Green's Theorem on Hall Algebras

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ABSTRACT. Let k be a finite field and Λ a hereditary finitary k-algebra. Let $\mathcal{P}$ be the set of isomorphism classes of finite Λ-modules. We define a multiplication on the $\mathbb{Q}$-space with basis $\mathcal{P}$ by counting the number of submodules U of a given module V with prescribed isomorphism classes both of V/U and U. In this way we obtain the so called Hall algebra $\mathcal{H}{=}\mathcal{H}(\Lambda,\mathbb{Q})$ with coefficients in $\mathbb{Q}$. Besides $\mathcal{H}$, we are also interested in the subalgebra $\mathcal{C}$ generated by the the subset I of all isomorphism classes of simple Λ-modules; this subalgebra is called the corresponding composition algebra, since it encodes the number of composition series of all Λ-modules.

Recently, J. A. Green has introduced on $\mathcal{H}$ (and on $\mathcal{C}$) a comultiplication δ so that it becomes nearly a bialgebra. Here, "nearly" means that $\delta\colon \mathcal{H}{\to}\mathcal{H}{\otimes}\mathcal{H}$ is in general not an algebra homomorphism for the usual (componentwise) multiplication on $\mathcal{H}{\otimes}\mathcal{H}$; instead, there is a slightly twisted multiplication on $\mathcal{H}{\otimes}\mathcal{H}$ which has to be considered. The appearance of this twist should not be considered as a disadvantage: on the contrary, it is the main ingredient for Drinfeld and Lusztig in order to define the positive part U^+ of a quantum group for an arbitrary Cartan datum. It follows that U^+ is isomorphic to a twisted generic composition algebra.

Green's compatibility theorem for multiplication and comultiplication on $\mathcal{H}$ is expressed in a marvelous formula which deals with pairs of submodules of a given module. This formula should be considered as a detailed investigation of the $3{\times}3$ Lemma in homological algebra.

This is the written version of two lectures given at the workshop preceding ICRA VII, held at U.N.A.M., Mexico, D.F. We present the complete proof of Green's formula and outline its consequences, following closely the presentation in the original paper [G1].

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Part I. Green's Formula

Let Λ be a ring. Let $\mathcal{P}$ be the set of isomorphism classes of Λ-modules of finite cardinality. For every $\alpha \in \mathcal{P}$ let V_α be a representative in α. We denote by 0 both the zero module and its isomorphism class.

We recall that Λ is said to be *finitary,* provided $\operatorname{Ext}^1(M,N)$ is a finite set for any pair M,N of finite Λ-modules (clearly, it is sufficient to check this for finite, simple Λ-modules M,N). Throughout these notes, we usually will assume that the ring Λ is finitary.

Also, Λ is said to be hereditary, provided $\operatorname{Ext}^2(M,N) = 0$ for all Λ-modules M,N. It is well-known that an algebra is hereditary if and only if submodules of projective modules are projective – and this explains the name.

1. The formula

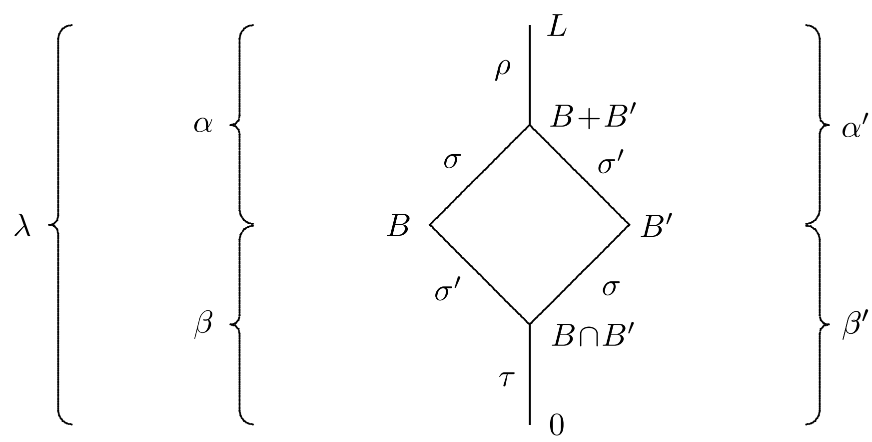
Given $\alpha \in \mathcal{P}$, let a_α be the order of the automorphism group of V_α.

Given α, β, λ in $\mathcal{P}$, let $g^\lambda_{\alpha\beta}$ be the number of submodules B of V_λ such that V_λ/B is isomorphic to V_α, whereas B is isomorphic to V_β.

Theorem (J. A. Green). *Let k be a finite field and Λ a k-algebra which is hereditary and finitary. Let $\alpha, \beta, \alpha', \beta'$ be elements of $\mathcal{P}$. Then*

$$a_\alpha a_\beta a_{\alpha'} a_{\beta'} \sum_\lambda g^\lambda_{\alpha\beta} g^\lambda_{\alpha'\beta'}\, a_\lambda^{-1} = \sum_{\rho,\sigma,\sigma',\tau} \frac{|\operatorname{Ext}^1(V_\rho, V_\tau)|}{|\operatorname{Hom}(V_\rho, V_\tau)|}\, g^\alpha_{\rho\sigma} g^{\alpha'}_{\rho\sigma'} g^\beta_{\sigma'\tau} g^{\beta'}_{\sigma\tau}\, a_\rho a_\sigma a_{\sigma'} a_\tau.$$

Green's theorem exhibits a universal formula for pairs of submodules: We consider submodules B and B' of a module L, where $L/B \in \alpha$, $L/B' \in \alpha'$, $B \in \beta$ and $B' \in \beta'$ (these four isomorphism classes $\alpha, \beta, \alpha', \beta'$ are given). The left hand side counts the numbers of such pairs of submodules in arbitrary modules $L = V_\lambda$, always weighted with a_λ^{-1}. The right hand side involves the four subfactors $L/(B+B')$, $B/(B\cap B')$, $B'/(B\cap B')$, and $B\cap B'$; their isomorphism classes are denoted by ρ, σ', σ and τ, respectively.



We stress that the right hand side of the formula only involves small modules (namely modules which occur as submodules or factor modules of the given modules $V_\alpha, V_\beta, V_{\alpha'}, V_{\beta'}$), whereas the left hand side deals with modules V_λ, where $\dim_k V_\lambda = \dim_k V_\alpha + \dim_k V_\beta$.

2. Examples

We usually denote the number of elements of k by q_k; in this section, we just write q.

A. Consider two non-isomorphic simple modules $S(1), S(2)$ with

$$\mathrm{End}(S(1)) = \mathrm{End}(S(2)) = k \quad \text{and} \quad \dim_k \mathrm{Ext}^1(S(1), S(2)) = n.$$

For example, we may consider the path algebra Λ of a quiver with two vertices and precisely n arrows, all of which start at the same vertex and end at the other one.

Let $\alpha = \alpha'$ be the isomorphism class of $S(1)$ and $\beta = \beta'$ the isomorphism class of $S(2)$.

On the left side, all possible isomorphism classes λ have to be considered. There is the semisimple module $S(1) \oplus S(2)$ with automorphism group $k^* \times k^*$ of order $(q-1)^2$. In addition, there are the indecomposable modules of length 2 with top $S(1)$ and socle $S(2)$. They have automorphism group k^* of order $q-1$. The number of isomorphism classes of such modules is $\dfrac{q^n-1}{q-1}$. Altogether we have $1 + \dfrac{q^n-1}{q-1}$ isomorphism classes λ. Always, we have $g^\lambda_{\alpha\beta} = g^\lambda_{\alpha'\beta'} = 1$. Thus the left hand side is

$$\begin{aligned} a_\alpha a_\beta a_{\alpha'} a_{\beta'} \sum_\lambda g^\lambda_{\alpha\beta} g^\lambda_{\alpha'\beta'}\, a_\lambda^{-1} & \\ &= (q-1)^4 \Big(1 \cdot 1 \cdot \frac{1}{(q-1)^2} + \frac{q^n-1}{q-1} \cdot 1 \cdot 1 \cdot \frac{1}{q-1}\Big) \\ &= (q-1)^2 q^n. \end{aligned}$$

The right hand side has only one non-zero summand, namely for ρ the isomorphism class of $S(1)$ and τ the isomorphism class of $S(2)$ (and $\sigma = \sigma' = 0$). Since the order of $\mathrm{Ext}^1(S(1), S(2))$ is q^n, whereas $\mathrm{Hom}(S(1), S(2)) = 0$ and the order of the zero group 0 is 1, we get:

$$\begin{aligned} \frac{|\operatorname{Ext}^1(V_\rho, V_\tau)|}{|\operatorname{Hom}(V_\rho, V_\tau)|}\, g^\alpha_{\rho\sigma} g^{\alpha'}_{\rho\sigma'} g^\beta_{\sigma'\tau} g^{\beta'}_{\sigma\tau}\, a_\rho a_\sigma a_{\sigma'} a_\tau & \\ &= q^n \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot (q-1) \cdot 1 \cdot 1 \cdot (q-1). \end{aligned}$$

(The number n of arrows of the quiver appears in the right hand term only in the extra factor $|\operatorname{Ext}^1(V_\rho, V_\tau)|$, the remaining factors are independent of n. In the left hand term, n determines the number of summands.)

B. Consider a fixed simple module S with $\operatorname{End}(S) = k$, and assume that $\dim_k \operatorname{Ext}^1(S,S) = n$. Let $\alpha = \alpha' = \beta = \beta'$ be the isomorphism class of S.

On the left side, the possible isomorphism classes λ are as follows: V_λ may be semisimple, thus of the form $S \oplus S$; then its automorphism group is $\mathrm{Gl}_2(k)$, of order $q(q^2-1)(q-1)$, and we have $g^\lambda_{\alpha\beta} = g^\lambda_{\alpha'\beta'} = q+1$. Or else V_λ is an indecomposable module of length 2 with top S and socle S. Then the automorphism group is of order $q(q-1)$ and $g^\lambda_{\alpha\beta} = g^\lambda_{\alpha'\beta'} = 1$. Note that the number of isomorphism classes of such indecomposable length 2 modules is again $\dfrac{q^n-1}{q-1}$. Thus the left hand side ist

$$\begin{aligned} a_\alpha a_\beta a_{\alpha'} a_{\beta'} \sum_\lambda g^\lambda_{\alpha\beta} g^\lambda_{\alpha'\beta'}\, a_\lambda^{-1} \\ &= (q-1)^4\Big((q+1)\cdot(q+1)\cdot\frac{1}{q(q^2-1)(q-1)} + \frac{q^n-1}{q-1} 1\cdot 1\cdot \frac{1}{q(q-1)}\Big) \\ &= (q-1)^4 \cdot \frac{q^n+q}{q(q-1)^2} = (q^{n-1}+1)(q-1)^2. \end{aligned}$$

The right hand side has two non-zero summands: One is given for ρ and τ the isomorphism class of S, (and $\sigma = \sigma' = 0$). Here we have to take into account that the order of $\operatorname{Ext}^1(S,S)$ is q^n, while the order of $\operatorname{Hom}(S,S) = k$ is q. The second summand to be considered occurs for $\rho = \tau = 0$ and $\sigma = \sigma'$ the isomorphism class of S. We get:

$$\begin{aligned} \sum \frac{|\operatorname{Ext}^1(V_\rho, V_\tau)|}{|\operatorname{Hom}(V_\rho, V_\tau)|}\, g^\alpha_{\rho\sigma} g^{\alpha'}_{\rho\sigma'} g^\beta_{\sigma'\tau} g^{\beta'}_{\sigma\tau}\, a_\rho a_\sigma a_{\sigma'} a_\tau \\ &= \frac{q^n}{q}(q-1)^2 + \frac{1}{1}(q-1)^2 = (q^{n-1}+1)(q-1)^2. \end{aligned}$$

3. Preliminary results

We exhibit several elementary counting formulae:

Let A, B, L be Λ-modules. Let $\mathcal{E}^L_{AB}$ be the set of pairs (a,b) of Λ-homomorphisms such that

$$0 \to B \xrightarrow{b} L \xrightarrow{a} A \to 0$$

is an exact sequence.

If $\dim_k L \neq \dim_k A + \dim_k B$, then $\mathcal{E}^L_{AB}$ is empty. If $\dim_k L = \dim_k A + \dim_k B$, then $\mathcal{E}^L_{AB}$ consists of all pairs (a,b) where $a\colon L \to A$ is an epimorphism, $b\colon B \to L$ is a monomorphism and $ab = 0$.

Proposition 1. *Let A, B, L be Λ-modules and α, β, λ their isomorphism classes, respectively. Then*

$$|\mathcal{E}^L_{AB}| = g^\lambda_{\alpha\beta} a_\alpha a_\beta.$$

Proof: Let U be the image of $b\colon B \to L$, thus $b = u\underline{b}$, where $\underline{b}\colon B \to U$ is an isomorphism and $u\colon U \to L$ is the inclusion map. Similarly, write $a = \underline{a}p$, where $p\colon L \to L/U$ is the canonical projection and $\underline{a}\colon L/U \to A$ is an isomorphism. The set of tripels $(U, \underline{a}, \underline{b})$ obtained in this way consists of a submodule U of L such that U belongs to the isomorphism class β, whereas L/U belongs to the isomorphism class α, and arbitrary isomorphisms $\underline{a}\colon L/U \to A$ and $\underline{b}\colon B \to U$. Thus the number of such tripels is precisely $g^{\lambda}_{\alpha\beta} a_\alpha a_\beta$.

A Λ-module is given by a pair (V, γ), where V is a k-space and $\gamma\colon \Lambda \to \mathrm{End}_k(V)$ is a k-algebra homomorphism.

Proposition 2. *Let $\lambda \in \mathcal{P}$. Let V be a k-space of dimension $\dim_k V_\lambda$. The number of pairs (V, γ), where $\gamma\colon \Lambda \to \mathrm{End}_k(V)$ is a k-algebra homomorphism such that the Λ-module (V, γ) belongs to λ, is*

$$|\,\mathrm{Aut}_k(V)| \cdot a_\lambda^{-1}.$$

Proof: Observe that $G = \mathrm{Aut}_k(V)$ operates on the set of k-algebra homomorphisms γ as follows: $g * \gamma = g\gamma g^{-1}$ for $g \in G$ (that means: given $r \in \Lambda$, let $(g * \gamma)(r) = g\gamma(r)g^{-1}$). The orbits under this action are just the isomorphism classes, whereas the stabilizer G_γ of a fixed element γ is just the automorphism group of the Λ-module (V, γ). The assertion follows from the well-known fact that the number of elements in the orbit of γ is just $|G| \cdot |G_\gamma|^{-1}$.

Proposition 3. *Consider the following commutative diagram of Λ-module homomorphisms, with exact rows*

$$\begin{array}{ccccccccc}
 & & X' & \xrightarrow{x} & X & \xrightarrow{x'} & X'' & \longrightarrow & 0 \\
 & & {\scriptstyle d'}\downarrow & & \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle d''} & & \\
0 & \longrightarrow & Y' & \xrightarrow{y} & Y & \xrightarrow{y'} & Y''. & &
\end{array}$$

Then the number of possible homomomorphisms d (with all the other maps x, x', y, y', d', d'' fixed) is equal to $|\,\mathrm{Hom}_\Lambda(X'', Y')|$.

Proof: Let $\mathcal{D}$ be the set of all $d \in \mathrm{Hom}_\Lambda(X, Y)$ which satisfy $dx = yd'$ and $y'd = d''x'$. Fix some element $d_0 \in \mathcal{D}$.

Given $z \in \mathrm{Hom}_\Lambda(X'', Y')$, then also $d_0 + yzx'$ belongs to $\mathcal{D}$, since $x'x = 0$ and $y'y = 0$. Since y is injective and x' is surjective, we see that the map $\mathrm{Hom}_\Lambda(X'', Y') \to \mathcal{D}$, which sends z to $d_0 + yzx'$ is injective. The map is also surjective: Given $d \in \mathcal{D}$, then $(d - d_0)x = 0$, thus we can factor the homomorphism

$d - d_0$ through the cokernel of x, thus $d - d_0 = z'x'$ for some $z'\colon X'' \to Y$. Now $y'z'x' = y'(d-d_0) = 0$, the surjectivity of x' yields $y'z' = 0$, and therefore z' can be factored through the kernel y of y'. We obtain $z\colon X'' \to Y'$ with $yz = z'$. Altogether we have $d - d_0 = z'x' = yzx'$.

Let A, B, L be Λ-modules, ε an element of $\mathrm{Ext}^1_\Lambda(A,B)$, and let $\mathcal{E}^L(\varepsilon)$ be the set of pairs (a,b) of Λ-homomorphisms such that

$$0 \to B \xrightarrow{b} L \xrightarrow{a} A \to 0$$

is an exact sequence and belongs to the class ε.

Proposition 4. *Let A, B, L be Λ-modules. Let λ be the isomorphism class of L. Let ε be an element of $\mathrm{Ext}^1_\Lambda(A,B)$. Then the cardinality of $\mathcal{E}^L(\varepsilon)$ is either zero or else*

$$\frac{a_\lambda}{|\operatorname{Hom}_\Lambda(A,B)|}.$$

Proof: Assume that there exists an exact sequence

$$0 \to B \xrightarrow{b} L \xrightarrow{a} A \to 0,$$

which belongs to ε, thus the pair (a,b) belongs to $\mathcal{E}^L(\varepsilon)$.

Given an automorphism h of L, let

$$h * (a,b) = (ah^{-1}, hb).$$

The commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "b"] \arrow[d, equal] & L \arrow[r, "a"] \arrow[d, "h"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, "hb"] & L \arrow[r, "ah^{-1}"] & A \arrow[r] & 0
\end{tikzcd}$$

shows that the pair (ah^{-1}, hb) again belongs to $\mathcal{E}^L(\varepsilon)$. In this way, the group $\mathrm{Aut}_\Lambda(L)$ operates on the set $\mathcal{E}^L(\varepsilon)$. By the definition of the equivalence relation which gives $\mathrm{Ext}^1(A,B)$, the operation by $\mathrm{Aut}_\Lambda(L)$ on $\mathcal{E}^L(\varepsilon)$ is transitive. Thus, it remains to calculate the stabilizer of an element. According to Proposition 3, the order of the stabilizer of (a,b) is equal to $|\operatorname{Hom}_\Lambda(A,B)|$. This completes the proof.

Given a k-space V, an exact sequence of the form

$$0 \to B \to (V,\gamma) \to A \to 0,$$

with a k-algebra homomorphism $\gamma\colon \Lambda \to \operatorname{End}_k(V)$ will be called a *V-sequence.* Let ε be an element of $\operatorname{Ext}^1_\Lambda(A,B)$. Let $\mathcal{E}(\varepsilon)$ be the set of V-sequences in ε, where V is a fixed k-space of dimension $\dim_k A + \dim_k B$.

Proposition 5. *Let A, B be Λ-modules. Let ε be an element of $\operatorname{Ext}^1_\Lambda(A,B)$. Then*

$$|\mathcal{E}(\varepsilon)| = \frac{|\operatorname{Aut}_k(V)|}{|\operatorname{Hom}_\Lambda(A,B)|}.$$

Proof: Let

$$0 \to B \xrightarrow{b} L \xrightarrow{a} A \to 0$$

be an element of ε. Let λ be the isomorphism class of L. The number of algebra homomorphisms $\gamma\colon \Lambda \to \operatorname{End}(V)$ such that (V,γ) belongs to λ is given by $\frac{|\operatorname{Aut}_k(V)|}{a_\lambda}$, according to Proposition 2.

For any such γ the number of pairs (a,b) such that (γ,a,b) belongs to $\mathcal{E}(\varepsilon)$ is given by $\frac{a_\lambda}{|\operatorname{Hom}_\Lambda(A,B)|}$, according to Proposition 4 (note that for any γ, there is at least one such triple, since any isomorphism $h\colon L \to (V,\gamma)$ yields a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r, "b"] \arrow[d, equal] & L \arrow[r, "a"] \arrow[d, "h"] & A \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & B \arrow[r, "hb"] & (V,\gamma) \arrow[r, "ah^{-1}"] & A \arrow[r] & 0,
\end{tikzcd}$$

and this shows that the lower exact sequence again belongs to ε.)

Altogether, we see that

$$|\mathcal{E}(\varepsilon)| = \frac{|\operatorname{Aut}_k(V)|}{a_\lambda} \cdot \frac{a_\lambda}{|\operatorname{Hom}_\Lambda(A,B)|} = \frac{|\operatorname{Aut}_k(V)|}{|\operatorname{Hom}_\Lambda(A,B)|}.$$

4. Factorizations of homomorphisms

Let $f\colon Y \to X$ be a homomorphism of Λ-modules and M the image of f. Let V be a k-space of dimension $\dim_k X + \dim_k Y - \dim_k M$.

For $L = (V,\gamma)$, let $\mathcal{F}(f;L)$ be the set of all pairs (c,d) where $c\colon L \to X$ is an epimorphism, $d\colon Y \to L$ a monomorphism, and $cd = f$. Let $\mathcal{F}(f)$ be the set of tripels (γ,c,d) where $\gamma\colon \Lambda \to \operatorname{End}_k(V)$ is an algebra homomorphism and (c,d) belongs to $\mathcal{F}(f;(V,\gamma))$.

If we denote by $y\colon Y \to M$ the projection and by $x\colon M \to X$ the inclusion map (thus $f = xy$), then we are looking for all possible factorizations $f = cd$ such that the commutative diagram

$$\begin{tikzcd}[column sep=small] & M \arrow[dr, "x"] & \\ Y \arrow[ur, "y"] \arrow[dr, "d"'] & & X \\ & L \arrow[ur, "c"'] & \end{tikzcd}$$

is both a pushout and a pullback diagram.

Proposition. *Let $f\colon Y \to X$ be a homomorphism of Λ-modules. Let T be the kernel and R the cokernel of f. Assume that $\operatorname{Ext}^2(R,T)=0$. Then*

$$|\mathcal{F}(f)| = |\operatorname{Aut}_k(V)| \frac{|\operatorname{Ext}^1_\Lambda(R,T)|}{|\operatorname{Hom}_\Lambda(R,T)|}.$$

Remark. This is the only calculation where the vanishing of Ext^2 enters. It is required to ensure that $\mathcal{F}(f)$ is non-empty. For a short proof that $\mathcal{F}(f) \neq \emptyset$, we refer to [HR].

Proof of Proposition: We denote by $y'\colon T \to Y$ and $x\colon M \to X$ the inclusion maps, by $y\colon Y \to M$ and $x'\colon X \to R$ the canonical projections; in particular, we have $f = xy$. We denote by ε_0 the equivalence class of the exact sequence

$$0 \to M \xrightarrow{x} X \xrightarrow{x'} R \to 0$$

in $\operatorname{Ext}^1_\Lambda(R,M)$.

The exact sequence

$$0 \longrightarrow T \xrightarrow{\ y'\ } Y \xrightarrow{\ y\ } M \longrightarrow 0$$

gives rise to a long exact sequence

$$\begin{array}{llllll} 0 & \longrightarrow \operatorname{Hom}_\Lambda(R,T) & \longrightarrow & \operatorname{Hom}_\Lambda(R,Y) & \longrightarrow & \operatorname{Hom}_\Lambda(R,M) \\ & \longrightarrow \operatorname{Ext}^1_\Lambda(R,T) & \xrightarrow{\operatorname{Ext}^1_\Lambda(R,y')} & \operatorname{Ext}^1_\Lambda(R,Y) & \xrightarrow{\operatorname{Ext}^1_\Lambda(R,y)} & \operatorname{Ext}^1_\Lambda(R,M) \\ & \longrightarrow \quad 0 & & & & \end{array}$$

where we use that $\operatorname{Ext}^2_\Lambda(R,T)=0$. We denote by E the image of $\operatorname{Ext}^1_\Lambda(R,y')$, thus we have an exact sequence

$$0 \to \operatorname{Hom}_\Lambda(R,T) \to \operatorname{Hom}_\Lambda(R,Y) \to \operatorname{Hom}_\Lambda(R,M) \to \operatorname{Ext}^1_\Lambda(R,T) \to E \to 0,$$

and consequently

$$|E| = |\operatorname{Hom}_\Lambda(R,T)|^{-1} \cdot |\operatorname{Hom}_\Lambda(R,Y)| \cdot |\operatorname{Hom}_\Lambda(R,M)|^{-1} \cdot |\operatorname{Ext}^1_\Lambda(R,T)|.$$

Given $(\gamma, c, d) \in \mathcal{F}(f)$, the following diagram with $d' = x'c$ is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{d} & (V,\gamma) & \xrightarrow{d'} & R & \longrightarrow & 0 \\
 & & {\scriptstyle y}\big\downarrow & & \big\downarrow{\scriptstyle c} & & \big\| & & \\
0 & \longrightarrow & M & \xrightarrow{x} & X & \xrightarrow{x'} & R & \longrightarrow & 0.
\end{array}$$

Also, its upper row is exact: since both c and x' are surjective, also d' is surjective; on the other hand, we have $d'd = x'cd = x'f = x'xy = 0$, thus the exactness of the upper row follows from the equality $\dim_k V = \dim_k X + \dim_k Y - \dim_k M = \dim_k Y + \dim_k R$.

We define a map $\omega\colon \mathcal{F}(f) \to \operatorname{Ext}^1_\Lambda(R, Y)$, sending the tripel (γ, c, d) to the equivalence class $\omega(\gamma, c, d)$ of the exact sequence

$$0 \to Y \xrightarrow{d} (V,\gamma) \xrightarrow{x'c} R \to 0.$$

We claim that the image of the map ω is just the inverse image of ε_0 under $\operatorname{Ext}^1_\Lambda(R, y)$. On the one hand, the diagram above shows that ε_0 is induced from $\omega(\gamma, c, d)$ via the homomorphism y. Conversely, assume a V-sequence ε in $\operatorname{Ext}^1_\Lambda(R, Y)$ induces ε_0 via the homomorphism y. Thus, there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y & \xrightarrow{d} & (V,\gamma) & \xrightarrow{d'} & R & \longrightarrow & 0 \\
 & & {\scriptstyle y}\big\downarrow & & \big\downarrow{\scriptstyle c} & & \big\| & & \\
0 & \longrightarrow & M & \xrightarrow{x} & X & \xrightarrow{x'} & R & \longrightarrow & 0
\end{array}$$

and ε is the equivalence class of the upper row. Then (γ, c, d) belongs to $\mathcal{F}(f)$ and $d' = x'c$, thus $\varepsilon = \omega(\gamma, c, d)$.

Since the image of ω is the inverse image of ε_0 under $\operatorname{Ext}^1_\Lambda(R, y)$, and since $\operatorname{Ext}^1_\Lambda(R, y)$ is surjective, the image of ω is a residue class E' modulo the kernel of $\operatorname{Ext}^1_\Lambda(R, y)$. But this kernel is just E, thus the image of ω has order $|E'| = |E|$. For any element ε in E', the number of V-sequences in ε is

$$\frac{|\operatorname{Aut}_k(V)|}{|\operatorname{Hom}_\Lambda(R, Y)|}$$

according to Proposition 5 of Section 3. For a given V-sequence

$$0 \to Y \xrightarrow{d} (V,\gamma) \xrightarrow{d'} R \to 0,$$

the number of possible maps c satisfying $cd = xy$ and $x'c = d'$ is $|\operatorname{Hom}_\Lambda(R, M)|$ according to Proposition 3 of Section 3. Altogether, we see that

$$\begin{aligned}
|\mathcal{F}(f)| &= |E| \frac{|\operatorname{Aut}_k(V)|}{|\operatorname{Hom}_\Lambda(R, Y)|} |\operatorname{Hom}_\Lambda(R, M)| \\
&= |\operatorname{Aut}_k(V)| \frac{|\operatorname{Ext}^1_\Lambda(R, T)|}{|\operatorname{Hom}_\Lambda(R, T)|}.
\end{aligned}$$

This completes the proof.

* * *

We consider now Green's formula in detail. First, we interpret the left hand side, then the right hand side of the formula (both multiplied with a fixed factor, namely $|\operatorname{Aut}_k(V)|$, where V is a k-space). We will see that the left hand side counts suitable "crosses", while the right hand side counts pairs consisting of a "frame" and the factorization of a map derived from it. Here, the frames and the crosses are parts of the 3×3 configuration in Homological Algebra (consisting of three rows and three columns all of which are short exact sequences). It will be easy to attach to every cross a corresponding frame. The main interest lies in the fact that one can count the number of crosses which yield the same frame.

We fix elements α, β, α' and β' of $\mathcal{P}$. We want to derive Green's formula. We can assume that

$$\dim_k V_\alpha + \dim_k V_\beta = \dim_k V'_\alpha + \dim_k V'_\beta,$$

since otherwise the formula is trivially satisfied: both sides are zero (since $g^\lambda_{\alpha\beta} \neq 0$ implies that $\dim_k V_\alpha + \dim_k V_\beta = \dim_k V_\lambda$).

Fix Λ-modules $A \in \alpha$, $B \in \beta$, $A' \in \alpha'$ and $B' \in \beta'$. Also, fix a k-space V of dimension $\dim_k A + \dim_k B$.

5. Green's formula: The left hand side

Let $\gamma\colon \Lambda \to \operatorname{End}_k(V)$ be a k-algebra homomorphism and $L = (V,\gamma)$. We consider the set

$$\mathcal{Q}(L) = \mathcal{E}^L_{AB} \times \mathcal{E}^L_{A'B'},$$

and we denote by $\mathcal{Q}$ the union of the sets $\mathcal{Q}(V,\gamma)$. Thus $\mathcal{Q}$ is the set of quintets (γ, a, b, a', b') where $\gamma\colon \Lambda \to \operatorname{End}_k(V)$ is a k-algebra homomorphism, and a, b, a', b' are Λ-module homomorphisms such that we obtain exact sequences

$$0 \longrightarrow B \xrightarrow{\ b\ } (V,\gamma) \xrightarrow{\ a\ } A \longrightarrow 0,$$
$$0 \longrightarrow B' \xrightarrow{\ b'\ } (V,\gamma) \xrightarrow{\ a'\ } A' \longrightarrow 0.$$

The data given by the elements of $\mathcal{Q}$ may be visualized as forming the following crosses:

$$\begin{array}{ccccc} & & B & & \\ & & \Big\downarrow{\scriptstyle b} & & \\ B' & \xrightarrow{\ b'\ } & (V,\gamma) & \xrightarrow{\ a'\ } & A' \\ & & \Big\downarrow{\scriptstyle a} & & \\ & & A & & \end{array}$$

Proposition.

$$|\mathcal{Q}| = |\operatorname{Aut}_k(V)| \cdot a_\alpha a_\beta a_{\alpha'} a_{\beta'} \sum_\lambda g^\lambda_{\alpha\beta} g^\lambda_{\alpha'\beta'}\, a_\lambda^{-1}.$$

Proof: Consider some $L = (V, \gamma)$. According to Proposition 1 of Section 3, we know

$$|\mathcal{E}^L_{AB} \times \mathcal{E}^L_{A'B'}| = g^\lambda_{\alpha\beta} a_\alpha a_\beta \cdot g^\lambda_{\alpha'\beta'} a_{\alpha'} a_{\beta'}.$$

According to Proposition 2 of Section 3, the number of choices for γ with $(V, \gamma) \in \lambda$ is $|\operatorname{Aut}_k(V)| \cdot a_\lambda^{-1}$. Thus the number of elements of $\mathcal{Q}$ is equal to

$$\sum_\lambda |\operatorname{Aut}_k(V)| \cdot a_\lambda^{-1} \cdot g^\lambda_{\alpha\beta} a_\alpha a_\beta \cdot g^\lambda_{\alpha'\beta'} a_{\alpha'} a_{\beta'}.$$

6. Green's formula: The right hand side

We introduce the set $\mathcal{R}$ of all sextets $(S, T, e_1, e_2, e_3, e_4)$, where S is a submodule of A, where T is a submodule of B, where (e_3, e_1) belongs to $\mathcal{E}^{B'}_{ST}$, and (e_4, e_2) belongs to $\mathcal{E}^{A'}_{A/S,B/T}$.

The data given by elements of $\mathcal{R}$ may be visualized as frames of the following kind:

$$\begin{tikzcd}
T \arrow[r, "u_T"] \arrow[d, "e_1"'] & B \arrow[r, "q_T"] & B/T \arrow[d, "e_2"] \\
B' \arrow[d, "e_3"'] & & A' \arrow[d, "e_4"] \\
S \arrow[r, "u_S"'] & A \arrow[r, "q_S"'] & A/S
\end{tikzcd}$$

For the submodule S of A, we have denoted the inclusion map $S \to A$ by u_S, and the canonical projection $A \to A/S$ by q_S. Similarly, for the submodule T of B, we have denoted by $u_T \colon T \to B$ the inclusion and by $q_T \colon B \to B/T$ the projection.

Let B, B' be Λ-modules and $e \colon T \to B'$ a monomorphism, where T is a submodule of B. We denote by $B \sqcup_e B'$ the pushout of the inclusion map $u_T \colon T \to B$ and the map e, thus

$$B \sqcup_e B' = B \oplus B' / \{(m, -e(m)) \,|\, m \in T\}.$$

We denote by $u_B\colon B \to B \underset{e}{\sqcup} B'$ and $u_{B'}\colon B' \to B \underset{e}{\sqcup} B'$ the canonical inclusion maps. Thus, we have the following commutative square:

$$\begin{array}{ccc} T & \xrightarrow{u_T} & B \\ {\scriptstyle e}\big\downarrow & & \big\downarrow{\scriptstyle u_B} \\ B' & \xrightarrow[u_{B'}]{} & B \underset{e}{\sqcup} B'. \end{array}$$

We also consider the dual situation. We start with Λ-modules A and A', and an epimorphism $e'\colon A' \to A/S$, where S is a submodule of A. We denote by $A \overset{e'}{\sqcap} A'$ the pullback of the projection map $q_S\colon A \to A/S$ and e', thus

$$A \overset{e'}{\sqcap} A' = \{(m, m') \,|\, m \in A,\ m' \in A', m + S = e'(m')\}.$$

We denote by $p_A\colon A \overset{e'}{\sqcap} A' \to A$ and $p_{A'}\colon A \overset{e'}{\sqcap} A' \to A'$ the canonical projections.

Given $\Xi = (S, T, e_1, e_2, e_3, e_4) \in \mathcal{R}$, we define

$$Y = Y(\Xi) = B \underset{e_1}{\sqcup} B', \qquad X = X(\Xi) = A \overset{e_4}{\sqcap} A'.$$

Lemma. *Given $\Xi = (S, T, e_1, e_2, e_3, e_4) \in \mathcal{R}$, there exists a unique map $f = f(\Xi)\colon Y(\Xi) \to X(\Xi)$ satisfying*

$$\begin{aligned} p_A f u_B &= 0, & p_A f u_{B'} &= u_S e_3, \\ p_{A'} f u_B &= e_2 q_T, & p_{A'} f u_{B'} &= 0. \end{aligned}$$

The kernel of f is isomorphic to T and the cokernel of f is isomorphic to A/S.

$$\begin{tikzcd}
T \arrow[rr, "u_T"] \arrow[dd, "e_1"'] & & B \arrow[rr, "q_T"] \arrow[dl, "u_B"'] & & B/T \arrow[dd, "e_2"] \\
& Y \arrow[ddrr, "f(\Xi)"] & & & \\
B' \arrow[ur, "u_{B'}"] \arrow[dd, "e_3"'] & & & & A' \arrow[dd, "e_4"] \\
& & & X \arrow[ur, "p_{A'}"'] \arrow[dl, "p_A"] & \\
S \arrow[rr, "u_S"'] & & A \arrow[rr, "q_S"'] & & A/S
\end{tikzcd}$$

Proof: Since

$$0 \cdot u_T = 0 = u_S e_3 \cdot e_1,$$

and since Y is the pushout of u_T and e_1, there exists $f_A \colon Y \to A$ satisfying

$$f_A u_B = 0 \quad \text{and} \quad f_A u_{B'} = u_S e_3.$$

Similarly, since

$$e_2 q_T \cdot u_T = 0 = 0 \cdot e_1,$$

there exists $f_{A'} \colon Y \to A'$ satisfying

$$f_{A'} u_B = e_2 q_T \quad \text{and} \quad f_{A'} u_{B'} = 0.$$

Now we have

$$\begin{aligned} q_S f_A u_B = 0 = e_4 e_2 q_T = e_4 f_{A'} u_B, \\ q_S f_A u_{B'} = q_S u_S e_3 = 0 = e_4 f_{A'} u_{B'}. \end{aligned}$$

As a consequence, since X is the pullback of q_S and e_4, there exists $f \colon Y \to X$ such that

$$p_A f = f_A, \quad \text{and} \quad p_{A'} f = f_{A'}.$$

It follows that f satisfies the required equations.

The uniqueness of f comes from the fact that $[u_B \ u_{B'}] \colon B \oplus B' \to Y$ is an epimorphism and $\begin{bmatrix} p_A \\ p_{A'} \end{bmatrix} \colon X \to A \oplus A'$ is a monomorphism.

Finally, we have to determine kernel and cokernel of f. We claim that the following sequence is exact:

$$0 \longrightarrow T \xrightarrow{u_B u_T} Y \xrightarrow{f} X \xrightarrow{q_S p_A} A/S \longrightarrow 0.$$

First of all, observe that $f u_B u_T = 0$, since

$$\begin{aligned} p_A f u_B u_T &= 0 \cdot u_T = 0, \\ p_{A'} f u_B u_T &= e_2 q_T u_T = 0. \end{aligned}$$

Similarly, we see that $q_S p_A f = 0$. On the other hand, the homomorphism

$$\begin{bmatrix} p_A \\ p_{A'} \end{bmatrix} \cdot f \cdot [u_B \ u_{B'}] = \begin{bmatrix} 0 & u_S e_3 \\ e_2 q_T & 0 \end{bmatrix} = \begin{bmatrix} 0 & u_S \\ e_2 & 0 \end{bmatrix} \cdot \begin{bmatrix} q_T & 0 \\ 0 & e_3 \end{bmatrix}$$

has image isomorphic to $B/T \oplus S$. Since $\begin{bmatrix} p_A \\ p_{A'} \end{bmatrix}$ is a monomorphism, whereas $[u_B \ u_{B'}]$ is an epimorphism, also the image of f is isomorphic to $B/T \oplus S$. But we have

$$\begin{aligned} \dim_k Y &= \dim_k B + \dim_k B' - \dim_k T \\ &= \dim_k (B/T \oplus S) + \dim_k T. \end{aligned}$$

This shows that the sequence is exact at Y. A similar argument shows that the sequence is exact at X.

For any Λ-module L, let $\mathcal{O}(L)$ be the set of all octets $(S, T, e_1, e_2, e_3, e_4, c, d)$, where $\Xi = (S, T, e_1, e_2, e_3, e_4)$ belongs to $\mathcal{R}$ such that (c, d) belongs to $\mathcal{F}(f(\Xi); L)$.

Let $\mathcal{O}$ be the set of all tuples $(\gamma, S, T, e_1, e_2, e_3, e_4, c, d)$ where $\gamma\colon \Lambda \to \operatorname{End}_k(V)$ is a k-algebra homomorphism, and $(S, T, e_1, e_2, e_3, e_4, c, d)$ belongs to $\mathcal{O}(V, \gamma)$.

Proposition.

$$|\mathcal{O}| = |\operatorname{Aut}_k(V)| \cdot \sum_{\rho, \sigma, \sigma', \tau} \frac{|\operatorname{Ext}^1(V_\rho, V_\tau)|}{|\operatorname{Hom}(V_\rho, V_\tau)|}\, g^{\alpha}_{\rho\sigma} g^{\alpha'}_{\rho\sigma'} g^{\beta}_{\sigma'\tau} g^{\beta'}_{\sigma\tau}\, a_\rho a_\sigma a_{\sigma'} a_\tau.$$

Proof: Fix submodules S of A and T of B, let τ be the isomorphism class of T, σ the isomorphism class of S, σ' the isomorphism class of B/T and ρ the isomorphism class of A/S. Then the number of sextets of the form $\Xi = (S, T, e_1, e_2, e_3, e_4)$ in $\mathcal{R}$ (with fixed S, T) is equal to

$$g^{\alpha'}_{\rho\sigma'} g^{\beta'}_{\sigma\tau}\, a_\rho a_\sigma a_{\sigma'} a_\tau,$$

according to Proposition 1 of Section 3.

The previous Lemma asserts that the kernel of $f(\Xi)$ is isomorphic to V_τ, while its cokernel is isomorphic to V_ρ. According to Section 4, the number of pairs (c, d) in $\mathcal{F}(f(\Xi))$ is

$$|\operatorname{Aut}_k(V)| \cdot \frac{|\operatorname{Ext}^1(V_\rho, V_\tau)|}{|\operatorname{Hom}(V_\rho, V_\tau)|}.$$

Of course, the number of choices (S, T) with fixed $\rho, \sigma, \sigma', \tau$ is

$$g^\alpha_{\rho\sigma} g^\beta_{\sigma'\tau}.$$

Adding up all the possibilities, we obtain the formula as stated.

7. More about pushouts and pullbacks

Proposition. *Let B, B', L be Λ-modules. There is a bijection η between the set of pairs (b, b') where $b\colon B \to L$ and $b'\colon B' \to L$ are monomorphisms, and the set of triples (T, e, d) where T is a submodule of B, such that both $e\colon T \to B'$ and $d\colon B \sqcup_e B' \to L$ are monomorphisms.*

Here, $\eta(b, b') = (T, e, d)$, where $T = b^{-1}b'(B')$. The map e is defined by $e(m) = (b')^{-1}b(m)$ for $m \in T$, and d is defined by $du_B = b,\ du_{B'} = b'$.

Moreover, η^{-1} is defined by $\eta^{-1}(T, e, d) = (du_B, du_{B'})$.

Proof: Let $b\colon B \to L$ and $b'\colon B' \to L$ be monomorphisms. Of course, $T = b^{-1}b'(B')$ is a submodule of B, and $e(m) = (b')^{-1}b(m)$ for $m \in T$ defines a monomorphism $e\colon T \to B'$. Thus, the pushout $B \sqcup_e B'$ is defined. Since we have the following commutative diagram

$$\begin{tikzcd} T \arrow[r, "u_T"] \arrow[d, "e"'] & B \arrow[d, "b"] \\ B' \arrow[r, "b'"'] & L, \end{tikzcd}$$

there exists a unique $d\colon B \sqcup_e B' \to L$ such that $du_B = b,\ du_{B'} = b'$.

Consider the following diagram:

$$\begin{tikzcd} T \arrow[rr, "u_T"] \arrow[dd, "e"'] & & B \arrow[dl, "u_B"'] \arrow[dd, "b"] \\ & Y \arrow[dr, "d"] & \\ B' \arrow[ur, "u_{B'}"] \arrow[rr, "b'"'] & & L \end{tikzcd}$$

where $Y = B \underset{e}{\sqcup} B'$.

In order to see that d is a monomorphism, note that the map $\begin{bmatrix} u_T \\ -e \end{bmatrix} : T \to B \oplus B'$ is the kernel of $[b\ b']\colon B \oplus B' \to L$, and the map $[u_B\ u'_B]\colon B \oplus B \to B \underset{e}{\sqcup} B'$ is the cokernel of $\begin{bmatrix} u_T \\ -e \end{bmatrix}$.

The converse recipe is easier: given a triple (T, e, d), where T is a submodule of B, and both maps $e\colon T \to B'$ and $d\colon B \underset{e}{\sqcup} B' \to L$ are monomorphisms, we form the pair $\eta'(T, e, d) = (du_B, du_{B'})$. Since the maps $d, u_B, u_{B'}$ are monomorphisms, also du_B and $du_{B'}$ are monomorphisms.

We claim that η is bijective with inverse η'. Of course, we trivially have $\eta'\eta(b, b') = (b, b')$. Consider a triple (T, e, d) and let $b = du_B,\ b' = du_{B'}$. One easily checks that $T = b^{-1}b'(B')$, and that $e(m) = (b')^{-1}b(m)$ for $m \in T$. As a consequence, $\eta(b, b') = (T, e, d)$. This completes the proof.

We present a dual version of the previous Proposition:

Dual Proposition. *Let A, A', L be Λ-modules. There is a bijection η between the set of pairs (a, a') where $a\colon L \to A$ and $a'\colon L \to A'$ are epimorphisms, and the set of triples (S, e', c), where S is a submodule of A and $e'\colon A' \to A/S$ and $c\colon L \to A \overset{e'}{\sqcap} A'$ are epimorphisms.*

Here, $\eta(a, a') = (S, e', c)$, where $S = ab'(B')$, the map e' being defined by $e'(m') = a(a')^{-1}(m') + S$ for $m' \in A'$, and c the unique homomorphism $L \to A \overset{e'}{\sqcap} A'$ satisfying $p_A c = a,\ p_{A'}c = a'$.

Moreover, η^{-1} is defined by $\eta^{-1}(S, e', c) = (p_A c, p_{A'}c)$.

Note that $ab'(B') = a(\operatorname{Ker} a')$.

Consider the following diagram:

$$\begin{tikzcd}
L \arrow[rr, "a'"] \arrow[dr, "c"'] \arrow[dd, "a"'] & & A' \arrow[dd, "e'"] \\
& X \arrow[ur, "p_{A'}"'] \arrow[dl, "p_A"] & \\
A \arrow[rr, "q_S"'] & & A/S
\end{tikzcd}$$

where $X = A \overset{e'}{\sqcap} A'$. Of course, the composition $p = q_S p_A (= e' p_{A'})$ is an epimorphism $p\colon A \overset{e'}{\sqcap} A' \to A/S$ and its kernel is just $S \oplus \mathrm{Ker}(e')$.

8. Proof of Green's formula

Proposition. *There exists a bijection*

$$\eta\colon \mathcal{Q}(L) \longrightarrow \mathcal{O}(L).$$

Proof: Given $(a,b) \in \mathcal{E}^L_{AB}$ and $(a',b') \in \mathcal{E}^L_{A'B'}$, let $\eta(b,b') = (T, e_1, d)$ and $\eta(a,a') = (S, e_4, c)$, as defined above. Thus, we have diagrams which are pushouts and pullbacks

$$\begin{array}{ccc} T & \xrightarrow{u_T} & B \\ {\scriptstyle e_1}\downarrow & & \downarrow{\scriptstyle u_B} \\ B' & \xrightarrow[u_{B'}]{} & Y \end{array} \qquad\qquad \begin{array}{ccc} X & \xrightarrow{p_{A'}} & A' \\ {\scriptstyle p_A}\downarrow & & \downarrow{\scriptstyle e_4} \\ A & \xrightarrow[q_S]{} & A/S \end{array}$$

and maps $d\colon Y \to L$, $c\colon L \to X$ such that

$$\begin{aligned} du_B &= b, & p_A c &= a, \\ du_{B'} &= b', & p_{A'} c &= a'. \end{aligned}$$

We denote by $q_T\colon B \to B/T$ the canonical projection, and by $u_S\colon S \to A$ the inclusion.

Note that u_T is the kernel of $a'b$, thus there exists a monomorphism $e_2\colon B/T \to A'$ such that

$$a'b = e_2 q_T.$$

Similarly, q_S is the cokernel of ab', thus there exists an epimorphism $e_3\colon B' \to S$ such that

$$ab' = u_S e_3,$$

We have

$$e_4 e_2 = 0 \quad \text{and} \quad e_3 e_1 = 0$$

(for example, $e_4 e_2 q_T = e_4 a' b = q_S a b = 0$). Actually, both sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B/T & \xrightarrow{e_2} & A' & \xrightarrow{e_4} & A/S & \longrightarrow & 0, \\ 0 & \longrightarrow & T & \xrightarrow{e_1} & B' & \xrightarrow{e_3} & S & \longrightarrow & 0 \end{array}$$

are exact: since e_1, e_2 are monomorphisms, e_3, e_4 are epimorphisms and $e_4e_1 = 0 = e_3e_2$, we know that

$$\begin{aligned}\dim_k A' &\geq \dim_k B/T + \dim_k A/S \quad \text{and}\\ \dim_k B' &\geq \dim_k T + \dim_k S.\end{aligned}$$

Adding up these two inequalities, we obtain

$$\begin{aligned}\dim_k L &= \dim_k A' + \dim_k B'\\ &\geq \dim_k B/T + \dim_k A/S + \dim_k T + \dim_k S\\ &= \dim_k A + \dim_k B = \dim_k L.\end{aligned}$$

In this way, we see that

$$\begin{aligned}\dim_k A' &= \dim_k B/T + \dim_k A/S, \quad \text{and}\\ \dim_k B' &= \dim_k T + \dim_k S.\end{aligned}$$

Altogether we see that (e_3, e_1) belongs to $\mathcal{E}^{B'}_{ST}$, and (e_4, e_2) to $\mathcal{E}^{A'}_{A/S,B/T}$, thus $\Xi = (S, T, e_1, e_2, e_3, e_4)$ belongs to $\mathcal{R}$.

Remark. Here, we use the following result (see [Mi], Proposition I.16.5.): The diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & T \arrow[r, "u_T"] \arrow[d, "e_1"'] & B \arrow[r, "q_T"] \arrow[d, "b"] & B/T \arrow[r] \arrow[d, "e_2"] & 0 \\
0 \arrow[r] & B' \arrow[r, "b'"] \arrow[d, "e_3"'] & L \arrow[r, "a'"] \arrow[d, "a"] & A' \arrow[r] \arrow[d, "e_4"] & 0 \\
0 \arrow[r] & S \arrow[r, "u_S"'] \arrow[d] & A \arrow[r, "q_S"'] \arrow[d] & A/S \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

is commutative with exact rows and exact columns if and only if the upper left square is a pullback, the lower right square is a pushout, q_T is an epimorphism, e_2 is a monomorphism with $e_2q_T = a'b$, while e_3 is an epimorphism, and u_S is a monomorphism with $u_Se_3 = ab'$.

Insert into this diagram also

$$Y = B \underset{e_1}{\sqcup} B' \quad \text{and} \quad X = A \overset{e_4}{\sqcap} A'$$

and the corresponding maps u_B, u'_B, d and p_A, p'_A, c :

$$\begin{tikzcd}
T \arrow[rr, "u_T"] \arrow[dd, "e_1"'] & & B \arrow[rr, "q_T"] \arrow[dl, "u_B"'] \arrow[dd, "b"] & & B/T \arrow[dd, "e_2"] \\
& Y \arrow[dr, "d"] & & & \\
B' \arrow[ur, "u_{B'}"] \arrow[rr, "b'"'] \arrow[dd, "e_3"'] & & L \arrow[rr, "a'"] \arrow[dr, "c"'] \arrow[dd, "a"'] & & A' \arrow[dd, "e_4"] \\
& & & X \arrow[ur, "p_{A'}"'] \arrow[dl, "p_A"] & \\
S \arrow[rr, "u_S"'] & & A \arrow[rr, "q_S"'] & & A/S
\end{tikzcd}$$

We have

$$\begin{aligned} p_A c d u_B &= ab = 0 & \qquad p_A c d u_{B'} &= ab' = u_S e_3 \\ p_{A'} c d u_B &= a'b = e_2 q_T & \qquad p_{A'} c d u_{B'} &= a'b' = 0. \end{aligned}$$

Thus $cd = f(\Xi)$. It follows that the octet $\eta(a, b, a', b') = (S, T, e_1, e_2, e_3, e_4, c, d)$ belongs to $\mathcal{O}(L)$.

Conversely, given an octet $(S, T, e_1, e_2, e_3, e_4, c, d)$ in $\mathcal{O}(L)$, form $\eta^{-1}(T, e_1, d) = (b, b')$ and $\eta^{-1}(S, e_4, c) = (a, a')$. We claim that

$$ab = 0, \quad \text{and} \quad a'b' = 0.$$

Namely, let $f = f(S, T, e_1, e_2, e_3, e_4)$. Then

$$ab = p_A c d u_B = p_A f u_B = 0 \quad \text{and} \quad a'b' = p_{A'} c d u_{B'} = p_{A'} f u_{B'} = 0.$$

This shows that (a, b) belongs to $\mathcal{E}^L_{AB}$ and that (a', b') belongs to $\mathcal{E}^L_{A'B'}$. Let $\eta'(S, T, e_1, e_2, e_3, e_4, c, d) = (a, b, a', b')$. As we have seen, this element belongs to $\mathcal{Q}(L)$.

By construction, we have $\eta'\eta(a, b, a', b') = (a, b, a', b')$ and it is easy to see that also $\eta\eta'(S, T, e_1, e_2, e_3, e_4, c, d) = (S, T, e_1, e_2, e_3, e_4, c, d)$. This completes the proof.

Proof of Green's Theorem: According to Section 8, we have $|\mathcal{Q}| = |\mathcal{O}|$, and by Sections 5 and 6, this yields the formula, with both sides multiplied by $|\operatorname{Aut}_k(V)|$.

Remark. Also the following converse of Green's theorem is true: In case Green's formula is satisfied for all finite Λ-modules, then the category of finite Λ-modules is hereditary.

9. Green's formula rewritten

Remember that Λ is a k-algebra, where k is a finite field, say with q_k elements. We denote by $K_0(\Lambda)$ the Grothendieck group of all finite Λ-modules modulo exact sequences. For any finite Λ-module M, we denote by $\mathbf{dim}\, M$ the corresponding element of $K_0(\Lambda)$. Let $I \subseteq \mathcal{P}$ be the set of isomorphism classes of simple Λ-modules. For $i \in I$, a representative of the isomorphism class will usually be denoted by $S(i)$, instead of V_i.

The Jordan-Hölder Theorem asserts that $K_0(\Lambda)$ is the free abelian group on the set I of isomorphism classes of simple Λ-modules, or better, on the set of dimension vectors $\mathbf{dim}\, S(i)$, and we may write

$$\mathbf{dim}\, M = \sum_{i \in I} [M : S(i)]\, \mathbf{dim}\, S(i),$$

for any finite Λ-module M. Here, $[M : S(i)]$ is the Jordan-Hölder multiplicity of $S(i)$ in M. It is the number of factors isomorphic to $S(i)$ in any fixed composition series of M.

Given Λ-modules M, N, let

$$\langle M, N \rangle = \dim_k \operatorname{Hom}(M, N) - \dim_k \operatorname{Ext}^1(M, N).$$

Since Λ is hereditary, $\langle M, N \rangle$ depends only on $\mathbf{dim}\, M, \mathbf{dim}\, N$. Thus, we can write

$$\langle \mathbf{dim}\, M, \mathbf{dim}\, N \rangle = \langle M, N \rangle,$$

and in this way, we obtain a bilinear form on $K_0(\Lambda)$ with values in $\mathbb{Z}$. Also, if we consider elements $\alpha, \beta \in \mathcal{P}$, then we may write

$$\langle \alpha, \beta \rangle = \langle V_\alpha, V_\beta \rangle.$$

Of course, since $|k| = q_k$, we have

$$q_k^{\langle M, N \rangle} = \frac{|\operatorname{Hom}(M, N)|}{|\operatorname{Ext}^1(M, N)|}.$$

In particular, the extra factor $|\operatorname{Ext}^1(V_\rho, V_\tau)| \cdot |\operatorname{Hom}(V_\rho, V_\tau)|^{-1}$ appearing in Green's formula may be written in the form

$$\frac{|\operatorname{Ext}^1(V_\rho, V_\tau)|}{|\operatorname{Hom}(V_\rho, V_\tau)|} = q_k^{-\langle V_\rho, V_\tau \rangle}.$$

Now, let v_k be a square root of q_k. The symmetrization of $\langle -,-\rangle$ will be denoted by $\bullet$, thus

$$\mathbf{x}\bullet\mathbf{y} = \langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{y},\mathbf{x}\rangle$$

for $\mathbf{x},\mathbf{y} \in K_0(\Lambda)$. Let

$$\overline{g}^{\lambda}_{\alpha\beta} = v_k^{\langle\alpha,\beta\rangle} g^{\lambda}_{\alpha\beta}.$$

Green's formula may be rewritten as follows:

Theorem* (J. A. Green). *Let k be a finite field and Λ a k-algebra which is hereditary and finitary. Let $\alpha,\beta,\alpha',\beta'$ be elements of $\mathcal{P}$. Then*

$$a_\alpha a_\beta a_{\alpha'} a_{\beta'} \sum_\lambda \overline{g}^{\lambda}_{\alpha\beta}\overline{g}^{\lambda}_{\alpha'\beta'}\, a_\lambda^{-1} = \sum_{\rho,\sigma,\sigma',\tau} {v_k}^{\sigma\bullet\sigma'}\; \overline{g}^{\alpha}_{\rho\sigma}\overline{g}^{\alpha'}_{\rho\sigma'}\overline{g}^{\beta}_{\sigma'\tau}\overline{g}^{\beta'}_{\sigma\tau}\, a_\rho a_\sigma a_{\sigma'} a_\tau.$$

Proof: The left hand term differs from the left hand term of Green's first formula just by $v_k^{\langle\alpha,\beta\rangle+\langle\alpha',\beta'\rangle}$.

We consider now the right hand side:

$$\begin{aligned}
&\sum_{\rho,\sigma,\sigma',\tau} {v_k}^{\sigma\bullet\sigma'}\; \overline{g}^{\alpha}_{\rho\sigma}\overline{g}^{\alpha'}_{\rho\sigma'}\overline{g}^{\beta}_{\sigma'\tau}\overline{g}^{\beta'}_{\sigma\tau}\, a_\rho a_\sigma a_{\sigma'} a_\tau \\
= &\sum_{\rho,\sigma,\sigma',\tau} {v_k}^{\sigma\bullet\sigma'+e} g^{\alpha}_{\rho\sigma} g^{\alpha'}_{\rho\sigma'} g^{\beta}_{\sigma'\tau} g^{\beta'}_{\sigma\tau}\, a_\rho a_\sigma a_{\sigma'} a_\tau,
\end{aligned}$$

where

$$e = \langle\rho,\sigma\rangle + \langle\rho,\sigma'\rangle + \langle\sigma',\tau\rangle + \langle\sigma,\tau\rangle.$$

Consider a non-zero summand. It follows from $g^{\alpha}_{\rho\sigma} g^{\alpha'}_{\rho\sigma'} g^{\beta}_{\sigma'\tau} g^{\beta'}_{\sigma\tau} \neq 0$ that we have

$$\begin{aligned}
\mathbf{dim}\, V_\alpha &= \mathbf{dim}\, V_\rho + \mathbf{dim}\, V_\sigma, \\
\mathbf{dim}\, V_{\alpha'} &= \mathbf{dim}\, V_\rho + \mathbf{dim}\, V_{\sigma'}, \\
\mathbf{dim}\, V_\beta &= \mathbf{dim}\, V_{\sigma'} + \mathbf{dim}\, V_\tau, \\
\mathbf{dim}\, V_{\beta'} &= \mathbf{dim}\, V_\sigma + \mathbf{dim}\, V_\tau,
\end{aligned}$$

and therefore

$$\begin{aligned}
\langle\alpha,\beta\rangle &= \langle\rho,\sigma'\rangle + \langle\rho,\tau\rangle + \langle\sigma,\sigma'\rangle + \langle\sigma,\tau\rangle, \\
\langle\alpha',\beta'\rangle &= \langle\rho,\sigma\rangle + \langle\rho,\tau\rangle + \langle\sigma',\sigma\rangle + \langle\sigma',\tau\rangle.
\end{aligned}$$

Thus,

$$\begin{aligned}
\sigma\bullet\sigma' + e &= \langle\sigma,\sigma'\rangle + \langle\sigma',\sigma\rangle + e \\
&= \langle\alpha,\beta\rangle + \langle\alpha',\beta'\rangle - 2\langle\rho,\tau\rangle.
\end{aligned}$$

Altogether, we see that the right hand term is of the form

$$\sum_{\rho,\sigma,\sigma',\tau} {v_k}^{\sigma\bullet\sigma'}\ \overline{g}^{\alpha}_{\rho\sigma}\overline{g}^{\alpha'}_{\rho\sigma'}\overline{g}^{\beta}_{\sigma'\tau}\overline{g}^{\beta'}_{\sigma\tau}\, a_\rho a_\sigma a_{\sigma'} a_\tau$$
$$= v_k^{\langle\alpha,\beta\rangle+\langle\alpha',\beta'\rangle} \sum_{\rho,\sigma,\sigma',\tau} v_k^{-2\langle\rho,\tau\rangle} g^{\alpha}_{\rho\sigma} g^{\alpha'}_{\rho\sigma'} g^{\beta}_{\sigma'\tau} g^{\beta'}_{\sigma\tau}\, a_\rho a_\sigma a_{\sigma'} a_\tau.$$

This is just the right hand term of Green's first formula, multiplied by $v_k^{\langle\alpha,\beta\rangle+\langle\alpha',\beta'\rangle}$. Altogether, we see that the second formula is a direct consequence of the first one.

Part II. Hall algebras

Let Λ be a finitary ring. Below, we recall that we may use the numbers $g^{\lambda}_{\alpha\beta}$ as structure constants for a $\mathbb{Z}$-algebra with basis $\mathcal{P}$; this algebra is called the Hall algebra $\mathcal{H}(\Lambda)$ of Λ with coefficients in $\mathbb{Z}$.

More generally, fix some commutative ring K (with unit element 1_K). We may consider also the Hall algebra $\mathcal{H}(\Lambda, K)$ of Λ with coefficients in K; it is the free K-module $K\mathcal{P}$ with a basis indexed by $\mathcal{P}$, and again with structure constants $g^{\lambda}_{\alpha\beta}$.

On the other hand, it also will be of interest to consider a suitable coalgebra structure on $K\mathcal{P}$. Let us assume that Λ is a k-algebra, where k is a finite field of order q_k. We will see below that the numbers

$$h^{\alpha\beta}_{\lambda} = g^{\lambda}_{\alpha\beta} a_\alpha a_\beta a_\lambda^{-1},$$

will belong to $\mathbb{Z}[q_k^{-1}]$, thus in case $q_k \cdot 1_K$ is invertible in K, we may use these elements $h^{\alpha\beta}_{\lambda}$ as structure constants for a comultiplication.

In case Λ is a hereditary k-algebra, Green's formula expresses a compatibility relation between the described multiplication and comultiplication: we obtain what we call a (K, v, χ)-bialgebra.

1. Preliminaries: Twisting the multiplication of a graded algebra.

Let K be a commutative ring. If I is a set, we denote by KI the free K-module with basis I, or better, with a basis indexed by I. In particular, $\mathbb{Z}I$ is the free abelian group with basis I. The elements of $\mathbb{Z}I$ can be written in the form $x = (x_i)_i$, with integers x_i, so that $x_i = 0$ for almost all $i \in I$. We write $x \geq 0$ provided $x_i \geq 0$ for all $i \in I$. Also, we write $x > 0$, provided we have both $x \geq 0$ and $x \neq 0$.

We are going to consider $\mathbb{Z}I$-graded algebras (and later also $\mathbb{Z}I$-graded coalgebras). We start with a $\mathbb{Z}I$-graded K-module

$$A = \bigoplus_{x \in \mathbb{Z}I} A_x$$

(thus, this is a direct decomposition of K-modules); this will be the underlying set. We say that A is *positive* provided that $A_x \neq 0$ only for $x \geq 0$, and that $A_0 = K$.

For $A = (A, \mu)$ to be a $\mathbb{Z}I$-graded K-algebra, we require that μ is an associative multiplication on A with a unit element, and that μ respects the grading (thus μ maps $A_x \otimes A_y$ into A_{x+y}, for $x, y \in \mathbb{Z}I$; in particular, the unit element has to belong to A_0.) We usually will just write ab instead of $\mu(a, b)$.

Let A be a $\mathbb{Z}I$-graded K-algebra. Given a bilinear form ϕ on $\mathbb{Z}I$ and an invertible element $v \in K$, we are going to define a twisted multiplication $*$ on A, and we denote the new algebra by $A_{[v,\phi]}$.

In order to form $A_{[v,\phi]}$, we will not change the underlying K-module nor the grading, thus

$$A_{[v,\phi]} = \bigoplus_{x \in \mathbb{Z}I} A_x.$$

On this K-module, we introduce a new multiplication $*$ as follows: For $a \in A_x$, $b \in A_y$, where $x, y \in \mathbb{Z}I$, we define

$$a * b = v^{\phi(x,y)}\ ab;$$

(instead of $\phi(x,y)$ we also will write $\phi(a,b)$, where, as before, $a \in A_x$, and $b \in A_y$ are homogeneous elements of A). The new multiplication $*$ again is associative, as one easily checks (using the bilinearity of ϕ); in fact, one calculates for homogeneous elements a, b, c both $a * (b * c)$ and $(a * b) * c$ and shows in this way:

$$a * b * c = v^{\phi(a,b)+\phi(a,c)+\phi(b,c)}\ abc,$$

more generally: if $a_1, a_2, \dots, a_n$ are homogeneous elements of A, then

$$a_1 * a_2 * \cdots * a_n = v^{\sum_{i<j} \phi(a_i,a_j)}\ a_1 a_2 \cdots a_n.$$

Also, the unit element 1 of A (which belongs to A_0) will be a unit element for the new multiplication $*$ as well, since $\phi(0,x) = 0 = \phi(x,0)$ for all $x \in \mathbb{Z}I$.

Lemma. *Let $f\colon A \to B$ be a homomorphism of $\mathbb{Z}I$-graded K-algebras. If we consider f as a map $A_{[v,\phi]} \to B_{[v,\phi]}$, then this is also a homomorphism of $\mathbb{Z}I$-graded K-algebras.*

Proof. Let $a, a' \in A$ be homogeneous. Then $\phi(a,a') = \phi(f(a), f(a'))$, thus we have

$$f(a * a') = f(v^{\phi(a,a')} aa') = v^{\phi(a,a')} f(aa') = v^{\phi(a,a')} f(a) f(a') = f(a) * f(a').$$

2. The (K, v, χ)-bialgebras.

Let K be a commutative ring. We are going to consider besides $\mathbb{Z}I$-graded K-algebras also $\mathbb{Z}I$-graded K-coalgebras.

For a $\mathbb{Z}I$-graded K-coalgebra $A = (A, \delta)$, we again assume that there is given a $\mathbb{Z}I$-graded K-module

$$A = \bigoplus_{x \in \mathbb{Z}I} A_x$$

and we require that

$$\delta\colon A \to A \otimes_K A$$

is a coassociative comultiplication on A with a counit, and that δ respects the grading (thus $\delta(A_z) \subseteq \bigoplus_{x+y=z} A_x \otimes_K A_y$). We often will use the socalled Sweedler notation: we write $\delta(a) = \sum a_1 \otimes a_2$, instead of $\delta(a) = \sum_i a_{i1} \otimes a_{i2}$. We usually will assume that $A_0 = K$ so that the counit will just be the projection onto A_0.

In order to describe a possible interrelation between a multiplication and a comultiplication, we assume that there are given two bilinear forms χ', χ'' on $\mathbb{Z}I$ with values in $\mathbb{Z}$ and in addition an invertible element $v \in K$. Let A be a $\mathbb{Z}I$-graded algebra.

Given the pair $\chi = (\chi', \chi'')$ of bilinear forms on $\mathbb{Z}I$ and $v \in K$, we consider a corresponding map $(\mathbb{Z}I)^4 \to \mathbb{Z}$, which we also denote by χ and which is defined as follows:

$$\chi(x_1, x_2, x_3, x_4) = \chi'(x_1, x_4) + \chi''(x_2, x_3),$$

we say that χ *is given by the pair* (χ', χ''). Note that this map χ is a bilinear form on $(\mathbb{Z}I)^2$. (A characterization of the maps $(\mathbb{Z}I)^4 \to \mathbb{Z}$ which are given by pairs (χ', χ'') will be presented at the beginning of Part IV.)

Of course, we may consider $A \otimes A$ as a $(\mathbb{Z}I)^2$-graded algebra, where for $x, y \in \mathbb{Z}I$, we have $(A \otimes A)_{(x,y)} = A_x \otimes A_y$. Thus, given a pair $\chi = (\chi', \chi'')$ of bilinear forms on $\mathbb{Z}I$, we may use the corresponding bilinear form χ on $(\mathbb{Z}I)^2$ in order to twist the multiplication of $A \otimes A$ and we obtain in this way the algebra $(A \otimes A)_{[v,\chi',\chi'']} = (A \otimes A)_{[v,\chi]}$. To repeat: the algebra $(A \otimes A)_{[v,\chi',\chi'']}$ is given by $A \otimes A$ together with the multiplication $*$ defined by

$$(a_1 \otimes a_2) * (b_1 \otimes b_2) = v^{\chi'(a_1,b_2)+\chi''(a_2,b_1)}\, a_1 b_1 \otimes a_2 b_2.$$

By definition, a (K, v, χ)-*bialgebra* is of the form $A = (A, \mu, \delta)$, where A is a K-module with a direct decomposition $A = \bigoplus_{x \in \mathbb{Z}I} A_x$, such that (A, μ) is a $\mathbb{Z}I$-graded algebra, (A, δ) is a $\mathbb{Z}I$-graded coalgebra and such that on the one hand, the counit ϵ satisfies $\epsilon(1) = 1$, and, on the other hand,

$$\delta \colon A \to (A \otimes A)_{[v,\chi]}$$

is an algebra homomorphism. The (K, v, χ)-bialgebra A is said to be *positive* provided A is a positive $\mathbb{Z}I$-graded K-module. (In dealing with $(K, v, (\chi', \chi''))$-bialgebras, we will delete the inner brackets and speak of (K, v, χ', χ'')-bialgebras.)

Lemma 1. *Let A be a positive (K, v, χ)-bialgebra. Then, for $a \in A_i$ with $i \in I$, we have $\delta(a) = a \otimes 1 + 1 \otimes a$.*

Proof. Let $i \in I$. We have $(A \otimes A)_i = A_i \otimes A_0 + A_0 \otimes A_i = A_i \otimes K + K \otimes A_i$. Now, given $a \in A_i$, the element $\delta(a)$ belongs to $(A \otimes A)_i$, thus $\delta(a) = a' \otimes 1 + 1 \otimes a''$, where a', a'' are suitable elements of A_i. The fact that the projection $A \to A_0$ is the counit implies that $a' = a = a''$.

Lemma 2. *Let A, B be positive (K, v, χ)-bialgebras. Assume that A is generated as a K-algebra by $\bigoplus_{i \in I} A_i$. Then any homomorphism $f \colon A \to B$ of $\mathbb{Z}I$-graded algebras is also a homomorphism of coalgebras.*

Proof: Let $f\colon A \to B$ be a homomorphism of $\mathbb{Z}I$-graded algebras. Then $f\otimes f\colon A\otimes A \to B\otimes B$ is a homomorphism of $\mathbb{Z}I$-graded algebras, and therefore also $f\otimes f\colon (A\otimes A)_{[v,\chi]} \to (B\otimes B)_{[v,\chi]}$ is a homomorphism of $\mathbb{Z}I$-graded algebras. Consider the following diagram of homomorphisms of $\mathbb{Z}I$-graded algebras and algebra homomorphisms:

$$\begin{tikzcd} A \arrow[r, "\delta"] \arrow[d, "f"'] & (A\otimes A)_{[v,\chi]} \arrow[d, "f\otimes f"] \\ B \arrow[r, "\delta"'] & (B\otimes B)_{[v,\chi]} \end{tikzcd}$$

In order to show that it commutes, we only have to consider a generating set of A. Let $a\in A_i$ with $i\in I$. Then we know that $\delta(a) = a\otimes 1 + 1\otimes a$. Also, $f(a)\in B_i$, and therefore $\delta(f(a)) = f(a)\otimes 1 + 1\otimes f(a)$. Also, $(f\otimes f)\delta(a) = f(a)\otimes 1 + 1\otimes f(a)$ thus $(f\otimes f)\delta(a) = \delta f(a)$, and therefore $(f\otimes f)\delta = \delta f$.

3. The Hall algebra $\mathcal{H}(\Lambda, K)$.

Let Λ be a ring. Recall that $\mathcal{P}$ denotes the set of isomorphism classes of Λ-modules of finite cardinality and let I be the set of all isomorphism classes of finite simple Λ-modules, it is a subset of $\mathcal{P}$. Let us consider the free K-module $\mathcal{H}(\Lambda,K) = K\mathcal{P}$ with basis $\{u_\alpha \mid \alpha\in\mathcal{P}\}$.

In order to be able to define a multiplication on $\mathcal{H}(\Lambda,K)$, we want to assume that Λ is finitary, and we use the numbers $g^\lambda_{\alpha\beta}$ as structure constants: thus, for $\alpha,\beta\in\mathcal{P}$, let

$$u_\alpha u_\beta = \sum_\lambda g^\lambda_{\alpha\beta}\, u_\lambda;$$

note that this is a finite sum, since we assume that Λ is finitary.

We also want to see that $\mathcal{H}(\Lambda,K)$ is a $\mathbb{Z}I$-graded algebra. For $x\in\mathbb{Z}I$, let $\mathcal{P}(x)$ be the set of isomorphism classes α such that $\mathbf{dim}\, V_\alpha = x$. Of course, $\mathcal{P}(x)$ is empty unless $x\geq 0$ and $\mathcal{P}(0)$ just consists of the isomorphism class of the zero module.

Let $\mathcal{H}(\Lambda,K)_x = K\mathcal{P}(x)$, the free K-module with basis elements u_α, where $\alpha\in\mathcal{P}(x)$.

Lemma. *$\mathcal{H}(\Lambda,K)$ is a $\mathbb{Z}I$-graded algebra. Its unit element is $1 = u_0$.*

Proof: The multiplication respects the grading, since $g^\lambda_{\alpha\beta}\neq 0$ implies that $\mathbf{dim}\, V_\lambda = \mathbf{dim}\, V_\alpha + \mathbf{dim}\, V_\beta$.

The multiplication is associative: Namely, given elements $\alpha,\beta,\gamma,\lambda$ in $\mathcal{P}$, let $g^\lambda_{\alpha\beta\gamma}$ be the number of filtrations

$$U'\subseteq U\subseteq V_\lambda \quad \text{such that} \quad V_\lambda/U\in\alpha,\ U/U'\in\beta,\ U'\in\gamma.$$

Then it is easy to see that we have

$$\sum_\nu g^\lambda_{\alpha\nu} g^\nu_{\beta\gamma} = g^\lambda_{\alpha\beta\gamma} = \sum_\mu g^\mu_{\alpha\beta} g^\lambda_{\mu\gamma}.$$

Of course, the left hand term is the coefficient of u_λ in $u_\alpha(u_\beta u_\gamma)$, the right hand term is the coefficient of u_λ in $(u_\alpha u_\beta)u_\gamma$.

Finally, u_0 is a right unit for this multiplication, since every module M has a unique submodule isomorphic to the zero module, and the corresponding factor module is isomorphic to M. Similarly, u_0 is a left unit.

As mentioned above, we will denote the Hall algebra of Λ with coefficients in $\mathbb{Z}$ just by $\mathcal{H}(\Lambda)$.

4. The comultiplication.

Let k be a finite field of order q_k, let Λ be a k-algebra. We now assume that the element $q_k \cdot 1_K$ is invertible in K. We will write $\mathbb{Z}_k = \mathbb{Z}[q_k^{-1}]$.

In order to define a comultiplication on $\mathcal{H}(\Lambda, K) = K\mathcal{P}$, we need the following consideration: We consider the numbers

$$h_\lambda^{\alpha\beta} = g^\lambda_{\alpha\beta} a_\alpha a_\beta a_\lambda^{-1},$$

we want to use them as structure constants for a comultiplication. The definition of $h_\lambda^{\alpha\beta}$ involves the inverse of a_λ. Let us show that actually it is sufficient to require (as we do) that the element $q_k \cdot 1_K$ is invertible in K.

Given Λ-modules A, B, L, let us denote by $\operatorname{Ext}^1_\Lambda(A,B)^L$ the set of equivalence classes in $\operatorname{Ext}^1_\Lambda(A,B)$ of exact sequences of the form

$$0 \longrightarrow B \xrightarrow{b} L \xrightarrow{a} A \longrightarrow 0.$$

The following observation seems to be due to Riedtmann [Ri] and Peng [P].

Proposition. *Let $\alpha, \beta, \lambda \in \mathcal{P}$. Then*

$$h_\lambda^{\alpha\beta} = \frac{|\operatorname{Ext}^1_\Lambda(V_\alpha, V_\beta)^{V_\lambda}|}{|\operatorname{Hom}_\Lambda(V_\alpha, V_\beta)|}.$$

Proof. Let $A = V_\alpha,\ B = V_\beta,\ L = V_\lambda$. Consider the set $\mathcal{E}^L_{AB}$ of exact sequences of the form

$$0 \longrightarrow B \xrightarrow{b} L \xrightarrow{a} A \longrightarrow 0.$$

According to Propositions 1 and 4 in Section I.3, we have

$$g^{\lambda}_{\alpha\beta}a_\alpha a_\beta = |\mathcal{E}^L_{AB}| = \sum_{\varepsilon\in \operatorname{Ext}^1_\Lambda(A,B)^L} |\mathcal{E}^L(\varepsilon)| = |\operatorname{Ext}^1_\Lambda(A,B)^L| \frac{a_\lambda}{|\operatorname{Hom}_\Lambda(A,B)|}.$$

Corollary. *The coefficients $h^{\alpha\beta}_\lambda$ belong to $\mathbb{Z}_k$.*

Proof: Let $n = \dim_k \operatorname{Hom}_\Lambda(V_\alpha, V_\beta)$. Then

$$q_k^n \cdot h^{\alpha\beta}_\lambda = |\operatorname{Ext}^1_\Lambda(V_\alpha,V_\beta)^{V_\lambda}| \ \in \mathbb{Z}.$$

Since we assume that $q_k\cdot 1_K$ is invertible in K, we can define the comultiplication δ on $\mathcal{H}(\Lambda,K)$ by

$$\delta\colon \mathcal{H}(\Lambda,K) \longrightarrow \mathcal{H}(\Lambda,K)\otimes_K \mathcal{H}(\Lambda,K)$$

given by

$$\delta(u_\lambda) = \sum_{\alpha,\beta} h^{\alpha\beta}_\lambda\, u_\alpha\otimes u_\beta \quad \Big(= \sum_{\alpha,\beta} g^{\lambda}_{\alpha\beta}a_\alpha a_\beta a_\lambda^{-1}\, u_\alpha\otimes u_\beta\Big).$$

Note that this always is a finite sum, since for every $\lambda\in\mathcal{P}$, the number of pairs (α,β) with $g^\lambda_{\alpha\beta}\neq 0$ is finite. (Thus, in order to consider the comultiplication, we do not need the assumption on Λ to be finitary.)

Let us mention the following special case: in case $V_\lambda = V_i$ is a simple module, we have

$$\delta(u_i) = u_i\otimes 1 + 1\otimes u_i.$$

As before, we consider $\mathcal{H}(\Lambda,K)$ as a $\mathbb{Z}I$-graded module. Of course, $\mathcal{H}(\Lambda,K)_0 = K$, and we denote by ϵ the projection modulo the ideal $\bigoplus_{x>0}\mathcal{H}(\Lambda,K)_x$ onto $\mathcal{H}(\Lambda,K)_0$.

Lemma. *$\mathcal{H}(\Lambda,K)$ is a $\mathbb{Z}I$-graded coalgebra with counit ϵ.*

It is clear that δ respects the grading: namely, if $h^{\alpha\beta}_\lambda\neq 0$, then we must have $\mathbf{dim}\, V_\lambda = \mathbf{dim}\, V_\alpha + \mathbf{dim}\, V_\beta$.

Let us show that this comultiplication is coassociative:

$$\begin{aligned}(1\otimes\delta)\delta(u_\lambda) &= (1\otimes\delta)\Big(\sum_{\alpha,\nu} g^\lambda_{\alpha\nu}\, a_\alpha a_\nu a_\lambda^{-1}\, u_\alpha\otimes u_\nu\Big)\\ &= \sum_{\alpha,\nu}\sum_{\beta,\gamma} g^\lambda_{\alpha\nu}a_\alpha a_\nu a_\lambda^{-1}\, g^\nu_{\beta\gamma}a_\beta a_\gamma a_\nu^{-1}\, u_\alpha\otimes u_\beta\otimes u_\gamma\\ &= \sum_{\alpha,\beta,\gamma} g^\lambda_{\alpha\beta\gamma}a_\alpha a_\beta a_\gamma a_\lambda^{-1}\, u_\alpha\otimes u_\beta\otimes u_\gamma\end{aligned}$$

and similarly

$$(\delta\otimes 1)\delta(u_\lambda)=\sum_{\alpha,\beta,\gamma} g^\lambda_{\alpha\beta\gamma}a_\alpha a_\beta a_\gamma a_\lambda^{-1}\; u_\alpha\otimes u_\beta\otimes u_\gamma .$$

Also, it is easy to see that the K-linear map $\epsilon\colon \mathcal{H}(\Lambda,K)\to K$ is a counit.

5. The compatibility of multiplication and comultiplication.

Let k be a finite field of order q_k, let Λ be a k-algebra which is hereditary and finitary. Let K be a commutative ring with $q=q_k\cdot 1_K$ being invertible in K.

Let us write $\mathcal{H}=\mathcal{H}(\Lambda,K)$. We consider also the tensor product $\mathcal{H}\otimes\mathcal{H}$ (over K) as an algebra with multiplication $*$ defined as follows:

$$(u_\alpha\otimes u_\beta)*(u_{\alpha'}\otimes u_{\beta'})=\frac{|\operatorname{Ext}^1(V_\alpha,V'_\beta)|}{|\operatorname{Hom}(V_\alpha,V'_\beta)|}\,\mu(u_\alpha\otimes u_{\alpha'})\otimes\mu(u_\beta\otimes u_{\beta'});$$

thus, we deal with $(\mathcal{H}\otimes\mathcal{H})_{[q,-\langle -,-\rangle,0]}$.

Corollary 1 to Green's formula. *The map $\delta\colon\mathcal{H}\to(\mathcal{H}\otimes\mathcal{H})_{[q,-\langle -,-\rangle,0]}$ is an algebra homomorphism.*

Proof: Consider the following diagram

$$\begin{array}{ccc}
\mathcal{H}\otimes\mathcal{H} & \xrightarrow{\mu} & \mathcal{H}\\
\Big\downarrow{\scriptstyle \delta\otimes\delta} & & \\
\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H} & & \Big\downarrow{\scriptstyle\delta}\\
\zeta' \searrow & & \\
\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H}\otimes\mathcal{H} & \xrightarrow[\mu\otimes\mu]{} & \mathcal{H}\otimes\mathcal{H}
\end{array}$$

where ζ' is defined by

$$\zeta'(u_\alpha\otimes u_\beta\otimes u_{\alpha'}\otimes u_{\beta'})=\frac{|\operatorname{Ext}^1(V_\alpha,V'_\beta)|}{|\operatorname{Hom}(V_\alpha,V'_\beta)|}\,u_\alpha\otimes u_{\alpha'}\otimes u_\beta\otimes u_{\beta'}.$$

Green's formula expresses the fact that this diagram commutes: Indeed, consider $u_\alpha\otimes u_\beta\ \in\ \mathcal{H}\otimes\mathcal{H}$. Then

$$\begin{aligned}
\delta\mu\big(u_\alpha\otimes u_\beta\big)&=\delta\Big(\sum_\lambda g^\lambda_{\alpha\beta}u_\lambda\Big)\\
&=\sum_\lambda g^\lambda_{\alpha\beta}\sum_{\alpha',\beta'} g^\lambda_{\alpha'\beta'}a_{\alpha'}a_{\beta'}a_\lambda^{-1}\;u_{\alpha'}\otimes u_{\beta'}\\
&=\sum_{\alpha',\beta'}\ \Big(a_{\alpha'}a_{\beta'}\sum_\lambda\ g^\lambda_{\alpha\beta}g^\lambda_{\alpha'\beta'}a_\lambda^{-1}\Big)\,u_{\alpha'}\otimes u_{\beta'}
\end{aligned}$$

whereas

$$
\begin{aligned}
&(\mu\otimes\mu)\,\zeta'\,(\delta\otimes\delta)\big(u_\alpha\otimes u_\beta\big)\\
&\quad=(\mu\otimes\mu)\,\zeta'\Big(\sum_{\rho,\sigma,\sigma',\tau} g^\alpha_{\rho\sigma}g^\beta_{\sigma'\tau}\,a_\rho a_\sigma a_{\sigma'}a_\tau a_\alpha^{-1}a_\beta^{-1}u_\rho\otimes u_\sigma\otimes u_{\sigma'}\otimes u_\tau\Big)\\
&\quad=(\mu\otimes\mu)\Big(\sum_{\rho,\sigma,\sigma',\tau}\frac{|\operatorname{Ext}^1(V_\rho,V_\tau)|}{|\operatorname{Hom}(V_\rho,V_\tau)|}\,g^\alpha_{\rho\sigma}g^\beta_{\sigma'\tau}\,a_\rho a_\sigma a_{\sigma'}a_\tau a_\alpha^{-1}a_\beta^{-1}u_\rho\otimes u_{\sigma'}\otimes u_\sigma\otimes u_\tau\Big)\\
&\quad=\sum_{\rho,\sigma,\sigma',\tau}\frac{|\operatorname{Ext}^1(V_\rho,V_\tau)|}{|\operatorname{Hom}(V_\rho,V_\tau)|}\,g^\alpha_{\rho\sigma}g^\beta_{\sigma'\tau}\,a_\rho a_\sigma a_{\sigma'}a_\tau a_\alpha^{-1}a_\beta^{-1}\Big(\sum_{\alpha'}g^{\alpha'}_{\rho\sigma'}u_{\alpha'}\Big)\otimes\Big(\sum_{\beta'}g^{\beta'}_{\sigma\tau}u_{\beta'}\Big)\\
&\quad=\sum_{\alpha',\beta'}\Big(a_\alpha^{-1}a_\beta^{-1}\sum_{\rho,\sigma,\sigma',\tau}\frac{|\operatorname{Ext}^1(V_\rho,V_\tau)|}{|\operatorname{Hom}(V_\rho,V_\tau)|}\,g^\alpha_{\rho\sigma}g^\beta_{\sigma'\tau}g^{\alpha'}_{\rho\sigma'}g^{\beta'}_{\sigma\tau}\,a_\rho a_\sigma a_{\sigma'}a_\tau\Big)\,u_{\alpha'}\otimes u_{\beta'}.
\end{aligned}
$$

Of course, $(\mu\otimes\mu)\,\zeta'$ is just the multiplication map of $(\mathcal H\otimes\mathcal H)_{[q,-\langle -,-\rangle,0]}$. The commutativity of the diagram expresses the fact that

$$\delta\colon \mathcal H\to(\mathcal H\otimes\mathcal H)_{[q,-\langle -,-\rangle,0]}$$

is an algebra homomorphism.

We may read the diagram also in another way: We may consider $\mathcal H\otimes\mathcal H$ as a coalgebra using the comultiplication

$$\delta'=\zeta'\,(\delta\otimes\delta);$$

since we use the same twist, we will denote this coalgebra by $(\mathcal H\otimes\mathcal H)_{[q,-\langle -,-\rangle,0]}$, again. We may reformulate the commutativity of the diagram as follows:

Corollary 2. *The multiplication map $\mu\colon(\mathcal H\otimes\mathcal H)_{[q,-\langle -,-\rangle,0]}\to\mathcal H$ is a coalgebra homomorphism.*

In the case of the ordinary Hall algebra $\mathcal H(\widehat{\mathbb Z}_p)$, where p is a prime number, the comultiplication has been considered by Geissinger and Zelevinsky (see [Z] p.116).

Theorem. *Let k be a finite field of order q_k. Let Λ be a k-algebra which is hereditary and finitary. Let K be a commutative ring such that $q=q_k\cdot 1_K$ is invertible in K. Then $\mathcal H(\Lambda,K)$ is a $(K,q,-\langle -,-\rangle,0)$-bialgebra.*

6. The twisted Hall algebras.

Again, we start with a k-algebra Λ, where k is a finite field. Let us assume now that K contains an invertible element v such that $v^2=q_k\cdot 1_K$, where $q_k=|k|$.

Let $\mathcal{H}_*(\Lambda, K) = K\mathcal{P}$. In case Λ is finitary, we endow $\mathcal{H}_*(\Lambda, K)$ with the multiplication

$$u_\alpha * u_\beta = \sum_\lambda \overline{g}^\lambda_{\alpha\beta}\, u_\lambda;$$

we recall that

$$\overline{g}^\lambda_{\alpha\beta} = v^{\langle\alpha,\beta\rangle} g^\lambda_{\alpha\beta}.$$

Similarly, let

$$\overline{h}^{\alpha\beta}_\lambda = v^{\langle\alpha,\beta\rangle} h^{\alpha\beta}_\lambda = v^{\langle\alpha,\beta\rangle} g^\lambda_{\alpha\beta} a_\alpha a_\beta a_\lambda^{-1}.$$

Without any assumption on Λ, we may consider $\mathcal{H}_*(\Lambda, K)$ as a coalgebra using the comultiplication

$$\delta_*(u_\lambda) = \sum_{\alpha,\beta} \overline{h}^{\alpha\beta}_\lambda\, u_\alpha \otimes u_\beta \quad \Big(= \sum_{\alpha,\beta} v^{\langle\alpha,\beta\rangle} g^\lambda_{\alpha\beta} a_\alpha a_\beta a_\lambda^{-1}\, u_\alpha \otimes u_\beta\Big).$$

Recall that $\bullet$ denotes the symmetrization of the bilinear form $\langle -, - \rangle$ on the Grothendieck group $\mathbb{Z}I = K_0(\Lambda)$ of Λ.

Theorem. *Let k be a finite field of order q_k. Let Λ be a k-algebra which is hereditary and finitary. Let K be a commutative ring which contains an invertible element v such that $v^2 = q_k \cdot 1_K$. Then $\mathcal{H}_*(\Lambda, K)$ is a $(K, v, 0, \bullet)$-bialgebra.*

Proof: Similar to the case of $\mathcal{H}(\Lambda, K)$.

Of particular interest is the case where K is equal to $\mathbb{Z}_{*k} = \mathbb{Z}[v_k, v_k^{-1}]$, here v_k is a root of $q_k = |k|$. We write $\mathcal{H}_*(\Lambda)$ instead of $\mathcal{H}_*(\Lambda, \mathbb{Z}_{*k})$.

7. The bilinear form on $\mathcal{H}(\Lambda, K)$ and $\mathcal{H}_*(\Lambda, K)$.

We define a symmetric bilinear form on the free K-module $K\mathcal{P}$ as follows: For $\alpha \in \mathcal{P}$, let $t_\alpha = \frac{|V_\alpha|}{a_\alpha}$. Let

$$(u_\alpha, u_\beta) = \begin{cases} t_\alpha & \text{if } \alpha = \beta \\ 0 & \text{otherwise.} \end{cases}$$

In case we deal with a $\mathbb{Z}I$-graded algebra A which is endowed with a bilinear form $(-, -)$, we also will consider a corresponding bilinear form on $A \otimes A$, again denoted by $(-, -)$ and defined by

$$(a_1 \otimes a_2, b_1 \otimes b_2) = (a_1, b_1)(a_2, b_2)$$

(thus, the form $(-, -)$ is defined componentwise, without any twist). A bilinear form $(-, -)$ on A will be said *to respect the $\mathbb{Z}I$-grading* provided we have $(a, b) = 0$ for homogeneous elements $a, b \in A$ with $|a| \neq |b|$.

Proposition. *For all elements a, b, c in $\mathcal{H}(\Lambda, K)$, we have*

$$(a, bc) = (\delta(a), b \otimes c).$$

We can reformulate the result as follows: The comultiplication δ is left adjoint to the multiplication.

Proof: It is sufficient to consider the case $a = u_\alpha,\ b = u_\beta,\ c = u_\gamma$. By definition, $bc = \sum_\lambda g^\lambda_{\beta\gamma} u_\lambda$, thus

$$\begin{aligned}(u_\alpha, u_\beta u_\gamma) &= (u_\alpha, \sum_\lambda g^\lambda_{\beta\gamma} u_\lambda) \\ &= g^\alpha_{\beta\gamma}\ (u_\alpha, u_\alpha) \\ &= g^\alpha_{\beta\gamma}\ t_\alpha.\end{aligned}$$

On the other hand, $\delta(u_\alpha) = \sum_{\mu\nu} g^\alpha_{\mu\nu} a_\mu a_\nu a_\alpha^{-1} u_\mu \otimes u_\nu$, thus

$$\begin{aligned}(\delta(u_\alpha), u_\beta \otimes u_\gamma) &= (\sum_{\mu\nu} g^\alpha_{\mu\nu} a_\mu a_\nu \frac{1}{a_\alpha} u_\mu \otimes u_\nu, u_\beta \otimes u_\gamma) \\ &= g^\alpha_{\beta\gamma} a_\beta a_\gamma \frac{1}{a_\alpha} (u_\beta \otimes u_\gamma, u_\beta \otimes u_\gamma) \\ &= g^\alpha_{\beta\gamma} a_\beta a_\gamma \frac{1}{a_\alpha} t_\beta t_\gamma \\ &= g^\alpha_{\beta\gamma} a_\beta a_\gamma \frac{1}{a_\alpha} \frac{|V_\beta|}{a_\beta} \frac{|V_\gamma|}{a_\gamma} \\ &= g^\alpha_{\beta\gamma}\ \frac{|V_\alpha|}{a_\alpha} \\ &= g^\alpha_{\beta\gamma}\ t_\alpha.\end{aligned}$$

Here, we have used that for $g^\alpha_{\beta\gamma} \neq 0$ we must have $|V_\alpha| = |V_\beta| \cdot |V_\gamma|$.

There is the analogous result for $\mathcal{H}_*(\Lambda, K)$.

Proposition*. *For all elements a, b, c in $\mathcal{H}_*(\Lambda, K)$, we have*

$$(a, b * c) = (\delta_*(a), b \otimes c).$$

Again, we may reformulate the result: The comultiplication δ_* is left adjoint to the multiplication of $\mathcal{H}_*(\Lambda, K)$.

Remark. Instead of using the coefficients $t_\alpha = |V_\alpha| {a_\alpha}^{-1}$ one may consider (as Green [G1] does) $t'_\alpha = {a_\alpha}^{-1}$. Of course, the corresponding calculations will become easier. However, our choice will give precisely the bilinear form used by Lusztig on U^+ (at least for a basic k-algebra Λ), since for $i \in I$ we get

$$t_i = \frac{|V_i|}{a_i} = \frac{{q_k}^{d_i}}{{q_k}^{d_i} - 1},$$

where $d_i = \dim_k \operatorname{End}(V_i)$, and $q_k = |k|$. The coefficients t_α play a prominent role in Lusztig's description of the canonical basis of U^+ in terms of the Lusztig form on U^+, see [L], Theorem 14.2.3.

Part III. The composition algebras

Let Λ be a finitary k-algebra. We have defined above the Hall algebra $\mathcal{H}(\Lambda)$. By definition, the elements u_α, with $\alpha \in \mathcal{P}$ form a $\mathbb{Z}$-basis of $\mathcal{H}(\Lambda)$. Recall that I denotes the subset of $\mathcal{P}$ given by the isomorphism classes of the simple Λ-modules, and we may consider the subring $\mathcal{C}(\Lambda)$ generated by the elements u_i with $i \in I$. This subalgebra is called the *composition algebra* of Λ with coefficients in $\mathbb{Z}$. We are going to study properties of this subring.

We want to give an interpretation of this subring $\mathcal{C}(\Lambda)$. Let $(i_1, i_2, \dots, i_n)$ be a sequence of elements from I. Then we can form the product $u_{i_1}u_{i_2}\cdots u_{i_n}$ in $\mathcal{H}(\Lambda)$, thus this product is a linear combination

$$u_{i_1}u_{i_2}\cdots u_{i_n} = \sum_{\lambda\in\mathcal{P}} c^\lambda u_\lambda,$$

and clearly, c^λ counts the number of composition series of V_λ which are of the form

$$V_\lambda = L_0 \supset L_1 \supset L_2 \supset \cdots \supset L_n = 0$$

with L_{j-1}/L_j belonging to the isomorphism class i_j. We see that $\mathcal{C}(\Lambda)$ encodes the numbers of possible composition series of arbitrary Λ-modules, this explains the name.

We want to derive some relations between the generators u_i. In order to do so, we need to recall some notation.

1. Quantum binomial coefficients.

Consider the ring $\mathbb{Z}[\tilde v, \tilde v^{-1}]$, with $\tilde v$ a variable. The elements of $\mathbb{Z}[\tilde v, \tilde v^{-1}]$ will be called *Laurent polynomials* in the variable $\tilde v$. Given such a Laurent polynomial $\varphi = \varphi(\tilde v)$, and an integer d, one denotes by φ_d the Laurent polynomial which is obtained from φ by inserting $\tilde v^d$, thus $\varphi_d(\tilde v) = \varphi(\tilde v^d)$. We denote $\tilde q = \tilde v^2$.

In this section, we will write v instead of $\tilde v$ and q instead of $\tilde q$.

For $0 \le t \le n$, let

$$\begin{aligned}
[n] &= \frac{v^n - v^{-n}}{v - v^{-1}} \quad \Big(= v^{n-1} + v^{n-3} + \dots v^{-n+1}\Big),\\
[n]! &= \prod\nolimits_{t=1}^{n} [t]\,,\\
\begin{bmatrix} n \\ t \end{bmatrix} &= \frac{[n]!}{[t]![n-t]!}.
\end{aligned}$$

and also

$$\begin{aligned}
\left| n \right] &= \frac{q^n-1}{q-1} \quad \left(= q^{n-1}+\cdots+q+1\right) &&= v^{n-1}[n]\,,\\
\left| n \right]! &= \prod\nolimits_{t=1}^{n} \left| t \right] &&= v^{\binom{n}{2}}[n]!\,,\\
\left|\begin{matrix} n\\ t\end{matrix}\right] &= \frac{\left| n \right]!}{\left| t \right]!\cdot \left| n-t \right]!} &&= v^{t(n-t)}\begin{bmatrix} n\\ t\end{bmatrix}.
\end{aligned}$$

Lemma. *For any element z in a $\mathbb{Z}[q]$-algebra, we have*

$$\prod_{j=0}^{n-1}\left(1+q^j z\right) = \sum_{t=0}^{n} q^{\binom{t}{2}} \left|\begin{matrix} n\\ t\end{matrix}\right] z^t.$$

The proof is rather straight-forward, using induction. See for example [J].

Corollary 1. *The elements* $\left|\begin{matrix} n\\ t\end{matrix}\right]$ *belong to* $\mathbb{Z}[q]$.

Proof: The left hand side of the formula exhibited in Lemma can be written in the form $\sum_{t=0}^{n} c_t z^t$, where c_t is a polynomial in the variable q. This coefficient c_t is the sum of all products $q^{j_1}q^{j_2}\cdots q^{j_t}$, where $0 \le j_1 < j_2 < \cdots < j_t \le n-1$. In particular, such a product is always divisible by $q^{\binom{t}{2}}$. Thus, $c_t = q^{\binom{t}{2}} c'_t$, where c'_t is a polynomial in the variable q. The lemma asserts that $c'_t = \left|\begin{matrix} n\\ t\end{matrix}\right]$.

Corollary 2. *The elements* $\begin{bmatrix} n\\ t\end{bmatrix}$ *belong to* $\mathbb{Z}[v, v^{-1}]$.

Corollary 3. *There is the following equality:*

$$\sum_{t=0}^{n}(-1)^t q^{\binom{t}{2}} \left|\begin{matrix} n\\ t\end{matrix}\right] = 0.$$

Proof. Take $z=-1$. Then we obtain $(1+q^0(-1)) = 0$ as a factor on the left hand side, namely as the factor with index 0, thus the product is zero.

Interpretation. Let k be a finite field with q_k elements. Recall that the expressions

$$\left| n \right], \quad \left| n \right]! \quad \text{and} \quad \left|\begin{matrix} n\\ t\end{matrix}\right]$$

are polynomials in the variable q with integer coefficients, thus we may insert for q integers, for example the number q_k; as usual, we will write $|n](c)$ for the evaluation of the polynomial $|n] = |n](q)$ at the number c. *Note that* $|n]$, $|n]!$, $\left|\begin{matrix} n \\ t \end{matrix}\right]$ *usually will be considered as polynomials in the variable* q *(in contrast to* $[n]$, $[n]!$, $\begin{bmatrix} n \\ t \end{bmatrix}$ *which are Laurent polynomials in the variable* v *and* $q = v^2$*). The convention* $\phi_d(v) = \phi(v^d)$ *will be used only for Laurent polynomials in the variable* v*, thus we write* $[n]_2$ *instead of* $\frac{v^{2n}-v^{-2n}}{v^2-v^{-2}}$*, but we write* $|n](q^2)$ *instead of* $\frac{q^{2n}-1}{q^2-1}$.

Let V be an n-dimensional k-space. Then $|n](q_k)$ is the number of one-dimensional subspaces of V, and $|n]!(q_k)$ is the number of complete linear flags in V (a complete linear flag in V is a sequence of subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ where V_i is of dimension i for all $0 \leq i \leq n$). Finally, $\left|\begin{matrix} n \\ m \end{matrix}\right](q_k)$ is the number of m-dimensional subspaces of V.

2. The fundamental relations for $\mathcal{H}(\Lambda)$ and $\mathcal{H}_*(\Lambda)$

Consider elements $i \neq j$ in I, and let us assume that $\operatorname{Ext}^1(S(j), S(i)) = 0$. Then we define

$$\begin{aligned}
d_i &= \dim_k \operatorname{End}(S(i)), \\
e(i,j) &= \dim_k \operatorname{Ext}^1(S(i), S(j)), \\
n(i,j) &= d_i^{-1} e(i,j) + 1, \\
n(j,i) &= d_j^{-1} e(i,j) + 1.
\end{aligned}$$

Note that in case also the opposite assumption $\operatorname{Ext}^1(S(i), S(j)) = 0$ is satisfied, then we will have $e(i,j) = 0 = e(j,i)$, and therefore $n(i,j) = 1 = n(j,i)$.

Proposition. *Consider elements* $i \neq j$ *in* I*. We assume that* $\operatorname{Ext}^1(S(j), S(i)) = 0$*. If we also have* $\operatorname{Ext}^1(S(i), S(i)) = 0$*, then the following relation*

$$\sum_{t=0}^{n(i,j)} (-1)^t q_k^{d_i \binom{t}{2}} \left|\begin{matrix} n(i,j) \\ t \end{matrix}\right] (q_k^{d_i})\, u_i^{n(i,j)-t} u_j u_i^t = 0$$

is satisfied in $\mathcal{H}(\Lambda)$.

If we have $\operatorname{Ext}^1(S(j), S(j)) = 0$*, then the relation*

$$\sum_{t=0}^{n(j,i)} (-1)^t q_k^{d_j \binom{t}{2}} \left|\begin{matrix} n(j,i) \\ t \end{matrix}\right] (q_k^{d_j})\, u_j^t u_i u_j^{n(j,i)-t} = 0$$

is satisfied in $\mathcal{H}(\Lambda)$.

Proof: We indicate the proof of the second relation as presented in [R2], the proof of the first one is, of course, similar.

We fix the pair (i,j). As abbreviation, let $d = d_j$ and $n = n(j,i)$. Observe that $e(i,j) = d(n-1)$. First, we have to calculate the product $u_j^t u_i u_j^{n-t}$ in $\mathcal{H}(\Lambda)$. This product has to be written as a linear combination $c_t^\lambda u_\lambda$ with coefficients $c_t^\lambda \in \mathbb{Z}$ and $\lambda \in \mathcal{P}$. Then we have to show that for every $\lambda \in \mathcal{P}$, we have

$$\sum_{t=0}^{n} (-1)^t q_k^{d\binom{t}{2}} \left|\begin{matrix} n \\ t \end{matrix}\right] (q_k^d)\ c_t^\lambda = 0.$$

Thus, we fix some $\lambda \in \mathcal{P}$. Let L be a Λ-module in the isomorphism class λ. In order that $c_t^\lambda \neq 0$, we see that L must have a composition series with n factors of the form $S(j)$ and one factor isomorphic to $S(i)$. Let N be a direct summand of L of minimal length which has a composition factor $S(i)$. Then $L = N \oplus S(j)^m$ for some m. Here, we use that $\mathrm{Ext}^1(S(j), S(j)) = 0$. Using again this condition and the fact that $\mathrm{Ext}^1(S(j), S(i)) = 0$, we see that the radical of N is of the form $S(j)^{n-m}$, and N modulo its radical is isomorphic to $S(i)$. In particular, it follows that L modulo its radical is of the form $S(i) \oplus S(j)^m$. Finally, since $\dim_k \mathrm{Ext}^1(S(i), S(j)) < dn$, it follows that $m \geq 1$.

The information obtained so far is sufficient to calculate c_t^λ. Namely, we have to find the number of composition series

$$L = L_0 \supset L_1 \supset \cdots \supset L_{n+1} = 0$$

such that the first t factors are of the form $S(j)$, the next one is of the form $S(i)$, and the last $n-t$ ones are again of the form $S(j)$. Clearly, if $t > m$, then $c_t^\lambda = 0$, since L modulo its radical is of the form $S(i) \oplus S(j)^m$.

Thus, let $t \leq m$. If $(L_i)_i$ is a composition series of L as required, then we must have $L_t \supseteq N$. Thus, we first choose a submodule L_t/N of L/N of length $m-t$, and then in L/L_t a complete flag of submodules. The module L_t will have a unique submodule L_{t+1} with factor module isomorphic to $S(i)$, thus, finally, we have to choose a complete flag of submodules in L_{t+1}. The number of possible submodules of the form L_t is the number of $(m-t)$-dimensional subspaces in an m-dimensional vectorspace over the field $k' = \mathrm{End}(S(j))$, thus equal to $\left|\begin{matrix} m \\ m-t \end{matrix}\right] ({q_k}^d)$. The number of complete flags in L/L_t is $|\,t\,]!({q_k}^d)$ and the number of complete flags in L_{t+1} is $|\,n-t\,]!({q_k}^d)$. This shows that c_t^λ is the evaluation at q_k^d of the following polynomial:

$$\left|\begin{matrix} m \\ m-t \end{matrix}\right] \cdot |\,t]! \cdot |\,n-t]! = \frac{|\,m\,]!}{|\,m-t\,]!|\,t\,]!} |\,t\,]!|\,n-t\,]! = \frac{|\,m\,]!|\,n-t\,]!}{|\,m-t\,]!}.$$

Thus:

$$\sum_{t=0}^{n}(-1)^t q_k^{d\binom{t}{2}} \left|\begin{matrix} n \\ t\end{matrix}\right] (q_k^d)\ c_t^\lambda = \sum_{t=0}^{m}(-1)^t q_k^{d\binom{t}{2}} \left(\frac{|\,n\,]!}{|\,t\,]!|\,n-t\,]!}\cdot\frac{|\,m\,]!|\,n-t\,]!}{|\,m-t\,]!}\right)(q_k^d)$$

$$= |\,n\,]!(q_k^d)\ \sum_{t=0}^{m}(-1)^t q_k^{d\binom{t}{2}} \frac{|\,m\,]!}{|\,t\,]!|\,m-t\,]!}(q_k^d)$$

$$= |\,n\,]!(q_k^d)\ \sum_{t=0}^{m}(-1)^t q_k^{d\binom{t}{2}} \left|\begin{matrix} m \\ t\end{matrix}\right] (q_k^d) = 0$$

This completes the proof.

* * *

Proposition. *Consider elements $i \neq j$ in I. We assume that* $\mathrm{Ext}^1(S(j),S(i)) = 0$. *If we also have* $\mathrm{Ext}^1(S(i),S(i)) = 0$, *then the relation*

$$\sum_{t=0}^{n(i,j)} (-1)^t \begin{bmatrix} n(i,j) \\ t \end{bmatrix}_{d_i} u_i^{n(i,j)-t} u_j u_i^t = 0$$

is satisfied in $\mathcal{H}_*(\Lambda)$.

If we have $\mathrm{Ext}^1(S(j),S(j)) = 0$, *then the relation*

$$\sum_{t=0}^{n(j,i)} (-1)^t \begin{bmatrix} n(j,i) \\ t \end{bmatrix}_{d_j} u_j^t u_i u_j^{n(i,j)-t} = 0$$

is satisfied in $\mathcal{H}_*(\Lambda)$.

Proof: We only have to rewrite the multiplication in $\mathcal{H}_*(\Lambda)$ using the multiplication of $\mathcal{H}(\Lambda, \mathbb{Z}_{*k})$ and the twisting factors. See [R4].

3. The composition algebra and the twisted composition algebra.

Recall that the composition algebra $\mathcal{C}(\Lambda)$ of Λ with coefficients in $\mathbb{Z}$ is the subring of $\mathcal{H}(\Lambda)$ which is generated by the elements u_i with $i \in I$. We note the following: *The composition algebra $\mathcal{C}(\Lambda)$ is a free $\mathbb{Z}$-module,* since submodules of free $\mathbb{Z}$-modules are free.

As an abelian group, the composition algebra $\mathcal{C}(\Lambda)$ is generated by the set of monomials $u_{i_1}u_{i_2}\dots u_{i_n}$ with $i_1, i_2, \dots, i_n \in I$ and $n \geq 0$. Since the generating elements u_i are homogeneous elements, the composition algebra $\mathcal{C}(\Lambda)$ again is a $\mathbb{Z}I$-graded algebra, and we have

$$\mathcal{C}(\Lambda) = \bigoplus_{x\in\mathbb{Z}I} \mathcal{C}(\Lambda)_x.$$

For $x \in \mathbb{Z}I$, the $\mathbb{Z}$-module $\mathcal{C}(\Lambda)_x$ is generated by the monomials $u_{i_1}u_{i_2}\dots u_{i_n}$ of degree x. Of course, the monomial $u_{i_1}u_{i_2}\dots u_{i_n}$ has degree $x = (x_i)_i$ if and only if the element i occurs precisely x_i times in the sequence $(i_1, i_2, \dots, i_n)$.

Consider now a commutative ring K. We define

$$\mathcal{C}(\Lambda, K) = \mathcal{C}(\Lambda) \otimes_{\mathbb{Z}} K.$$

Since $\mathcal{C}(\Lambda)$ is a free $\mathbb{Z}$-module, it follows that $\mathcal{C}(\Lambda, K)$ is a free K-module. Also, $\mathcal{C}(\Lambda, K)$ is again a $\mathbb{Z}I$-graded algebra, and we have

$$\mathcal{C}(\Lambda, K) = \bigoplus_{x \in \mathbb{Z}I} \mathcal{C}(\Lambda, K)_x$$

where $\mathcal{C}(\Lambda, K)_x$ is generated, as a K-module, by the monomials $u_{i_1}u_{i_2}\dots u_{i_n}$ of degree x. We call $\mathcal{C}(\Lambda, K)$ the *composition algebra of* Λ *with coefficients in* K.

Let us assume now that $q_k \cdot 1_K$ is invertible in K. We consider first the special case $\mathbb{Z}_k = \mathbb{Z}[q_k^{-1}]$. The inclusion map $\iota\colon \mathcal{C}(\Lambda) \to \mathcal{H}(\Lambda)$ induces an injective map

$$\iota_k = \iota \otimes 1\colon \mathcal{C}(\Lambda, \mathbb{Z}_k) \longrightarrow \mathcal{H}(\Lambda) \otimes_{\mathbb{Z}} \mathbb{Z}_k = \mathcal{H}(\Lambda, \mathbb{Z}_k),$$

thus we may consider $\mathcal{C}(\Lambda, \mathbb{Z}_k)$ as a subalgebra of $\mathcal{H}(\Lambda, \mathbb{Z}_k)$. Also, since both $\mathcal{C}(\Lambda, \mathbb{Z}_k)$ and $\mathcal{H}(\Lambda, \mathbb{Z}_k)$ are free $\mathbb{Z}_k$-modules, it follows that $\iota_k \otimes \iota_k$ again is injective, thus we also may consider $\mathcal{C}(\Lambda, \mathbb{Z}_k) \otimes_{\mathbb{Z}_k} \mathcal{C}(\Lambda, \mathbb{Z}_k)$ as subalgebra of $\mathcal{H}(\Lambda, \mathbb{Z}_k) \otimes_{\mathbb{Z}_k} \mathcal{H}(\Lambda, \mathbb{Z}_k)$.

As we have seen above, $\mathcal{H}(\Lambda, \mathbb{Z}_k)$ is a coalgebra via the comultiplication δ with

$$\delta(u_\lambda) = \sum_{\alpha,\beta} g^{\lambda}_{\alpha\beta} a_\alpha a_\beta a_\lambda^{-1}\, u_\alpha \otimes u_\beta.$$

We have $\delta(u_i) = u_i \otimes 1 + 1 \otimes u_i$, thus δ maps $\mathcal{C}(\Lambda, \mathbb{Z}_k)$ into the $\mathbb{Z}_k$-subalgebra of $\big(\mathcal{H}(\Lambda, \mathbb{Z}_k) \otimes \mathcal{H}(\Lambda, \mathbb{Z}_k)\big)_{[q_k, -\langle -,-\rangle, 0]}$ generated by the elements $u_i \otimes 1$ and $1 \otimes u_i$ with $i \in I$. But the latter is just $\big(\mathcal{C}(\Lambda, \mathbb{Z}_k) \otimes \mathcal{C}(\Lambda, \mathbb{Z}_k)\big)_{[q_k, -\langle -,-\rangle, 0]}$; thus we see that δ induces a comultiplication

$$\mathcal{C}(\Lambda, \mathbb{Z}_k) \longrightarrow \big(\mathcal{C}(\Lambda, \mathbb{Z}_k) \otimes \mathcal{C}(\Lambda, \mathbb{Z}_k)\big)_{[q_k, -\langle -,-\rangle, 0]}$$

again denoted by δ. In this way, $\mathcal{C}(\Lambda, \mathbb{Z}_k)$ becomes a $(\mathbb{Z}_k, q_k, -\langle -,-\rangle, 0)$-bialgebra.

For a general commutative ring K with $q = q_k \cdot 1_K$ invertible, we note that $\mathcal{C}(\Lambda, K) = \mathcal{C}(\Lambda, \mathbb{Z}_k) \otimes_{\mathbb{Z}_k} K$. Since $\mathcal{C}(\Lambda, \mathbb{Z}_k)$ is a $(\mathbb{Z}_k, q_k, -\langle -,-\rangle, 0)$-bialgebra, it follows that $\mathcal{C}(\Lambda, K)$ is a $(K, q, -\langle -,-\rangle, 0)$-bialgebra.

Theorem. *Let* k *be a finite field of order* q_k*. Let* Λ *be a* k*-algebra which is hereditary and finitary. Let* K *be a commutative ring such that* $q = q_k \cdot 1_K$ *is* *invertible in K. Then $\mathcal{C}(\Lambda, K)$ is a $(K, q, -\langle -,-\rangle, 0)$-bialgebra. As a K-module, $\mathcal{C}(\Lambda, K)$ is free.*

* * *

We introduce now the corresponding twisted composition algebra. We consider only commutative rings K which contain an invertible element v such that $v^2 = q_k \cdot 1_K$. In particular, we will have to deal with the commutative ring $\mathbb{Z}_{*k} = \mathbb{Z}[v_k, v_k^{-1}]$, where v_k denotes a root of $q_k = |k|$.

As before, Λ is a finitary k-algebra. Let $\mathcal{C}_*(\Lambda)$ be the $\mathbb{Z}_{*k}$-subalgebra of $\mathcal{H}_*(\Lambda)$ generated by the elements u_i, with $i \in I$, it will be said to be the *twisted composition algebra* of Λ (with coefficients in $\mathbb{Z}_{*k}$). As in the case of the composition algebra itself, we see: *The twisted composition algebra $\mathcal{C}_*(\Lambda)$ is a free $\mathbb{Z}_{*k}$-module,* since also $\mathbb{Z}_{*k}$ is a principal ideal domain.

As a $\mathbb{Z}_{*k}$-module, the composition algebra $\mathcal{C}_*(\Lambda)$ is generated by the set of monomials $u_{i_1}u_{i_2}\dots u_{i_n}$ with $i_1, i_2, \dots, i_n \in I$ and $n \geq 0$. Since the generating elements u_i are homogeneous elements, the composition algebra $\mathcal{C}_*(\Lambda)$ again is a $\mathbb{Z}I$-graded algebra, and we have

$$\mathcal{C}_*(\Lambda) = \bigoplus_{x \in \mathbb{Z}I} \mathcal{C}_*(\Lambda)_x;$$

of course, for $x \in \mathbb{Z}I$, the $\mathbb{Z}_{*k}$-module $\mathcal{C}_*(\Lambda)_x$ is generated by the monomials $u_{i_1}u_{i_2}\dots u_{i_n}$ of degree x.

Next, we want to see that $\mathcal{C}_*(\Lambda)$ also is a coalgebra. Consider the inclusion map $\iota_* \colon \mathcal{C}_*(\Lambda) \to \mathcal{H}_*(\Lambda)$. Since both $\mathcal{C}_*(\Lambda)$ and $\mathcal{H}_*(\Lambda)$ are free $\mathbb{Z}_{*k}$-modules, it follows that $\iota_* \otimes \iota_*$ again is injective, thus we also may consider $\mathcal{C}_*(\Lambda) \otimes_{\mathbb{Z}_{*k}} \mathcal{C}_*(\Lambda)$ as subalgebra of $\mathcal{H}_*(\Lambda) \otimes_{\mathbb{Z}_{*k}} \mathcal{H}_*(\Lambda)$.

As we have seen above, $\mathcal{H}_*(\Lambda)$ is a coalgebra via the comultiplication δ_* with

$$\delta_*(u_\lambda) = \sum_{\alpha,\beta} \overline{g}^{\lambda}_{\alpha\beta} a_\alpha a_\beta a_\lambda^{-1}\, u_\alpha \otimes u_\beta.$$

We have $\delta_*(u_i) = u_i \otimes 1 + 1 \otimes u_i$, thus δ_* maps $\mathcal{C}_*(\Lambda)$ into the $\mathbb{Z}_{*k}$-subalgebra of $\big(\mathcal{H}_*(\Lambda) \otimes \mathcal{H}_*(\Lambda)\big)_{[v_k,0,\bullet]}$ generated by the elements $u_i \otimes 1$ and $1 \otimes u_i$ with $i \in I$. But this subalgebra is just $\big(\mathcal{C}_*(\Lambda) \otimes \mathcal{C}_*(\Lambda)\big)_{[v_k,0,\bullet]}$; thus we see that δ_* induces a comultiplication

$$\mathcal{C}_*(\Lambda) \longrightarrow \big(\mathcal{C}_*(\Lambda) \otimes \mathcal{C}_*(\Lambda)\big)_{[v_k,0,\bullet]}$$

again denoted by δ_*. In this way, $\mathcal{C}_*(\Lambda)$ becomes a $(\mathbb{Z}_{*k}, v_k, 0, \bullet)$-bialgebra.

Consider now a commutative ring K which contains an element v such that $v^2 = q_k \cdot 1_K$ and we consider K as a $\mathbb{Z}_{*k}$-algebra, with $v_k \cdot 1_K = v$. We define

$$\mathcal{C}_*(\Lambda, K) = \mathcal{C}_*(\Lambda) \otimes_{\mathbb{Z}_{*k}} K.$$

Since $\mathcal{C}_*(\Lambda)$ is a free $\mathbb{Z}_{*k}$-module, it follows that $\mathcal{C}_*(\Lambda, K)$ is a free K-module. Also, $\mathcal{C}_*(\Lambda, K)$ is again a $\mathbb{Z}I$-graded algebra, and we have

$$\mathcal{C}_*(\Lambda, K) = \bigoplus_{x \in \mathbb{Z}I} \mathcal{C}_*(\Lambda, K)_x$$

where $\mathcal{C}_*(\Lambda, K)_x$ is generated, as a K-module, by the monomials $u_{i_1} u_{i_2} \dots u_{i_n}$ of degree x. We call $\mathcal{C}_*(\Lambda, K)$ the *twisted composition algebra of* Λ *with coefficients in* K*.*

Theorem*. *Let k be a finite field of order q_k. Let Λ be a k-algebra which is hereditary and finitary. Let K be a commutative ring which contains an invertible element v such that $v^2 = q_k \cdot 1_K$. Then $\mathcal{C}_*(\Lambda, K)$ is a $(K, v, 0, \bullet)$-bialgebra. As a K-module, $\mathcal{C}_*(\Lambda, K)$ is free.*

4. Euler forms and Cartan data.

We are going to consider some special bilinear forms on a free abelian group $\mathbb{Z}I$ which occur in the representation theory of algebras. For simplicity, we assume that I is a finite set. Note that a bilinear form on $\mathbb{Z}I$ is uniquely determined by an arbitrary function $I \times I \to \mathbb{Z}$.

An *Euler form* is a pair (ω, d) consisting of a bilinear form ω on $\mathbb{Z}I$ with values in $\mathbb{Z}$ and a function $d\colon I \to \mathbb{N}_1$ such that the following three properties are satisfied for all $i, j \in I$:

(a) $\omega(i, j)$ is divisible by both d_i and d_j;

(b) $\omega(i, j) \le 0$ for $i \ne j$;

(c) $d_i^{-1}\omega(i, i) \le 1$.

An *Euler datum* is a pair $(\bullet, d)$ consisting of a symmetric bilinear form $\bullet$ on $\mathbb{Z}I$ with values in $\mathbb{Z}$ and a function $d\colon I \to \mathbb{N}_1$ such that the following two properties are satisfied:

(a) $\dfrac{i \bullet i}{2d_i} \in \{1, 0, -1, -2, \dots\}$, for all $i \in I$.

(b) $\dfrac{i \bullet j}{d_i} \in \{0, -1, -2, \dots\}$, for all $i \ne j$ in I.

Let (ω, d) be an Euler form, let $\bullet$ be the symmetrization of ω, thus $x \bullet y = \omega(x, y) + \omega(y, x)$. Then $(\bullet, d)$ is an Euler datum.

Conversely, let $(\bullet, d)$ be an Euler datum. We fix some total ordering $\prec$ on the set I. We denote by $\omega = \omega_{(\bullet, \prec)}$ the following bilinear form on $\mathbb{Z}I$

$$\omega(i, j) = \begin{cases} i \bullet j & \text{if } i \prec j, \\ \frac{1}{2}\, i \bullet i & \text{if } i = j, \\ 0 & \text{if } i \succ j. \end{cases}$$

Then (ω, d) is an Euler form, and its symmetrization is the given Euler datum.

The Euler form (ω, d) is said to be *without short cycles,* provided we have, first of all, $\omega(i,i) = d_i$, and secondly, $\omega(i,j)\omega(j,i) = 0$ for $i \neq j$. Of course, given an Euler form (ω, d) without short cycles, the function d is uniquely determined by ω, thus we do not have to mention d.

Remark. For a general Euler form (ω, d), the function d is not uniquely determined, for example consider the zero bilinear form ω on $\mathbb{Z}$ itself. A more interesting example is given for $I = \{1, 2\}$, by $\omega(1,2) = -2$ and $\omega(i,j) = 0$ otherwise. In this case, any one of d_1, d_2 may be equal to either 1 or 2.

An Euler datum $(\bullet, d)$ with $i{\bullet}i = 2d_i$ for all i is said to be a *Cartan datum.* Of course, dealing with a Cartan datum, the function d is uniquely determined by $(\bullet)$, thus we do not have to mention d. A symmetric bilinear form $(\bullet)$ is a Cartan datum if and only if the following two conditions are satisfied:

(a′) $i{\bullet}i \in \{2, 4, 6, \dots\}$, for all $i \in I$.

(b′) $2\frac{i\bullet j}{i\bullet i} \in \{0, -1, -2, \dots\}$, for all $i \neq j$ in I.

We have shown above:

Lemma. *Given an Euler form ω without short cycles, its symmetrization will be a Cartan datum, and any Cartan datum arises in this way.*

Remark. Cartan data occur as basic ingredient in the book of Lusztig [L]; there is a direct correspondence between the set of Cartan data and the set of pairs consisting of a symmetrizable generalized Cartan matrices and a corresponding symmetrization, as considered in the book of Kac [K]. For any Euler datum $(\bullet, d)$, the set I is used only as an index set, and we may replace it by any other set of the same cardinality. In particular, if I is a set with n elements, we may replace I by the set $\{1, 2, \dots, n\}$. Note that *the Cartan data $(\bullet)$ with set $I = \{1, \dots, n\}$ correspond bijectively to the (symmetrizable) generalized Cartan matrices with a given symmetrization.* We recall that a *generalized Cartan matrix $(a_{ij})_{ij}$ of size n with a symmetrization $d = (d_1, \dots, d_n)$* is given by n^2 integers a_{ij} and n positive integers d_i (with $1 \leq i, j \leq n$), such that

(1) $a_{ii} = 2$ for all i;

(2) $a_{ij} \leq 0$ for all $i \neq j$; and

(3) $d_i a_{ij} = d_j a_{ji}$ for all $i \neq j$.

Given a Cartan datum $(\{1, \dots, n\}, \bullet)$, then we define $d_i = \frac{i\bullet i}{2}$, and $a_{ii} = 2$, for all $1 \leq i \leq n$, whereas, for $i \neq j$, we define $a_{ij} = 2\frac{i\bullet j}{i\bullet i}$. In this way, we obtain a generalized Cartan matrix $(a_{ij})_{ij}$ of size n with a symmetrization $d = (d_1, \dots, d_n)$. Conversely, let $(a_{ij})_{ij}$ be a generalized Cartan matrix of size n with a symmetrization $d = (d_1, \dots, d_n)$. Then we define $i{\bullet}j = d_i a_{ij}$ for all i, j.

Proposition. *Let k be a field. Let Λ be a k-algebra with only a finite set I of isomorphism classes of simple Λ-modules. For any $i \in I$, let $S(i)$ be a module in the* *isomorphism class i. We assume that for $i, j \in I$, the k-spaces* $\mathrm{Hom}_\Lambda(S(i), S(j))$ *and* $\mathrm{Ext}^1_\Lambda(S(i), S(j))$ *are finite-dimensional. For $i, j \in I$, let*

$$\begin{aligned} \langle i, j\rangle &= \dim_k \mathrm{Hom}_\Lambda(S(i), S(j)) - \dim_k \mathrm{Ext}^1_\Lambda(S(i), S(j)), \\ d_i &= \dim_k \mathrm{End}(S(i)). \end{aligned}$$

Then $E(\Lambda) = (\langle -, -\rangle, d)$ is an Euler form.

The symmetrization of $E(\Lambda)$ is a Cartan datum if and only if $\mathrm{Ext}^1_\Lambda(S(i), S(i)) = 0$, *for all i.*

Also, $E(\Lambda)$ has no short cycles, provided we have first of all $\mathrm{Ext}^1_\Lambda(S(i), S(i)) = 0$, *for all i, and secondly, if* $\mathrm{Ext}^1_\Lambda(S(i), S(j)) \neq 0$ *for some pair $i \neq j$, then we have* $\mathrm{Ext}^1_\Lambda(S(j), S(i)) = 0$.

Proof: Since the modules $S(i)$ are simple and pairwise non-isomorphic, we see that $\mathrm{End}_\Lambda(S(i))$ is a division ring, for any i, and that $\mathrm{Hom}_\Lambda(S(i), S(j)) = 0$, for $i \neq j$.

Let e_{ij} be the k-dimension of $\mathrm{Ext}^1_\Lambda(S(i), S(j))$. For all i, j, we may consider $\mathrm{Ext}^1_\Lambda(S(i), S(j))$ as an $\mathrm{End}_\Lambda(S(j))$-$\mathrm{End}_\Lambda(S(i))$-bimodule. In this way, we see that e_{ij} is divisible both by d_i and by d_j.

We have $\langle i, i\rangle = d_i - e_{ii}$, thus

$$\frac{\langle i, i\rangle}{d_i} = 1 - \frac{e_{ii}}{d_i}.$$

Finally, it is clear that $i \bullet i = 2d_i$ if and only if $\mathrm{Ext}^1_\Lambda(S(i), S(i)) = 0$.

Remark. Assume that the simple Λ-modules $S(i), S(j)$ are finite-dimensional. Then clearly $\mathrm{Hom}_\Lambda(S(i), S(j))$ is a finite-dimensional k-space. But one should keep in mind that there are obvious examples of (even hereditary) k-algebras Λ with only finite-dimensional simple Λ-modules, such that $\mathrm{Ext}^1_\Lambda(M, N)$ may not be finite-dimensional for some pair M, N of finite-dimensional Λ-modules. For example, take $\Lambda = \begin{bmatrix} k & V \\ 0 & k \end{bmatrix}$, where V is an infinite-dimensional k-space (with k acting centrally on V). However, the required finiteness condition will be satisfied in case we start with a k-algebra Λ which is finite-dimensional:

Corollary. *Let Λ be a finite-dimensional hereditary k-algebra. Let I be the set of isomorphism classes of simple Λ-modules. Then $E(\Lambda)$ is defined, and its symmetrization is a Cartan datum.*

Proof. Since Λ is finite-dimensional, all simple Λ-modules are finite-dimensional, and $\mathrm{Ext}^1_\Lambda(M, N)$ is finite-dimensional for all finite-dimensional Λ-modules M, N. This shows that $E(\Lambda)$ is defined. Since Λ is in addition also hereditary, one knows that $\mathrm{Ext}^1_\Lambda(S, S) = 0$ for all simple Λ-modules S. Thus, the symmetrization of $E(\Lambda)$ is a Cartan datum.

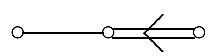
Example. Consider the algebra $\Lambda = \begin{bmatrix} k & 0 & 0 \\ k & k & 0 \\ k & k & k \end{bmatrix}$ of all lower triangular (3×3)-matrices with coefficients in k. As index set, we have $I = \{1,2,3\}$, and we have first of all, $\langle i,i\rangle = 1$, for all i; secondly, we have $\langle 1,2\rangle = -1 = \langle 2,3\rangle$; and finally $\langle i,j\rangle = 0$ for the remaining pairs i,j. Of course, this is usually called the case A_3, with diagram

In general, given a quiver Q without loops, and consider $\Lambda = kQ$, the path algebra of Q. If Q does not have oriented cycles, then Λ is finite dimensional, and the vertex set of the quiver Q may be used as index set for the isomorphism classes of the simple Λ-modules. It follows that in this case $E(\Lambda) = (\omega, d)$, where we have $d_i = 1$ for all i, and where $-\omega(i,j)$ is the number of arrows $i \to j$, for $i \neq j$.

If the quiver Q has oriented cycles, then we have to consider instead of the path algebra kQ its completion $\Lambda = \overline{kQ}$ at the ideal generated by the arrows; again, the vertex set of the quiver Q may be used as index set for the isomorphism classes of the simple Λ-modules, and $E(\Lambda) = (\omega, d)$ is given by $d_i = 1$ for all i, and by $-\omega(i,j)$ being the number of arrows $i \to j$, for $i \neq j$.

This shows that it is easy to construct for an arbitrary Euler form (ω, d) with $d_i = 1$ for all $i \in I$ an algebra Λ with $E(\Lambda) = (\omega, d)$.

As we will see, all Euler forms (ω, d) can be realized in the form $E(\Lambda)$ for some k-algebra Λ, at least for suitable fields k. Note that if k is algebraically closed, then we always will have $d_i = 1$ for all i, thus the only Euler forms (ω, d) which can be realized, are those with $d_i = 1$, for all $i \in I$. On the other hand, suppose there is given a field extension $k \subseteq K$ with $[K : k] = n$, and consider the k-algebra $\Lambda = \begin{bmatrix} k & 0 & 0 \\ k & k & 0 \\ K & K & K \end{bmatrix}$. As index set, we again have $I = \{1,2,3\}$, and we have $\langle i,i\rangle = 1$, for $i = 1,2$ and $\langle 3,3\rangle = n$; secondly, we have $\langle 1,2\rangle = -1$, and $\langle 2,3\rangle = n$; and finally $\langle i,j\rangle = 0$ for the remaining pairs i,j. Of course, for $n = 2$, this is called the case C_3, with diagram



We are interested in the case when k is a finite field. In this case, any Euler form can be realized in the form $E(\Lambda)$, where Λ is some k-algebra:

Proposition. *Let (ω, d) be an Euler form. Let k be a finite field. Then there exists a hereditary k-algebra Λ such that $E(\Lambda) = (\omega, d)$.*

Proof. We denote by $\overline{k}$ an algebraic closure of k. For any $n \in \mathbb{N}_1$, let $F(n)$ be the subfield of $\overline{k}$ with $[F(n) : k] = n$. (Of course, $F(n)$ is the splitting field of the equation $X^{q_k^n} - X$, where q_k is the cardinality of k, thus $F(n)$ is uniquely determined.) Note that for $n = n_1 n_2$, we have $F(n_1) \subseteq F(n)$ and $[F(n) : F(n_1)] = n_2$.

In particular, if $m = n_1 n_2 m'$, where n_1, n_2, m' are positive integers, then $F(m)$ is an $F(n_1)$-$F(n_2)$-bimodule (using the field multiplication as scalar multiplications). For $m = 0$, let $F(m)$ be the zero module. Then we have the general assertion: if $m = n_1 n_2 m'$, where n_1, n_2 are positive integers and m' is a non-negative integer, $F(m)$ is an $F(n_1)$-$F(n_2)$-bimodule.

We define a k-species $\mathcal{S} = (F_i, {}_iM_j)_{i,j\in I}$ as follows: For $i \neq j$, let $e_{ij} = -\omega(i,j)$; and let $e_{ii} = -\omega(i,i) + d_i$. Let

$$F_i = F(d_i) \quad \text{and} \quad {}_iM_j = F(e_{ij}).$$

The tensor algebra $T = T(\mathcal{S})$ is a hereditary k-algebra. For $i \in I$, we consider the corresponding simple T-module $S(i)$ (here, $S(i)$ is the representation of $\mathcal{S}$ which is given by the field $F(i)$ at the vertex i, by zero at the remaining vertices, and by using only zero maps). In case T is infinite-dimensional, there will be additional simple T-modules, thus we have to choose for Λ a suitable localisation of T so that the modules $S(i)$ are the only simple Λ-modules. We claim that $E(\Lambda) = (\omega, d)$.

Clearly, we have $\mathrm{End}(S(i)) = F(i)$, and we can identify

$$\mathrm{Ext}^1(S(i), S(j)) = \mathrm{Hom}_k({}_iM_j, k).$$

This finishes the proof.

Remark. Our assumption on I to be finite, is not essential. However, in order to realize homologically bilinear forms on $\mathbb{Z}I$ where I is infinite, it seems natural to consider hereditary length categories in general. Recall that a length category is an abelian category where all objects have finite length. Of course, given any ring Λ, the category of all Λ-modules of finite length is a length category.

5. The generic composition algebras.

For any Euler form (ω, d), we are going to construct a socalled generic composition algebra with coefficients in $K_0 = \mathbb{Z}[\tilde{q}, \tilde{q}^{-1}]$, where $\tilde{q}$ is a variable.

Let $\mathcal{K}$ be a set of finite fields and assume that the order of these fields is not bounded. For any $k \in \mathcal{K}$, there exists a k-algebra Λ_k such that $E(\Lambda_k) = (\omega, d)$. Note that the simple modules for these algebras are indexed by our fixed set I.

We consider the product

$$\Pi = \prod_{k\in\mathcal{K}} \mathcal{H}(\Lambda_k, \mathbb{Z}_k);$$

its elements will be written in the form $a = (a^{(k)})_k$ with $a^{(k)} \in \mathcal{H}(\Lambda_k, \mathbb{Z}_k)$; we call $a^{(k)}$ the coefficient of a with index k. Consider the central element $\tilde{q}$ with $\tilde{q}^{(k)} = q_k$ and its inverse $\tilde{q}^{-1}$ (its coefficient with index (k) is q_k^{-1}). Note that in this way Π may be regarded as a $\mathbb{Z}[\tilde{q},\tilde{q}^{-1}]$-algebra. (Note that the ring homomorphism $\mathbb{Z}[\tilde{q},\tilde{q}^{-1}] \to \Pi$ which sends $\tilde{q}$ to $\tilde{q}$ is obviously injective; this has allowed us to denote the element $(q_k)_k$ of Π just by $\tilde{q}$.)

We define some additional elements in Π. For $i \in I$, let $\tilde{u}_i = (\tilde{u}_i^{(k)})_k$ where $\tilde{u}_i^{(k)} = u_i$ (the generator corresponding to the simple Λ_k-module with index i). We denote by $\mathcal{C}(\omega,d)$ the subring of Π generated by the elements $\tilde{q},\tilde{q}^{-1}$ and $\tilde{u}_i$ with $i \in I$. The $\mathbb{Z}[\tilde{q},\tilde{q}^{-1}]$-algebra $\mathcal{C}(\omega,d)$ will be called *a generic composition algebra of type* (ω,d).

* * *

In order to construct a twisted generic composition algebra, again we start with an Euler form (ω,d). Let $K_{*0} = \mathbb{Z}[\tilde{v},\tilde{v}^{-1}]$, where also $\tilde{v}$ is a variable. (Of course, one may ask why we distinguish the rings $\mathbb{Z}[\tilde{v},\tilde{v}^{-1}]$ and $\mathbb{Z}[\tilde{q},\tilde{q}^{-1}]$: the reason is that we want to have $\tilde{q} = \tilde{v}^2$).

The twisted generic compositon algebra will be a K_{*0}-algebra. Again, let $\mathcal{K}$ be a set of finite fields and assume that the order of these fields is not bounded, and for any $k \in \mathcal{K}$, we take a k-algebra Λ_k such that $E(\Lambda_k) = (\omega,d)$.

We consider now the product

$$\Pi_* = \prod_{k\in\mathcal{K}} \mathcal{H}_*(\Lambda_k);$$

(note that $\mathcal{H}_*(\Lambda_k) = \mathcal{H}_*(\Lambda_k,\mathbb{Z}_{*k})$). Consider this time the central element $\tilde{v}$ with $\tilde{v}^{(k)} = v_k$ and its inverse $\tilde{v}^{-1}$ (the coefficient with index (k) being v_k^{-1}). Note that in this way Π_* may be regarded as a K_{*0}-algebra.

As above, we define the elements $\tilde{u}_i$ in Π_*, for $i \in I$, thus $\tilde{u}_i = (\tilde{u}_i^{(k)})_k$ where $\tilde{u}_i^{(k)} = u_i$ is the generator corresponding to the simple Λ_k-module with index i. And we denote by $\mathcal{C}_*(\omega,d)$ the K_{*0}-subalgebra of Π_* generated by these elements $\tilde{u}_i$ with $i \in I$. The K_{*0}-algebra $\mathcal{C}_*(\omega,d)$ will be called *a twisted generic composition algebra of type* (ω,d).

Lemma. *There is a K_{*0}-algebra homomorphism*

$$\big(\mathcal{C}(\omega,d)\otimes_{K_0} K_{*0}\big)_{[\tilde{v},\omega]} \longrightarrow \mathcal{C}_*(\omega,d)$$

which maps $\tilde{u}_i \otimes 1$ to $\tilde{u}_i$.

Proof. On the one hand, consider for any $k \in \mathcal{K}$ the composition of the canonical ring homomorphisms

$$\mathcal{C}(\omega,d) \to \Pi \to \mathcal{H}(\Lambda_k) \to \mathcal{H}(\Lambda_k)\otimes\mathbb{Z}_{*k}$$

and also

$$K_{*0} \xrightarrow{\pi_k} \mathbb{Z}_{*k} \to \mathcal{H}(\Lambda_k)\otimes\mathbb{Z}_{*k},$$

where $\pi_k(\tilde{v}) = v_k$. They obviously combine to a ring homomorphism

$$\mathcal{C}(\omega, d) \otimes_{\mathbb{Z}} K_{*0} \longrightarrow \mathcal{H}(\Lambda_k) \otimes \mathbb{Z}_{*k},$$

since the images of the two compositions commute. Since the images both of $\tilde{q} \otimes 1$ and $1 \otimes \tilde{q}$ are equal, namely $q_k \otimes 1 = 1 \otimes q_k$, we see that we can replace the tensor product sign $\otimes_{\mathbb{Z}}$ by $\otimes_{K_0}$.

According to Lemma II.1, the map

$$\mathcal{C}(\omega, d) \otimes_{K_0} K_{*0} \longrightarrow \mathcal{H}(\Lambda_k) \otimes \mathbb{Z}_{*k}$$

may also be considered as an algebra homomorphism

$$\big(\mathcal{C}(\omega, d) \otimes_{K_0} K_{*0}\big)_{[\tilde{v},\omega]} \longrightarrow \big(\mathcal{H}(\Lambda_k) \otimes \mathbb{Z}_{*k}\big)_{[v_k,\omega]} = \mathcal{H}_*(\Lambda_k).$$

The family of these maps yields a corresponding map

$$\big(\mathcal{C}(\omega, d) \otimes_{K_0} K_{*0}\big)_{[\tilde{v},\omega]} \to \prod_{k \in K} \mathcal{H}_*(\Lambda_k)$$

and its image is the K_{*0}-algebra generated by the elements $\tilde{u}_i$, thus just $\mathcal{C}_*(\omega, d)$.

Part IV. Quantum Groups

The following are given: First of all, a set I and a pair $\chi = (\chi', \chi'')$ of bilinear forms on $\mathbb{Z}I$ with values in $\mathbb{Z}$. Second, let K be a commutative ring and let $v \in K$ be an invertible element.

We will consider the free K-algebra $F = K\langle I\rangle$ generated by the set I; the generator corresponding to $i \in I$ will be denoted by θ_i. Note that F is a $\mathbb{Z}I$-graded algebra, with θ_i of degree i.

Depending on v and χ, we will define on F a comultiplication $\delta\colon F \to F \otimes F$, so that (F, δ) is a $\mathbb{Z}I$-graded coalgebra. The definition follows Lusztig [L], and is arranged in such a way that F becomes a (K, v, χ)-bialgebra. All the results presented in this part are based on the book [L] of Lusztig.

1. Preliminaries on bilinear forms.

We are going to construct some (K, v, χ)-bialgebras. Before we do this, let us consider pairs of bilinear forms on a free abelian group. Thus, let us assume that there are given a set I and a pair $\chi = (\chi', \chi'')$ of bilinear forms on $\mathbb{Z}I$ with values in $\mathbb{Z}$.

Recall that $\chi\colon (\mathbb{Z}I)^4 \to \mathbb{Z}$ is defined by

$$\chi(x_1, x_2, x_3, x_4) = \chi'(x_1, x_4) + \chi''(x_2, x_3),$$

and that we say that χ is given by the pair (χ', χ'').

In general, let $\chi\colon (\mathbb{Z}I)^n \to \mathbb{Z}$ be a (set) map. Given $1 \le r < s \le n$, we denote by $L(r, s)$ the following property: If x_i, y_i, z_i are integers such that $x_r = y_r + z_r,\ x_s = y_s + z_s$, and $x_i = y_i = z_i$ for the remaining i, then

$$\chi(x_1, \dots, x_n) = \chi(y_1, \dots, y_n) + \chi(z_1, \dots, z_n),$$

The maps $\chi\colon (\mathbb{Z}I)^4 \to \mathbb{Z}$ which are given by pairs of bilinear forms on $\mathbb{Z}I$ are characterized by the following conditions: $L(1,2)$, $L(1,3)$, $L(2,4)$, $L(3,4)$.

As an example, let us show that $\chi\colon (\mathbb{Z}I)^4 \to \mathbb{Z}$ given by the pair (χ', χ'') satisfies the property $L(1, 2)$:

$$\begin{aligned}\chi(y_1 + z_1, y_2 + z_2, x_3, x_4) &= \chi'(y_1 + z_1, x_4) + \chi''(y_2 + z_2, x_3)\\ &= \chi'(y_1, x_4) + \chi'(z_1, x_4) + \chi''(y_2, x_3) + \chi''(z_2, x_3)\\ &= \chi(y_1, y_2, x_3, x_4) + \chi(z_1, z_2, x_3, x_4)\end{aligned}$$

Similarly, χ also satisfies the conditions $L(1, 3), L(2, 4), L(3, 4)$. Conversely, assume that $\chi\colon (\mathbb{Z}I)^4 \to \mathbb{Z}$ satisfies the properties $L(1, 2), L(1, 3), L(2, 4), L(3, 4)$. Then $L(1, 2)$ and $L(3, 4)$ yield

$$\begin{aligned}\chi(x_1, x_2, x_3, x_4) &= \chi(x_1, 0, x_3, x_4) + \chi(0, x_2, x_3, x_4)\\ &= \chi(x_1, 0, x_3, 0) + \chi(x_1, 0, 0, x_4) + \chi(0, x_2, x_3, 0) + \chi(0, x_2, 0, x_4);\end{aligned}$$

using $L(2,4)$ we see that $\chi(x_1,0,x_3,0) = \chi(x_1,0,x_3,0) + \chi(x_1,0,x_3,0)$, and therefore $\chi(x_1,0,x_3,0) = 0$; similarly, we see that $L(1,3)$ yields $\chi(0,x_2,0,x_4) = 0$. As a consequence,

$$\chi(x_1,x_2,x_3,x_4) = \chi(x_1,0,0,x_4) + \chi(0,x_2,x_3,0),$$

and clearly the two terms on the right are bilinear forms on $\mathbb{Z}I$.

The conditions $L(1,2)$ and $L(3,4)$ just mean that χ is a bilinear form on $(\mathbb{Z}I)^2$. Thus: *If $\chi\colon (\mathbb{Z}I)^4 \to \mathbb{Z}$ is given by pairs of bilinear forms on $\mathbb{Z}I$, then χ is a bilinear form on $(\mathbb{Z}I)^2$.*

If χ is a bilinear form on $(\mathbb{Z}I)^2$, then $\tilde{\chi}$ defined by

$$\tilde{\chi}(x_1,x_2,x_3;y_1,y_2,y_3) = \chi(x_1,x_2;y_1,y_2) + \chi(x_1,x_3;y_1,y_3) + \chi(x_2,x_3;y_2,y_3)$$

for $x_1,x_2,x_3,y_1,y_2,y_3 \in \mathbb{Z}I$ is a bilinear form on $(\mathbb{Z}I)^3$. Again, the proof is straightforward.

We extend a previous convention as follows: Given any $\mathbb{Z}I$-graded algebra A, a (set) map $\chi\colon (\mathbb{Z}I)^n \to \mathbb{Z}$, and elements $a_1,\dots,a_n$ such that a_i is homogeneous of degree x_i, then we write $\chi(a_1,\dots,a_n) = \chi(x_1,\dots,x_n)$, in case this is well-defined and not confusing. Sometimes it will be helpful to denote the degree of a homogeneous element $a \in A$ by $|a|$, thus $|a| \in \mathbb{Z}I$ and $a \in A_{|a|}$ (of course, $|a|$ is well-defined only in case $a \neq 0$).

2. The free K-algebra on I as a (K, v, χ)-bialgebra.

Recall that a fixed set I is given. We denote by $\langle I\rangle$ the free semigroup (with 1) generated by I, the generator corresponding to $i \in I$ will be denoted by θ_i. Thus, the elements of $\langle I\rangle$ are words in the letters θ_i ($i \in I$): there is the empty word which is denoted by 1, and there are the words $\theta_{i_1}\theta_{i_2}\cdots\theta_{i_n}$ of length $n \geq 1$, with $i_1, i_2, \dots, i_n \in I$. The multiplication in $\langle I\rangle$ is just the concatenation of words. The free K-algebra $F = K\langle I\rangle$ generated by I is just the semigroup algebra of $\langle I\rangle$ over K; it is the free K-module with basis $\langle I\rangle$, and the multiplication of these base elements is the concatenation of words.

We consider $F = K\langle I\rangle$ as a $\mathbb{Z}I$-graded algebra, with the generator θ_i being of degree i. We denote by $\epsilon\colon F \to F_0 = K$ the canonical projection.

In order to define the comultiplication δ, we consider the algebra $(F\otimes F)_{[v,\chi]}$; recall that it is given by the tensor product $F\otimes F$ with the multiplication

$$(a_1\otimes a_2) * (b_1 \otimes b_2) = v^{\chi(a_1,a_2,b_1,b_2)} a_1b_1 \otimes a_2b_2.$$

As we have seen above, this is an associative multiplication, since χ is a bilinear form on $(\mathbb{Z}I)^2$.

Let δ be the algebra homomorphism $F \to (F \otimes F)_{[v,\chi]}$ defined by $\delta(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i$.

Lemma. *With this comultiplication δ the K algebra $F = K\langle I\rangle$ becomes a (K, v, χ)-bialgebra; its counit is ϵ.*

We have to verify the following: first, the comultiplication is coassociative; and secondly, ϵ is the corresponding counit; after these verifications, it will be trivially true that we have obtained a (K, v, χ)-bialgebra.

Recall that the bilinear form χ on $(\mathbb{Z}I)^2$ gives rise to a corresponding bilinear form $\tilde{\chi}$ on $(\mathbb{Z}I)^3$. Thus, given a $\mathbb{Z}I$-graded K-algebra, we may consider the $(\mathbb{Z}I)^2$-graded K-algebra $A \otimes A$ (with componentwise multiplication $(a_1 \otimes a_2)(b_1 \otimes b_2) = a_1b_1 \otimes a_2b_2$) and its twisted version $(A \otimes A)_{[v,\chi]}$ (as considered in the last chapter), as well as the $(\mathbb{Z}I)^3$-graded algebra $A \otimes A \otimes A$ (again with componentwise multiplication) and its twisted version $(A \otimes A \otimes A)_{[v,\tilde{\chi}]}$. Of course, all these algebras $A \otimes A$, $(A \otimes A)_{[v,\chi]}$, $A \otimes A \otimes A$ and $(A \otimes A \otimes A)_{[v,\tilde{\chi}]}$ can be considered also as $\mathbb{Z}I$-graded algebras using the "total degree": For example, for $z \in \mathbb{Z}I$, we let

$$(A \otimes A)_z = \bigoplus_{x+y=z} A_x \otimes A_y.$$

Lemma. *Let A be a $\mathbb{Z}I$-graded K-algebra. Let $\delta\colon A \to (A \otimes A)_{[v,\chi]}$ be a homomorphism of $\mathbb{Z}I$-graded algebras. Then also $1 \otimes \delta$ and $\delta \otimes 1$ are algebra homomorphisms $(A \otimes A)_{[v,\chi]} \to (A \otimes A \otimes A)_{[v,\tilde{\chi}]}$.*

Proof: Consider elements $a_1, a_2, b_1, b_2 \in A$. Let

$$\delta(a_2) = \sum a_{21} \otimes a_{22},$$
$$\delta(b_2) = \sum b_{21} \otimes b_{22},$$

thus

$$(1 \otimes \delta)(a_1 \otimes a_2) = \sum a_1 \otimes a_{21} \otimes a_{22},$$
$$(1 \otimes \delta)(b_1 \otimes b_2) = \sum b_1 \otimes b_{21} \otimes b_{22}.$$

The product of these two elements is

$$\begin{aligned}&(1\otimes\delta)(a_1 \otimes a_2) \cdot (1 \otimes \delta)(b_1 \otimes b_2)\\ &\quad = \sum v^{\chi(a_1,a_{21},b_1,b_{21})+\chi(a_1,a_{22},b_1,b_{22})+\chi(a_{21},a_{22},b_{21},b_{22})} a_1b_1 \otimes a_{21}b_{21} \otimes a_{22}b_{22}.\end{aligned}$$

On the other hand

$$\begin{aligned}&(1 \otimes \delta)\big((a_1 \otimes a_2)(b_1 \otimes b_2)\big)\\ &\quad = (1 \otimes \delta)v^{\chi(a_1,a_2,b_1,b_2)}(a_1b_1 \otimes a_2b_2)\\ &\quad = v^{\chi(a_1,a_2,b_1,b_2)}\Big(a_1b_1 \otimes \sum (a_{21} \otimes a_{22})(b_{21} \otimes b_{22})\Big)\\ &\quad = \sum v^{\chi(a_1,a_2,b_1,b_2)+\chi(a_{21},a_{22},b_{21},b_{22})} a_1b_1 \otimes a_{21}b_{21} \otimes a_{22}b_{22}.\end{aligned}$$

We have to show that the two exponents of v are equal. Note that the last summands of the exponents are equal, thus it remains to see that

$$\chi(a_1, a_2, b_1, b_2) = \chi(a_1, a_{21}, b_1, b_{21}) + \chi(a_1, a_{22}, b_1, b_{22}),$$

but this is just the linearity condition $L(2,4)$.

This shows that $1 \otimes \delta$ is multiplicative; similarly, one shows that $\delta \otimes 1$ is multiplicative.

Proof of Lemma: The maps $(\delta \otimes 1)\delta, (1 \otimes \delta)\delta$ from F to $(F \otimes F \otimes F)_{[v,\tilde{\chi}]}$ are algebra homomorphisms and they coincide on the generators θ_i (both send θ_i to $\theta_i \otimes 1 \otimes 1 + 1 \otimes \theta_i \otimes 1 + 1 \otimes 1 \otimes \theta_i$), thus they are equal. This shows the coassociativity.

Next, we show that ϵ is a counit. One easily checks that $\epsilon \otimes 1$ and $1 \otimes \epsilon$ are algebra homomorphisms $(F \otimes F)_{[v,\chi]} \to F$ since $F_x \neq 0$ only for $x \geq 0$. It follows that the algebra maps $(\epsilon \otimes 1)\delta$ and $(1 \otimes \epsilon)\delta$ both are equal to the identity, since they map any θ_i to itself.

3. Lusztig forms on F.

Given a graded $\mathbb{Z}I$-module $A = \bigoplus_{x \in \mathbb{Z}I} A_x$, a bilinear form $(-,-) : A \otimes A \to K$ is said to respect the grading provided we have $(a, b) = 0$ for $a \in A_x,\ b \in A_y$ and $x \neq y$ in $\mathbb{Z}I$.

Proposition. *Assume that for any $i \in I$, there is given an element $t_i \in K$. There exists a unique bilinear form $(-,-)_t$ on the (K, v, χ)-bialgebra $F = K\langle I \rangle$ with the following properties:*

(0) *The bilinear form $(-,-)_t$ respects the grading.*

(1) *We have $(\theta_i, \theta_i)_t = t_i$ for all $i \in I$,*

(2) *For $\delta(a) = \sum a_1 \otimes a_2$, we have*

$$(a, bc)_t = \sum (a_1, b)_t (a_2, c)_t.$$

We write just:

$$(a, bc)_t = (\delta(a), b \otimes c)_t,$$

where we introduce a corresponding bilinear form on $F \otimes F$ which works componentwise and we denote it again by $(-,-)_t$. Thus, for $a_1, a_2, b_1, b_2 \in F$, we have $(a_1 \otimes a_2, b_1 \otimes b_2)_t = (a_1, b_1)_t (a_2, b_2)_t$.

This result has been shown (at least for special pairs χ) by Lusztig [L]. We call $(-,-)_t$ the *Lusztig form on $F = K\langle I \rangle$ with respect to v, χ and t.*

Proof of the existence:

The algebra $F = \bigoplus_{x\in\mathbb{Z}I} F_x$ is a graded algebra, and the parts F_x are finitely generated free k-modules. We may consider the corresponding dual k-modules F_x^* and we define

$$F^* = \bigoplus_{x\in\mathbb{Z}I} F_x^*,$$

with multiplication given by δ and with comultiplication given by the multiplication of F. Let $\phi_i,\ i \in I$ be the dual basis (inside $\bigoplus_{i\in I} F_i^*$) for the generating set $\theta_i,\ i \in I$. Given an element $\phi \in F_x^*$, we denote the image of $a \in F_x$ under ϕ by $(a)\phi$.

Remark. The elements $\phi_i,\ i \in I$ usually will not form a generating set of F^*; in contrast, one should be interested in the subalgebra generated by the elements ϕ_i. The algebra F^* is called the *shuffle algebra for I* by J.A.Green, see [G2].

Consider the algebra homomorphism

$$\alpha\colon F \to F^* \quad \text{defined by} \quad (\theta_i)\alpha = \phi_i t_i.$$

Define

$$(a, b)_t = (a)(b)\alpha$$

(this means: apply α to b, this gives a linear form $(b)\alpha$, and now apply this linear form to a; in case a, b are homogeneous and have different degrees, then this is by definition 0.)

Clearly, $(-,-)_t$ is a bilinear form, with the following properties: if a, b are homogeneous, and $(a, b)_t \neq 0$, then a, b have the same degree, thus $(-,-)_t$ respects the grading of F. Also, we have $(\theta_i, \theta_i)_t = (\theta_i)\phi_i t_i = t_i$, thus also condition (1) is satisfied. In addition, we have the following equality:

$$(a, bc)_t = a(bc)\alpha = a(b\alpha * c\alpha)$$

where $*$ is the multiplication in F^* (let us recall how the multiplication in F^* is defined: let ϕ, ψ belong to F^*. Then $\phi * \psi$ maps $a \in F$ onto $\delta(a)(\phi \otimes \psi)$, where δ is the comultiplication of F.) Now assume that $\delta(a) = \sum a_1 \otimes a_2$. We can continue our calculation

$$a(b\alpha * c\alpha) = \sum (a_1)(b)\alpha \cdot (a_2)(c)\alpha = \sum (a_1, b)_t (a_2, c)_t.$$

This shows that also condition (2) is satisfied.

The proof of the unicity will use the following Lemma:

Lemma. *Assume that A is a $\mathbb{Z}I$-graded k-algebra. Let $\delta\colon A \to (A \otimes A)_{[v,\chi]}$ be a homomorphism of $\mathbb{Z}I$-graded algebras. Assume that there exists a bilinear form $(-,-)$ on A with values in k such that*

$$(a, bc) = (\delta(a), b \otimes c)$$

for all $a, b, c \in A$.

Let a, a' *be homogeneous elements of* A *and let*

$$\delta(a) = \sum a_1 \otimes a_2, \qquad \delta(a') = \sum a'_1 \otimes a'_2.$$

Then, given $b, b' \in A$, *we have*

$$(aa', bb') = \sum v^{\chi(a_1,a_2,a'_1,a'_2)}(a_1a'_1, b)(a_2a'_2, b').$$

Proof: Since δ preserves the $\mathbb{Z}I$-grading, the various elements a_1, a_2, a'_1, a'_2 all are homogeneous, again. Since δ is multiplicative,

$$\begin{aligned}\delta(aa') &= \delta(a)\delta(a') \\ &= \sum (a_1 \otimes a_2) \cdot (a'_1 \otimes a'_2) \\ &= \sum v^{\chi(a_1,a_2,a'_1,a'_2)} a_1a'_1 \otimes a_2a'_2,\end{aligned}$$

and therefore

$$\begin{aligned}(aa', bb') &= (\delta(aa'), b \otimes b') \\ &= \sum v^{\chi(a_1,a_2,a'_1,a'_2)}(a_1a'_1 \otimes a_2a'_2, b \otimes b') \\ &= \sum v^{\chi(a_1,a_2,a'_1,a'_2)}(a_1a'_1, b)(a_2a'_2, b').\end{aligned}$$

Proof of the unicity of the form $(-,-)_t$. Let $(-,-)$ and $(-,-)'$ be two bilinear forms with the properties (0), (1), (2). If a, b belong to $A_0 \oplus \bigoplus_{i\in I} A_i$, then by assumption $(a, b) = (a, b)'$. Consider now A_x where $x = (x_i)_i$ belongs to $\mathbb{Z}I$, all $x_i \geq 0$, and $\sum x_i \geq 2$. We show that the forms $(-,-)$ and $(-,-)'$ agree on A_x. Let a, a', b, b' be homogeneous of non-zero degree such that aa' and bb' belong to A_x. Let

$$\delta(a) = \sum a_1 \otimes a_2, \qquad \delta(a') = \sum a'_1 \otimes a'_2.$$

The Lemma asserts that

$$(aa', bb') = \sum v^{\chi(a_1,a_2,a'_1,a'_2)}(a_1a'_1, b)(a_2a'_2, b');$$

and similarly

$$(aa', bb')' = \sum v^{\chi(a_1,a_2,a'_1,a'_2)}(a_1a'_1, b)'(a_2a'_2, b')'.$$

By induction, we know that

$$\begin{aligned}(a_1a'_1, b) &= (a_1a'_1, b)' \quad \text{and} \\ (a_2a'_2, b') &= (a_2a'_2, b')'.\end{aligned}$$

Thus we see that $(aa', bb') = (aa', bb')'$.

4. Scalar extension.

Let $\eta\colon K \to K'$ be a homomorphism of commutative rings. Let v be invertible in K and $\eta(v) = v'$. Let $(-,-)_1$ be the bilinear form on $K\langle I\rangle$ for v, χ and $1 = (1)_i$. Let $(-,-)'_t$ be the bilinear form on $K'\langle I\rangle$ for v', χ and $t = (t_i)_i$. If $x = (x_i)_i$ belongs to $\mathbb{Z}I$ (thus $x_i \in \mathbb{Z}$, and almost all of these coefficients are zero), then we put $t^x = \prod_{i\in I} t_i^{x_i}$.

We define $\eta\colon K\langle I\rangle \to K'\langle I\rangle$ by $\eta(v) = v'$ and $\eta(\theta_i) = \theta_i$.

Corollary 1. *Let $x \in \mathbb{Z}I$. Then, for all $a, b \in K\langle I\rangle_x$, we have*

$$(\eta(a), \eta(b))'_t = \eta\big((a,b)_1\big) \cdot t^x.$$

Proof: Let $F = K\langle I\rangle$ and $F' = K'\langle I\rangle$. First of all, we note that

$$\eta \otimes \eta\colon \big(F \otimes_K F\big)_{[v,\chi]} \to \big(F' \otimes_{K'} F'\big)_{[v',\chi]}$$

is an algebra homomorphism. It follows that

$$\delta'\eta = (\eta \otimes \eta)\delta,$$

since these two maps coincide on the generators θ_i.

Since F' is the free K'-module with basis $\langle I\rangle$, we may define a K'-bilinear form $(-,-)''$ on F' by just specifying the values on pairs from $\langle I\rangle$. For $x \in \mathbb{Z}I$, we denote by $\langle I\rangle_x$ the set of words of degree x. Recall that the monomial $\theta_{i_1}\theta_{i_2}\ldots\theta_{i_n}$ has degree $x = (x_i)_i$ if and only if the element i occurs precisely x_i times in the sequence $(i_1, i_2, \ldots, i_n)$.

The set $\langle I\rangle$ will be considered as a K-basis of F; if θ is an element of $\langle I\rangle$, then the corresponding element of F' will be denoted by $\eta(\theta)$. Of course, the element $\eta(\theta)$ is just the "same" element as θ, but, in this way, we distinguish the basis elements of F and F'.

If θ, θ' are monomials of degree x for some $x \in \mathbb{Z}I$, let

$$(\eta(\theta), \eta(\theta'))'' = \eta\big((\theta, \theta')_1\big) \cdot t^x$$

(note that the last product has only finitely many factors different from 1); for $x \neq y$ in $\mathbb{Z}I$ and $\theta \in \langle I\rangle_x$, $\theta' \in \langle I\rangle_y$, we define $(\eta(\theta), \eta(\theta'))'' = 0$. We claim that $(-,-)'' = (-,-)'_t$.

First of all, the bilinear form $(-,-)''$ respects the grading. Second, we clearly have $(\eta(\theta_i), \eta(\theta_i))'' = t_i$. Thus it remains to consider the adjunction property

$(a, bc)'' = (\delta'(a), b \otimes c)''$, for $a, b, c \in F'$ (note that we will denote the comultiplication of F' by δ'). It is sufficient to verify this equality for a, b, c being monomials. Let $|a| = x$, $|b| = y$, $|c| = z$. We can assume that $x = y + z$. Let $\delta(a) = \sum a_1 \otimes a_2$. Then $\delta'\eta(a) = \sum \eta(a_1) \otimes \eta(a_2)$. We see:

$$\begin{aligned}
(\delta'\eta(a), \eta(b) \otimes \eta(c))''\cdot &= \sum (\eta(a_1), \eta(b))''(\eta(a_2), \eta(c))'' \\
&= \sum \eta\big((a_1, b)_1\big)t^y \cdot \eta\big((a_2, c)_1\big)t^z \\
&= \eta\big(\sum (a_1, b)_1 (a_2, c)_1\big) \cdot t^x \\
&= \eta\big((a_1, b)_1 (a_2, c)_1\big) \cdot t^x \\
&= \eta\big((a, bc)_1\big) \cdot t^x \\
&= (\eta(a), \eta(b)\eta(c))''.
\end{aligned}$$

Since $(-,-)''$ satisfies all the defining properties for $(-,-)_t$, we conclude that these forms are equal.

There are two special cases which should be mentioned.

Corollary 2. *Let $x \in \mathbb{Z}I$. The values on F_x of the Lusztig form $(-,-)_t$ with respect to v, χ and t are given by*

$$(a, b)_t = (a, b)_1 \cdot t^x.$$

Of particular interest is (as J.A.Green has pointed out) the case $K_0 = \mathbb{Z}[\tilde{q}, \tilde{q}^{-1}]$, where $\tilde{q}$ is a variable; the corresponding Lusztig form on $K_0\langle I\rangle$ with respect to $\tilde{q}, \chi$ and 1 will be called the *generic Lusztig form for χ*.

Corollary 3. *Let $K_0 = \mathbb{Z}[\tilde{q}, \tilde{q}^{-1}]$. Let $(-,-)_1$ be the generic Lusztig form on $K_0\langle I\rangle$. Let $\eta\colon K_0 \to K$ be a ring homomorphism, and let $(-,-)_1'$ the Lusztig form on $K\langle I\rangle$ with respect to $\eta(\tilde{q}), \chi$ and 1. Then we have for $a, b \in K_0\langle I\rangle$*

$$(\eta(a), \eta(b))_1' = (a, b)_1(\eta(\tilde{q})).$$

Note that $(a, b)_1(\eta(\tilde{q}))$ means the following: $(a, b)_1$ belongs to $\mathbb{Z}[\tilde{q}, \tilde{q}^{-1}]$, is thus a Laurent polynomial in the variable $\tilde{q}$ and we insert $\eta(\tilde{q}) \in K$ into this Laurent polynomial.

5. The definition of $U^+(K,v,\chi)$.

As before, let K be a commutative ring, $v \in K$ an invertible element. Let I be a set and $\chi = (\chi', \chi'')$ a pair of bilinear forms on $\mathbb{Z}I$.

Let $(-,-) = (-,-)_1$ be the Lusztig form on $F = K\langle I\rangle$ with respect to v, χ and $t_i = 1$ for all $i \in I$. Let $\mathcal{I}(K,v,\chi)$ be the set of elements $b \in F$ such that $(a,b) = 0$ for all $a \in F$.

Lemma. *$\mathcal{I}(K,v,\chi)$ is an ideal of F.*

Proof. Clearly, $\mathcal{I}(K,v,\chi)$ is a K-submodule. Let $b \in \mathcal{I}(K,v,\chi)$ and $c \in F$. Then, for $a \in F$ with $\delta(a) = \sum a_1 \otimes a_2$ we have

$$\begin{aligned}(a, bc) &= (\delta(a), b \otimes c) = \sum (a_1, b)(a_2, c) = 0,\\ (a, cb) &= (\delta(a), c \otimes b) = \sum (a_1, c)(a_2, b) = 0.\end{aligned}$$

Definition: Let

$$U^+(K,v,\chi) = F/\mathcal{I}(K,v,\chi).$$

For special values of χ, we obtain in this way just the positive part of a corresponding "quantum group", see Lusztig [L]. Typical cases will be considered in the last section.

Proposition 1. *Let A be a positive (K,v,χ)-bialgebra and assume that A is generated by elements $a_i \in A_i$, with $i \in I$. In addition, let $(-,-)$ be a bilinear form on A which respects the grading and such that $(a,bc) = (\delta(a), b \otimes c)$ for all $a,b,c \in A$, and let us assume that no element $t_i = (a_i,a_i)$ is a zero-divisor in K.*

Then the kernel of the K-algebra homomorphism $\xi\colon K\langle I\rangle \to A$ with $\xi(\theta_i) = a_i$ is contained in the ideal $\mathcal{I}(K,v,\chi)$ and thus ξ induces an algebra homomorphism $\overline{\xi}\colon A \to U^+(K,v,\chi)$.

Proof: Write $F = K\langle I\rangle$. According to section II.1, we know that $\xi\colon F \to A$ is also a coalgebra homomorphism. We define a bilinear form $(-,-)'$ on F by

$$(a,b)' = (\xi(a), \xi(b)) \quad \text{for} \quad a,b \in F.$$

This bilinear form respects the grading of F, and we have

$$\begin{aligned}(a,bc)' &= (\xi(a), \xi(bc))\\ &= (\xi(a), \xi(b)\xi(c))\\ &= (\delta\xi(a), \xi(b) \otimes \xi(c))\\ &= (\xi\delta(a), \xi(b) \otimes \xi(c))\\ &= (\delta(a), b \otimes c)'.\end{aligned}$$

The unicity of the Lusztig form on F with respect to v, χ and t implies that $(-,-)' = (-,-)_t$.

We want to show that $\operatorname{Ker}\xi \subseteq \mathcal{I}(K, v, \chi)$. Since ξ respects the grading, it is sufficient to consider $b \in F_x \cap \operatorname{Ker}\xi$. Let $a \in F_x$. Then

$$(a, b)_1 \cdot t^x = (a, b)_t = (a, b)' = (\xi(a), \xi(b)) = 0.$$

This shows that b belongs to $\mathcal{I}(K, v, \chi)$.

* * *

In general, the Lusztig forms on F will not be symmetric. We show the following:

Proposition 2. *If χ'' is symmetric, then the Lusztig form $(-,-)_1$ on F is also symmetric.*

Proof: We assume that χ'' is symmetric and we show that $(-,-)_1$ also satisfies

$$(aa', c)_1 = (a \otimes a', \delta(c))_1$$

for all $a, a', c \in A$.

This equality is easily verified in case c belongs to $\mathcal{F}_0 \oplus \bigoplus_{i \in I} \mathcal{F}_i$, thus, it is sufficient to consider the case where $c = bb'$ with homogeneous elements b, b' of non-zero degree. Write

$$\begin{aligned} \delta(a) &= \sum a_1 \otimes a_2, & \delta(a') &= \sum a_1' \otimes a_2', \\ \delta(b) &= \sum b_1 \otimes b_2, & \delta(b') &= \sum b_1' \otimes b_2'. \end{aligned}$$

Using the Lemma and induction, we have

$$\begin{aligned} (aa', bb')_1 &= \sum v^{\chi(a_1, a_2, a_1', a_2')} (a_1 a_1', b)_1 (a_2 a_2', b')_1 \\ &= \sum v^{\chi(a_1, a_2, a_1', a_2')} (a_1 \otimes a_1', \delta(b))_1 (a_2 \otimes a_2', \delta(b'))_1 \\ &= \sum v^{\chi(a_1, a_2, a_1', a_2')} (a_1 \otimes a_1', b_1 \otimes b_2)_1 (a_2 \otimes a_2', b_1' \otimes b_2')_1 \\ &= \sum v^{\chi(a_1, a_2, a_1', a_2')} (a_1, b_1)_1 (a_1', b_2)_1 (a_2, b_1')_1 (a_2', b_2')_1. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} \delta(bb') &= \delta(b)\delta(b') \\ &= \sum (b_1 \otimes b_2) \cdot (b_1' \otimes b_2') \\ &= \sum v^{\chi(b_1, b_2, b_1', b_2')} b_1 b_1' \otimes b_2 b_2', \end{aligned}$$

and therefore

$$\begin{aligned}
(a\otimes a',\delta(bb'))_1 &= \sum v^{\chi(b_1,b_2,b_1',b_2')}(a\otimes a', b_1b_1'\otimes b_2b_2')_1\\
&= \sum v^{\chi(b_1,b_2,b_1',b_2')}(a, b_1b_1')_1(a', b_2b_2')_1\\
&= \sum v^{\chi(b_1,b_2,b_1',b_2')}(\delta(a), b_1\otimes b_1')_1(\delta(a'), b_2\otimes b_2')_1\\
&= \sum v^{\chi(b_1,b_2,b_1',b_2')}(a_1\otimes a_2, b_1\otimes b_1')_1(a_1'\otimes a_2', b_2\otimes b_2')_1\\
&= \sum v^{\chi(b_1,b_2,b_1',b_2')}(a_1,b_1)_1(a_2,b_1')_1(a_1',b_2)_1(a_2',b_2')_1\\
&= \sum v^{\chi(b_1,b_2,b_1',b_2')}(a_1,b_1)_1(a_1',b_2)_1(a_2,b_1')_1(a_2',b_2')_1.
\end{aligned}$$

We claim that for non-zero $(a_1,b_1)_1(a_1',b_2)_1(a_2,b_1')_1(a_2',b_2')_1$, the coefficients

$$v^{\chi(a_1,a_2,a_1',a_2')} \quad \text{and} \quad v^{\chi(b_1,b_2,b_1',b_2')}$$

coincide. However $(a_1,b_1)_1(a_1',b_2)_1(a_2,b_1')_1(a_2',b_2')_1 \neq 0$ implies that

$$|a_1| = |b_1|, \quad |a_1'| = |b_2|, \quad |a_2| = |b_1'|, \quad |a_2'| = |b_2'|$$

and by assumption

$$\chi(a_1,a_2,a_1',a_2') = \chi(b_1,b_1',b_2,b_2') = \chi(b_1,b_2,b_1',b_2').$$

We see that also the form $(-,-)'$ defined by $(a,b)' = (b,a)_1$ satisfies the conditions (0), (1) (2) of a Lusztig form, thus $(-,-)' = (-,-)_1$, thus we have $(a,b)_1 = (b,a)_1$ for all a,b.

Part V. The Isomorphism

Let $\bullet$ be a Cartan datum which is defined on $\mathbb{Z}I$. We can consider it as the symmetrization of an Euler form ω without short cycles. In Part III, we have constructed to (ω, d) generic composition algebras and twisted generic composition algebras. In Part IV, we have seen that given any pair of bilinear forms on $\mathbb{Z}I$, for example $(-\omega, 0)$ or $(0, \bullet)$, one may endow the corresponding free algebra $K\langle I\rangle$ with a symmetric bilinear form whose radical is an ideal, and we have denoted the corresponding factor algebra by U^+. The aim of this final part is to show that under some constraints the same algebras will have been constructed in these two different ways.

Let $\mathbb{Q}(\tilde{v})$ be the field of rational functions in one variable $\tilde{v}$. Let $\tilde{q} = \tilde{v}^2$. As before, we consider the following subrings of $\mathbb{Q}(\tilde{v})$

$$K_0 = \mathbb{Z}[\tilde{q}, \tilde{q}^{-1}] \quad \text{and} \quad K_{*0} = \mathbb{Z}[\tilde{v}, \tilde{v}^{-1}].$$

1. Quantum Serre relations.

In Part III, the fundamental relations for $\mathcal{H}(\Lambda)$ and $\mathcal{H}_*(\Lambda)$ have been exhibited. These relations only involve the generators of the corresponding composition algebras, thus, we obtain in this way relations which are valid for the generators $\tilde{u}_i$ of the generic composition algebra and the twisted generic composition algebra.

Let ω be an Euler form without short cycles. (Thus, $\omega\colon \mathbb{Z}I \times \mathbb{Z}I \to \mathbb{Z}$ is a bilinear form, $d_i = \omega(i,i)$ is positive and divides both $\omega(i,j)$ and $\omega(j,i)$ and, for $i \neq j$, we have $\omega(i,j)\omega(j,i) = 0$.) We define for $i \neq j$

$$n(i,j) = \begin{cases} -d_i^{-1}\omega(i,j) + 1 & \text{if} \quad \omega(j,i) = 0, \\ -d_i^{-1}\omega(j,i) + 1 & \text{if} \quad \omega(i,j) = 0, \end{cases}$$

Let $\bullet$ be the symmetrization of ω. Then we have

$$n(i,j) = -d_i^{-1}\, i{\bullet}j + 1,$$

thus $n(i,j)$ only depends on $\bullet$.

First, let us consider the case of the generic composition algebra $\mathcal{C}(\omega)$. By definition, $\mathcal{C}(\omega)$ may be considered as a factor algebra of the free algebra $K_0\langle I\rangle$.

We consider a general commutative ring K with a fixed invertible element $q \in K$ as a K_0-algebra, with $\tilde{q}$ acting on K by multiplying with q. Then also the free algebra $K\langle I\rangle$ becomes a K_0-algebra and we may consider the ideal $\mathcal{J}(K,q,\omega)$ in $K\langle I\rangle$ generated by the elements

$$\sum_{t=0}^{n(i,j)} (-1)^t q^{d_i\binom{t}{2}} \left|\begin{matrix} n(i,j) \\ t \end{matrix}\right] (q^{d_i})\, \theta_i^{n(i,j)-t}\theta_j\theta_i^t,$$

$$\sum_{t=0}^{n(j,i)} (-1)^t q^{d_j\binom{t}{2}} \left|\begin{matrix} n(j,i) \\ t \end{matrix}\right] (q^{d_j})\, \theta_j^t\theta_i\theta_j^{n(j,i)-t},$$

for all pairs $i \neq j$ with $\omega(j,i) = 0$. We recall that the coefficients $\left|\begin{matrix} n(i,j) \\ t \end{matrix}\right] (q^{d_i})$ belong to K_0 (they even belong to $\mathbb{Z}[\tilde{q}]$), according to III.1. The generating elements of $\mathcal{J}(K,q,\omega)$ will be said to be *quantum Serre relations*. We define

$$\widehat{U}^+(K,q,\omega) = K\langle I\rangle/\mathcal{J}(K,q,\omega).$$

Remark. The quantum Serre relations exhibited above are certain homogeneous elements in the free algebra $K_0\langle I\rangle$, they are linear combinations of monomials of the form $\theta_i^{t'}\theta_j\theta_i^t$ with fixed $t+t'$, and the coefficients are in K_0, thus Laurent polynomials in the variable $\tilde{q}$. If we evaluate $\tilde{q}$ at 1, we obtain usual Serre relations as they are used in order to obtain the Serre presentation of the positive part of a semisimple complex Lie algebra or, more generally, the positive part of a Kac-Moody Lie algebra.

We see the following: The kernel of the canonical projection

$$K\langle I\rangle \longrightarrow \mathcal{C}(\omega)\otimes_{K_0} K$$

which sends θ_i to $\tilde{u}_i \otimes 1$ contains the twosided ideal $\mathcal{J}(K,q,\omega)$. Thus:

Proposition. *Let ω be an Euler form without short cycles. Let K be a commutative ring and $q \in K$. Then there exists a homomorphism of K-algebras*

$$\pi^K\colon \widehat{U}^+(K,q,\omega) \to \mathcal{C}(\omega)\otimes_{K_0} K$$

which maps the residue class of θ_i onto $\tilde{u}_i \otimes 1$.

* * *

Assume now that there is given an invertible element v in K, so that we may consider K as a K_{*0}-algebra, with $\tilde{v}$ operating on K by the multiplication with v.

We denote by $\mathcal{J}_*(K,v,\bullet)$ the twosided ideal of $K\langle I\rangle$ generated by the elements

$$\sum_{t=0}^{n(i,j)} (-1)^t \begin{bmatrix} n(i,j) \\ t \end{bmatrix}_{d_i} \theta_i^{n(i,j)-t}\theta_j\theta_i^t$$

for all $i \neq j$. Again, we call these generators *quantum Serre relations* and we put

$$\widehat{U}_*^+(K, v, \bullet) = K\langle I\rangle / \mathcal{J}_*(K, v, \bullet).$$

Remark. The second kind of quantum Serre relations which have been exhibited as generators of $\mathcal{J}_*(K, v, \bullet)$ are homogeneous elements in the free algebra $K_{*0}\langle I\rangle$, thus linear combinations of monomials of the form $\theta_i^{t'}\theta_j\theta_i^t$ with fixed $t+t'$, and the coefficients are in K_{*0}, thus Laurent polynomials in the variable $\tilde{v}$. Again, the evaluation of $\tilde{v}$ at 1 yields usual Serre relations.

Proposition*. *Let ω be an Euler form without short cycles. Let $\bullet$ be its symmetrization. Let K be a commutative ring and $v \in K$ invertible. There exists a homomorphism of K-algebras*

$$\pi_*^K : \widehat{U}_*^+(K, v, \bullet) \to \mathcal{C}_*(\omega) \otimes_{K_{*0}} K$$

which maps the residue class of θ_i onto $\tilde{u}_i \otimes 1$.

Again, this follows from part III: The kernel of the canonical map

$$K\langle I\rangle \to \mathcal{C}_*(\omega) \otimes_{K_*0} K$$

which sends θ_i to $\tilde{u}_i \otimes 1$ contains the twosided ideal $\mathcal{J}_*(K, v, \bullet)$.

* * *

Finally, let us compare the two algebras $\widehat{U}^+(K, v^2, \omega)$ and $\widehat{U}_*^+(K, v, \bullet)$.

Lemma. *Let ω be an Euler form without short cycles. Let $\bullet$ be its symmetrization. Let K be a commutative ring and $v \in K$ invertible. Then there exists an isomorphism of K-algebras*

$$\widehat{U}_*^+(K, v, \bullet) \to \widehat{U}^+(K, v^2, \omega)_{[v,\omega]}$$

which maps the residue class of θ_i modulo $\mathcal{J}_*(K, v, \bullet)$ to the residue class of the same element θ_i modulo $\mathcal{J}(K, v^2, \omega)$.

Proof: We consider the K-algebra $A = K\langle I\rangle_{[v,\omega]}$. Since $K\langle I\rangle$ is a free algebra, there exists an algebra homomorphism

$$\eta\colon K\langle I\rangle \longrightarrow A$$

with $\eta(\theta_i) = \theta_i$. Since ω has no short cycles, we can assume that $\omega(j,i) = 0$, therefore $n(i,j) = -d_i^{-1}\omega(i,j) + 1$, and thus $\omega(i,j) = -d_i(n(i,j) - 1)$. We write $n = n(i,j)$. Let us calculate $\eta(\theta_i^{n-t}\theta_j\theta_i^t)$. We have

$$\begin{aligned}
\eta(\theta_i^{n-t}\theta_j\theta_i^t) &= v^{\binom{n}{2}\omega(i,i)+(n-t)\omega(i,j)}\ \theta_i^{n-t}\theta_j\theta_i^t \\
&= v^{\binom{n}{2}\omega(i,i)-(n-t)d_i(n-1)}\ \theta_i^{n-t}\theta_j\theta_i^t \\
&= v^{\binom{n}{2}\omega(i,i)-nd_i(n-1)}v^{d_i t(n-1)}\ \theta_i^{n-t}\theta_j\theta_i^t.
\end{aligned}$$

It follows that the quantum Serre relation

$$\sum_{t=0}^{n}(-1)^t \begin{bmatrix} n \\ t \end{bmatrix}_{d_i} \theta_i^{n-t}\theta_j\theta_i^t$$

is mapped unter η onto a multiple of

$$\sum_{t=0}^{n}(-1)^t \left(\begin{bmatrix} n \\ t \end{bmatrix} v^{t(n-1)} \right)_{d_i} \theta_i^{n-t}\theta_j\theta_i^t = \sum_{t=0}^{n}(-1)^t \left(\begin{vmatrix} n \\ t \end{bmatrix} v^{t(t-1)} \right)_{d_i} \theta_i^{n-t}\theta_j\theta_i^t$$

(the scalar factor $v^{\binom{n}{2}\omega(i,i)-nd_i(n-1)}$ does not depend on t). It follows that η yields a K-algebra homomorphism

$$\widehat{U}_*^+(K,v,\bullet) \to \widehat{U}^+(K,v^2,\omega)_{[v,\omega]}$$

and this has to be an isomophism, since we can use the same argument in order to construct a corresponding K-algebra homomorphism

$$\widehat{U}^+(K,v^2,\omega) \to \widehat{U}_*^+(K,v,\bullet)_{[v,-\omega]}.$$

This completes the proof.

2. Green's map from $\mathcal{C}(\omega,d)$ or $\mathcal{C}_*(\omega,d)$ onto some U^+.

Proposition. *Let (ω,d) be an Euler form on $\mathbb{Z}I$. There exists a homomorphism of K_0-algebras*

$$\psi\colon \mathcal{C}(\omega,d) \to U^+(K_0,\tilde{q},-\omega,0)$$

with $\tilde{u}_i$ being mapped to the residue class of θ_i.

Proof. Let $\mathcal{K}$ be a family of finite fields k with $q_k = |k|$ being unbounded. For every field k, let Λ_k be a k-algebra with $E(\Lambda_k) = (\omega,d)$.

We recall that the generic composition algebra $\mathcal{C}(\omega,d)$ is the K_0-subalgebra of $\prod_{k\in\mathcal{K}}\mathcal{H}(\Lambda_k,\mathbb{Z}_k)$ generated by the elements $\tilde{u}_i$. Thus, there exists a K_0-algebra homomorphism

$$\xi\colon K_0\langle I\rangle \to \prod_{k\in\mathcal{K}}\mathcal{H}(\Lambda_k,\mathbb{Z}_k)$$

such that $\xi(\theta_i) = \tilde{u}_i$; its image is $\mathcal{C}(\omega,d)$.

We may compose ξ with the canonical projections onto the factors $\mathcal{H}(\Lambda_k,\mathbb{Z}_k)$ and obtain corresponding maps

$$\xi_k\colon K_0\langle I\rangle \to \mathcal{H}(\Lambda_k,\mathbb{Z}_k).$$

Note that we have

$$\operatorname{Ker}\xi = \bigcap_{k\in\mathcal{K}} \operatorname{Ker}\xi_k.$$

We claim that

$$\operatorname{Ker}\xi \;\subseteq\; \mathcal{I}(K_0,\tilde{q},-\omega,0).$$

This then implies that $\mathcal{C}(\omega,d) = K_0\langle I\rangle/\operatorname{Ker}\xi$ maps onto $U^+(K_0,\tilde{q},-\omega,0)$, with the residue class of θ_i modulo $\operatorname{Ker}\xi$ being mapped onto the residue class of θ_i modulo the ideal $\mathcal{I}(K_0,\tilde{q},-\omega,0)$.

We can factor the map ξ_k as follows:

$$K_0\langle I\rangle \xrightarrow{\ \eta_k\ } \mathbb{Z}_k\langle I\rangle \xrightarrow{\ \xi_k'\ } \mathcal{H}(\Lambda_k,\mathbb{Z}_k),$$

where η_k is the scalar extension map (it sends θ_i to θ_i), and where ξ_k' is the $\mathbb{Z}_k$-algebra homomorphism which sends θ_i to u_i. The map ξ_k' is the one considered in Section IV.5, thus we know that

$$\operatorname{Ker}\xi_k' \;\subseteq\; \mathcal{I}(\mathbb{Z}_k,q_k,-\omega,0).$$

On the other hand, according to IV.4, for any pair of elements $a,b\in K_0\langle I\rangle$, we know that

$$(\eta_k(a),\eta_k(b))_1^{(k)} = (a,b)_1(q_k),$$

where $(-,-)_1$ is the generic Lusztig form on $K_0\langle I\rangle$ for $(-\omega,0)$, whereas $(-,-)_1^{(k)}$ is the Lusztig form on $\mathbb{Z}_k\langle I\rangle$ with respect to $q_k,(-\omega,0)$ and 1.

Now assume there is given $b\in\operatorname{Ker}\xi$, thus $\eta_k(b)$ belongs to $\operatorname{Ker}\xi_k'$ and therefore to $\mathcal{I}(\mathbb{Z}_k,q_k,-\omega,0)$. If now $a\in K_0\langle I\rangle$ is an arbitrary element, then the Laurent polynomial $(a,b)_1$ satisfies

$$(a,b)_1(q_k) = (\eta_k(a),\eta_k(b))_1^{(k)} = 0$$

for all $k\in\mathcal{K}$. Since there are infinitely many such q_k, it follows that $(a,b)_1$ is the zero element in K_0. This completes the proof.

* * *

Proposition*. *Let (ω,d) be an Euler form on $\mathbb{Z}I$. Let $(\bullet,d)$ be the corresponding symmetrization. There exists a homomorphism of K_{*0}-algebras*

$$\psi_*\colon \mathcal{C}_*(\omega,d)\to U^+(K_{*0},v,0,\bullet)$$

with $\tilde{u}_i$ being mapped to the residue class of θ_i.

Proof. Let $\mathcal{K}$ be a family of finite fields k with $q_k=|k|$ being unbounded. For every field k, let Λ_k be a k-algebra with $E(\Lambda_k)=(\omega,d)$.

The twisted generic composition algebra $\mathcal{C}_*(\omega,d)$ is the K_{*0}-subalgebra of $\prod_{k\in\mathcal{K}}\mathcal{H}_*(\Lambda_k)$ generated by the elements $\tilde{u}_i$. Thus, there exists a K_{*0}-algebra homomorphism

$$\xi_*\colon K_{*0}\langle I\rangle \to \prod_{k\in\mathcal{K}}\mathcal{H}_*(\Lambda_k)$$

such that $\xi_*(\theta_i)=\tilde{u}_i$; its image is $\mathcal{C}_*(\omega,d)$

We compose ξ_* with the canonical projection onto the factors $\mathcal{H}_*(\Lambda_k)$ and obtain

$$\xi_{*k}\colon K_{*0}\langle I\rangle \to \mathcal{H}_*(\Lambda_k).$$

We have

$$\operatorname{Ker}\xi_* = \bigcap_{k\in\mathcal{K}}\operatorname{Ker}\xi_{*k}.$$

We claim that

$$\operatorname{Ker}\xi_* \subseteq \mathcal{I}(K_{*0},\tilde{v},0,\bullet).$$

This then implies that $\mathcal{C}_*(\omega,d)=K_{*0}\langle I\rangle/\operatorname{Ker}\xi_*$ maps onto $U^+(K_{*0},\tilde{v},0,\bullet)$, with the residue class of θ_i modulo $\operatorname{Ker}\xi_*$ being mapped onto the residue class of θ_i modulo the ideal $\mathcal{I}(K_{*0},\tilde{v},0,\bullet)$.

We can factor the map ξ_{*k} as follows:

$$K_{*0}\langle I\rangle \xrightarrow{\eta_k} \mathbb{Z}_{*k}\langle I\rangle \xrightarrow{\xi'_{*k}} \mathcal{H}_*(\Lambda_k),$$

where η_k is the scalar extension map (it sends θ_i to θ_i), and where ξ'_{*k} is the $\mathbb{Z}_{*k}$-algebra homomorphism which sends θ_i to u_i. The map ξ'_{*k} is the one considered in section IV.5, thus we know that

$$\operatorname{Ker}\xi'_{*k} \subseteq \mathcal{I}(\mathbb{Z}_{*k},v_k,0,\bullet).$$

On the other hand, according to IV.4, for any pair of elements $a,b\in K_{*0}\langle I\rangle$, we know that

$$(\eta_k(a),\eta_k(b))_1^{(k)} = (a,b)_1(v_k),$$

where $(-,-)_1$ is the generic Lusztig form on $K_{*0}\langle I\rangle$ for $(0,\bullet)$, whereas $(-,-)_1^{(k)}$ is the Lusztig form on $\mathbb{Z}_{*k}\langle I\rangle$ with respect to $v_k,(0,\bullet)$ and 1.

Now assume there is given $b\in\operatorname{Ker}\xi_*$, thus $\eta_k(b)$ belongs to $\operatorname{Ker}\xi'_{*k}$ and therefore to $\mathcal{I}(\mathbb{Z}_{*k},v_k,0,\bullet)$. If now $a\in K_{*0}\langle I\rangle$ is an arbitrary element, then the Laurent polynomial $(a,b)_1$ satisfies

$$(a,b)_1(v_k) = (\eta_k(a),\eta_k(b))_1^{(k)} = 0$$

for all $k\in\mathcal{K}$. Since there are infinitely many such v_k, it follows that $(a,b)_1$ is the zero element in K_{*0}. This completes the proof.

3. From $\widehat{U}^+$ to $\mathcal{C}$ to U^+.

Let ω be an Euler form without short cycles and with symmetrization $\bullet$. Recall that we obtain in this way an arbitrary Cartan datum $\bullet$.

On the one hand, we have the $\mathbb{Q}(\tilde{v})$-algebra homomorphism $\pi = \pi^{\mathbb{Q}(\tilde{v})}$

$$\pi\colon \widehat{U}^+(\mathbb{Q}(\tilde{v}), \tilde{q}, \omega) \to \mathcal{C}(\omega) \otimes_{K_0} \mathbb{Q}(\tilde{v}),$$

which sends the residue class of θ_i to $\tilde{u}_i \otimes 1$, for any $i \in I$. On the other hand, there is given Green's map

$$\psi\colon \mathcal{C}(\omega) \to U^+(K_0, \tilde{q}, -\omega, 0);$$

tensoring with the field of rational functions $\mathbb{Q}(\tilde{v})$, we obtain

$$\psi \otimes 1\colon \mathcal{C}(\omega) \otimes_{K_0} \mathbb{Q}(\tilde{v}) \to U^+(K_0, \tilde{q}, -\omega, 0) \otimes_{K_0} \mathbb{Q}(\tilde{v})$$

Also, we may consider the canonical map

$$\mu\colon U^+(K_0, \tilde{q}, -\omega, 0,) \otimes_{K_0} \mathbb{Q}(\tilde{v}) \to U^+(\mathbb{Q}(v), \tilde{q}, -\omega, 0)$$

given by multiplication.

Altogether we have three surjective $\mathbb{Q}(\tilde{v})$-algebra homomorphisms

$$\begin{array}{c}
\widehat{U}^+(\mathbb{Q}(\tilde{v}), \tilde{q}, \omega) \\
\Big\downarrow{\scriptstyle \pi} \\
\mathcal{C}(\omega) \otimes_{K_0} \mathbb{Q}(\tilde{v}) \\
\Big\downarrow{\scriptstyle \psi\otimes 1} \\
U^+(K_0, \tilde{q}, -\omega, 0,) \otimes_{K_0} \mathbb{Q}(\tilde{v}) \\
\Big\downarrow{\scriptstyle \mu} \\
U^+(\mathbb{Q}(\tilde{v}), \tilde{q}, -\omega, 0)
\end{array}$$

and the composition sends the residue class of θ_i (modulo the ideal $\mathcal{J}(\mathbb{Q}(\tilde{v}), \tilde{q}, \omega)$ generated by the quantum Serre relations) to the residue class of θ_i (modulo the ideal $\mathcal{I}(\mathbb{Q}(\tilde{v}), \tilde{q}, -\omega, 0)$).

* * *

In the twisted case, we have corresponding maps, as follows. First of all, there is $\pi_* = \pi_*^{\mathbb{Q}(\tilde{v})}$

$$\pi_*\colon \widehat{U}_*^+(\mathbb{Q}(\tilde{v}), \tilde{v}, \bullet) \to \mathcal{C}_*(\omega) \otimes_{K_{*0}} \mathbb{Q}(\tilde{v}),$$

which sends the residue class of θ_i to $\tilde{u}_i \otimes 1$, for any $i \in I$. On the other hand, there is given Green's map

$$\psi_*\colon \mathcal{C}_*(\omega) \to U^+(K_{*0}, \tilde{v}, 0, \bullet);$$

tensoring with the field of rational functions $\mathbb{Q}(\tilde{v})$, we obtain

$$\psi_* \otimes 1\colon \mathcal{C}_*(\omega) \otimes_{K_{*0}} \mathbb{Q}(\tilde{v}) \to U^+(K_{*0}, \tilde{v}, 0, \bullet) \otimes_{K_{*0}} \mathbb{Q}(\tilde{v})$$

Finally, we consider again the multiplication map

$$\mu_*\colon U^+(K_{*0}, \tilde{v}, 0, \bullet) \otimes_{K_{*0}} \mathbb{Q}(\tilde{v}) \to U^+(\mathbb{Q}(\tilde{v}), \tilde{v}, 0, \bullet).$$

Altogether, we get

$$\begin{array}{c} \widehat{U}_*^+(\mathbb{Q}(\tilde{v}), \tilde{v}, \bullet) \\ \Big\downarrow{\scriptstyle \pi_*} \\ \mathcal{C}_*(\omega) \otimes_{K_{*0}} \mathbb{Q}(\tilde{v}) \\ \Big\downarrow{\scriptstyle \psi_*\otimes 1} \\ U^+(K_{*0}, \tilde{v}, 0, \bullet) \otimes_{K_{*0}} \mathbb{Q}(\tilde{v}) \\ \Big\downarrow{\scriptstyle \mu_*} \\ U^+(\mathbb{Q}(\tilde{v}), \tilde{v}, 0, \bullet) \end{array}$$

and the composition sends the residue class of θ_i (modulo the ideal $\mathcal{J}_*(\mathbb{Q}(\tilde{v}), \tilde{v}, \bullet)$ generated by the quantum Serre relations) to the residue class of θ_i (modulo the ideal $\mathcal{I}(\mathbb{Q}(\tilde{v}), \tilde{v}, 0, \bullet)$).

4. Lusztig's Theorem.

We recall the following:

Theorem (Lusztig). *Let $\bullet$ be a Cartan datum. Then the ideal $\mathcal{I}(\mathbb{Q}(\tilde{v}), \tilde{v}, 0, \bullet)$ is generated by the quantum Serre relations.*

This means that

$$\mathcal{I}(\mathbb{Q}(\tilde{v}), \tilde{v}, 0, \bullet) = \mathcal{J}_*(\mathbb{Q}(\tilde{v}), \tilde{v}, \bullet).$$

For the proof, we refer to Lusztig [L], Theorem 33.1.3. The proof is based on the representation theory of Kac-Moody Lie-algebras; it uses some fundamental results in the book of Kac [K], namely the Kac-Weyl character formula 10.4 and its corollary 10.4.

5. Green's Theorem.

Theorem. *Let ω be an Euler form without short cycles. Then the three $\mathbb{Q}(\tilde{v})$-algebra maps*

$$\begin{array}{ccc} \widehat{U}^+_*(\mathbb{Q}(\tilde{v}),\tilde{v},\bullet) & \xrightarrow{\pi_*} & \mathcal{C}_*(\omega)\otimes_{K_{*0}}\mathbb{Q}(\tilde{v}) \\ \mathcal{C}_*(\omega)\otimes_{K_{*0}}\mathbb{Q}(\tilde{v}) & \xrightarrow{\psi_*\otimes 1} & U^+(K_{*0},\tilde{v},0,\bullet)\otimes_{K_{*0}}\mathbb{Q}(\tilde{v}) \\ U^+(K_{*0},\tilde{v},0,\bullet)\otimes_{K_{*0}}\mathbb{Q}(\tilde{v}) & \xrightarrow{\mu_*} & U^+(\mathbb{Q}(\tilde{v}),\tilde{v},0,\bullet) \end{array}$$

are isomorphisms.

Proof: All the maps mentioned are surjective and the composition is the identity. Thus, all these maps must be bijective.

In section III.5, we have constructed a map

$$\big(\mathcal{C}(\omega,d)\otimes K_{*0}\big)_{[\tilde{v},\omega]}\to\mathcal{C}_*(\omega,d)$$

of K_{*0}-algebras. Scalar extension yields a corresponding map

$$\tau\colon \big(\mathcal{C}(\omega,d)\otimes_{K_0}\mathbb{Q}(\tilde{v})\big)_{[\tilde{v},\omega]}\to\mathcal{C}_*(\omega,d)\otimes_{K_*0}\mathbb{Q}(\tilde{v})$$

Corollary. *Let ω be an Euler form without short cycles. Then the following $\mathbb{Q}(\tilde{v})$-algebra maps*

$$\begin{array}{ccc} \widehat{U}^+(\mathbb{Q}(\tilde{v}),\tilde{q},\omega) & \xrightarrow{\pi} & \mathcal{C}(\omega)\otimes_{K_0}\mathbb{Q}(\tilde{v}) \\ \big(\mathcal{C}(\omega)\otimes_{K_0}\mathbb{Q}(\tilde{v})\big)_{[\tilde{v},\omega]} & \xrightarrow{\tau} & \mathcal{C}_*(\omega)\otimes_{K_*0}\mathbb{Q}(\tilde{v}) \end{array}$$

are isomorphisms.

Proof: According to Lemma II.1, we may consider π also as an algebra homomorphism

$$\pi\colon \widehat{U}^+(\mathbb{Q}(\tilde{v}),\tilde{q},\omega)_{[\tilde{v},\omega]}\to\big(\mathcal{C}(\omega)\otimes_{K_0}\mathbb{Q}(\tilde{v})\big)_{[\tilde{v},\omega]}.$$

In addition, consider the isomorphism of $\mathbb{Q}(\tilde{v})$-algebras

$$\eta\colon \widehat{U}^+_*(\mathbb{Q}(\tilde{v}),\tilde{v},\bullet)\to\widehat{U}^+(\mathbb{Q}(\tilde{v}),\tilde{v}^2,\omega)_{[\tilde{v},\omega]}$$

constructed in V.1. Then, the composition $\tau\pi\eta$ is just the map π_* considered in Theorem, and there we have seen that π_* is an isomorphism. Since all three maps τ,π,η are surjective, it follows that they also are isomorphisms. This completes the proof.

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