A Partition Formula for Fibonacci Numbers

Philipp Fahr and Claus Michael Ringel
Fakultät für Mathematik
Universität Bielefeld
POBox 100 131
D-33501 Bielefeld, Germany
philfahr@math.uni-bielefeld.de
ringel@math.uni-bielefeld.de

Abstract

We are going to present a partition formula for the even Fibonacci numbers. The formula is motivated by the appearance of these Fibonacci numbers in the representation theory of the socalled 3-Kronecker quiver.

1 Introduction

Let $f_0, f_1, \dots$ be the Fibonacci numbers, with $f_0 = 0, f_1 = 1$ and $f_{i+1} = f_i + f_{i-1}$ for $i \geq 1$. The aim of this note is to present a partition formula for the even index Fibonacci numbers f_{2n}.

We start with the 3-regular tree (T, E) with vertex set T and edge set E (3-regularity means that every vertex of T has precisely 3 neighbours). Fix a vertex x_0 as base point. Let T_r be the sets of vertices of T which have distance r to x_0, thus $T_0 = \{x_0\}$, T_1 are the neighbours of x_0, and so on. The vertices in T_r will be called even or odd, depending on r being even or odd. Note that $|T_0| = 1$ and

$$|T_r| = 3 \cdot 2^{r-1} \qquad \text{for } r \geq 1. \tag{1}$$

(The proof for equation (1) is by induction: T_1 consists of the three neighbours of x_0. For $r \geq 1$, any element in T_r has precisely one neighbour in T_{r-1}, thus two neighbours in T_{r+1}, therefore $|T_{r+1}| = 2|T_r|$.)

Given a set S, let $\mathbb{Z}[S]$ be the set of functions $S \to \mathbb{Z}$ with finite support.

We are interested in certain elements $a_t \in \mathbb{Z}[T]$. Any vertex $y \in T$ yields a reflection σ_y on $\mathbb{Z}[T]$, defined for $b \in \mathbb{Z}[T]$ by

$$(\sigma_y b)(x) = \begin{cases} b(x) & \\ & \text{provided} \\ -b(y) + \sum_{z \in N(y)} b(z) & \end{cases} \begin{matrix} x \neq y \\ \\ x = y, \end{matrix}$$

here $N(y)$ denotes the set of neighbours of y in (T, E).

Denote by Φ_0 the product of the reflections σ_y with y even; this product is independent of the order, since these reflections commute: even vertices never are neighbours of each other. Similarly, we denote by Φ_1 the product of the reflections σ_y with y odd.

The elements of $\mathbb{Z}[T]$ we are interested in, are labelled a_t with $t \in \mathbb{N}_0$. We will start with the characteristic function a_0 of T_0 (thus, $a_0(x) = 0$ unless $x = x_0$ and $a_0(x_0) = 1$) and look at the functions

$$a_t = (\Phi_0\Phi_1)^t a_0, \quad \text{with } t \geq 0.$$

Clearly, a_t is constant on T_r, for any $r \geq 0$, so we may define $a[r] : \mathbb{N}_0 \to \mathbb{Z}$ by

$$a[r]_t = a_t(x) \quad \text{for } r \in \mathbb{N}_0 \text{ and } x \in T_r.$$

We define

$$\begin{aligned} \overline{a}_t(0) &= \sum_{r \text{ even}} |T_r| \cdot a[r]_t = a[0]_t + 3\sum_{m\geq 1} 2^{2m-1} a[2m]_t, \\ \overline{a}_t(1) &= \sum_{r \text{ odd}} |T_r| \cdot a[r]_t = 3\sum_{m\geq 0} 2^{2m} a[2m+1]_t, \end{aligned}$$

these are finite sums, since $a[r]_t = 0$ for $r > 2t$. Note that we have used the equality (1).

2 The Partition Formula

$$\boxed{f_{4t} = \overline{a}_t(1), \qquad f_{4t+2} = \overline{a}_t(0).}$$

For example, for $t = 3$, we obtain the following two equalities:

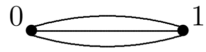
$$\begin{aligned} 144 = f_{12} &= \overline{a}_3(1) = 3 \cdot 12 + 12 \cdot 5 + 48 \cdot 1 \\ 377 = f_{14} &= \overline{a}_3(0) = 29 + 6 \cdot 18 + 24 \cdot 6 + 96 \cdot 1. \end{aligned}$$

Here is the table for $t \leq 6$.

t	$a[0]_t$	$a[1]_t$	$a[2]_t$	$a[3]_t$	$a[4]_t$	$a[5]_t$	$a[6]_t$	$a[7]_t$	$a[8]_t$	$a[9]_t$	$a[10]_t$	$\overline{a}_t(0)$	$\overline{a}_t(1)$
0	1											1	0
1	2	1	1									8	3
2	7	3	4	1	1							55	21
3	29	12	18	5	6	1	1					377	144
4	130	53	85	25	33	7	8	1	1			2584	987
5	611	247	414	126	177	42	52	9	10	1	1	17711	6765
6	2965	1192	2062	642	943	239	313	63	75	11	$\cdots$	121393	46368

3 Proof of the partition formula

In order to relate the functions $\overline{a}_t(0), \overline{a}_t(1)$ and the Fibonacci numbers f_{2n}, we consider the multigraph $(\overline{T}, \overline{E})$ with two vertices $0, 1$ and three edges between 0 and 1.



Any element $c \in \mathbb{Z}[\overline{T}]$ is determined by the values $c(0)$ and $c(1)$, thus we will write $c = [c(0), c(1)]$.

Claim: $\qquad [\overline{a}_t(0), \overline{a}_t(1)] = [f_{4t+2}, f_{4t}]$.
The claim is obviously true for $t = 0$, since $[\overline{a}_0(0), \overline{a}_0(1)] = [1, 0] = [f_2, f_0]$. The general assertion will follow from the following recursion formulae

$$[\overline{a}_t(0), \overline{a}_t(1)] \begin{bmatrix} 8 & 3 \\ -3 & -1 \end{bmatrix} = [\overline{a}_{t+1}(0), \overline{a}_{t+1}(1)],$$

$$[f_{m+2}, f_m] \begin{bmatrix} 8 & 3 \\ -3 & -1 \end{bmatrix} = [f_{m+6}, f_{m+4}].$$

For the Fibonacci numbers, this is well-known:

$$\begin{aligned} [f_{m+6}, f_{m+4}] &= [f_{m+5}, f_{m+4}] \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ &= [f_{m+1}, f_m] \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^4 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \\ &= [f_{m+2}, f_m] \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^4 \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \end{aligned}$$

and the product of the matrices in the last line is just $\begin{bmatrix} 8 & 3 \\ -3 & -1 \end{bmatrix}$, which is the Coxeter matrix of the 3-Kronecker quiver.

It remains to deal with $\overline{a}_t$. We may consider (T, E) as the universal covering of $(\overline{T}, \overline{E})$, say with a covering map

$$\pi : (T, E) \to (\overline{T}, \overline{E}),$$

where $\pi(x) = 0$ if x is an even vertex of (T, E), and $\pi(x) = 1$ if x is an odd vertex. Such a covering map will provide a bijection between the three edges in E which contain a given vertex x with the edges in $\overline{E}$.

Given $b \in \mathbb{Z}[T]$, define $\overline{b} \in \mathbb{Z}[\overline{T}]$ by

$$\overline{b}(i) = \sum\nolimits_{x \in \pi^{-1}(i)} b(x) \qquad \text{for } i = 0, 1.$$

Observe that in this way we obtain from $a_t \in \mathbb{Z}[T]$ precisely $\overline{a}_t \in \mathbb{Z}[\overline{T}]$ as defined above.

On $\mathbb{Z}[\overline{T}]$, we also consider reflections, but we have to take into account the multiplicity of edges: There are the two reflections σ_0, σ_1, with

$$\begin{aligned} (\sigma_0 c)(0) &= -c(0) + 3c(1), & (\sigma_0 c)(1) &= c(1), \\ (\sigma_1 c)(0) &= c(0), & (\sigma_1 c)(1) &= -c(1) + 3c(0), \end{aligned}$$

for $c \in \mathbb{Z}[\overline{T}]$. Note that this implies that

$$[c(0), c(1)] \begin{bmatrix} 8 & 3 \\ -3 & -1 \end{bmatrix} = [(\sigma_0\sigma_1 c)(0), (\sigma_0\sigma_1 c)(1)]$$

We finish the proof by observing that $\overline{a}_t = (\sigma_0\sigma_1)^t \overline{a}_0$. This follows directly from the following consideration: Let $b \in \mathbb{Z}[T]$, then

$$\overline{\Phi_0 b} = \sigma_0 \overline{b}, \quad \overline{\Phi_1 b} = \sigma_1 \overline{b},$$

as it is easily verified.

4 Remarks

1. The partition formula presented here seems to be very natural. We were surprised to see that the integer sequence $a[0]_t$ has not yet been recorded in *The On-Line Encyclopedia of Integral Sequences*. There seems to be numerical evidence that the sequence $a[1]_t$ is just the sequence A110122 of the On-Line Encyclopedia (it counts the Delannoy paths of length t with no EE's crossing the line $y = x$), but we do not see why these are the same sequences.

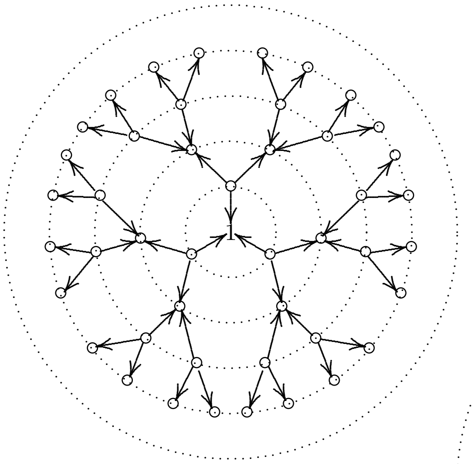
2. Our interest in the even index Fibonacci numbers comes from the fact that the pairs $[f_{4t+2}, f_{4t}]$ are just the dimension vectors of the exceptional representations of 3-Kronecker quiver. These are the preprojective and preinjective representations with dimension vectors $(1, 0), (3, 1), (8, 3) \ldots,$ and $(0, 1), (1, 3), (3, 8), \ldots$ respectively. A similar method as presented here can be used to obtain a new partition formula for the odd index Fibonacci numbers.

3. The considerations presented here can be generalized to the n-regular tree and the multigraph with two vertices and n edges.

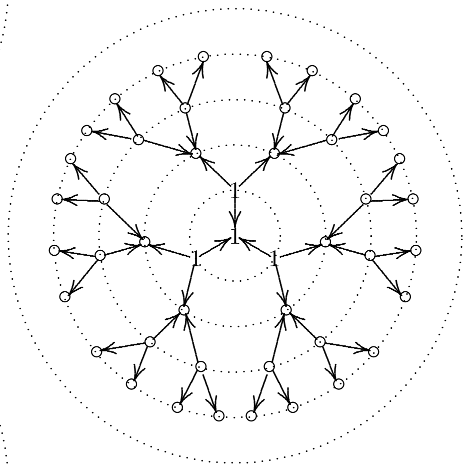
5 Illustrations

The following illustrations show the functions a_t for $t = 0, 1, 2$ as well as the corresponding functions $\Phi_1 a_t$. Here, (T, E) has been endowed with an orientation such that the edges point to the even vertices. In this way, the even vertices are sinks, the odd vertices are sources. The dotted circles indicate the various sets T_r, the center is just $T_0 = \{x_0\}$.

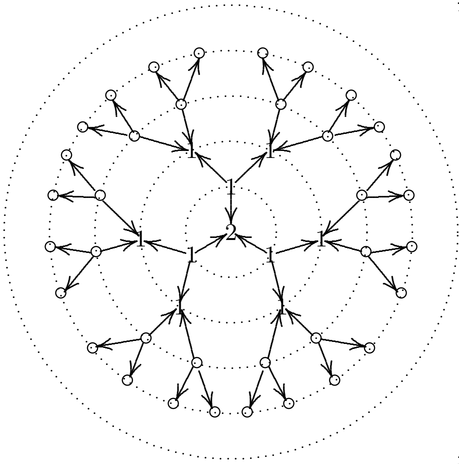
a_0



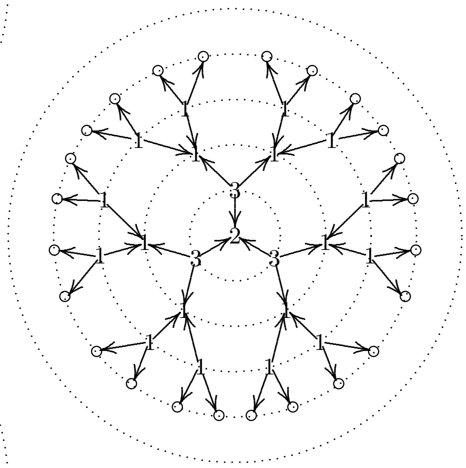
$\Phi_1 a_0$



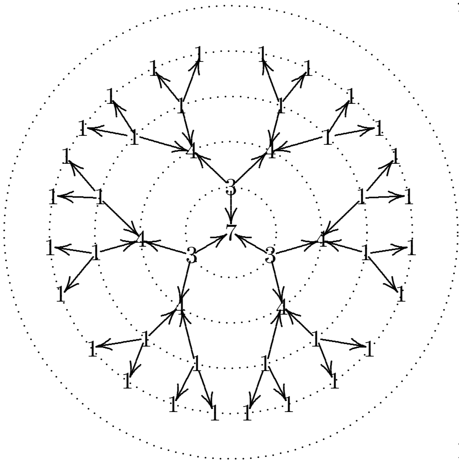
a_1



$\Phi_1 a_1$



a_2



$\Phi_1 a_2$