Free Submodules of Indecomposable Modules

Claus Michael Ringel

ABSTRACT. Let Λ be an artin algebra which has only finitely many isomorphism classes of non-faithful indecomposable modules. We show that large indecomposable Λ-modules contain large free submodules.

1. Main Proposition.

Let Λ be an artin algebra. The modules to be considered will be finitely generated left Λ-modules.

Proposition. *Let Λ be an artin algebra which has only finitely many isomorphism classes of non-faithful indecomposable modules. Then, for every natural number t, there is a bound b_t such that any indecomposable Λ-module of length at least b_t contains a free submodule of rank t.*

We may formulate some consequences of the assertion:

If M is an indecomposable module of length at least b_t, and N is any module of length t, then N occurs as a subfactor of M (this means that there are submodules $M'' \subseteq M' \subseteq M$ with M'/M'' isomorphic to N).

Namely, if N is a module of length t, then N is a factor module of ${}_\Lambda\Lambda^t$, thus if the submodule M' of M is free of rank t, there is $M'' \subseteq M'$ such that M'/M'' is isomorphic to N.

Assume that we know that there are infinitely many isomorphism classes of modules of length t. Then any indecomposable module of length at least b_t has infinitely many isomorphism classes of subfactors.

It is the last assertion which seems to be of interest, since it belongs to the focus of the Brauer-Thrall conjectures: to the problem of relating families of indecomposable modules of a fixed length to large indecomposable modules. We will provide further remarks on this relationship in section 3.

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2. Proof.

The proof of Proposition will be given in this section. We need some preparations. The length of a module M will be denoted by $|M|$, let $d = |{}_\Lambda\Lambda|$.

We say that a module M has *height* at least $h \geq 1$ provided there is a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_h = M$$

such that all the subfactors M_i/M_{i-1} (with $1 \leq i \leq h$) are faithful (and M has height 0, provided M is not faithful). The height $h(M)$ of M is by definition the maximum of all numbers h such that M has height at least h; note that the height $h(M)$ is obviously bounded by the length $|M|$.

Lemma 1. *Assume that $h(M) \geq d$. Then ${}_\Lambda\Lambda$ can be embedded into M.*

Proof: We show by induction on r the following: if the height of M is at least r, and $r \leq d$, then there is a map $u\colon {}_\Lambda\Lambda \to M$ with image of length at least r. The induction starts with $r = 0$ and u the zero map. Assume that the assertion has been shown for some $r < d$. Consider a module M of height $r+1$, say with a filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_{r+1}$ such that all the factors M_i/M_{i-1}, for $1 \leq i \leq r+1$, are faithful. Now M/M_1 has height at least r, thus there is a map $u\colon {}_\Lambda\Lambda \to M/M_1$ with image I of length at least r. Since ${}_\Lambda\Lambda$ is a free module, there exists a lifting $u'\colon {}_\Lambda\Lambda \to M$ such that $\pi u' = u$, where $\pi\colon M \to M/M_1$ is the canonical epimorphism with kernel M_1. Consider the image I' of u'. Case 1: $I' \cap M_1 \neq 0$. Then $|I'| > |I|$, thus $|I'| \geq r+1$. Case 2: $I' \cap M_1 = 0$, thus $|I'| = |I| \geq r$. If $|I| > r$, then we are done, thus assume $|I| = r$. Let $u'(1) = x$, thus $u'(\lambda) = \lambda x$. Since $|I| = r < d = |{}_\Lambda\Lambda|$, we see that there is some left ideal L of Λ such that $L \neq 0$, but $Lx = 0$. Take a non-zero element $\lambda_0 \in L$. Since M_1 is faithful, there is $y \in M_1$ with $\lambda_0 y \neq 0$, thus λ_0 is not contained in the kernel of the map $v\colon {}_\Lambda\Lambda \to M_1$ defined by $v(1) = y$. Now, consider the map $u'' = \begin{bmatrix} u' \\ v \end{bmatrix} \colon {}_\Lambda\Lambda \to I \oplus M_1 \subseteq M$. The kernel L'' of u'' is the intersection of the kernel L of u' and the kernel of v. Since the kernel of v does not contain λ_0 (and $\lambda_0 \in L$), we see that L'' is a proper subset of L, thus the image of u'' has length greater than r. This completes the proof.

Corollary. *Assume that $h(M) \geq dt$. Then the free Λ-module of rank t can be embedded into M.*

Proof: Let $0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$ be a filtration, where any factor M_i/M_{i-1} for $1 \leq i \leq t$ has height at least d. According to the previous lemma, there is an embedding $u_i\colon {}_\Lambda\Lambda \to M_i/M_{i-1}$, and, since ${}_\Lambda\Lambda$ is a free module, we can lift the embedding to an embedding $u_i'\colon {}_\Lambda\Lambda \to M_i$. Let I_i be the image of u_i'. Since the composite map ${}_\Lambda\Lambda \to M_i \to M_i/M_{i-1}$ still is injective, we see that $I_i \cap M_{i-1} = 0$. But this means that $\sum_{i=1}^t I_i$ actually is a direct sum, thus isomorphic to ${}_\Lambda\Lambda^t$.

Let k be the center of Λ, it is a commutative artinian ring (and Λ is a finitely generated k-module). For any pair of Λ-modules M, N, the group $\operatorname{Ext}^1(M,N)$ is a k-module of finite length, where k is the center of Λ, and we write $\dim \operatorname{Ext}^1(M,N)$ in order to denote its length as a k-module.

Ext-Lemma. *Let N be an indecomposable module. Let*

$$0 \to N^n \oplus N' \xrightarrow{u} X \to Q \to 0$$

be an exact sequence of Λ-modules. If $\dim \operatorname{Ext}^1(Q,N) < n$, then for at least one of the canonical inclusions $u_i\colon N \to N^n \oplus N'$, the composition uu_i is a split monomorphism.

This is a basic and well-known result in homological algebra, it is one of the ingredients used by Roiter in [Ro], see also [Ri].

We assume from now on that Λ is an artin algebra which has only finitely many isomorphism classes of non-faithful indecomposable modules, let $N_1, \dots, N_m$ be such modules, one from each isomorphism class. Let

$$\begin{aligned} e_i &= \max\{\dim \operatorname{Ext}^1(S, N_i) \mid S \text{ a simple } \Lambda\text{-module}\}, \quad \text{and} \\ e_i' &= \max\{\dim \operatorname{Ext}^1(N_i, S) \mid S \text{ a simple } \Lambda\text{-module}\}. \end{aligned}$$

We can use these numbers in order to bound other Ext-groups. For example, for every module M we see by induction on the length of M that

$$\dim \operatorname{Ext}^1(N_i, M) \le |M| e_i'.$$

Also, let us introduce the following numbers:

$$c = 1 + \sum e_i |N_i| \quad \text{and} \quad c' = 1 + \sum e_i' |N_i|.$$

Call M *f-minimal* provided M is indecomposable and faithful, and no proper indecomposable submodule is faithful.

Lemma 2. *An f-minimal module M has length at most c.*

Proof: Let M a be f-minimal. Let M' be a maximal submodule of M. Our assumption implies that any indecomposable direct summands of M' is of the form N_i, thus $M' = \bigoplus_{i=1}^m N_i^{x_i}$. According to the Ext-Lemma, $x_i \le e_i$, since otherwise M would have a direct summand isomorphic to N_i, but M is indecomposable and faithful. This shows that $|M| = |M/M'| + |M'| = 1 + \sum x_i|N_i| \le 1 + e_i|N_i| = c$.

Lemma 3. *Any module M of height $h(M) = t$ has a filtration*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_s \subseteq M$$

with $s \le t$ such that the factors M_i/M_{i-1} for $1 \le i \le s$ are f-minimal and M/M_s is a direct sum of non-faithful modules.

Proof: We are going to define M_i inductively, as follows: Let $M_0 = 0$. Assume that we have constructed already $0 = M_0 \subset M_1 \subset \cdots \subset M_r$ such that all the factors M_i/M_{i-1} are f-minimal, for $1 \le i \le r$. If M/M_r is a direct sum of non-faithful modules, let $s = r$. Thus assume M/M_r has an indecomposable direct summand which is faithful, say U. Then there is a submodule U' of U which is f-minimal. Write $U' = M_{r+1}/M_r$, where $M_r \subset M_{r+1} \subseteq M$. This procedure will stop eventually, thus we obtain the filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_s \subseteq M$ where M_i/M_{i-1} for $1 \le i \le s$ are f-minimal and M/M_s is a direct sum of non-faithful modules. Since there are s factors which are faithful, we see that $s \le h(M) = t$.

Lemma 4. *Assume that M is indecomposable and faithful. Then*

$$|M| \leq cc'h(M).$$

Proof. According to Lemma 3, there is a filtration

$$0 = M_0 \subset M_1 \subset \cdots \subset M_s \subseteq M$$

with $s \leq t$ such that the factors M_i/M_{i-1} for $1 \leq i \leq s$ are f-minimal and M/M_s is a direct sum of non-faithful modules. By Lemma 2, $|M_i/M_{i-1}| \leq c$, for $1 \leq i \leq s$, thus $|M_s| \leq cs$.

Write $M/M_s = \bigoplus N_i^{x_i}$. According to the dual of the Ext-Lemma, we have

$$x_i \leq \dim \operatorname{Ext}^1(N_i, M_s) \leq |M_s|e_i' \leq cse_i'$$

Thus

$$\begin{aligned} |M| = |M_s| + |M/M_s| &\leq cs + \sum x_i|N_i| \leq cs + \sum cse_i'|N_i| \\ &= cs(1 + \sum e_i'|N_i|) = cc's \leq cc't. \end{aligned}$$

This completes the proof.

Proof of Proposition. Let $b_t = cc'dt$. If $|M| \geq b_t$, then

$$h(M) \geq \frac{1}{cc'}|M| \geq \frac{1}{cc'}b_t = dt,$$

where the first inequality sign comes from Lemma 4. Corollary of Lemma 1 implies that ${}_\Lambda\Lambda^t$ can be embedded into M, as we wanted to show.

3. Remarks concerning the second Brauer-Thrall conjecture.

In his famous solution of the first Brauer-Thrall conjecture [Ro], Roiter has shown how infinite families of pairwise non-isomorphic modules with fixed length can be used in order to construct large indecomposable modules (see also [Ri]). Of course, infinite families of pairwise non-isomorphic modules with fixed length can only exist in case Λ is a ring with infinitely many elements (or, equivalently, in case Λ is a module-finite k-algebra, where k is a commutative ring with infinitely many elements). Under this assumption, one may conversely ask whether it is possible to construct infinite families of pairwise non-isomorphic modules with fixed length, starting from large indecomposable modules. The second Brauer-Thrall conjecture (known to hold for finite-dimensional k-algebras where k is an infinite perfect field) asserts that any representation infinite k-algebra has infinite families of pairwise non-isomorphic modules with fixed length, however it does not predict how to get hold of such a family. We propose the following conjecture:

Conjecture. *Assume that Λ is an infinite artin algebra which is of infinite representation type. Then there exists a bound b such that any indecomposable* *module of length greater than b contains infinitely many isomorphism classes of subfactors.*

Let us observe that for some well-known algebras, such a bound can easily be calculated (and it turns out that b often may be chosen to be amazingly small). For example, for the Kronecker quiver take $b = 2$: *Any indecomposable Kronecker module of length greater than 2 contains infinitely many isomorphism classes of indecomposable subfactors.* Namely, any indecomposable Kronecker module M of length at least 3 contains an indecomposable module of length 3 either as a submodule (in case M is preprojective or regular) or else as a factor module (in case M is preinjective or regular). There are two isomorphism classes of indecomposable modules N of length 3: the module N may be either projective or else injective. The indecomposable projective module of length 3 has infinitely many isomorphism classes of factor modules, the indecomposable injective module of length 3 has infinitely many isomorphism classes of submodules. Actually, the same bound $b = 2$ works even for all the generalized Kronecker algebras (these are the path algebras of the quivers with two vertices a, b and a finite number of arrows $a \to b$).

In case Λ is an arbitrary tame hereditary algebra, the conjecture can be shown to hold; the bounds b can be calculated in all these cases, they are larger than in the Kronecker case, but still quite small.

If Λ is a string algebra, then $b = 2sp$ works, where $s = s(\Lambda)$ is the number of isomorphism classes of simple Λ-modules and $p = p(\Lambda)$ the maximal length of an indecomposable projective Λ-module. For a string algebra Λ, the indecomposable Λ-modules are well-known and easy to describe, see for example [BR] for the terminology used here. Consider a string module M of length greater than b. Its socle has length at least $2s + 1$, thus one of the simple modules occurs in the socle with multiplicity at least 3. The module $M = M(w)$ is described by some word w, and there are the following three possibilities: first, w contains (at least) twice a subword of the form $\beta^{-1}\alpha$, second, w starts with α and contains $\beta^{-1}\alpha$ as a subword, or third, w ends with β^{-1} and contains $\beta^{-1}\alpha$ as a subword. Always, we obtain all the primitive band modules $M(w', \lambda)$ a factor modules of M, where w' is a primitive cyclic word starting with α and ending with β^{-1}. A similar argument can be used in case we consider a band module of length greater than b.

Let us return to the second Brauer-Thrall conjecture, thus we assume that we consider an artin algebra Λ with infinitely many elements. Recall that there is one class of algebras which can be treated quite easily: those with an infinite set of two-sided ideals. As Jans [J] has shown in this situation (at least if Λ is a finite dimensional algebra over a field k) there exists a one-parameter family of isomorphism classes of indecomposable modules which are local and say of length q (they are annihilated by pairwise different ideals, in particular, none of these modules is faithful); and one can form self-extensions of these modules in order to obtain corresponding one-parameter families of indecomposable modules of length nq, for all $n \in \mathbb{N}_1$. Thus, the interesting class of algebras which have to be considered are those with only finitely many two-sided ideals.

An artin algebra which is not of finite representation type will be said to be *representation-infinite.* In case Λ is representation-infinite, but any proper factor algebra is of finite representation type, then Λ will be said to be *minimal representation-infinite.*

Lemma. *Let Λ be an artin algebra with infinitely many elements. The following properties are equivalent:*

(i) Λ *is minimal representation-infinite and has only finitely many two-sided ideals.*

(ii) Λ *is representation-infinite and has only finitely many isomorphism classes of non-faithful indecomposable modules.*

Proof: (i) $\implies$ (ii). If there are only finitely many two-sided ideals, and if for every non-zero two-sided ideal I, the factor algebra Λ/I is of finite representation type, then there are only finitely many isomorphism classes of non-faithful indecomposable modules.

(ii) $\implies$ (i). Note that condition (ii) immediately implies that Λ is minimal representation-infinite. But (ii) also implies that there are only finitely many ideals, as Jans [J] has shown.

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Fakultät für Mathematik, Universität Bielefeld, POBox 100 131, D-33 501 Bielefeld
E-mail address: `ringel@mathematik.uni-bielefeld.de`