Gabriel-Roiter inclusions
and Auslander-Reiten theory.

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ABSTRACT. Let Λ be an artin algebra. The aim of this paper is to outline a strong relationship between the Gabriel-Roiter inclusions and the Auslander-Reiten theory. If X is a Gabriel-Roiter submodule of Y, then Y is shown to be a factor module of an indecomposable module M such that there exists an irreducible monomorphism $X \to M$. We also will prove that the monomorphisms in a homogeneous tube are Gabriel-Roiter inclusions, provided the tube contains a module whose endomorphism ring is a division ring.

Let Λ be an artin algebra, and $\operatorname{mod}\Lambda$ the category of Λ-modules of finite length. The basic notion of Auslander-Reiten theory is that of an irreducible map: these are the maps in the radical of $\operatorname{mod}\Lambda$ which do not belong to the square of the radical. They are used in order to define the Auslander-Reiten quiver $\Gamma(\Lambda)$: its vertices are the isomorphism classes $[X]$ of indecomposable Λ-modules X, and one draws an arrow $[X] \to [Y]$ provided there exists an irreducible map $X \to Y$. In 1975, Auslander and Reiten have shown the existence of Auslander-Reiten sequences: for any indecomposable non-injective module X, there exists an exact sequence $0 \to X \to Y \to Z \to 0$ such that both maps $X \to Y$ and $Y \to Z$ are irreducible; here, Z is indecomposable (and not projective) and any indecomposable non-projective module occurs in this way. The sequence is uniquely determined both by X and by Z and one writes $X = \tau Z$, and $Z = \tau^{-1}X$ and calls τ the Auslander-Reiten translation. The middle term Y is not necessarily indecomposable: the indecomposable direct summands Y' of Y are precisely the modules with an arrow $[X] \to [Y']$, and also precisely the modules with an arrow $[Y'] \to [Z]$.

The *Gabriel-Roiter measure* $\mu(M)$ of a Λ-module M is a rational number defined inductively as follows: For the zero module $M = 0$, one sets $\mu(0) = 0$. If $M \neq 0$ is decomposable, then $\mu(M)$ is the maximum of $\mu(M')$ where M' is a proper submodule of M, whereas for an indecomposable module M, one sets

$$\mu(M) = 2^{-|M|} + \max_{M' \subset M} \mu(M').$$

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It is obvious that calculating the maximum of $\mu(M')$, with M' a proper submodule of M, one may restrict to look at indecomposable submodules M' of M. If M is indecomposable and not simple, then there always exists an indecomposable submodule $M' \subset M$ such that $\mu(M) - \mu(M') = 2^{-|M|}$, such submodules are called *Gabriel-Roiter submodules* of M, and the inclusion map $M' \subset M$ is called a *Gabriel-Roiter inclusion.* Note that M may have non-isomorphic Gabriel-Roiter submodules, however all Gabriel-Roiter submodules of M have at least the same length. Inductively, we obtain for any indecomposable module M a chain of indecomposable submodules

$$M_1 \subset M_2 \subset \cdots \subset M_{t-1} \subset M_t = M$$

such that M_1 is simple and all the inclusions $M_{i-1} \subset M_i$ for $2 \leq i \leq t$ are Gabriel-Roiter inclusions, such a sequence is called a *Gabriel-Roiter filtration.* Given such a Gabriel-Roiter filtration, we have (by definition)

$$\mu(M) = \sum\nolimits_{j=1}^{t} 2^{-|M_j|},$$

and it will sometimes be convenient to call also the set $I = \{|M_1|, \dots, |M_t|\}$ the Gabriel-Roiter measure of M. Thus the Gabriel-Roiter measure $\mu(M)$ of a module M will be considered either as a finite set I of natural numbers, or else as the rational number $\sum_{i \in I} 2^{-i}$, whatever is more suitable.

The paper is divided into two parts. The first part comprises sections 1 to 3; here we will discuss in which way Gabriel-Roiter inclusions are related to Auslander-Reiten sequences.

Theorem A. *Let X be a Gabriel-Roiter submodule of Y. Then there is an irreducible monomorphism $X \to M$ with M indecomposable and an epimorphism $M \to Y$ such that the composition $X \to M \to Y$ is injective* (and therefore also a Gabriel-Roiter inclusion.)

This result may be reformulated as follows: *Let X be a Gabriel-Roiter submodule of Y. Then there is an irreducible embedding $X \subset M$ with M indecomposable and a submodule U of M with $X \cap U = 0$ such that M/U is isomorphic to Y.*

The proof of Theorem A will be given in section 1. Section 2 will exhibit applications, in particular we will derive some results concerning the existence of indecomposable submodules of a given module. Section 3 will use Theorem A in order to discuss the so-called take-off part of a bimodule algebra.

There will be an intermediate section 4 where we will introduce the notion of a piling submodule; this notion will be helpful for the further discussions. Note that looking at the piling submodules of a module M corresponds to the process of constructing inductively Gabriel-Roiter filtrations, starting with the simple submodules of M and going upwards.

The second part, sections 5 and 6, deals with modules belonging to homogeneous tubes, or, more generally, to modules which have a suitable filtration such that all the factors are isomorphic to a given indecomposable module M. Recall that a component of the Auslander-Reiten quiver of Λ is called a *homogeneous tube* provided it is of the form $\mathbb{Z}A_\infty/\tau$. The indecomposable modules belonging to a homogeneous tube $\mathcal{T}$ will always be labeled as $M[t]$ with $t \in \mathbb{N}_1$ such that $M[1]$ is of smallest possible length in $\mathcal{T}$ and any $M[t]$ has a filtration with t factors isomorphic to $M[1]$; the module $M[1]$ will be called the *boundary module of* $\mathcal{T}$. Also, a module is said to be a *brick* provided its endomorphism ring is a division ring.

Theorem B. *Let $\mathcal{T}$ be a homogeneous tube with indecomposable modules $M[t]$, where $t \in \mathbb{N}_1$. Let $m \geq 2$.*

(a) *Given a Gabriel-Roiter filtration of $M[m]$, then there is a submodule of $M[m]$ isomorphic to $M[1]$ which occurs in the filtration.*

(b) *If $M[1]$ is a brick, then $M[m]$ has a unique Gabriel-Roiter submodule, namely the unique submodule of $M[m]$ which is isomorphic to $M[m-1]$.*

The proof of Theorem B will be given in sections 5 and 6. Note that there is a wealth of artin algebras with homogeneous tubes: according to Crawley-Boevey, any tame k-algebra with k an algebraically closed field has homogeneous tubes, but there are also many wild artin algebras having homogeneous tubes. The case of tame hereditary algebras has been studied very carefully by Bo Chen, in particular see [C, Corollary 4.5] which provides a proof of Theorem B in this case.

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1. Proof of Theorem A.

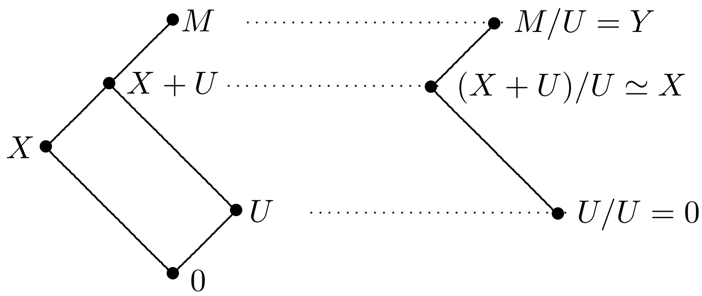
Here is a more precise statement.

1.1. Theorem. *Let X be a Gabriel-Roiter submodule of Y. Let $\phi_i\colon X \to N_i$ be irreducible maps with N_i indecomposable such that $\phi = (\phi_i)\colon X \to \bigoplus_{i=1}^t N_i$ is a source map for X. Then there is an index i and an epimorphism $\psi\colon N_i \to Y$ such that $\psi\phi_i$ is a monomorphism* (and thus a Gabriel-Roiter inclusion).

Second formulation. *Let X be a Gabriel-Roiter submodule of Y. Let $\phi_i\colon X \to N_i$ be irreducible maps with N_i indecomposable such that $\phi = (\phi_i)\colon X \to \bigoplus_{i=1}^t N_i$ is a source map for X. Then there is an index i such that ϕ_i is injective, and there is a submodule $U \subset N_i$ with $\phi_i(X) \cap U = 0$ such that N_i/U is isomorphic to Y.*

Relevance. Recall that any indecomposable module Y which is not simple has a Gabriel-Roiter submodule X and X is indecomposable again. Thus, the theorem asserts that in order to construct all the indecomposable modules Y, one can proceed inductively as follows, starting with the simple modules. In order to find indecomposable modules Y

which are not simple, we consider an indecomposable module X already constructed, an irreducible monomorphism $f\colon X \to M$, and an epimorphism $g\colon M \to Y$ such that the composition $gf\colon X \to Y$ is injective. Of course, we can assume that f is an embedding. The epimorphism g is determined by its kernel U, thus by a submodule U of M such that $X \cap U = 0$. The picture to have in mind is the following:



In this way we obtain all the possible Gabriel-Roiter inclusions, and thus all the indecomposable modules Y. According to 1.1, we only have to look at finitely many irreducible embeddings $X \to M_i = M$ (and this information is stored in the Auslander-Reiten quiver of Λ). The new datum required is the submodule U of M with $X \cap U = 0$. (Unfortunately, we do not know any criterion on U which tells us whether we obtain a Gabriel-Roiter inclusion, not even whether we get an indecomposable module $Y = M/U$).

Anyway, we should add that the irreducibility of an embedding $X \to M$ yields that for any proper submodule M' of M with $X \subseteq M'$, the embedding $X \subseteq M'$ splits, thus there is a submodule U of M with $M' = X \oplus U$. In this way, the submodules U to be considered correspond to the proper submodules of M/X.

Also, let us stress that here we deal with a quite unusual conjunction of indecomposable modules: as we know, the modules X, M, Y are indecomposable. Since both embeddings of X into M and into Y are mono-irreducible, also the factor modules M/X and $Y/X \simeq M/(X + U)$ are indecomposable.

Proof of Theorem 1.1. Let $u\colon X \to Y$ be a Gabriel-Roiter inclusion. We denote by $\mathrm{Sing}(X, Y)$ the set of maps $f\colon X \to Y$ which are not monomorphisms and we know that $\mathrm{Sing}(X, Y)$ is closed under addition, see [R4] or [R5].

Let $\phi = (\phi_i)\colon X \to \bigoplus_{i=1}^{t} N_i$ be the source map, with all N_i indecomposable. We obtain maps $\psi_i\colon N_i \to Y$ such that $\sum_i \psi_i\phi_i = u$. Since u is a monomorphism and $\mathrm{Sing}(X, Y)$ is closed under addition, we see that at least one of the maps $\psi_i\phi_i$, say $\psi_1\phi_1$ has to be a monomorphism. As a consequence, also ϕ_1 is a monomorphism. We claim that ψ_1 is surjective. Assume for the contrary that the image Z of ψ_1 is a proper submodule of Y, write $\psi_1 = \nu\psi_1'$ with $\psi_1'\colon N_1 \to Z$ and $\nu\colon Z \to Y$ the inclusion map. Let X' be the image of $\psi_1\phi_1$, it is contained in Z and we write $\psi_1'\phi_1 = \nu' g$ with $g\colon X \to X'$ and $\nu'\colon X' \to Z$ the inclusion map. Note that g is an isomorphism, since $\psi_1\phi_1$ is a monomorphism. With X also X' is a Gabriel-Roiter submodule of Y and the inclusions $X' \subseteq Z \subset Y$ show that X' is a direct summand of Z, thus there is a retraction $r\colon Z \to X'$ (with $r\nu' = 1_{X'}$). But then

$$g^{-1} r \psi_1' \phi_1 = g^{-1} r \nu' g = g^{-1} g = 1_X$$

shows that ϕ_1 is a split monomorphism. This contradicts the fact that ϕ_1 is irreducible. Therefore ψ_1 has to be surjective. In the statement of Theorem 1.1, we write $\psi = \psi_1$.

This kind of argumentation can be used inductively:

1.2. Theorem. *Assume that $X = M_0$ is a Gabriel-Roiter submodule of Y. Then either*

(a) *there is a natural number $n \geq 1$ such that there are indecomposable modules M_{i+1} and irreducible maps $f_i\colon M_i \to M_{i+1}$ with $0 \leq i < n$ and $M_n = Y$ such that the map $f_{n-1}\cdots f_0$ is a monomorphism (and thus a Gabriel-Roiter inclusion) and $f_{n-1}\cdots f_1$ is an epimorphism, or else:*

(b) *there is an infinite sequence of indecomposable modules M_{i+1}, of irreducible maps $f_i\colon M_i \to M_{i+1}$ with $i \geq 0$ and of maps $g_i\colon M_i \to Y$ with $i \geq 1$ such that for any $m \geq 1$ the composition $g_m f_{m-1}\cdots f_0$ is a monomorphism (and thus a Gabriel-Roiter inclusion) and $g_m f_{m-1}\cdots f_1$ is an epimorphism.*

Proof. By induction on $m \geq 1$, we try to construct an indecomposable module M_m, an irreducible map $f_{m-1}\colon M_{m-1} \to M_m$ and a map $g_m\colon M_m \to Y$ such that $g_m f_{m-1}\cdots f_0$ is a monomorphism and $g_m f_{m-1}\cdots f_1$ is an epimorphism. For $m = 1$, this has been done in 1.1.

Now consider the case $m \geq 2$ and assume that appropriate modules $M_1, \dots, M_{m-1}$ and maps $f_0, \dots, f_{m-2}$ as well as $g_1, \dots, g_{m-1}$ have been constructed. If g_{m-1} is an isomorphism, then we replace f_{m-2} by $g_{m-1}f_{m-2}\colon M_{m-2} \to Y$. Of course, with f_{m-2} also $g_{m-1}f_{m-2}$ is irreducible, therefore we obtain the case (a) with $n = m-1$.

Thus, we can assume that g_{m-1} is not an isomorphism. Since we know by induction that $g_{m-1}f_{m-2}\cdots f_1$ and therefore also g_{m-1} is an epimorphism, we see that g_{m-1} is not a monomorphism. It follows that g_{m-1} can be factored through the source map of M_{m-1}, say $g_{m-1} = \sum_i \psi_i\phi_i$, with irreducible maps $\phi_i\colon M_{m-1} \to N_i$, maps $\psi_i\colon N_i \to Y$, and indecomposable modules N_i. It follows that

$$g_{m-1}f_{m-2}\cdots f_0 = \sum_i \psi_i\phi_i f_{m-2}\cdots f_0.$$

Here, we see that a Gabriel-Roiter inclusion is written as a sum of maps, thus there is an index i, such that $\psi_i\phi_i f_{m-2}\cdots f_0$ is injective. We can assume that $i = 1$ and we let $M_m = N_1$, $f_{m-1} = \phi_1\colon M_{m-1} \to M_m$ and $g_m = \psi_1\colon M_m \to Y$. By construction, M_m is indecomposable and f_{m-1} is irreducible. Also, the composition $g_m f_{m-1}\cdots f_0$ is a monomorphism.

It remains to be shown that $g_m f_{m-1}\cdots f_1$ is an epimorphism. It not, then its image Z is a proper submodule of Y which contains the image X' of the Gabriel-Roiter inclusion $g_m f_{m-1}\cdots f_1 f_0$. But then X' would be a direct summand of Z and this would imply that f_0 is a split monomorphism, in contrast to the fact that f_0 is irreducible. This completes the proof.

2. Applications.

An obvious consequence of Theorem A is the following:

2.1. Corollary. *If X is a Gabriel-Roiter submodule of some module Y, then there exists an irreducible monomorphism $X \to M$ with M indecomposable.*

This shows that a lot of modules cannot be Gabriel-Roiter submodules of other modules. For example

(1) Injective modules (of course).
(2) Let Λ be the path algebra of the n-Kronecker quiver: this is the quiver with 2 vertices a, b and n arrows from a to b. If $n \geq 2$, then Λ is representation-infinite and has a preinjective component. If X is an indecomposable preinjective module, then no irreducible map $X \to Y$ with Y indecomposable is a monomorphism.
(3) Consider the n-subspace quiver for some $n \geq 1$ (this is the quiver with $n+1$ vertices, such that there is a unique sink whereas the remaining vertices are sources, and such that there is precisely one arrow from any source to the sink). Let X be an indecomposable module of the form $\tau^{-t}P$ where $t \geq 1$, where P is the unique simple projective module. Then there is no irreducible map $X \to Y$ with Y indecomposable, which is a monomorphism, thus X cannot occur as a Gabriel-Roiter submodule. For example, for $n = 4$ this concerns all the indecomposable preprojective modules X of length $6t + 1$ with $t \geq 1$.

Less trivial are the following consequences of Theorem A:

Let p be the maximal length of an indecomposable projective module, let q be the maximal length of an indecomposable injective module.

2.2. Corollary. *Let $X \to Y$ be a Gabriel-Roiter inclusion. Then $|Y| \leq pq|X|$.*

Proof: Of course, X cannot be injective. It is well-known that for an indecomposable non-injective module X, one has $|\tau^{-1}(X)| \leq (pq-1)|X|$, thus the middle term X' of the Auslander-Reiten sequence starting in X has length at most $pq|X|$. Theorem 1 asserts that Y is a factor module of X', thus also $|Y| \leq pq|X|$.

This result is already mentioned in [R4], as a corollary to Lemma 3.1 of [R4]. Also, there we have shown that 2.2 implies the "successor lemma". Here are two further consequences.

2.3. Corollary. *Let M be an indecomposable module and $1 \leq a < |M|$ a natural number. Then there exists an indecomposable submodule M' of M with length in the interval $[a+1, pqa]$.*

Proof: Take a Gabriel-Roiter filtration $M_1 \subset \cdots \subset M_n = M$. Let i be maximal with $|M_i| \leq a$. Then $1 \leq i < n$, thus M_{i+1} exists and $a < |M_{i+1}| \leq pq|M_i| \leq pqa$.

2.4. Corollary. *Let M be an indecomposable module and assume that all indecomposable proper submodules of M are of length at most b. Then $|M| \leq pqb$.*

Proof: Let X be a Gabriel-Roiter submodule of M. By assumption, $|X| \leq b$, thus $|M| \leq pq|X| \leq pqb$.

Reformulation: Let $\mathcal{N}$ be a class of indecomposable modules. Recall that a module M is said to be $\mathcal{N}$-critical provided it does not belong to $\operatorname{add}\mathcal{N}$, but any proper indecomposable submodule of M belongs to $\mathcal{N}$. Corollary 2.4 asserts the following: if all the modules in $\mathcal{N}$ are of length at most b, then any $\mathcal{N}$-critical module is of length at most pqb.

Observe that the last two corollaries do not refer at all to Gabriel-Roiter notions.

3. The take-off part of a bimodule algebra.

A rational number (or a finite set of natural numbers) will be said to be a Gabriel-Roiter measure for Λ, provided there is an indecomposable Λ-module with this measure. For any Gabriel-Roiter measure J, we denote by $\mathcal{A}(J)$ the set of isomorphism classes of indecomposable modules with measure J (or representatives of these isomorphism classes). Recall from [R3] the following: If Λ is a representation-infinite artin algebra, there is a countable sequence of Gabriel-Roiter measures $I_1 < I_2 < \cdots$ (the so-called "take-off measures") for Λ such that any other Gabriel-Roiter measure I for Λ satisfies $I_t < I$ for all t. Obviously, $I_1 = \{1\}$ and it is easy to see that $I_2 = \{1, t\}$, where t is the largest possible length of a local Λ-module of Loewy length 2.

In this section, we consider the finite dimensional hereditary algebras with s = 2, where s denotes the number of simple modules, thus we deal with representations of a bimodule ${}_FM_G$, where F, G are division rings. We assume that $\Lambda = \begin{bmatrix} F & M \\ 0 & G \end{bmatrix}$ is representation-infinite. Of course, we require that Λ is an artin algebra, thus there is a commutative field k contained in the center both of F and of G and acting centrally on M and such that $\dim_k M$ is finite. Let $a = \dim {}_FM, b = \dim M_G$. The assumption that Λ is representation-infinite means that $ab \geq 4$.

Often we will present an indecomposable module by just writing down its dimension vector (note that a representation of the bimodule ${}_FM_G$ is a triple $({}_FX, {}_GY, \gamma)$, where $\gamma\colon {}_FM_G \otimes_G Y \to {}_FX$ is F-linear, its dimension vector is the pair $(\dim {}_FX, \dim {}_GY)$).

Let $P_1, P_2, \dots$ be the sequence of preprojective modules, with non-zero maps $P_i \to P_{i+1}$.

$$\begin{tikzcd}[column sep=small]
 & P_2=(a,1) \arrow[dr] & \cdots\cdots & P_4=(a^2b-2a,ab-1) \arrow[dr] & \cdots\cdots & \cdots \\
P_1=(1,0) \arrow[ur] & \cdots\cdots & P_3=(ab-1,b) \arrow[ur] & \cdots\cdots & P_5 & \cdots
\end{tikzcd}$$

with $\mathrm{End}(P_{2n-1}) = F$, and $\mathrm{End}(P_{2n}) = G$, for all n.

Note that always $\mathcal{A}(I_1)$ consists of the simple Λ-modules. Here we show that the remaining take-off modules are the modules P_n with $n \geq 2$.

Proposition. *For $n \geq 2$, $\mathcal{A}(I_n) = \{P_n\}$.*

For $n = 2$, the assertion is true according to the general description of I_2. For $n > 2$, we use induction. We have to consider three cases:

Case 1. Consider first a bimodule ${}_FM_G$ with $a, b \geq 2$. Then all the non-zero maps $P_n \to P_{n+1}$ are monomorphisms. Also, since all the irreducible maps ending in P_n are monomorphisms, *the monomorphisms $P_{n-1} \to P_n$ are Gabriel-Roiter inclusions.*

Consider some $n > 2$ and assume that the assertion is true for $n-1$. Since there is a Gabriel-Roiter inclusion $P_{n-1} \to P_n$, it follows that $I_n = I_{n-1} \cup \{t\}$ with $t \geq |P_n|$. Thus let Y be indecomposable with $\mu(Y) = I_n$, let X be a Gabriel-Roiter submodule of Y. Then $\mu(X) = I_{n-1}$, thus by induction $X = P_{n-1}$. But now we can apply theorem 1.1

above which shows that Y is a factor module of P_n. Since $|Y| = t \ge |P_n|$, we see that $Y = P_n$.

Case 2. $F \subset G$, and $M = {}_F G_G$, thus $a = [G : F]$. Then we deal with the preprojective modules

$$\begin{tikzcd}[column sep=small]
 & P_2 = (a,1) \arrow[dr] \arrow[rr, dotted, no head] & & P_4 = (a^2-2a, a-1) \arrow[dr] \arrow[rr, dotted, no head] & & \cdots \\
P_1 = (1,0) \arrow[ur] \arrow[rr, dotted, no head] & & P_3 = (a-1,1) \arrow[ur] \arrow[rr, dotted, no head] & & P_5 \arrow[r, dotted, no head] & \cdots
\end{tikzcd}$$

- *The non-zero maps $P_{2n-1} \to P_{2n}$ are injective and are Gabriel-Roiter inclusions.*
- The non-zero maps $P_{2n} \to P_{2n+1}$ are surjective.
- *The non-zero maps $P_{2n-1} \to P_{2n+1}$ are injective and are Gabriel-Roiter inclusions.*

Consider some $2n$ and assume that the assertion is true for $2n-1$. The argument is the same as in Case 1, using theorem 1.1.

Also, consider some $2n+1$ and assume that the assertion is true for $2n-1$ and $2n$. Since the irreducible maps starting in P_{2n} are epi, we see that I_{2n+1} cannot start with I_{2n}. Since there are Gabriel-Roiter inclusions $P_{2n-1} \to P_{2n+1}$, we see that $I_{2n+1} = I_{2n-1} \cup \{t\}$ with $|P_{2n}| > t \ge |P_{2n+1}|$.

Thus let Y be indecomposable with $\mu(Y) = I_{2n+1}$, let X be a Gabriel-Roiter submodule of Y. Then $\mu(X) = I_{2n-1}$, thus by induction $X = P_{2n-1}$. But now we can apply 1.2. It shows that Y is a factor module of P_{2n+1}. Since $|Y| = t \ge |P_{2n+1}|$, we see that $Y = P_{2n+1}$.

Case 3. $G \subset F$, and $M = {}_F F_G$, thus $b = [F : G]$. Then we deal with the preprojectives

$$\begin{tikzcd}[column sep=small]
 & P_2 = (1,1) \arrow[dr] \arrow[rr, dotted, no head] & & P_4 = (b-2, b-1) \arrow[dr] \arrow[rr, dotted, no head] & & \cdots \\
P_1 = (1,0) \arrow[ur] \arrow[rr, dotted, no head] & & P_3 = (b-1,b) \arrow[ur] \arrow[rr, dotted, no head] & & P_5 \arrow[r, dotted, no head] & \cdots
\end{tikzcd}$$

The non-zero maps $P_{2n-1} \to P_{2n}$ are surjective, for $n \ge 2$, whereas $P_1 \to P_2$ is injective (and this is a Gabriel-Roiter inclusion).

- *The non-zero maps $P_{2n} \to P_{2n+1}$ are injective and are Gabriel-Roiter inclusions.*
- *The non-zero maps $P_{2n} \to P_{2n+2}$ are injective and are Gabriel-Roiter inclusions.*

Proof: As in case 2, but taking into account the additional Gabriel-Roiter inclusion $P_1 \to P_2$.

4. Piling submodules.

We call an indecomposable submodule U of some module Y *piling*, provided $\mu(V) \le \mu(U)$ for all indecomposable submodules V of Y with $|V| \le |U|$; actually, it is sufficient to check the condition for the indecomposable submodules V of Y with $|V| < |U|$. Namely, if there exists an indecomposable submodule V of Y with $|V| = |U|$ and $\mu(V) > \mu(U)$, then there exists a proper submodule V' of V with $\mu(V') > \mu(U)$). There is the following alternative description:

4.1. Lemma. *An indecomposable submodule U of Y is piling if and only if $\mu(Y)$ starts with $\mu(U)$* (this means that $\mu(U) = \mu(Y) \cap \{1, 2, \dots, |U|\}$).

Proof: Let U be an indecomposable submodule of Y, let $U_1 \subset U_2 \subset \cdots \subset U_s = U$ and $Y_1 \subset Y_2 \subset \cdots \subset Y_t$ be Gabriel-Roiter filtrations, where $Y_t \subseteq Y$ is an indecomposable submodule with $\mu(Y_t) = \mu(Y)$.

First, assume that U is piling in Y. We claim that $|U_i| = |Y_i|$ for $1 \le i \le s$. If not, then there is some minimal i with $|U_i| \neq |Y_i|$ and since $\mu(U) \le \mu(Y)$, we must have $|U_i| > |Y_i|$. But then $V = Y_i$ is a submodule of Y with $|V| < |U|$ and $\mu(V) = \{|Y_1|, \dots, |Y_i|\} > \mu(U)$, a contradiction to the assumption that U is piling in Y.

Second, assume that $\mu(U) = \mu(Y) \cap \{1, 2, \dots, |U|\}$. Let V be an indecomposable submodule of Y with $|V| \le |U|$ and assume that $\mu(V) > \mu(U)$. If $V_1 \subset V_2 \subset \cdots \subset V_r = V$ is a Gabriel-Roiter filtration of V, then there must be some $1 \le j \le \min(s+1, r)$ such that $|U_i| = |V_i|$ for $1 \le i < j$ and either $j = s+1$ or else $|U_j| > |V_j|$. The case $j = s+1$ cannot happen, since otherwise $|U| = |U_s| = |V_s| < |V_{s+1}| \le |V|$, but $|V| \le |U|$. Thus we have $|V_j| < |U_j|$. But then $\mu(V) > \mu(Y)$, since $|V_i| = |U_i| = |Y_i|$ for $1 \le i < j$ and $|V_j| < |U_j| = |Y_j|$. This is impossible: a submodule V of Y always satisfies $\mu(V) \le \mu(Y)$.

Note that all the submodules in a Gabriel-Roiter filtration of an indecomposable module are piling, but usually there are additional ones: for example all simple submodules are piling. The fact that a submodule U of Y is piling depends only on the isomorphism class of U and the set of isomorphism classes of submodules V of Y with $|V| \le |U|$ (but for example not on the embedding of U into Y). Here are some further properties:

4.2. *Assume that $U \subseteq V \subseteq W$. If U is piling in V and V is piling in W, then U is piling in W.* Proof: If $\mu(W)$ starts with $\mu(V)$ and $\mu(V)$ starts with $\mu(U)$, then obviously $\mu(W)$ starts with $\mu(U)$.

4.3. *If $U \subseteq V \subseteq W$ and U is a piling submodule of W, then also of V.* Proof: Let X be a submodule of V with $|X| \le |U|$. Consider X as a submodule of W and conclude that $\mu(X) \le \mu(U)$.

4.4. *If $U \subseteq X \oplus Y$ is a piling submodule, then at least one of the maps $U \to X$ or $U \to Y$ is an embedding with piling image.* Recall the strong Gabriel property: Assume that $U, X_1, \dots, X_n$ are indecomposable modules and there are given maps $f_i\colon U \to X_i$ such that the map $f = (f_i)_i\colon U \to \bigoplus X_i = X$ is a monomorphism and its image is a piling submodule of X. Then at least one of the maps f_i is a monomorphism (and its image is a piling submodule of X_i). Now let $U \subseteq X \oplus Y$ be a piling submodule. According to the strong Gabriel property, one of the maps $U \to X$, $U \to Y$ is an embedding, say $f\colon U \to X$. Since U is piling in $X \oplus Y$, it follows that $f(U)$ is piling in X.

5. Piling submodules of modules with a homogeneous M-filtration.

5.1. Let M be an indecomposable module. An *M-filtration* of a module Y is a chain of modules

$$M = M[1] \subset M[2] \subset \cdots \subset M[m] = Y$$

such that $M[i]/M[i-1] \simeq M$ for all $2 \le i \le m$; the number m is called the *M-length* of Y. Observe that a module Y with an M-filtration is just an iterated self-extension of the module M.

In case all the modules $M[i]$ are indecomposable and the inclusion maps are mono-irreducible maps, we call this a *homogeneous* M-filtration. For example, if $\mathcal{T}$ is a homogeneous tube of the Auslander-Reiten quiver of Λ and M is the boundary module of $\mathcal{T}$, then any module belonging to $\mathcal{T}$ has a homogeneous M-filtration.

On the other hand, it is easy to construct modules M with self-extensions

$$0 \to M \to M[2] \to M \to 0$$

where $M[2]$ is indecomposable such that the inclusion $M \to M[2]$ is not mono-irreducible. For example, take the quiver with vertices a, b, c, one arrow $a \to b$, two arrows $b \to c$. Then any indecomposable module M with dimension vector $(1,1,1)$ is as required.

The aim of this section is to study Gabriel-Roiter filtrations of modules with a homogeneous M-filtration.

5.2. Theorem. *Let Y be a module with a homogeneous M-filtration. Let U be a piling submodule of Y.*

(a) *If $|U| \le |M|$, then U is isomorphic to a submodule of M.*

(b) *If $|U| \ge |M|$, then any Gabriel-Roiter filtration of U contains a module isomorphic to M.*

5.3. Corollary. *Let Y be a module with a homogeneous M-filtration. Then any Gabriel-Roiter filtration of Y contains a module isomorphic to M.*

This is the special case of (b) where $U = Y$. On the other hand, the assertion (a) of Theorem B is a direct application of 5.3: Given a homogeneous tube $\mathcal{T}$ with modules $M[t]$, then any module $M[t]$ has a homogeneous $M[1]$-filtration.

Before we start with the proof of Theorem 5.2, we insert a general observation:

5.4. *Let $X \to Y$ be a proper inclusion with $X \neq 0$ and Y indecomposable (for example a Gabriel-Roiter inclusion). Assume that Y' is a submodule of Y with $X + Y' = Y$. Then $|Y'| > |Y/X|$.*

Proof. The submodule Y' of Y maps onto $Y'/(X \cap Y') \simeq (X+Y')/X = Y/X$ with kernel $X \cap Y'$. If $|Y'| \le |Y/X|$, then this surjective map has to be an isomorphism, thus $X \cap Y' = 0$. But then $Y = X \oplus Y'$, whereas Y is indecomposable and $0 \neq X \neq Y$.

Proof of 5.2(a). Assume that $|U| \le |M|$. According to 5.4, we see that $M[m-1] + U$ is a proper submodule of Y, thus $M[m-1] + U = M[m-1] \oplus U'$ for some submodule U' of Y which is isomorphic to a proper submodule of M. Now U is a piling submodule of Y, thus also of $M[m-1] \oplus U'$, therefore of $M[m-1]$ or of U'. In the first case, use induction on m. In the second case, just recall that U' is isomorphic to a submodule of M.

Proof of 5.2(b). We can assume that M is not simple, since otherwise $M[m]$ is serial and nothing has to be shown.

Let $U_1 \subset U_2 \subset \cdots \subset U_s = U$ be a Gabriel-Roiter filtration of U. Assume that $|U_r| < |M|$ and $|U_{r+1}| \geq |M|$ by assumption, such an r must exist, since M is not simple and $|U_s| \geq |M|$. We apply (a) to the submodule U_r (as a piling submodule of U it is piling in Y) and see that U_r is isomorphic to a submodule M' of M. From the definition of a Gabriel-Roiter filtration it follows that $|U_{r+1}| \leq |M|$. Thus $|U_{r+1}| = |M|$. Now we apply (a) to the submodule U_{r+1} and see that U_{r+1} is isomorphic to M.

6. Modules with homogeneous M-filtrations, where M is a brick.

6.1. We will assume now in addition that the endomorphism ring of M is a division ring, thus that M is a brick. Then we can use the process of simplification [R1]: Let $\mathcal{F}(M)$ be the full subcategory of all modules which have an M-filtration. The new assumption implies that this category $\mathcal{F}(M)$ is an abelian category, even a length category, and M is its only simple object. Of course, the M-filtrations of an object Y are just the composition series of Y when considered as an object of $\mathcal{F}(M)$. Thus, if Y has an M-filtration

$$M = M[1] \subset M[2] \subset \cdots \subset M[m] = Y$$

with all $M[i]$ for $1 \leq i \leq m$ indecomposable, then Y considered as an object of $\mathcal{F}(M)$ is uniform, thus M is the only submodule of Y isomorphic to M.

In case Y has a unique M-filtration

$$M = M[1] \subset M[2] \subset \cdots \subset M[m] = Y$$

then Y, considered as an object in $\mathcal{F}(M)$ is even serial and then all the factors $M[t]/M[s]$ with $0 \leq s < t \leq m$ are indecomposable (here, $M[0] = 0$).

Conversely, if Y has the M-filtration

$$M = M[1] \subset M[2] \subset \cdots \subset M[m] = Y$$

and all the factor modules $M[i]/M[i-2]$ with $2 \leq i \leq m$ are indecomposable (again, we set $M[0] = 0$), then Y has only one M-filtration.

6.2. Theorem. *Assume that M is a brick. Let Y be a module with a homogeneous M-filtration and assume that*

$$M = M[1] \subset M[2] \subset \cdots \subset M[m] = Y$$

is the only M-filtration of Y. If U is a piling submodule of Y and $|U| \geq |M|$, then $U = M[j]$ for some $1 \leq j \leq m$.

Remark: The assumption $|U| \geq |M|$ is important. For example, if $|M| \geq 2$, then the socle of $M[m]$ is of length at least m, thus for $m \geq 2$, there are many simple submodules and all are piling.

Proof of 6.2. We can assume that M is not simple, since otherwise $M[m]$ is a serial module and the submodules $M[j]$ are the only non-zero submodules.

We use induction on m. For $m = 1$ nothing has to be shown. Thus let $m \geq 2$. Let U be a piling submodule of Y and $|U| \geq |M|$. We use a second induction, now on $|U|$ in order to show that $U = M[j]$ for some j. The induction starts with $|U| = |M|$. In this case 5.2(a) shows that U is isomorphic to M, but according to the process of simplification, any submodule of Y isomorphic to M is equal to M, thus $U = M = M[1]$.

Now let $|U| > |M|$. Let U' be a Gabriel-Roiter submodule of U.

First, assume that $|U'| < |M|$. Then, according to 5.2(a), there is a submodule M' of M which is isomorphic to U'. Using the definition of a Gabriel-Roiter sequence, and the fact that there is the inclusion $M' \subset M$ with M indecomposable, we see that $|U| \leq |M|$, but this contradicts the assumption $|U| > |M|$.

Next, consider the case $|U'| \geq |M|$. By the second induction, we see that $U' = M[j]$ for some $j \geq 1$. Of course, since U' is a proper submodule of Y, we see that $j < m$. The inclusion $M[j] \subset M[j+1]$ and the definition of a Gabriel-Roiter filtration shows that $|U| \leq |M[j+1]|$. Claim: We can assume that $M[m-1] + U$ is a proper submodule of $M[m]$.

Otherwise, we easily see that $M[m]/M[j] = M[m-1]/M[j] \oplus U/M[j]$. But the process of simplification shows that $M[m]/M[j]$ is indecomposable. Also, $U/M[j] = U/U'$ is non-zero, thus $M[m-1]/M[j] = 0$ and $U = M[m]$.

Since $M[m-1] + U$ is a proper submodule of $M[m]$, we see that $M[m-1] + U = M[m-1] \oplus C$ for some submodule C of Y and C is isomorphic to a proper submodule of M. Write the inclusion map $U \to M[m-1] \oplus C$ in the form $[f, f']^t$, where $f\colon U \to M[m-1]$ and $f'\colon U \to C$. Since U is a piling submodule of Y and $U \subseteq M[m-1] \oplus C \subset Y$, it follows that U is a piling submodule of $M[m-1] \oplus C$, thus either f or f' is an embedding with piling image. However $|U| > |M|$ whereas $|C| < |M|$, thus $f\colon U \to M[m-1]$ is an embedding, and its image is a piling submodule. Now we use the induction on m in order to conclude that $f(U) = M[i]$ for some i. In particular, U is isomorphic to $M[i]$. But there is no non-zero homomorphism $M[i] \to C$, since C is a proper submodule of M (using simplification). This shows that $f' = 0$ and therefore the embedding $U \to M[m-1] \oplus C$ is just the map $f\colon U \to M[m-1]$. It follows that $U = M[i]$.

6.3. Corollary. *Assume that M is a brick. Let Y be a module with a homogeneous M-filtration and assume that*

$$M = M[1] \subset M[2] \subset \cdots \subset M[m] = Y$$

is the only M-filtration of Y. If $m \geq 2$, then Y has precisely one Gabriel-Roiter submodule, namely $M[m-1]$.

Proof of 6.3. Let X be a Gabriel-Roiter submodule of $Y = M[m]$, where $m \geq 2$. A Gabriel-Roiter submodule is a piling submodule. If $|X| \leq |M|$, then by 5.2(a), X is isomorphic to a submodule M' of M. With X also M' is a Gabriel-Roiter submodule of Y, thus the inclusions $M' \subseteq M \subset M[2] \subseteq M[m]$ show that $M' = M$ (and $m = 2$). Thus X is isomorphic to M. However the process of simplification asserts that M is the only submodule of $M[m]$ isomorphic to M. This shows that $X = M[1]$.

Thus we can assume that $|X| > |M|$ and use theorem 6.2.

6.4. Proof of Theorem B (b). Let $\mathcal{T}$ be a homogeneous tube of the Auslander-Reiten quiver of Λ. Denote the modules in $\mathcal{T}$ by $M[m]$, where $|M[m]| = m|M|$ and $M = M[1]$. Assume in addition that M is a brick. Since M is a brick, we can consider the abelian category $\mathcal{F}(M)$ which contains all the modules $M[m]$. We have a chain of inclusions

$$M = M[1] \subset M[2] \subset \cdots$$

and for $m \geq 2$, the factor module $M[m]/M[m-2]$ is indecomposable (here again, we set $M[0] = 0$). This shows that any $M[m]$ has the unique M-filtration

$$M = M[1] \subset M[2] \subset \cdots \subset M[m],$$

thus we can use the previous corollary.

6.5. Remarks. In 6.3 as well as in Theorem B (b), the assumption $m \geq 2$ is important: the module $M = M[1]$ usually will have more than one Gabriel-Roiter submodules. For example consider the four-subspace-quiver and $\mathcal{T}$ a homogeneous tube containing a module M of length 6. The module M has 4 maximal submodules and all are Gabriel-Roiter submodules.

Well-known examples of homogeneous tubes such that the endomorphism ring of the boundary module M is a division ring are the homogeneous tubes of a tame hereditary algebra, of a tubular algebra or of a canonical algebra [R2]. As we have mentioned in the introduction, tame hereditary algebras have been considered by Bo Chen in [C].

For a tubular algebra, the boundary modules of homogeneous tubes are of unbounded length. The same is true in case Λ is a tame hereditary or a canonical k-algebra and the algebraic closure of k is not a finite field extension of k.

At the end of the paper, let us consider also modules of infinite lengths. We consider again a homogeneous tube $\mathcal{T}$ with indecomposable modules $M[m]$, where $t \in \mathbb{N}_1$. There is a chain of irreducible maps

$$M[1] \to M[2] \to \cdots \to M[m] \to M[m+1] \to \cdots,$$

and we denote by $M[\infty]$ the corresponding direct limit; such a module is called a *Prüfer module.*

6.6. Corollary. *Let $\mathcal{T}$ be a homogeneous tube with indecomposable modules $M[m]$ and assume that $M[1]$ is a brick. Then the Gabriel-Roiter measure of $M[\infty]$ is a rational number.*

Proof. The module $M[\infty]$ has a Gabriel-Roiter filtration starting with a Gabriel-Roiter filtration of $M[1]$ and then using precisely the modules $M[m]$. Let X be a Gabriel-Roiter submodule of $M[1]$. Note that the length of $M[m]$ is sm, with $s = |M[1]|$. Thus

$$\gamma(M[\infty]) = \gamma(X) + \sum_{m \geq 1} 2^{-|M[m]|} = \gamma(X) + \sum_{m \geq 1} 2^{-sm} = \gamma(X) + \frac{1}{2^s - 1}.$$

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