. We see that these Auslander-Reiten sequences categorify the formula (3).

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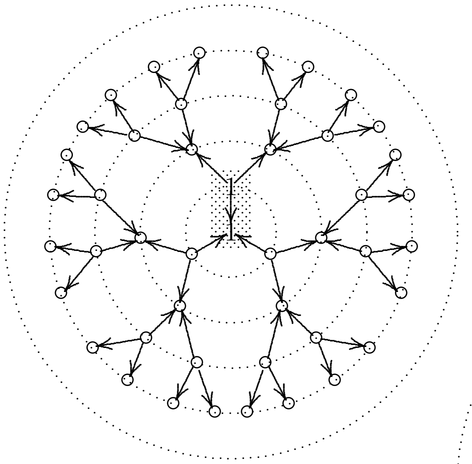
Keywords: Fibonacci numbers, quiver representations, universal cover, 3-regular tree, 3-Kronecker quiver, categorification.

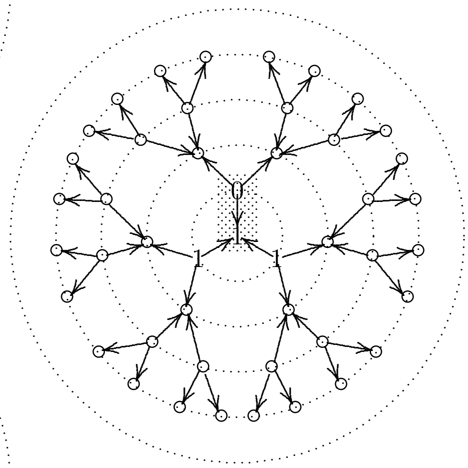
(Concerned with sequences A000045, A132262, A147316, A197956, A197957 of [15])

Illustrations. The following illustrations show the functions $r_t(x, y)$ for $t = 0, .., 5$, with x the center and y above x. The dotted circles indicate a fixed distance to x. We show (T, E) with bipartite orientation such that x is a sink (thus z is a sink if $d(x, y)$ is even, and a source if $d(x, z)$ is odd). The region containing the arrow $y \to x$ has been dotted. For the convenience of the reader we have indicated the corresponding Fibonacci pair $[f_{2t-1}, f_{2t+1}]$.

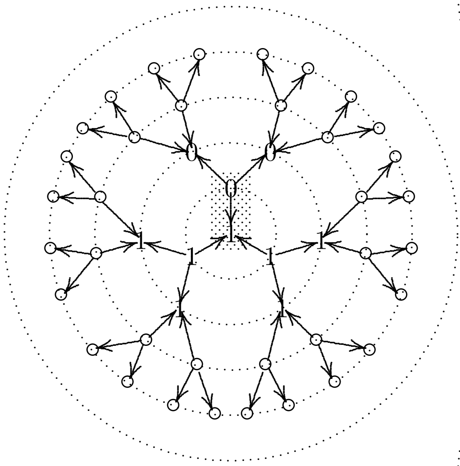
$r_0(x, y)$
$[1, 1]$



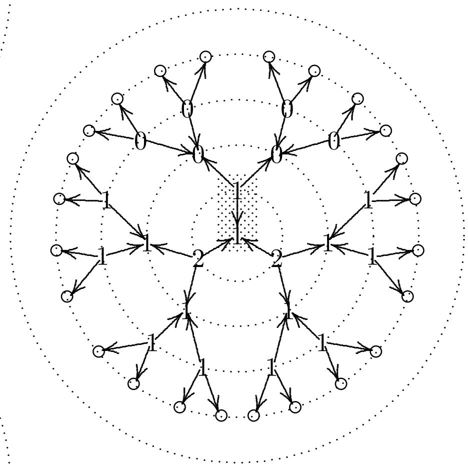
$r_1(x, y)$
$[1, 2]$



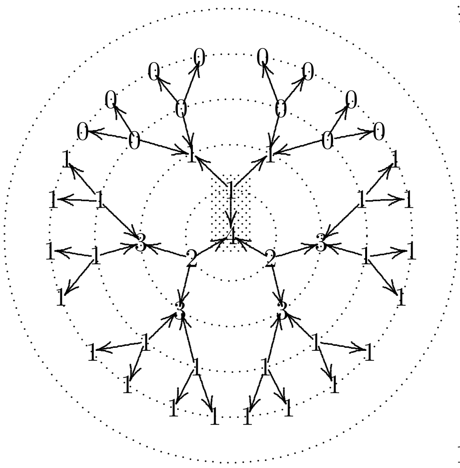
$r_2(x, y)$
$[2, 5]$



$r_3(x, y)$
$[5, 13]$



$r_4(x, y)$
$[13, 34]$



$r_5(x, y)$
$[34, 89]$