The repetitive algebra of a gentle algebra

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ABSTRACT. We are going to show that the repetitive algebra of a gentle algebra whose quiver has at least two cycles is non-domestic (thus even of non-polynomial growth).

1. Introduction.

Let k be a field. Given a quiver Q and arrows $\alpha\colon a \to b, \beta\colon b \to c$, the concatenation of α and β will be denoted by $\alpha\beta$, it is a path of length 2 starting in a and ending in c. Recall that a relation for the quiver Q is a non-zero linear combination of paths of length at least 2 having the same starting point and the same end point.

Let Q be a connected quiver and ρ a set of relations for Q. The pair (Q,ρ) is said to be *special biserial* provided the following conditions are satisfied:
(1) Every vertex is starting point of at most two arrows.
(1′) Every vertex is end point of at most two arrows.
(2) For every arrow β, there is at most one arrow α with $\alpha\beta \notin \rho$.
(2′) For every arrow β, there is at most one arrow γ with $\beta\gamma \notin \rho$.

The pair (Q,ρ) is said to be *gentle* provided besides (1), (1′), (2), (2′) also the following conditions are satisfied:
(3) All the relations in ρ are monomials of length two.
(4) For every arrow β, there is at most one arrow α' with $\alpha'\beta \in \rho$.
(4′) For every arrow β, there is at most one arrow γ' with $\beta\gamma' \in \rho$.

A k-algebra A is said to be *special biserial*, or *gentle*, provided A is isomorphic to the factor algebra $kQ/\langle\rho\rangle$, where (Q,ρ) is special biserial, or gentle respectively; here kQ denotes the path algebra of the quiver Q, and $\langle\rho\rangle$ the ideal generated by the set ρ. (The k-algebras which we consider will always have sufficiently many idempotents, but not necessarily a unit element; note that the path algebra of a quiver Q has a unit element if and only if Q has finitely many vertices. We are mainly interested in k-algebras A which are locally bounded: this means that for every primitive idempotent e, both Ae and eA are

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finite dimensional over k. Note that for a special biserial pair (Q, ρ), the algebra $kQ/\langle\rho\rangle$ is locally bounded if and only if for every vertex a of Q, almost all paths of Q starting or ending in a belong to $\langle\rho\rangle$. Of course, $kQ/\langle\rho\rangle$ is finite dimensional if and only if $kQ/\langle\rho\rangle$ is locally bounded and Q is finite.)

The finite dimensional representations of special biserial algebras are well understood. Methods of Gelfand and Ponomarev [GP] developed for an example have been found fruitful also in the general setting, see [SW]. In particular, a special biserial k-algebra is always tame, and either domestic or of non-polynomial growth, see [S]. (Since we deal with not necessarily finite dimensional algebras A, we should remark that we call A tame (or domestic) provided all finite dimensional factor algebras are tame (or domestic), and we say that A is of non-polynomial growth provided there is a finite dimensional factor algebra which is of non-polynomial growth.)

Given a finite dimensional k-algebra A, Hughes and Waschbüsch [HW, see also H] have introduced its repetitive algebra $\widehat{A}$. The repetitive algebras are always locally bounded. It is easy to see (see Assem-Skowroński [AS] and Pogorzaly-Skowroński [PG]) that a finite dimensional algebra A is gentle if and only if the repetitive algebra $\widehat{A}$ is special biserial. The construction of $\widehat{A}$ for a gentle algebra A will be reviewed below.

Let A be a finite dimensional algebra which is gentle. According to Nehring [N], the repetitive algebra $\widehat{A}$ is domestic provided the quiver $Q(A)$ of A has at most one cycle (by definition, the *number of cycles* of a connected quiver is the minimal number of arrows which have to be removed in order to obtain an oriented tree). The aim of the present note is to show the converse:

Theorem (1.1). *Let A be a finite dimensional algebra which is gentle and assume that $Q(A)$ has at least two cycles. Then $\widehat{A}$ is non-domestic (thus even of non-polynomial growth).*

Let us recall that Nehring [N] has shown that for a gentle algebra A with at least two cycles, the trivial extension $T(A)$ is non-domestic. Since the trivial extension $T(A)$ of A has $\widehat{A}$ as a Galois covering, our Theorem also yields a proof of this assertion.

The first part of the paper describes the relationship between the set $\mathcal{W}$ of words in the given quiver $Q(A)$ (here we consider all possible words, not taking into account the given set of relations), and a set $\widehat{\mathcal{W}}$ of words in the quiver of $\widehat{A}$ which do not involve certain paths. In fact, the set $\widehat{\mathcal{W}}$ is just the usual index set for the string modules of $\widehat{A}$ which are neither simple nor injective. We will construct a projection $\pi\colon \widehat{\mathcal{W}} \to \mathcal{W}$ and many sections σ^z and σ_z (with $z \in \mathbb{Z}$). Of particular interest is the following: words in $\mathcal{W}$ which are cyclic and of "cyclic defect zero" yield cyclic words in $\widehat{\mathcal{W}}$. All these considerations are well-known, but we hope that the formulations and proofs presented here will be found to be useful elsewhere. In particular, we introduce in Section 4 as a tool the "expansion" of a locally bounded gentle quiver, this is again a gentle quiver (but usually not locally bounded), and $\widehat{A}$ is obtained from it by adding further relations. The essential part of the paper are Sections 8, 9 and 10, where we consider the case of a gentle quiver with at least two cycles: there, we are going to construct sufficiently many cyclic words belonging to $\mathcal{W}$ which have cyclic defect zero. Using the maps σ^z and σ_z, these words yield corresponding cyclic words in $\widehat{\mathcal{W}}$.

The author is indebted to P. Dräxler and B. Roggon [Ro] for pointing out the problem. The result completes the classification of the socalled 'rpg-critical' algebras, see [Ro]. The author thanks J.Schröer for pointing out an inaccuracy in a previous version of the paper.

2. Words in a quiver.

Let Q be a quiver. The concatenation of paths in Q will be written by juxtaposition (the concatenation w_1w_2 of paths w_1, w_2 is defined only in case the path w_2 starts in the same vertex as w_1 ends). If v, w are paths in Q, then v is called a *subpath* of w provided there are paths v_1, v_2 in Q such that $w = v_1vv_2$.

We denote by Q^* the opposite quiver of Q. Let $\overline{Q} = Q \sqcup Q^*$ be the quiver obtained from the disjoint union of Q and Q^* by identifying the corresponding vertices of Q and Q^*, thus $\overline{Q}$ is obtained from Q by adding to every arrow $\alpha\colon a \to b$ a new arrow $\alpha^{-1}\colon b \to a$, and we write $(\alpha^{-1})^{-1} = \alpha$. The arrows of $\overline{Q}$ will be considered as letters in order to form words. The original arrows of Q will be said to be the *direct* letters, those of Q^* will be called the *inverse* letters.

We denote by $\mathcal{W} = \mathcal{W}(Q)$ the set of paths of length at least 1 in $\overline{Q}$ which do not contain any subpath of the form $\alpha\alpha^{-1}$ where α is a letter, the elements of $\mathcal{W}$ will be called *words* (using $\overline{Q}_1$ as letter set). (Note that we are excluding the paths of length zero; in this way, we deviate from a convention which otherwise seems to be useful, but which would force us to take extra care: in the sequel, paths of length zero usually will not be needed or will behave differently.) Let $w = l_1 \cdots l_n$ be a word with $l_i \in \overline{Q}_1$. Then $w^{-1} = l_n^{-1} \cdots l_1^{-1}$ is called its *inverse.* The word w is said to be *direct* or *inverse*, in case all the letters l_i are direct or inverse, respectively. A subpath of w of length at least 1 will be called a *subword.* If v, w are words, we will say that w does not *involve* v provided w has no subword equal to v or to v^{-1}.

Let $\mathcal{W}_c$ be the set of words such that the concatenation ww again is a word (this means, that the following two conditions are satisfied: first of all, the word w starts in the same vertex as it ends, and second: if l_1 is the first letter of w, and l_n the last one, then we must have $l_1 \neq l_n^{-1}$), the elements in $\mathcal{W}_c$ are called *cyclic words.* Given a cyclic word w, all the powers w^n with $n \geq 1$ are cyclic words. A cyclic word which is not a proper power of a cyclic word is said to be *primitive.* If $w = w_1w_2$ is a cyclic word, also w_2w_1 is a cyclic word; we say that it is obtained from w by *rotation.*

A cyclic word $w \in \mathcal{W}_c$ is said to be *elementary* provided $w = l_1 \cdots l_n$ with letters $l_i \in \overline{Q}_1$ such that the starting points of all the letters l_i are pairwise different. To say that a quiver Q has at least two cycles just means that there are at least two elementary cycles in $\mathcal{W}_c$ which cannot be obtained from each other by rotation and inversion.

Let Q be a quiver, let ρ be a set of paths of length at least 2 (such a set may be called a set of monomial relations). A word w in Q is said to be a *word in* (Q, ρ) provided w does not involve a path from ρ.

3. Monomial relations of length 2.

Let us assume that ρ is a set of paths in Q of length 2.

Given an element $w = l_1 \cdots l_n$ of $\mathcal{W}$, let $\delta'(w)$ be the number of indices $i \in \{1, \ldots, n-1\}$ such that l_i, l_{i+1} are direct letters and l_il_{i+1} belongs to ρ; similarly, let $\delta''(w)$ be the number of indices $i \in \{1, \ldots, n-1\}$ such that l_i, l_{i+1} are inverse letters and $l_{i+1}^{-1} l_i^{-1}$ belongs to ρ. We call $\delta(w) = \delta'(w) - \delta''(w)$ the *defect* of w. Given two elements $v, w \in \mathcal{W}$ such that vw is defined and belongs to $\mathcal{W}$, let

$$\delta(v, w) = \delta(vw) - \delta(v) - \delta(w),$$

(thus $\delta(v, w)$ detects relations involving the last letter of v and the first letter of w). For $w \in \mathcal{W}_c$, let

$$\delta_c(w) = \delta(w) + \delta(w, w),$$

we call $\delta_c(w)$ the *cyclic defect* of w. In particular, we will be interested in cyclic words w such that $\delta_c(w) = 0$ (the condition $\delta_c(w) = 0$ is sometimes called the *clock condition*: the number of "clockwise oriented" relations is equal to the number of "anti-clockwise oriented" relations).

Again, let us assume that ρ is a set of paths of length 2. Given any word w in $\mathcal{W}$, its *ρ-factorization* is $w = w_1 \cdots w_t$ provided the following conditions are satisfied:
(a) Any of the words w_i is either a direct or an inverse word in (Q, ρ),
(b) If w_i, w_{i+1} are both direct or both inverse, then $\delta(w_i, w_{i+1}) \neq 0$.

A path $\alpha_1 \cdots \alpha_n \colon a \to b$ in (Q, ρ) is said to be a *maximal* path in (Q, ρ) provided it cannot be prolonged on either side to a path in (Q, ρ) (thus, if β is an arrow starting in b, then $\alpha_n \beta$ belongs to ρ, if γ is an arrow ending in a, then $\gamma \alpha_1$ belongs to ρ; let us consider also the degenerate case $n = 0$ and therefore $a = b$: this path is maximal provided there is no arrow starting or ending in a).

Let (Q, ρ) be a gentle quiver (Q, ρ), and let w be a path of length at least one in (Q, ρ). We note the following: *There is at most one arrow α such that αw is a path in (Q, ρ). Similarly, there is at most one arrow β in Q such that $w\beta$ is a path in (Q, ρ).* We may call two arrows *path-equivalent* provided they belong to a path of (Q, ρ); note that this is an equivalence relation on Q_1.

As a consequence we see: *If (Q, ρ) is in addition locally bounded, then every path w of length at least one in (Q, ρ) is subpath of a unique maximal path of (Q, ρ).* Thus, for a locally bounded gentle quiver (Q, ρ), any equivalence class with respect to path-equivalence is just a set of arrows which combine to form a maximal path.

A connected gentle quiver (Q, ρ) is called *expanded*, provided Q_1 is not empty and for every arrow β, there are arrows α, γ such that the concatenations $\alpha\beta$ and $\beta\gamma$ do exist and both do not belong to ρ. Thus, a gentle quiver (Q, ρ) is expanded if and only if the vertices a are of the following two types: either a is a *crossing vertex*: there are precisely two arrows ending in a and two arrows starting in a, or else it is a *transition vertex*: there is just one arrow, say α ending in a and just one arrow, say β, starting in a, and $\alpha\beta$ does not belong to ρ. Of course, we see: *If (Q, ρ) is expanded, and w is any path of length at least one in (Q, ρ), then there is precisely one arrow α and precisely one arrow β such that $\alpha w \beta$ is a path in (Q, ρ).* Here we see: the equivalence classes with respect to path-equivalence are of two kinds: given a direct cyclic word w in $\mathcal{W}$ such that ww is a path in (Q, ρ), then the arrows occurring in w form such an equivalence classe; the remaining path-equivalence classes will be called threads: A *thread* is given by a set of pairwise different arrows α_z

indexed by the integers $z \in \mathbb{Z}$, such that the arrows α_z, α_{z+1} are composable and all the words $\alpha_z \cdots \alpha_{z'}$ with $z \leq z'$ are words in (Q, ρ).

Remark (3.1). A finite gentle quiver (Q, ρ) is locally bounded if and only if any arrow belongs to a maximal path, if and only if there is no cyclic path w in Q of length at least one such that w^2 is a word in (Q, ρ).

4. The expansion $\mathbb{Z}(Q, \rho)$ of a locally bounded gentle quiver (Q, ρ).

Assume now that (Q, ρ) is a locally bounded gentle quiver.

We form $\mathbb{Z}(Q, \rho) = (\widehat{Q}, \mathbb{Z}\rho)$ as follows: We take countably many copies of Q, indexed over the integers, thus for every vertex a of Q, there are the vertices $a[z]$ with $z \in \mathbb{Z}$, and for every arrow $\alpha : a \to b$, there are arrows $\alpha[z] : a[z] \to b[z]$. In addition, for every maximal path $p\colon a \to b$ in (Q, ρ), there are arrows $p'[z]\colon b[z{+}1] \to a[z]$ for all $z \in \mathbb{Z}$, they may be called the *connecting arrows.* In this way, we have defined the quiver $\widehat{Q}$ (in spite of the notation $\widehat{Q}$, this quiver not only depends on the quiver Q, but also on the set ρ). The relation set $\mathbb{Z}\rho$ contains all the paths $\alpha[z]\beta[z]$ with $\alpha\beta \in \rho$ and $z \in \mathbb{Z}$, and the following additional paths of length 2: Let $p = \alpha_1 \cdots \alpha_n\colon a \to b$ be a maximal path in (Q, ρ). If $\beta \neq \alpha_1$ is an arrow starting in a, then $p'[z]\beta[z]$ is supposed to belong to $\mathbb{Z}\rho$; if $\gamma \neq \alpha_n$ is an arrow ending in b, then $\gamma[z{+}1]p'[z]$ is supposed to belong to $\mathbb{Z}\rho$, for all $z \in \mathbb{Z}$. Finally, if q is a maximal path ending in a, then also the elements $p'[z]q'[z{-}1]$ are supposed to belong to $\mathbb{Z}\rho$.

We denote by ν the shift map given by $a[z] \mapsto a[z{+}1]$ and $\alpha[z] \mapsto \alpha[z{+}1]$. Of course, ν is an automorphism of $\mathbb{Z}(Q, \rho)$. Note that there is a canonical embedding $\iota\colon (Q, \rho) \to \mathbb{Z}(Q, \rho)$ given by $\iota(a) = a[0]$ and $\iota(\alpha) = \alpha[0]$ for any vertex a and any arrow α of Q. Whenever it will be convenient, we will identify (Q, ρ) with the image of $\iota\colon (Q, \rho) \to \mathbb{Z}(Q, \rho)$.

Proposition (4.1). *Let (Q, ρ) be a locally bounded gentle quiver. Then $\mathbb{Z}(Q, \rho)$ is an expanded gentle quiver. If $p = \alpha_1 \cdots \alpha_n$ is a maximal path in (Q, ρ), then the arrows $\alpha_{z(n+1)+i} = \alpha_i[-z]$ and $\alpha_{z(n+1)} = p'[-z]$ form a thread. All the threads are obtained in this way.*

Proof: We have to analyse the starting points and end points of the arrows $p'[z]$ where $p = \alpha_1 \cdots \alpha_n\colon a \to b$ is a maximal path in (Q, ρ). Using duality and the shift ν, it is sufficient to look at the end point of an arrow of the form $p'[0]\colon b[1] \to a[0]$.

First, let us assume that a is a source of Q, thus no connecting arrow will start in a. In case α_1 is the only arrow of Q starting in a, the path p is the only maximal path of (Q, ρ) starting in a, thus p' is the only arrow of $\widehat{Q}$ ending in a, and the path $p'\alpha_1$ does not belong to $\mathbb{Z}\rho$. Thus, the vertex a is a transition point. In case there is a second arrow $\beta \neq \alpha_1$ starting in a, then $p'[0]\beta[0]$ is supposed to belong to $\mathbb{Z}\rho$. There is a maximal path q in (Q, ρ) starting with the arrow β, thus there is the additional arrow $q'[0]$ in $\widehat{Q}$ ending in $a[0]$, and $q'[0]\alpha_1[0]$ belongs to $\mathbb{Z}\rho$. There are the two arrows β, α_1 of Q starting in a, thus there can be only two maximal paths of (Q, ρ) which start in a, and consequently, there are just two arrows of $\widehat{Q}$ which end in $a[0]$, and also precisely two relations in $\mathbb{Z}\rho$ which pass through $a[0]$, namely $p'[0]\beta[0]$ and $q'[0]\alpha_1[0]$. This shows that $a[0]$ is a crossing vertex.

Next assume that a is not a source in Q, thus there is at least one arrow, say γ of Q ending in a. Since (Q,ρ) is gentle and the path $p=\alpha_1\cdots\alpha_n$ starting in a is maximal, it follows that γ is the only arrow ending in a and that $\gamma\alpha_1$ belongs to ρ.

In case there is an arrow $\beta\neq\alpha_1$ starting in a, then three conclusions are of interest: by definition of $\mathbb{Z}\rho$, the path $p'[0]\beta[0]$ belongs to $\mathbb{Z}\rho$; second, p is the only maximal path of (Q,ρ) starting in a (any path with first letter β can be prolonged by multiplying from the left with γ); third, no maximal path of (Q,ρ) has γ as last arrow (since any path with last arrow γ can be prolonged by multiplying on the right with β), thus in $\mathbb{Z}(Q,\rho)$, no connecting arrow will start in a. We see that $a[0]$ is a crossing vertex with arrows $\gamma[0],p'[0]$ coming in, with arrows $\beta[0],\alpha_1[0]$ going out, and with relations $\gamma[0]\alpha_1[0],p'[0]\beta[0]$.

Finally, assume that α_1 is the only arrow starting in a. Since $\gamma\alpha_1$ belongs to ρ, there is a maximal path q in (Q,ρ) with last arrow being γ, thus in $\mathbb{Z}(Q,\rho)$ there is the connecting arrow $q'[-1]$ starting in $a[0]$, and $p'[0]q'[-1]$ belongs to $\mathbb{Z}\rho$. Again, we obtain a crossing vertex with arrows $\gamma[0],p'[0]$ coming in, with arrows $\alpha_1[0],q'[-1]$ going out and with relations $\gamma[0]\alpha_1[0],p'[0]q'[-1]$.

It follows that the maximal path p of (Q,ρ) yields a path-equivalence class of $\mathbb{Z}(Q,\rho)$ which consists of the arrows $\alpha[z]$, where α occurs in p, and the arrows $p'[z]$, and this path-equivalence class is a thread. Since (Q,ρ) is locally bounded, every arrow of Q belongs to a maximal path for (Q,ρ), thus all the arrows of $\mathbb{Z}(Q,\rho)$ belong to one of these threads. This completes the proof.

Let us formulate some consequences: Let (Q,ρ) be a locally bounded gentle quiver.

(a) The inclusion $\iota\colon(Q,\rho)\to\mathbb{Z}(Q,\rho)$ yields a bijection between the path-equivalence classes of (Q,ρ) and of $\mathbb{Z}(Q,\rho)$.

(b) The path-equivalence classes of $\mathbb{Z}(Q,\rho)$ are closed under ν.

A path in $\mathbb{Z}(Q,\rho)$ of the form $w_2[z+1]p'[z]w_1[z]$, where $p=w_1w_2$ is a maximal path of (Q,ρ) and $z\in\mathbb{Z}$, is said to be a *full path.* A full path ending in x starts in $\nu(x)$; note that in case Q has no cycles, the full paths are the only paths with this property. A path in $\mathbb{Z}(Q,\rho)$ of length at least 1 which does not contain a full path as a subpath will be said to be a *short path.* A word W in the quiver $\widehat{Q}$ will be said to be *short* provided w or w^{-1} is a short path.

(c) If w is a full path in $\mathbb{Z}(Q,\rho)$, then $\nu(w)w$ is a path in $\mathbb{Z}(Q,\rho)$.

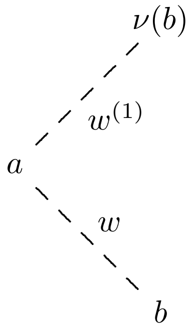
(d) Let $w\colon a[z]\to b[z']$ be a short path. Then either $z'=z$ or $z'=z-1$. A short path starting in $a[z]$ and ending in $b[z']$ will be said to be *even*, provided $z'=z$ and *odd*, provided $z'=z-1$.

(e) If w is a short path in $\mathbb{Z}(Q,\rho)$, then there exists a uniquely determined path w' such that $w'w$ is a full path. This path w' is a short path and it is also the uniquely defined path so that $\nu(w)w'$ is a full path.

5. The set $\widehat{\mathcal{W}}$.

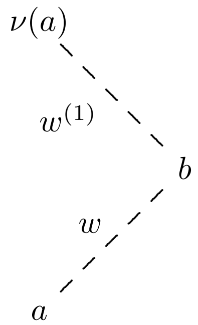
Let (Q,ρ) be a gentle quiver. We denote by $\widehat{\mathcal{W}}$ the set of words in $\mathcal{W}(\mathbb{Z}(Q,\rho))$ which do not involve any full path. Thus, the elements of $\widehat{\mathcal{W}}$ are those words $w\in\mathcal{W}(\widehat{Q})$ which neither involve paths form $\mathbb{Z}\rho$ nor full paths. If $w\in\widehat{\mathcal{W}}$, then its *standard factorization* is $w=w_1\cdots w_t$ where the subwords w_i are short paths and inverses of short paths; of every consecutive pair w_i, w_{i+1}, one of the words is a short path, the other one is the inverse of a short path.

For any short word $w \in \widehat{\mathcal{W}}$ and all $t \in \mathbb{Z}$, let us define a word $w^{(t)}$. First, consider a direct word $w\colon a \to b$ in $\widehat{\mathcal{W}}$. There is the word $w'\colon \nu(b) \to a$, and we consider the inverse $w^{(1)} = (w')^{-1}\colon a \to \nu(b)$.



(In these pictures, we use the following conventions: on the one hand, words are to be read from left to right; on the other hand, the dashed lines are paths or inverses of paths, with all the arrows pointing downwards. Observe that the convention for reading letters is opposite to the one used in [R].)

If w is an inverse word in $\widehat{\mathcal{W}}$, then we take the word $w^{(1)} = (w^{-1})'\colon \nu(a) \to b$.



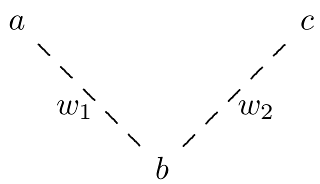
Always, we define $w^{(t)}$ for all integers t as follows:

$$w^{(2z)} = \nu^z(w), \quad \text{and} \quad w^{(2z+1)} = \nu^z(w^{(1)}).$$

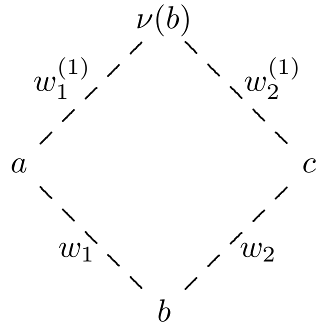
Lemma (5.1). *Let w be a word in $\widehat{\mathcal{W}}$ with standard factorization $w = w_1 \cdots w_n$. Then also $w_1^{(t)} \cdots w_n^{(t)}$, for any $t \in \mathbb{Z}$, belongs to $\widehat{\mathcal{W}}$ and this factorization is the standard one.*

Proof: The shifts ν^z with $z \in \mathbb{Z}$ are automorphisms of $\mathbb{Z}(Q,\rho)$; they map $\widehat{\mathcal{W}}$ to itself and preserve the standard factorizations. Thus, it is sufficient to assume that $t = 1$. Clearly, we only have to consider the case $n = 2$. One of the words w_1, w_2 is direct, the other is inverse, thus we have to consider two cases.

Case 1: Let $w_1\colon a \to b$ be direct, and $w_2\colon b \to c$ inverse.

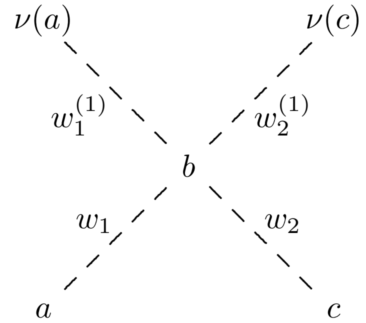


For $t = 1$, we obtain $w_1^{(1)} w_2^{(1)}$



Note that the two paths $(w_1^{(1)})^{-1} \colon \nu(b) \to a$ and $w_2^{(1)} \colon \nu(b) \to c$ both start in $\nu(b)$, but that they have different first letters: Since $w_1 w_2$ is a word, the paths w_1, w_2^{-1} have different last letters, thus they belong to different threads. In one of these threads, we can form the word $\nu(w_1)(w_1^{(1)})^{-1}$, in the other the word $\nu(w_2^{-1})w_2^{(1)}$. If the first letters of $(w_1^{(1)})^{-1}$ and of $w_2^{(1)}$ would coincide, also the last letters of $\nu(w_1)$ and $\nu(w_2^{-1})$ would coincide, impossible.

Case 2: Let $w_1 \colon a \to b$ be inverse, and $w_2 \colon b \to c$ direct. For $t = 1$, we have to consider $w_1^{(1)} w_2^{(1)}$. There are the paths $w_1^{(1)} \colon \nu(a) \to b$ and $(w_2^{(1)})^{-1} \colon \nu(c) \to b$.



Since the first letters of the paths w_1^{-1} and w_2 are different, the last letters of $w_1^{(1)}$ and $(w_2^{(1)})^{-1}$ have to be different, using again the argument that we deal with paths in different threads. Thus, we can form the word $w_1^{(1)} w_2^{(2)}$ in $\widehat{\mathcal{W}}$, and the given factorization is its standard factorization.

Lemma (5.2). *Let w be a short word in $\widehat{\mathcal{W}}$. Then there is a unique integer t such that $w^{(t)}$ starts and ends in vertices which belong to the image of $\iota \colon (Q, \rho) \to \mathbb{Z}(Q, \rho)$; all the arrows involved in $w^{(t)}$ belong to the image of ι.*

Proof: First, consider the case where w is direct, thus w is a short path. Let us assume that w starts in $a[z]$ and ends in $b[z']$, where a, b are vertices of Q and $z, z' \in \mathbb{Z}$. Then w and $w^{(1)}$ are the only words of the form $w^{(d)}$ with $d \in \mathbb{Z}$ which start in $a[z]$, thus $w^{(-2z)}$ and $w^{(-2z+1)}$ are the only words of the form $w^{(d)}$ which start in $a = a[0]$. There are two possibilities for z', namely $z' = z$ and $z' = z - 1$. In case $z' = z$, the path $w^{(-2z)}$ starts in a and ends in b, whereas the word $w^{(-2z+1)}$ ends in $b[1]$. In the second case, the word $w^{(-2z+1)}$ starts in a and ends in b, whereas the path $w^{(-2z)}$ ends in $b[-1]$.

Second, let w be inverse, thus w^{-1} is a short path. By the previous considerations, there is one and only one $t \in \mathbb{Z}$ such that $(w^{-1})^{(t)}$ starts and ends in vertices belonging to the image of ι. Then $w^{(t)} = ((w^{-1})^{(t)})^{-1}$ has the same property. This completes the proof.

Given a short word $w \in \widehat{\mathcal{W}}$, let $\pi(w)$ be the unique word of the form $w^{(t)}$ which belongs to $\mathcal{W}$. If w starts in $a[z]$, where a is a vertex of Q and $z \in \mathbb{Z}$, then

$$\pi(w) = \begin{cases} w^{(-2z)} & \text{provided } w \text{ is even,} \\ w^{(-2z+1)} & \text{provided } w \text{ is odd and direct,} \\ w^{(-2z-1)} & \text{provided } w \text{ is odd and inverse.} \end{cases}$$

In particular, for w even, we have $\pi(w) = w^{(-2z)} = \nu^{-z}(w)$.

6. The projection $\pi\colon \widehat{\mathcal{W}} \to \mathcal{W}$ and the sections σ^z and σ_z.

Given a short word $w \in \widehat{\mathcal{W}}$, we have defined $\pi(w) \in \mathcal{W}$. We are going to extend this definition in order to obtain a map

$$\pi\colon \widehat{\mathcal{W}} \to \mathcal{W}$$

as follows: If w belongs to $\widehat{\mathcal{W}}$ and $w = w_1 \cdots w_n$ is its standard factorization, then we want to define

$$\pi(w) = \pi(w_1) \cdots \pi(w_n).$$

We need the following lemma:

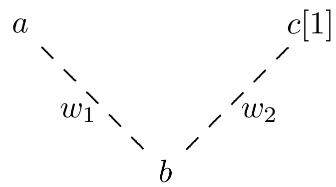
Lemma (6.1). *Assume that $w \in \widehat{\mathcal{W}}$ has the standard factorization $w = w_1 \cdots w_n$, then $\pi(w_1) \cdots \pi(w_n)$ is a word in $\mathcal{W}$ and this factorization is the ρ-factorization.*

Proof: It is sufficient to consider the case $n = 2$, thus there is given a standard factorization $w = w_1 w_2$ in $\widehat{\mathcal{W}}$. We have to distinguish whether these words w_1, w_2 are even or odd.

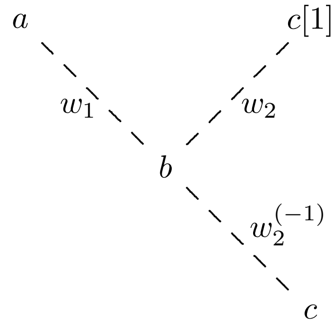
It is sufficient to consider the case of $\pi(w_1) = w_1$. Namely, let $\pi(w_1) = w_1^{(t)}$. Then we may consider $w_1^{(t)} w_2^{(t)}$ instead of $w_1 w_2$. According to Lemma (5.1), this is again the standard factorization of a word in $\widehat{\mathcal{W}}$ and we have $\pi(w_i) = \pi(w_i^{(t)})$.

Thus, we assume that w_1 is in the image of ι, say $w_1\colon a \to b$, where a, b are vertices of Q. If w_2 is even, then w_2 also belongs to the image of ι, thus $\pi(w_2) = w_2$ and $w_1 w_2$ is not only the standard factorization of a word in $\widehat{\mathcal{W}}$, but also its ρ-factorization.

Now assume that w_2 is odd. First, let us assume that w_1 is direct, thus w_2 is inverse. This is the given situation (where a, b, c are vertices of Q):

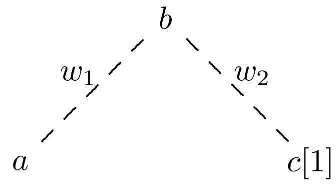


Note that $\pi(w_2) = w_2^{(-1)}\colon b \to c$, thus we deal with the following words:

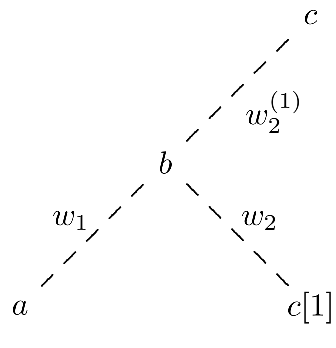


It remains to be seen that the concatenation of the last arrow of w_1 and the first arrow of $w_2^{(-1)}$ belongs to ρ. On the one hand, we know that the concatenation of the last arrow of w_2^{-1} and the first arrow of $w_2^{(-1)}$ does not belong to $\mathbb{Z}\rho$, since the path $w_2^{-1}w_2^{(-1)}$ is inside a thread. On the other hand, the last arrows of w_1 and w_2^{-1} are different, since w_1w_2 is a word. Thus, $w_1w_2^{(-1)}$ cannot lie inside a thread. This shows that $w_1w_2^{(-1)}$ is a ρ-factorization.

Second, let us assume that w_1 is inverse, thus w_2 is direct, thus we start with the following words (again, a, b, c are vertices in Q):



We have $\pi(w_2) = w_2^{(1)}\colon b \to c$, thus we deal with the following words:



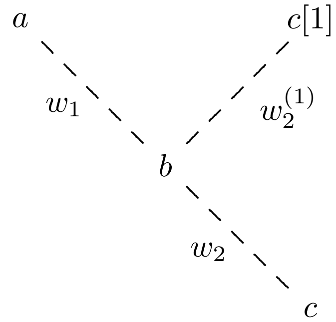
This time, we have to show that the concatenation of the last arrow of $(w_2^{(1)})^{-1}$ and the first arrow of w_1^{-1} belongs to ρ. On the one hand, we know that the concatenation of the last arrow of $(w_2^{(1)})^{-1}$ and the first arrow of w_2 does not belong to $\mathbb{Z}\rho$, since the path $(w_2^{(1)})^{-1}w_2$ is inside a thread. On the other hand, the first arrows of w_1^{-1} and w_2 are different, since w_1w_2 is a word. Thus, $w_1w_2^{(1)}$ cannot lie inside a thread. This shows that $w_1w_2^{(1)}$ is a ρ-factorization. In this way, we complete the proof.

The map $\pi\colon \widehat{\mathcal{W}} \to \mathcal{W}$ is rather useful. We are going to determine its fibers. First, let us reverse the considerations of Lemma (6.1).

Lemma (6.2). *Let $w \in \mathcal{W}$ be a word, with ρ-factorization $w = w_1w_2$. Given an integer t, let $t' = t + \delta(w_1w_2)$. Then $w_1^{(t)}w_2^{(t')}$ is a word in $\widehat{\mathcal{W}}$ and the factorization is the standard one.*

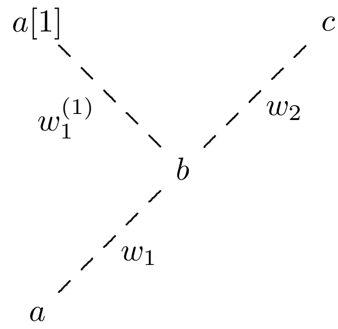
Proof: Using the shifts ν^z with $z \in \mathbb{Z}$, we see that it is sufficient to assume that $t = 0$ or $t = 1$. The words w_1, w_2 may be direct or inverse. If one of the words w_1, w_2 is direct, the other one inverse, then we are in the situation considered in Lemma (5.1), thus we can assume that both w_1, w_2 are direct, or both are inverse.

Let both $w_1\colon a \to b$ and $w_2\colon b \to c$ be direct. Since w_1w_2 is the ρ-factorization, we know that the concatenation of the last letter of w_1 and the first letter of w_2 belongs to ρ. In particular, we have $\delta(w_1w_2) = 1$. In order to settle the case $t = 0$, we have to show that $w_1^{(0)}w_2^{(1)}$ is in $\widehat{\mathcal{W}}$ and that this factorization is the standard one.



Note that the last letters of w_1 and $(w_2^{(1)})^{-1}$ are different: after all, the concatenation of the last letter of w_1 and the first letter of w_2 belongs to ρ, whereas $(w_2^{(1)})^{-1}w_2$ belongs to a thread. We obtain the case $t = 1$ from the case $t = 0$ using Lemma (5.1).

Next, let us assume that both words $w_1\colon a \to b$ and $w_2\colon b \to c$ are inverse. This time, $\delta(w_1w_2) = -1$. We first treat the case $t = 1$. Thus, we show that $w_1(1)w_2(0)$ is in $\widehat{\mathcal{W}}$ and that this factorization is the standard one.



Note that the last letters of $w_1^{(1)}$ and w_2^{-1} are different: the concatenation of the last letter of w_2^{-1} and the first letter of w_1^{-1} belongs to ρ, whereas $w_1^{(1)}w_1^{-1}$ belongs to a thread.

We obtain the case of arbitrary t from the case $t = 1$, using Lemma (5.1). This completes the proof.

Lemma (6.3). *Let $z \in \mathbb{Z}$. Let w be a word in $\mathcal{W}$ starting in a.*

There is a unique word $\sigma^z(w) \in \widehat{\mathcal{W}}$ with the following properties: $\sigma^z(w)$ starts in $a[z]$, we have $\pi\sigma^z(w) = w$ and the first letter of $\sigma^z(w)$ is direct.

There is a unique word $\sigma_z(w) \in \widehat{\mathcal{W}}$ with the following properties: $\sigma_z(w)$ starts in $a[z]$, we have $\pi\sigma_z(w) = w$ and the first letter of $\sigma_z(w)$ is inverse.

Proof. Let us assume that we deal with the word $w \in \mathcal{W}$ with canonical factorization $w = w_1 \cdots w_n$ in $\mathcal{W}(Q,\rho)$. Let

$$\epsilon^z(w_1) = \begin{cases} 2z & \text{if } w_1 \text{ is direct,} \\ 2z-1 & \text{if } w_1 \text{ is inverse.} \end{cases}$$

Similarly, let

$$\epsilon_z(w_1) = \begin{cases} 2z+1 & \text{if } w_1 \text{ is direct,} \\ 2z & \text{if } w_1 \text{ is inverse.} \end{cases}$$

Using these numbers $\epsilon^z(w_1)$ and $\epsilon_z(w_1)$, we define

$$\eta^z(t) = \epsilon^z(w_1) + \delta(w_1 \cdots w_t),$$
$$\eta_z(t) = \epsilon_z(w_1) + \delta(w_1 \cdots w_t).$$

We define $\sigma^z(w)$ and $\sigma_z(w)$ as follows:

$$\sigma^z(w) = w_1^{(\eta^z(1))} \cdots w_n^{(\eta^z(n))},$$
$$\sigma_z(w) = w_1^{(\eta_z(1))} \cdots w_n^{(\eta_z(n))}.$$

Let us show that these words $\sigma^z(w)$ and $\sigma_z(w)$ are defined in $\widehat{\mathcal{W}}$ and that the given factorization is the standard one.

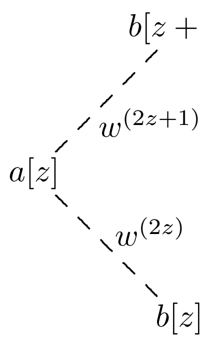
First, let us consider the case $n = 1$, thus we deal with a word $w = w_1$ in $\mathcal{W}(Q,\rho)$ which is direct or inverse. By definition,

$$\sigma^z(w) = w^{(\epsilon^z(w))} \quad \text{and} \quad \sigma_z(w) = w^{(\epsilon_z(w))}.$$

If w is direct, then

$$\sigma^z(w) = w^{(2z)} = \nu^z(w) \quad \text{and} \quad \sigma_z(w) = w^{(2z+1)} = \nu^z(w^{(1)}),$$

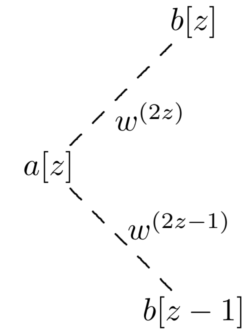
thus we are in the following situation:



If w is inverse, then

$$\sigma^z(w) = w^{(2z-1)} \quad \text{and} \quad \sigma_z(w) = w^{(2z)},$$

thus we deal with the following words:



It remains to observe that our recipes for $\sigma^z(w)$ and $\sigma_z(w)$ are just of the following kind: We have

$$\eta^z(i+1) = \eta^z(i) + \delta(w_i w_{i+1}) \quad \text{and} \quad \eta_z(i+1) = \eta_z(i) + \delta(w_i w_{i+1}),$$

thus Lemma (6.2) assures that all the concatenations are defined and yield standard factorizations.

Corollary (6.4). *For every word $w \in \mathcal{W}$, we have*

$$\pi^{-1}(w) = \{\sigma^z(w), \sigma_z(w) \mid z \in \mathbb{Z}\}.$$

Let α, β be arrows in Q. We denote by $\mathcal{W}(\alpha^{-1}, \beta)$ the set of words in $\mathcal{W}$ such that the first letter is α^{-1}, the last one is β. We denote by $\mathcal{W}(\alpha^{-1}, \beta)_d$ the set of words in $\mathcal{W}(\alpha^{-1}, \beta)$ of defect d. Similarly, let $\widehat{\mathcal{W}}(\alpha[z]^{-1}, \beta[z'])$ be the set of words in $\widehat{\mathcal{W}}$ with first letter $\alpha[z]^{-1}$ and last letter $\beta[z']$.

Corollary (6.5). *Let α, β be arrows of Q, let $d \in \mathbb{Z}$. Then*

$$\sigma_z(\mathcal{W}(\alpha^{-1}, \beta)_{2d}) = \widehat{\mathcal{W}}(\alpha[z]^{-1}, \beta[z+d]).$$

Let us now assume that α, β are different arrows of Q with the same end point and that $d = 0$. In this case the sets $\mathcal{W}(\alpha^{-1}, \beta)_0$ and $\widehat{\mathcal{W}}(\alpha[z]^{-1}, \beta[z])$ with $z \in \mathbb{Z}$ are semigroups with respect to concatenation.

Proposition (6.6). *Let $\alpha \neq \beta$ be arrows ending in the same vertex. Then σ_z is a semigroup isomorphism*

$$\sigma_z \colon \mathcal{W}(\alpha^{-1}, \beta)_0 \longrightarrow \widehat{\mathcal{W}}(\alpha[z]^{-1}, \beta[z])$$

with inverse π.

Proof: We know that the map σ_z is injective and that $\pi\sigma_z$ is the identity. The previous assertions yield that the image of $\mathcal{W}(\alpha^{-1},\beta)_0$ under σ_z is just $\widehat{\mathcal{W}}(\alpha[z]^{-1},\beta[z])$. It remains to observe that σ_z is compatible with concantenation, thus it is a semigroup isomorphism.

7. The repetitive algebra $\widehat{A}$ and its factor algebra $\overline{\widehat{A}}$.

Let (Q,ρ) be a gentle quiver and $A = kQ/\langle\rho\rangle$ the corresponding gentle algebra. Let $\widehat{A}$ be its repetitive algebra. As we know, $\widehat{A}$ is special biserial, thus there are two kinds of indecomposable $\widehat{A}$-modules, namely strings and bands. The set $\widehat{\mathcal{W}}$ is the index set for describing the string modules which are neither simple nor injective. Let us outline more details: The quiver $Q(\widehat{A})$ of the repetitive algebra $\widehat{A}$ is the quiver $\widehat{Q}$. As set $\widehat{\rho}$ of relations one takes the union of three sets: namely the set $\mathbb{Z}\rho$, second the set of all differences $w-w'$, where w,w' are full paths for $\mathbb{Z}(Q,\rho)$ with same starting point and same end point, and third, the set of all paths for $\mathbb{Z}(Q,\rho)$ which properly contain a full path.

Let us denote by $\overline{\widehat{\rho}}$ the union of the sets $\mathbb{Z}\rho$ and the set of all full paths for $\mathbb{Z}(Q,\rho)$. Note that $\widehat{A}$ is a self-injective algebra (this means that the indecomposable projective modules and the indecomposable injective modules coincide), and the factor algebra $\overline{\widehat{A}}$ of $\widehat{A}$ modulo its socle is given by the quiver $\widehat{Q}$ and the relation set $\overline{\widehat{\rho}}$. The indecomposable $\overline{\widehat{A}}$-modules are just the non-projective indecomposable $\widehat{A}$-modules. Thus we see that $\widehat{\mathcal{W}}$ describes the string modules for $\overline{\widehat{A}}$ which are not simple.

8. The x-y-Lemma.

Here we deal with an arbitrary quiver Q; there is no reference to any set of relations.

Given a vertex a in the quiver Q, we denote by $\mathcal{W}(a)$ the set of words starting in a and ending in a. Given a word $w = l_1\cdots l_n$ of length $n \ge 1$, we denote by $\phi(w) = l_1$ its first letter.

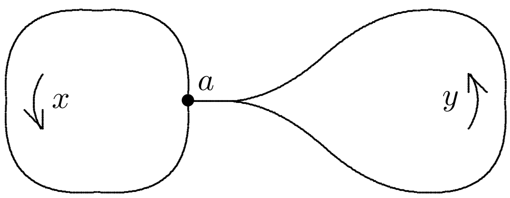
Lemma (8.1). *Let Q be a quiver with at least two cycles. Then there are a vertex a of Q and words $x,y \in \mathcal{W}(a)$ such that the letters*

$$l_1 = \phi(x),\ l_2 = \phi(x^{-1}),\ l_3 = \phi(y),\ l_4 = \phi(y^{-1}).$$

satisfy the following two properties:

(1) *The letters l_1,l_2,l_3 are pairwise different, whereas $l_3 = l_4$.*

(2) *The letters l_3 and l_3^{-1} do not occur in x, and they occur just once in y.*



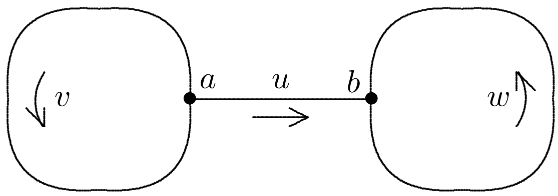
Proof: Since we assume that Q has at least two cycles, there are two elementary cycles v,w in Q such that w cannot be obtained from v by rotation and inversion.

We distinguish two cases:

Case 1. *There are two cycles v, w in Q which have no vertex in common.* We can assume that both cycles v, w are elementary. Since Q is connected, there exists a path from some vertex involved in v to a vertex involved in w and we choose such a path u of shortest possible length. Then clearly u belongs to $\mathcal{W}$. Let a be the starting point of u and b its end point. We can assume that v starts (and ends) in a and that w starts (and ends) in b. The minimality assures that all the paths

$$v^{\epsilon(v)}uw^{\epsilon(w)}$$

with $\epsilon(v), \epsilon(w) \in \{1, -1\}$ belong to $\mathcal{W}$.



Let

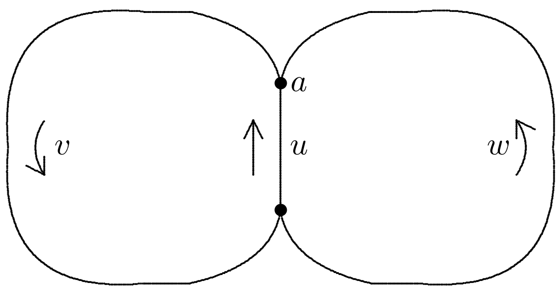
$$x = v \quad \text{and} \quad y = uwu^{-1}.$$

Since u is of length at least one, we have $\phi(y) = \phi(u) = \phi(y^{-1})$, and the three letters $\phi(x), \phi(x^{-1})$ and $\phi(y)$ are pairwise different. Also, the letter $\phi(u)$ will not occur as a letter in x, and it will occur only once as a letter of y.

Case 2. *Any two cycles v, w in Q have at least one vertex in common.* Let us choose two elementary paths v', w'. Let u be a path which occurs both in v' and w' and which is maximal with this property (note that u may have length zero). We may assume that v', w' have no other vertices in common. Thus: There are paths u, v, w such that all the paths

$$uv,\ vu,\ ,u^{-1}w,\ wu^{-1}, vw,\ wv$$

belong to $\mathcal{W}$; in case u has length zero, we will need that also the paths wv^{-1} and $v^{-1}w$ belong to $\mathcal{W}$. We deal with the following situation:



where u may have length zero. Let

$$x = vu \quad \text{and} \quad y = w^{-1}v^{-1}u^{-1}w.$$

Since w is of length at least one, we see that $\phi(y) = \phi(w^{-1}) = \phi(y^{-1})$. Also, the three letters $\phi(x), \phi(x^{-1})$ and $\phi(y)$ are pairwise different. In addition, the letter $\phi(u)$ will not occur as a letter in x, and it will occur only once as a letter of y.

This completes the proof.

9. Construction of cyclic words of defect zero.

Here and in the next section, we assume that a set ρ of paths of length 2 are given. This allows us to consider the cyclic defect of any cyclic word in Q. We assume that Q has at least two cycles, thus we know that there are elements x, y as constructed in the previous section.

Lemma (9.1). *The words*

$$x^n y x^{-n} y^{-1}$$

are primitive cyclic words in Q of zero cyclic defect.

Proof: The defect is calculated as follows:

$$\begin{aligned}\delta_c(x^n y x^{-n} y^{-1}) = \delta(x^n) + \delta(y) + \delta(x^{-n}) + \delta(y^{-1})\\ + \delta(x, y) + \delta(y, x^{-1}) + \delta(x^{-1}, y^{-1}) + \delta(y^{-1}, x)\end{aligned}$$

and there is pairwise cancellation: the rule $\delta(w^{-1}) = -\delta(w)$ with arbitrary w can be applied to the first four terms; for the last four terms, we use the rule $\delta(w^{-1}, v^{-1}) = -\delta(v, w)$ for words v, w with concatenation vw, and in addition that the first letter of y coincides with the first letter of y^{-1}, so that $\delta(x, y) = \delta(x, y^{-1}) = -\delta(y, x^{-1})$ and $\delta(x^{-1}, y^{-1}) = \delta(x^{-1}, y) = -\delta(y^{-1}, x)$.

The first letter of y appears just twice in the word $x^n y x^{-n} y^{-1}$. The first time, its predecessor is the last letter of x, the second time its predecessor is the last letter of x^{-1}. Since the last letters of x and x^{-1} are different, we see that we deal with a primitive cyclic word.

Lemma (9.2). *The words*

$$(xy)^n (x^{-1} y^{-1})^n$$

are primitive cyclic words of zero cyclic defect.

The calculation of the defect is as above. In order to see that all the words are primitive, consider again the first letter of y. It occurs $2n$ times. The first n times, its predecessor is the last letter of x, the last n times, its predecessor is the last letter of x^{-1}. Again we use that the last letters of x and x^{-1} are different.

Remark. All the words constructed above contain both direct and inverse letters (since both word x and x^{-1} are of length at least 1 and are subwords). Thus, we may rotate the words in order to obtain a cyclic word starting with an inverse letter α^{-1} and ending in a direct letter β (and $\alpha \neq \beta$ have the same end point). Of course, if a word w starts in the inverse letter α^{-1} and ends in the direct letter β, where $\alpha \neq \beta$ have the same end point, then w is a cyclic word and its cyclic defect coincides with its defect.

Thus we see that we have constructed many primitive elements belonging to $\mathcal{W}(\alpha^{-1},\beta)_0$. Proposition (6.6) shows that this completes the proof of Theorem (1.1).

10. Cyclic words of defect zero with subwords of large defect.

Lemma (10.1). *At least one of the words $x, xy, x^{-1}y$ has non-zero cyclic defect.*

Proof: As we have mentioned above, we can assume that there are given two paths x, y from a to a, with the following properties: let l_1^{-1} be the first letter and l_2 the last letter of x, let l_3^{-1} be the first letter and l_4 the last letter of y. Then $l_3 = l_4$ and the letters l_1, l_2, l_3 are pairwise different, of course with end point a. Let us assume that x has cyclic defect zero, thus $\delta(x) = -\delta(x,x)$.

We note the following: Since the letters l_1, l_2, l_3 are pairwise different and have the same end point, we can consider the three paths $l_1l_2^{-1}, l_2l_3^{-1}, l_3l_1^{-1}$. Since we deal with a gentle algebra, we see that two of these paths have defect zero, the remaining one has defect 1 or -1.

Consider the two cyclic words xy and $x^{-1}y$. Note that we have $\delta(x^{-1},y) = -\delta(y^{-1},x) = -\delta(y,x)$ since the last letters of y and y^{-1} are equal. As a consequence:

$$\begin{aligned}
\delta_c(xy) &= -\delta(x,x) + \delta(y) + \delta(x,y) + \delta(y,x),\\
\delta_c(x^{-1}y) &= -\delta(x^{-1},x^{-1}) + \delta(y) + \delta(x^{-1},y) + \delta(y,x^{-1})\\
&= +\delta(x,x) + \delta(y) - \delta(y,x) - \delta(x,y).
\end{aligned}$$

Thus we see that

$$\delta_c(xy) - \delta_c(x^{-1}y) = 2\left(\delta(x^{-1},x^{-1}) + \delta(x,y) + \delta(y,x)\right)$$

is either 2 or -2; here we use the remark above where l_1, l_2, l_3 are the last letters of x^{-1}, x, y, respectively. It follows that at least one of the words xy and $x^{-1}y$ has non-zero cyclic defect. This shows the assertion.

Corollary (10.2). *Let Q be a gentle quiver with at least two cycles. Then there are primitive cyclic words in $\mathcal{W}$ of zero cyclic defect which have subwords of arbitrarily large defect.*

Proof: In case x has non-zero cyclic defect, consider the words $x^nyx^{-n}y^{-1}$. Otherwise, we can assume that xy has non-zero cyclic defect (replacing, if necessary, x by x^{-1}), and then we consider the words $(xy)^n(x^{-1}y^{-1})^n$. In both cases, we obtain subwords w' with $|\delta(w')|$ arbitrarily large. Since the cyclic defect of w is zero, w will contain corresponding subwords with $\delta(w')$ being positive.

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