The Representation Theory of Dynkin Quivers.

Three Contributions.

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Abstract. The representations of the Dynkin quivers and the corresponding Euclidean quivers are treated in many books. These notes provide three building blocks for dealing with representations of Dynkin (and Euclidean) quivers. They should be helpful as part of a direct approach to study representations of quivers, and they shed some new light on properties of Dynkin and Euclidean quivers.

These notes are devoted to the representation theory of Dynkin and Euclidean quivers. The theory is treated in many books: It is well-known that the Dynkin quivers are representation-finite, that the dimension vectors of the indecomposable representations are just the corresponding positive roots and that any (not necessarily finite-dimensional) representation is the direct sum of indecomposable representations. The Euclidean quivers are the minimal tame quivers and also here, the category of all representations is well understood and various methods of proof are known. The aim of these notes is to provide three building blocks for dealing with representations of Dynkin (and Euclidean) quivers. They should be helpful as part of a direct approach to study representations of quivers, and they shed some new light on properties of Dynkin and Euclidean quivers. The presentations is based on lectures given at SJTU (Shanghai) in 2011 and 2015 and also at KAU (Jeddah) in 2012 and at IYET (Izmir) in 2014.

Part 1 considers the quivers of type $\mathbb{A}_n$. As one knows, any (not necessarily finite-dimensional) representation is the direct sum of thin representations. We provide a straight-forward arrangement of arguments in order to avoid indices and clumsy inductive considerations, but also avoiding somewhat fancy tools such as the recursive use of the Bernstein-Gelfand-Ponomarev reflection functors or bilinear forms and root systems. As we will see, the essential case to be considered is the case $\mathbb{A}_3$ as studied in any first year course in Linear Algebra.

2010 *Mathematics Subject Classification.* Primary 16G20, 16G60. Secondary: 05E10, 16D90, 16G70. Keywords. Quiver. Dynkin quiver. Euclidean quiver. The exceptional vertices of a Dynkin quiver. Representations of quivers. Thin representations. Filtrations of vector spaces. Conical representations of star quivers. Auslander-Reiten quiver. Thick subcategories. Perpendicular subcategories. One-point extension. Antichains of a poset. Antichains of an additive category. Simplification. Hammocks. The 2-4-8 property. The magic Freudenthal-Tits square.

In Part 2 we deal with the case $\mathbb{D}_n$. If Q is a quiver of type $\mathbb{D}_n$ with $n \geq 4$, the category of all representations of Q contains a full subcategory which is equivalent to the category of representations of a quiver of type $\mathbb{D}_{n-1}$ such that the remaining indecomposable representations are thin. This provides an inductive procedure starting with $\mathbb{D}_3 = \mathbb{A}_3$. No further linear algebra knowledge is required for dealing with the cases $\mathbb{D}_n$. The proof relies on some easily established equivalences of categories. As an afterthought, we give an interpretation in terms of Auslander-Reiten quivers.

Part 3 is devoted to the quivers Δ of type $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ and the categories $\operatorname{rep} \Delta$ of finite-dimensional representations. The cases $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ can be reduced to $\mathbb{A}_5, \mathbb{D}_6$ and $\mathbb{E}_7$, respectively, and there is a further reduction to $\mathbb{A}_2 \sqcup \mathbb{A}_2, \mathbb{A}_5, \mathbb{E}_6$, respectively. It seems to be of interest that this reduction scheme follows the rules of the magic Freudenthal-Tits square. In this way, we obtain a uniform way to construct the maximal indecomposable representation M in each of the cases $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$. Also, this provides a unified method to deal with the corresponding Euclidean quivers of type $\widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7$ and $\widetilde{\mathbb{E}}_8$, namely to construct the one-parameter families, as well as representatives of the τ-orbits of the simple regular representations.

Actually, a similar procedure can also be used for the cases $\mathbb{D}_n$ and $\widetilde{\mathbb{D}}_n$. For all the Dynkin quivers Δ of type $\mathbb{D}_n$ and $\mathbb{E}_m$ and for the corresponding Euclidean quivers $\widetilde{\Delta}$, we obtain a reduction to $\mathbb{D}_4$ and $\widetilde{\mathbb{D}}_4$, respectively.

Our approach in Part 3 may be formulated also as follows. Let Δ be a Dynkin quiver of type $\mathbb{D}_n$ or $\mathbb{E}_n$ and y its exceptional vertex (this is the uniquely determined vertex such that the corresponding Euclidean quiver $\widetilde{\Delta}$ is obtained from Δ by adding a vertex z as well as an arrow between y and z). We look at the quiver Δ' obtained from Δ by deleting y. The essential observation concerns the restriction $M|\Delta'$. As we will see, $M|\Delta'$ is the direct sum of three indecomposable representations; we call these representations $A(1), A(2), A(3)$ the special antichain triple of Δ. The thick subcategory generated by $A(1), A(2), A(3)$ and the simple representation $S(y)$ will be said to be the core of $\operatorname{rep} \Delta$. In case Δ is of type $\mathbb{E}_m$, it is also of interest to look at the quiver Δ'' obtained from Δ by deleting y and the neighbors of y. We will show that the restriction $M|\Delta''$ is the direct sum of four indecomposable representations and that its endomorphism ring is 6-dimensional.

The level of the presentation varies considerably and increases throughout the discussion. Whereas Part 1 is completely elementary, just based on some results in linear algebra, the further text is less self-contained. The inductive proof of Theorem 2.1 requires no prerequisites, but for a proper understanding one needs to have some knowledge about Auslander-Reiten quivers, the use of antichains and simplification as well as about perpendicular categories. In Part 3 we will use hammocks and one-point extensions. We have tried to incorporate most of the relevant definitions, but a neophyte may have to consult a standard reference such as [ARS], or also [R3]. Comments concerning the possible use of the material presented here can be found in remarks at the end of each part. The Panoramic View at the end of the paper shows how the three Parts can be completed for a full discussion of the borderline between representation-finite and representation-infinite quivers.

Historical comments. The present survey is devoted to what often is called Gabriel's theorem: that a representation-finite quiver is the disjoint union of Dynkin quivers of type $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ and that the indecomposable representations of such a Dynkin quiver correspond bijectively to the positive roots, see [G1]. One should be aware that Gabriel's interest was stimulated by earlier investigations of Yoshii [Y] who had aimed to classify the representation-finite algebras with radical square zero (a problem which is easily seen to be equivalent to the classification problem of representation-finite quivers). Yoshii's paper is difficult to read, and actually his answer is incorrect: he claimed that quivers of type $\widetilde{\mathbb{E}}_7$ are representation-finite. Henning Krause [Kr] has stressed that Gabriel himself referred to the result as Yoshii's theorem: Er habe *einen Satz von Yoshii ... neu hergeleitet und auch berichtigt* (in his German abstract of an Oberwolfach lecture). The observation that the dimension vectors of the indecomposable representations of a Dynkin quiver are just the positive roots has been reported to be due to Tits: he pointed this out when Gabriel exhibited the list of the dimension vectors in a seminar at Bonn. It is this bijection which has stirred up the further development.

The characterization of the representation-finite quivers was found independently also by Kleiner [Kl] and by Bäckström [Ba], working again in equivalent settings (namely dealing with representations of posets and with lattices over orders, respectively), but they had no Tits to explain them their findings. Thus a proper (but of course too long) name should be Theorem of Yoshii-Bäckström-Gabriel-Kleiner(-Tits).

The proofs of Bäckström, Gabriel and Kleiner use case-by-case considerations. The first conceptual proof is due to Bernstein-Gelfand-Ponomarev [BGP]; the paper introduces reflection functors in order to mimic the inductive construction of the positive roots starting with the simple roots. Let us also refer to [R6] (and [R7]) where the focus lies on the fact that indeomposable representations of a Dynkin quiver have no self-extensions. For quite a while, the relationship between the representation theory of quivers on the one hand and Lie theory on the other was solely based on the combinatorics of the finite root systems. For a direct way to construct Lie algebras starting with the representations of Dynkin quivers, see [R8] and many subsequent papers by various authors.

Infinite-dimensional representations of representation-finite algebras, and thus also of Dynkin quivers), have been studied in collaboration with Tachikawa, see Corollary 9.5 of [Ta], and independently, but later, by Auslander [Au].

The classification of the indecomposable representations of Euclidean quivers is due to Donovan-Freislich [DF] and Nazarova [N]. For the description of the corresponding module category, see the joint paper with Dlab [DR] and also [R1].

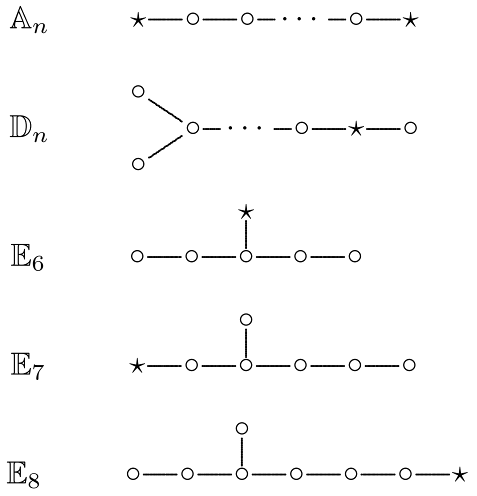
Acknowledgment. The author is indebted to Markus Schmidmeier for many helpful comments. In particular, he suggested to use the word conical. In first versions of part 3, we had restricted the attention to quivers of type $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$. On the basis of discussions with him, also the cases $\mathbb{D}_n$ are now treated in a parallel way.

Preliminaries.

We fix some field k. All vector spaces to be considered will be k-spaces, the algebras will be k-algebras, the categories will be k-categories. Note that in Part 1 and at the beginning of Part 2, the vector spaces to be considered may be infinite-dimensional. Later, we will restrict the discussion to finite-dimensional representations.

0.1. Quivers. We denote by $Q = (Q_0, Q_1, s, t)$ a finite quiver, thus Q_0, Q_1 are finite sets and $s, t\colon Q_1 \to Q_0$ are (set) maps. The elements of Q_0 are called vertices, the elements in Q_1 arrows. Usually, an arrow $\alpha \in Q_1$ will be drawn in the following way $s(\alpha) \overset{\alpha}{\to} t(\alpha)$ or $\alpha\colon s(\alpha) \to t(\alpha)$.

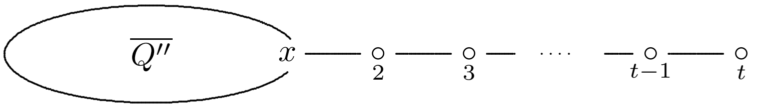
If Q is a quiver, its underlying graph $\overline{Q}$ is a graph (possibly with loops and multiple edges), it is obtained from Q by replacing the maps s, t by the map $\{s, t\}$ which sends $\alpha \in Q_0$ to the subset $\{s(\alpha), t(\alpha)\}$ of Q_1. The main concern of these notes are the *Dynkin quivers:* these are the quivers Q such that the underlying graph $\overline{Q}$ is one of the following Dynkin graphs $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$:



with vertices drawn as $\circ$ or $\star$. The index n in $\mathbb{A}_n, \mathbb{D}_n, \mathbb{E}_n$ is just the number of vertices. We call the vertices $\star$ *exceptional.* The exceptional vertices of a Dynkin graph (see [R2], p.6) can be characterized in many different ways, using for example the root system or the quadratic form attached to the graph. Note that the graphs of type $\mathbb{D}_n$ and $\mathbb{E}_m$ have a unique exceptional vertex.

If Δ is a Dynkin graph different from $\mathbb{A}_1$, the corresponding Euclidean graph $\widetilde{\Delta}$ is obtained from Δ by adding a vertex z and one edge between z and each exceptional vertex of Δ. If $\Delta = \mathbb{A}_1$, say with vertex y, we obtain the corresponding Euclidean graph $\widetilde{\mathbb{A}}_1$ by adding a vertex z and two edges between y and z. In addition, the graph with one vertex and one loop may also be considered as a Euclidean diagram, it will be labeled $\widetilde{\mathbb{A}}_0$. The Euclidean quivers of type $\widetilde{\mathbb{A}}_n$ will be exhibited at the end of this preliminary section, the remaining ones (of type $\widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8$) in Part 3.

An *arm* (Q', x) of length t is a pair consisting of a quiver Q' of type $\mathbb{A}_t$ and a vertex x of Q' which has at most one neighbor in Q' (such an arm is said to be *proper* provided its length is at least 2). We say that a quiver Q *has an arm* (Q', x) provided Q' is a full subquiver of Q and there are no arrows between the vertices of Q outside of Q' and the vertices in Q' different from x. Thus Q has an arm of length t provided $\overline{Q}$ has the form



for some quiver Q'' (we say that we have attached at x in Q'' an arm of length t).

A quiver Q is a *tree quiver* provided the underlying graph $\overline{Q}$ is a tree. Tree quivers can be constructed inductively: The quiver $\mathbb{A}_1$ (one vertex, no arrow) is a tree quiver. A quiver with at least two vertices is a tree quiver, provided it is obtained from a tree quiver by attaching an arm of length 2. Note that all the Dynkin quivers are tree quivers.

If X is a set of vertices of the quiver Q, we write $\langle X\rangle$ for the full subquiver of Q with vertices in X (and $\{X\}$ for the subquiver with vertex set X and no arrows). If the vertices of Q are labeled by the natural numbers $1, 2, \ldots, n$, and $1 \le i \le j \le n$, we write $[i, j]$ for $\langle i, i{+}1, \ldots, j\rangle$.

0.2. Representations. If Q is a quiver, a representation $M = (M_x, M_\alpha)$ is given by vector spaces M_x for $x \in Q_0$ and linear maps $M_\alpha \colon M_{s(\alpha)} \to M_{t(\alpha)}$ for $\alpha \in Q_1$; instead of M_α we often write just α. Note that the vector spaces considered here may be infinite-dimensional. We denote by $\mathbf{dim}\, M$ the dimension vector of M, with $(\mathbf{dim}\, M)_x$ being the k-dimension of the vector space M_x, for $x \in Q_0$. The *support* of a representation M is the set of vertices x with $M_x \neq 0$.

For any subquiver Q' of the quiver Q, we define the representation $M(Q')$ of Q by $M(Q')_x = k$ for all vertices $x \in Q'$ and $M(Q')_\alpha = 1_k$ for all arrows α in Q', whereas $M(Q')_x = 0$, if x is a vertex in Q, but not in Q', and $M(Q')_\alpha = 0$ if α is an arrow in Q, but does not belong to Q'. If x is a vertex of Q, let $S(x) = M(\{x\})$, this is called the simple representation corresponding to x. In case the quiver Q is finite and acyclic (that is, Q has no proper cyclic path), we denote by $P(x)$ the projective cover, by $I(x)$ the injective envelope of $S(x)$.

If M, M' are representations of a quiver Q, a homomorphism $f = (f_x) \colon M \to M'$ is given by linear maps $f_x \colon M_x \to M'_x$ for $x \in Q_0$ such that $M'_\alpha f_{s(\alpha)} = f_{t(\alpha)} M_\alpha$ holds for all $\alpha \in Q_1$. The composition gf of $f \colon M \to M'$ and $g \colon M' \to M''$ is defined by $(gf)_x = g_x f_x$ for all $x \in Q_0$. The category of representations of Q will be denoted by $\operatorname{Rep} Q$, the full subcategory of all finite-dimensional representations by $\operatorname{rep} Q$; these are abelian categories. If $M(i)$, $i \in I$, is a family of representations of a quiver Q, the direct sum $M = \bigoplus_{i\in I} M(i)$ is defined by $M_x = \bigoplus M(i)_x$ and $M_\alpha = \bigoplus M(i)_\alpha$. A representation M is said to be indecomposable provided it is not zero and for any direct sum decomposition $M = M' \oplus M''$, either $M' = 0$ or $M'' = 0$.

0.3. Thin representations of a quiver. A representation M is said to be *thin,* provided $\dim M_x \le 1$ for all vertices x. Of course, for any subquiver Q' of Q, the representation $M(Q')$ is thin.

The following properties of the thin representations will be relevant.

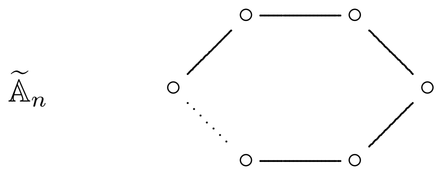
(T1) *Let Q be a quiver. Then $Q' \mapsto M(Q')$ provides an injective map from the set of connected subquivers of Q into the set of isomorphism classes of indecomposable thin representations of Q.*

(T2) *If Q is a tree quiver, then any thin indecomposable representation with support Q is isomorphic to $M(Q)$.*

As an immediate consequence of (T1) and (T2) we obtain:

(T3) *If Q is a tree quiver, then $Q' \mapsto M(Q')$ provides a bijection between the set of connected subquivers of Q and the set of isomorphism classes of indecomposable thin representations of Q.*

If Q is not the disjoint union of tree quivers, then there are indecomposable thin representations which are not isomorphic to representations of the form $M(Q')$. Here is the essential assertion: *Assume that the underlying graph of Q is a quiver of type $\widetilde{\mathbb{A}}_n$, thus its underlying graph is the cycle with $n+1$ vertices:*



Let α be one of the arrows. Let $\lambda \in k$ be different from 0 and 1. Define M as follows: $M_x = k$ for all vertices x, let $M_\alpha = \lambda$ and $M_\beta = 1$ for the remaining arrows β. Then M is thin, indecomposable, with support Q, but M is not isomorphic to $M(Q)$.

Part 1. The quivers of type $\mathbb{A}$.

Our aim is to provide a straightforward proof of the following well-known result:

1.1. Theorem. *Any representation of a quiver of type $\mathbb{A}$ is the direct sum of thin indecomposable representations.*

We should recall that according to (T2), any thin indecomposable representation is of the form $M(Q')$, where Q' is a connected subquiver of Q.

Our proof will rely on a detailed study of the quivers of type $\mathbb{A}_3$. For $n \geq 4$, we will first use induction and then we will invoke the knowledge about the case $\mathbb{A}_3$. Thus, let us look at the quivers of type $\mathbb{A}_3$.

The case $\mathbb{A}_3$. There are three different orientations a quiver of type $\mathbb{A}_3$ may have. We discuss these orientations one after the other. But the proof in all three cases is based on the same result: given two subspaces U, U' of a vector space V, there is a basis $\mathbf{B}$ of V which is compatible with each of the two subspaces (a basis $\mathbf{B}$ of a vector space V is said to be *compatible* with the subspace U of V provided $\mathbf{B} \cap U$ is a basis of U).

(a) The 2-subspace quiver. By definition, this is the following quiver:

$$\begin{tikzcd} \overset{1}{\circ} \arrow[r, "\alpha"] & \overset{2}{\circ} & \overset{3}{\circ} \arrow[l, "\beta"'] \end{tikzcd}$$

Starting with a representation M, let U be the image of M_α in $V = M_2$, and U' the image of M_β in V. Splitting off copies of $S(1)$, we can assume that M_α is the inclusion map of U in V; splitting off copies of $S(3)$, we can assume that M_β is the inclusion map of U' in V. Thus we deal with a vector space V with two subspaces U, U':

$$\begin{tikzcd} U \arrow[r] & V & U' \arrow[l] \end{tikzcd}$$

Using a basis $\mathbf{B}$ of V which is compatible with both U and U', we can write M as follows:

$$\begin{tikzcd} \bigoplus_{b\in\mathbf{B}\cap U} kb \arrow[r] & \bigoplus_{b\in\mathbf{B}} kb & \bigoplus_{b\in\mathbf{B}\cap U'} kb \arrow[l] \end{tikzcd} ,$$

thus we obtain a decomposition of M as a direct sum of copies of the following thin indecomposable representations:

$$\begin{tikzcd}
& k \arrow[r, "1"] & k & k \arrow[l, "1"'] & k \arrow[r, "1"] & k & 0 \arrow[l] & 0 \arrow[r] & k & k \arrow[l, "1"'] & 0 \arrow[r] & k & 0 \arrow[l] \\
\text{for any} & & b \in U \cap U' & & & b \in U \setminus U' & & & b \in U' \setminus U & & & b \notin U \cup U' &
\end{tikzcd}$$

(b) The quiver Q of type $\mathbb{A}_3$ with linear orientation:

$$\begin{tikzcd} \overset{1}{\circ} & \overset{2}{\circ} \arrow[l, "\alpha"'] & \overset{3}{\circ} \arrow[l, "\beta"'] \end{tikzcd}$$

Let $V = M_2$. Splitting off copies of $S(1)$, we can assume that M_α is the canonical projection $V \to V/U$, where U is the kernel of M_α. Splitting off copies of $S(3)$, we can assume that M_β is the inclusion of a subspace U' of V. Thus we deal with a vector space V with two subspaces U, U' and consider the corresponding representation of Q:

$$\begin{tikzcd} V/U & V \arrow[l] & U' \arrow[l] \end{tikzcd}$$

As in case (1), we take a basis $\mathbf{B}$ of V which is compatible with U, U', in order to write M as the direct sum of copies of the following (thin indecomposable) representations

$$\begin{tikzcd}
& 0 & k \arrow[l] & k \arrow[l, "1"'] & 0 & k \arrow[l] & 0 \arrow[l] & k & k \arrow[l, "1"'] & k \arrow[l, "1"'] & k & k \arrow[l, "1"'] & 0 \arrow[l] \\
\text{for any} & & b \in U \cap U' & & & b \in U \setminus U' & & & b \in U' \setminus U & & & b \notin U \cup U' &
\end{tikzcd}$$

(c) The 2-factor-space quiver. This is the quiver

$$\begin{tikzcd} \overset{1}{\circ} & \overset{2}{\circ} \arrow[l, "\alpha"'] \arrow[r, "\beta"] & \overset{3}{\circ} \end{tikzcd}$$

We may proceed as in the previous cases, by first splitting off copies of $S(1), S(3)$ so that we are left with a vector space V and two factor spaces of V, say with V/U and V/U' where U, U' are subspaces of V. A basis $\mathbf{B}$ of V which is compatible both with U and U' provides the desired direct sum decomposition with thin direct indecomposable summands.

But there is a second possibility for dealing with the case (c): We may use k-duality in order to reduce this case to the case (a).

Proof of Theorem 1.1. We deal with a quiver Q with the following underlying graph

For $1 \le s \le t \le n$, we have denoted by $[s,t]$ the full subquiver with vertices $s, s+1, \dots, t$. According to (T3), the representations $M([s,t])$ (with $1 \le s \le t \le n$) furnish a complete list of the indecomposable thin representations of Q, up to isomorphism. We have to show that any representation of Q is isomorphic to the direct sum of representations of the form $M([s,t])$.

As we know already, the assertion is true for $n \le 3$. Thus, consider now some $n \ge 4$. By induction, we may assume that any indecomposable representation of a quiver of type $\mathbb{A}_{n-1}$ is thin. We use this assertion first for $[1, n-1]$ and then for $[2, n]$, these are the first two steps of the proof.

We need another definition. Let c be a vertex of a tree quiver Δ. We say that an arrow $\alpha\colon x \to y$ *points to* c provided y and c belong to the same connected component of the quiver obtained from Δ by deleting α. A representation M of Δ is said to be c-*conical* provided M_α is injective for any arrow α of Δ which points to c, and M_β is surjective, for the remaining arrows β of Δ (an alternative terminology calls c a *peak* for M in case M is c-conical).

In our quiver Q, we use the ascending labels $1, 2, \dots, n$ for the vertices; the arrows pointing to c are the arrows of the form $\alpha\colon i \to i+1$ with $i < c$ and the arrows $\alpha\colon j{+}1 \to j$ with $c < j$). For example, the representation $M([s,t])$ of Q is c-conical provided $s \le c \le t$. If $1 \le i \le c \le j \le n$, and M is a representation of Q, we say that M is *conical on* $([i,j], c)$ provided the restriction $M|[i,j]$ is c-conical.

Step 1. *Any representation M of Q is a direct sum $M = M' \oplus M''$ such that M' is conical on $([1, n-1], n-1)$, whereas the support of M'' is contained in $[1, n-2]$.*

Proof. Consider the restriction $N = M|[1, n-1]$. By induction, we know that N can be written as a direct sum of representations of the form $M([s,t])$, with $1 \le s \le t \le n-1$. We decompose $N = N' \oplus N''$, where N' is a direct sum of representations of the form $M([s, n-1])$, and N'' a direct sum of representations of the form $M([s,t])$ with $t \le n-2$. Note that the representation N' of $[1, n{-}1]$ is $(n{-}1)$-conical.

Let us stress that $N'_{n-1} = M_{n-1}$ and $N''_{n-1} = 0$. We define a subrepresentation M' as follows:

$$M'|[1, n{-}1] = N' \quad \text{and} \quad M'|[n{-}1, n] = M|[n{-}1, n]$$

(since $N'_{n-1} = M_{n-1}$, we see that M' is well-defined). And we define a subrepresentation M'' by

$$M''|[1, n{-}1] = N'' \quad \text{and} \quad M''|[n{-}1, n] = 0$$

(since $N''_{n-1} = 0$, we see that also M'' is well-defined). Of course, we obtain in this way a direct decomposition $M = M' \oplus M''$. We have: $M'|[1, n{-}1] = N'$ is $(n{-}1)$-conical, and the support of M'' is equal to the support of N'', thus contained in $[1, n-2]$. This completes the proof.

Step 2. *Any representation M of Q is a direct sum $M = M' \oplus M''$ such that M' is conical on* $([2, n], 2)$*, whereas the support of M'' is contained in* $[3, n]$.

The proof is similar to the proof of step 1, but this time, we consider the subquiver $[2, n]$. Consider the restriction $N = M|[2, n]$. By induction, we know that N can be written as a direct sum of representation of the form $M([s, t])$, with $2 \leq s \leq t \leq n$. We decompose $N = N' \oplus N''$, where N' is a direct sum of representations of the form $M([2, t])$, and N'' a direct sum of representations of the form $M([s, t])$ with $3 \leq s$. Note that the representation N' of $[2, n]$ is 2-conical. And we stress that $N'_2 = M_2$ and $N''_2 = 0$. We define a subrepresentation M' as follows: $M'|[2, n] = N'$ and $M'|[1, 2] = M|[1, 2]$ (since $N'_2 = M_2$, we see that M' is well-defined). And we define a subrepresentation M'' by $M''|[2, n] = N''$ and $M'|[1, 2]$ the zero representation (since $N''_2 = 0$, we see that also M'' is well-defined). We obtain in this way a direct decomposition $M = M' \oplus M''$. We have: $M'|[2, n] = N'$ is 2-conical, and the support of M'' is equal to the support of N'', thus contained in $[3, n]$. This completes the proof.

Combining step 1 and step 2: *Any representation M of Q can be written as a direct sum $M = M' \oplus M''$, such that for any arrow γ in* $[2, n-1]$*, the map M'_γ is a vector space isomorphism, whereas M'' is a direct sum of representations of the form* $[s, t]$ *with* $3 \leq s$ *or with* $t \leq n - 2$.

Proof. According to step 1, we can assume that M is conical on $([1, n-1], n-1)$. We apply step 2 and obtain a direct decomposition $M = M' \oplus M''$, where M' is conical on $([2, n], 2)$ and M'' has support in $[3, n]$. Since M is conical on $([1, n-1], n-1)$, the same is true for any direct summand of M, thus M' is conical on $([1, n-1], n-1)$, as well as on $([2, n], 2)$. This means that for any arrow γ in $[2, n-1]$, the map M'_γ is both injective and surjective, thus a vector space isomorphism.

Step 3. *Let M be a representation of Q such that for any arrow γ in* $[2, n-1]$*, the map M'_γ is a vector space isomorphism. Then M is isomorphic to a representation M' with $M'_x = M_2$ for all $2 \leq x \leq n-1$ and such that for any arrow γ in* $[2, n-1]$*, the map M'_γ is the identity map.*

Proof. Let $M'|[1, 2] = M|[1, 2]$, let $M'_x = M_2$ for $2 \leq x \leq n-1$ and $M'_n = M_n$. for any arrow γ in $[2, n-1]$, let M'_γ be the identity map. We do not yet define the map M'_β, where β is the arrow between $n-1$ and n. But we start already to define an isomorphism $f\colon M \to M'$. We de this inductively, that means: we define $f|[1, x]$ starting with $x = 2$. For $x = 2$, we take as $f|[1, 2]$ the identity isomorphism. Now assume, we have defined $f|[1, x]$ for some $2 \leq x \leq n-2$ and we want to define f_{x+1}. Let γ be the arrow in-between x and $y = x + 1$. There are two possible orientations of γ: In case $\gamma\colon x \to y$, we define $f_y = f_x \cdot (M_\gamma)^{-1}$. In case $\gamma\colon y \to x$, we define $f_y = f_x \cdot M_\gamma$ In this way, we obtain an isomorphism $M|[1, y] \to M'|[1, y]$.

Now assume we have constructed f_x for $1 \leq x \leq n-1$ such these maps f_x combine to an isomorphism $M|[1, n-1] \to M'|[1, n-1]$. We now have to consider the arrow β in-between $n-1$ and n. Our aim is to define M'_β as a map between M_2 and M_n so that $f|[1, n-1]$ can be extended by $f_n = 1$ to an isomorphism $f\colon M \to M'$. Again, we have to distinguish the two possible orientations: In case $\beta\colon n-1 \to n$, we define $M'_\beta = M_\beta \cdot (f_{n-1})^{-1}$. In case $\beta\colon n \to n-1$, we define $M'_\beta = f_{n-1} \cdot M_\beta$. In this way, we obtain an isomorphism $f\colon M \to M'$.

Final step. Denote the arrow in $[1,2]$ by α, the arrow in $[n-1,n]$ by β. We can assume that we deal with a representation M with $M_x = M_2$ for all $2 \le x \le n-1$ and such that for any arrow γ in $[2, n-1]$, the map M_γ is the identity map.

We attach to M

$$M = \left(M_1 \overset{M_\alpha}{\rule[0.5ex]{2em}{0.4pt}} M_2 \overset{1}{=\!=\!=} M_2 \overset{1}{=\!=} \quad \cdots \quad \overset{1}{=\!=} M_2 \overset{M_\beta}{\rule[0.5ex]{2em}{0.4pt}} M_n \right)$$

the following representation N

$$N = \left(M_1 \overset{M_\alpha}{\rule[0.5ex]{2em}{0.4pt}} M_2 \overset{M_\beta}{\rule[0.5ex]{2em}{0.4pt}} M_n \right)$$

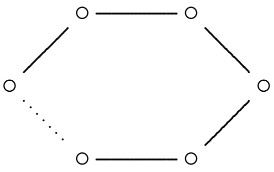
of a corresponding quiver of type $\mathbb{A}_3$, say with vertices labeled $1, 2, n$, and with arrows α, β as in the quiver Q.

As we know, we can write N as a direct sum of thin representations. Of course, such a direct decomposition leads to a corresponding direct decomposition of M. This completes the proof of Theorem 1.1.

1.2. The converse of Theorem 1.1. *Let Q be a finite connected quiver. If all indecomposable representations of Q are thin, then Q is of type $\mathbb{A}$.*

Proof: Let us assume that Q is connected and not of type $\mathbb{A}$. Then Q has a subquiver which is a cycle or which is of type $\mathbb{D}_4$. It is sufficient to show: *If Q is a quiver which is a cycle or of type $\mathbb{D}_4$, then there is an indecomposable representation which is not thin.*

First, consider the case where Q is a cycle, say



with vertices labeled $1, 2, \dots, n$ and an arrow $\alpha(i)$ in-between i and $i+1$, for $1 \le i < n$ and an arrow $\alpha(n)$ in-between n and 1. Without loss of generality, we can assume that $\alpha(n)\colon n \to 1$.

Let $M_i = k^2$, for all vertices i, take for all the arrows $\alpha(i)$ with $1 \le i < n$ the identity map $k^2 \to k^2$ and let $M_{\alpha(n)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. A straightforward calculation shows that M is indecomposable.

Second, we have to consider the quivers of type $\mathbb{D}_4$. It is easy to see (and will be shown in Part 2) that such a quiver has a unique indecomposable representation which is not thin. This completes the proof.

Let us mention several applications of Theorem 1.1.

1.3. Pairs of filtrations. The first application concerns vector spaces V with two filtrations $(U_i)_{1\le i\le n}$ and $(U'_j)_{1\le j\le m}$:

$$\begin{aligned}0 = U_0 \subseteq U_1 \subseteq U_2 \subseteq \cdots \subseteq U_n \subseteq U_{n+1} = V,\\ 0 = U'_0 \subseteq U'_1 \subseteq U'_2 \subseteq \cdots \subseteq U'_m \subseteq U'_{m+1} = V.\end{aligned}$$

Corollary (first formulation). *Let V be a vector space with two filtrations $(U_i)_{1\le i\le n}$ and $(U'_j)_{1\le j\le m}$. Then there is a basis $\mathbf{B}$ of V which is compatible with all the subspaces U_i, U'_j.*

Proof: Consider the quiver Q of type $\mathbb{A}_{n+m+1}$ with vertices labeled $1, \dots, n, \omega, m', \dots, 1'$ and the orientation

$$\begin{tikzcd}\overset{1}{\circ} \arrow[r] & \overset{2}{\circ} \arrow[r, dash] & \cdots \arrow[r] & \overset{n}{\circ} \arrow[r] & \overset{\omega}{\circ} & \overset{m'}{\circ} \arrow[l] & \cdots \arrow[l] & \overset{2'}{\circ} \arrow[l, dash] & \overset{1'}{\circ} \arrow[l]\end{tikzcd}$$

The two filtrations yield the following representation M of Q (all maps are inclusion maps):

$$\begin{tikzcd}U_1 \arrow[r] & U_2 \arrow[r, dash] & \cdots \arrow[r] & U_n \arrow[r] & V & U'_m \arrow[l] & \cdots \arrow[l] & U'_2 \arrow[l, dash] & U'_1 \arrow[l]\end{tikzcd}$$

If we write this representation as a direct sum of thin representations N, and choose in any of these direct summands a non-zero element $b \in N_\omega$, we obtain the required basis $\mathbf{B}$.

We may reformulate the Corollary as follows:

Corollary (second formulation). *Let V be a vector space with two filtrations $(U_i)_{1\le i\le n}$ and $(U'_j)_{1\le j\le m}$. For $1 \le i \le n+1$ and $1 \le j \le m+1$, let $C(i,j)$ be a subspace of $U_i \cap U'_j$ with*

$$\big((U_i \cap U'_{j-1}) + (U_{i-1} \cap U'_j)\big) \oplus C(i,j) = U_i \cap U'_j.$$

Then

$$V = \bigoplus_{\substack{1\le i\le n+1\\ 1\le j\le m+1}} C(i,j).$$

Here is the relationship between the assertions of the two formulations.

If $\mathbf{B}$ is a basis of V compatible with all the subspaces U_i, U'_j, let $\mathbf{B}(i,j)$ be the set of elements of $b \in \mathbf{B}$ with $b \in U_i \cap U'_j$ such that b does not belong to U_{i-1} nor to U'_{j-1}. Then $\mathbf{B}$ is the disjoint union of the subsets $\mathbf{B}(i,j)$ and $C(i,j)$ can be defined as the subspace generated by $\mathbf{B}(i,j)$.

Conversely, assume we have written V as the direct sum $\bigoplus C(i,j)$ with subspaces $C(i,j)$ of V such that $\big((U_i \cap U'_{j-1}) + (U_{i-1} \cap U'_j)\big) \oplus C(i,j) = U_i \cap U'_j$. Let $\mathbf{B}(i,j)$ be a basis of $C(i,j)$ and $\mathbf{B}$ the disjoint union of the sets $\mathbf{B}(i,j)$. Then $\mathbf{B}$ is a basis von V which is compatible with all the subspaces U_i, U'_j.

The Corollaries have been obtained by looking at a quiver Q of type $\mathbb{A}$ and writing a representation M of Q as a direct sum of thin representations, see the proof of Corollary 1.3. Conversely, we may use the assertions of the corollaries in order to recover the direct sum decomposition of M, as follows. For any pair i, j, let $M(i, j)$ be the thin indecomposable representation with support the vertices x between i and j'. Then

$$M = \bigoplus_{i,j} M(i, j) \otimes_k C(i, j).$$

1.4. Arms of a quiver. A second application concerns quivers with an arm. Assume that Q has an arm (Q', x), and that M is a representation of Q. We will say that M is *conical on* (Q', x) provided M_α is injective for any arrow α of Q' which points to x, and M_β is surjective, for the remaining arrows β of Q' (thus if the restriction of M to Q' is x-conical, as defined in the proof of Theorem 1.1).

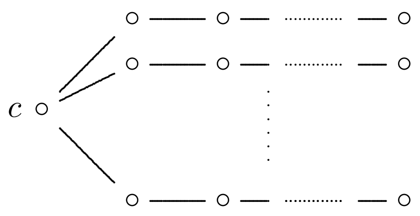
Corollary. *Assume that Q has an arm (Q', x). Then any representation M of Q can be decomposed $M = M' \oplus M''$, where M' is conical on (Q', x) and the support of M'' is contained in $Q' \setminus \{x\}$.*

In particular, this implies: *If Q has an arm (Q', x) and M is an indecomposable representation of Q such that $M_x \neq 0$, then M is conical on (Q', x).*

Proof of Corollary 1.4. Let M be a representation of Q. According to Theorem 1.1, we can write the restriction $M|Q'$ of M to Q' as a direct sum of thin indecomposable representations $X(i)$ with $i \in I$. Let I' be the set of indices $i \in I$ such that $X(i)_x \neq 0$ and let I'' be the set of indices $i \in I$ such that $X(i)_x = 0$. For any vertex $y \in Q'$, let $M'_y = \bigoplus_{i \in I'} X(i)_y$ and $M''_y = \bigoplus_{i \in I''} X(i)_y$. If y is a vertex of $Q \setminus Q'$, let $M'_y = M_y$, and $M''_y = 0$. It is easy to see that we obtain in this way a direct decomposition $M = M' \oplus M''$. The representations $X(i)$ with $i \in I'$ are conical on (Q', x), thus M' is conical on (Q', x). Since the representations $X(i)$ with $i \in I''$ satisfy $X(i)_x = 0$, it follows that the support of M'' is contained in $Q' \setminus \{x\}$.

Until now, we have tried to involve just knowledge from a Linear Algebra course. From now on, we will make references to some results in ring and module theory. On the one hand, the considerations which follow may be seen in this way in a broader setting. On the other hand, some of the further proofs do rely in an essential way on known results which have to be mentioned. The methods to be used will be the Auslander-Reiten theory including hammocks, one-point extensions as well as tilting theory and thick subcategories.

1.5. Star quivers. We say that a quiver Q is a *star quiver with center c and arms* $(Q(1), c), \dots, (Q(s), c)$ provided Q is the union of the quivers $Q(i)$, the pairs $(Q(i), c)$ with $1 \leq i \leq s$ are arms of Q and $Q(i) \cap Q(j) = \{c\}$ for all pairs $i \neq j$. Thus, the underlying graph of Q has the following shape:



If Q is a star quiver with center c and arms $(Q(i), c)$ of length $t(i)$, with $1 \leq i \leq s$, such that $t(1) \geq t(2) \geq \cdots \geq t(s) \geq 2$, then the sequence $(t(1), \cdots, t(s))$ is called the *type* of the star quiver Q (with respect to c); the type of the star quiver $\mathbb{A}_1$ is by definition the empty sequence (or the sequence (1)). A representation M of a star quiver with center c is said to be *conical* provided it is conical on all the arms. (In case we deal with a star quiver with at least 3 proper arms, the center of c is uniquely determined; otherwise we need the reference to the center c.)

Corollary. *Assume that Q is a star quiver with center c. Then any representation of Q is the direct sum $M = M' \oplus M''$ such that M' is conical and $M''_c = 0$.*

Note that the support of an indecomposable representation M of Q with $M_c = 0$ lies inside one of the arms. In particular, we see: *Assume that Q is a star quiver with center c. Then there are only finitely many indecomposable representations M of Q with $M_c = 0$ and any other indecomposable representation of Q is conical.* We denote by $\operatorname{Conic} Q$ the full subcategory of $\operatorname{Rep} Q$ given by all conical representations.

If all the arrows of a star quiver Q point to the center c, and M is a conical representation of Q, then we can assume that M is a subspace representation (meaning that all the maps M_α are inclusion maps of subspaces).

1.6. Proposition. *If Q, Q' are star quivers of the same type, then the categories $\operatorname{Conic} Q$ and $\operatorname{Conic} Q'$ are equivalent.*

The easiest way to prove this assertion seems to be to use tilting theory. Let Q be a star quiver with center c and Λ its path algebra. There is a unique multiplicity-free tilting Λ-module T such that $P(c)$ is a direct summand of T and such that $\operatorname{Hom}(P(c), T(i)) \neq 0$ for any indecomposable direct summand $T(i)$ of T (namely the slice module for the preprojective component of $\operatorname{Rep} Q$ for the slice with $P(c)$ as the only source). Let $\Lambda' = \operatorname{End}(T)^{\mathrm{op}}$, this is the path algebra of the quiver Q' with $\overline{Q} = \overline{Q'}$ and c the unique sink. The tilting functor $F = \operatorname{Hom}(T, -)\colon \operatorname{Rep} Q \to \operatorname{Rep} Q'$ provides an equivalence between $\mathcal{F}(T) = \{M \in \operatorname{Rep} Q \mid \operatorname{Ext}^1(T, M) = 0\}$ and $\mathcal{X}(T) = \{X \in \operatorname{Rep} Q' \mid \operatorname{Tor}_1(T, X) = 0\}$. It is easy to see that $\operatorname{Conic} \Lambda \subseteq \mathcal{F}(T)$, and $\operatorname{Conic} \Lambda' \subseteq \mathcal{X}(T)$, and that F sends $\operatorname{Conic} \Lambda$ onto $\operatorname{Conic} \Lambda'$.

1.7. Final remark for Part 1. As we have mentioned at the beginning, the aim of Part 1 is to provide a straightforward proof for the well-known result that all representations of a quiver of type $\mathbb{A}_n$ are direct sums of thin representations. Why should one care for such an approach? The representation theory of quivers has to be considered as a sort of higher linear algebra, thus any introductory course devoted to the representation theory of quivers should start with the discussion of quivers which describe basic settings in linear algebra. Such quivers are the subspace quivers, and more generally the star quivers, but also the quivers of type $\mathbb{A}$. Actually, for looking at star quivers, it is advisable to have some knowledge concerning the behavior of representations on the arms, thus about quivers of type $\mathbb{A}$. Thus, we believe that there are good reasons to start with quivers of type $\mathbb{A}$. Some techniques to work with quivers can be trained nicely in this way, and one has to be aware that the use of bases of vector spaces still is of interest in this setting, in contrast to the usual procedure when dealing with representations of quivers. Anyway, since the use of bases of vector spaces is omnipresent in any linear algebra course, the quivers of type $\mathbb{A}$ may be seen in this way as an intermediate step.

The usual approach to quiver representations starts with positive roots. But the general frame of Dynkin quivers hides the special features of the quivers of type $\mathbb{A}$, namely the predominance of thin representations and the possibility to work with bases. It seems to be worthwhile not to neglect these features since they are helpful in the more general setting of quivers of type $\widetilde{\mathbb{A}}$, or even of arbitrary special biserial algebras with their string and band modules, but also when looking at arms of quivers, and in particular at star quivers.

Part 2. The quivers of type $\mathbb{D}$.

We consider now the quivers with underlying graph of type $\mathbb{D}_n$. Whereas our interest in type $\mathbb{A}$ was devoted to thin indecomposable representations, here we are faced with categories for which the thin representations play a minor role. The relevant indecomposable representations will be collected in a full subcategory $\check{\mathcal{B}}$ which contains all the non-thin indecomposable representations and only few thin representations.

First, let us consider the following quiver $Q(n)$

$$\begin{tikzcd}
 & 1 & & & & \\
Q(n) & & 3 \arrow[ul, "\pi_1"'] \arrow[dl, "\pi_2"] & 4 \arrow[l] & \cdots \arrow[l] & n \arrow[l] \\
 & 2 & & & &
\end{tikzcd}$$

defined for $n \geq 3$. Of course, $Q(3)$ is a quiver of type $\mathbb{A}_3$ (the 2-factor-space quiver), thus our main interest will lie in the quivers $Q(n)$ with $n \geq 4$.

We denote by $\check{\mathcal{B}}$ the full subcategory of all representations M of $Q(n)$ such that the map $\left[\begin{smallmatrix}\pi_1\\ \pi_2\end{smallmatrix}\right] : M_3 \to M_1 \oplus M_2$ is bijective. Let $\mathcal{B} = \check{\mathcal{B}} \cap \operatorname{rep} Q(n)$.

Again, our theme is to present a proof of a well-known result, namely the following theorem.

2.1. Theorem. *Any representation of $Q(n)$ is the direct sum of a representation in $\check{\mathcal{B}}$ and thin representations.*

Proof. There are several steps of splitting off direct sums of thin representations in order to obtain a representation in $\check{\mathcal{B}}$. Two of the steps are based on Theorem 1.1.

(0) As a preliminary step, we consider the quiver $\sigma Q(n)$

$$\begin{tikzcd}
 & 1 \arrow[dr, "\mu_1"] & & & & \\
\sigma Q(n) & & 3 & 4 \arrow[l] & \cdots \arrow[l] & n \arrow[l] \\
 & 2 \arrow[ur, "\mu_2"'] & & & &
\end{tikzcd}$$

and denote by $\mathcal{N}$ the full subcategory of all representations of $\sigma Q(n)$ of the form

$$N = \left(\begin{tikzcd}
U_1 \arrow[dr, "\mu_1"] & & & & \\
 & N_3 & N_4 \arrow[l] & \cdots \arrow[l] & N_n \arrow[l] \\
U_2 \arrow[ur, "\mu_2"'] & & & &
\end{tikzcd}\right)$$

where U_1, U_2 are subspaces of N_3 and $\mu_i \colon U_i \to N_3$ are the inclusion maps.

Let N be a representation in $\mathcal{N}$. We look at the representation

$$\overline{N} = \left(U_1 \oplus U_2 \xrightarrow{[\,\mu_1\ \mu_2\,]} N_3 \leftarrow N_4 \leftarrow \cdots \leftarrow N_n \right)$$

of the following quiver of type $\mathbb{A}_{n-1}$

$$b \to 3 \leftarrow 4 \leftarrow \cdots \leftarrow n$$

and use the conification procedure of Corollary 1.3. In this way, we write $\overline{N}$ as the direct sum of a representation of the form

$$0 \to N_3'' \leftarrow N_4'' \leftarrow \cdots \leftarrow N_n''.$$

and a representation

$$U_1 \oplus U_2 \to N_3' \leftarrow N_4' \leftarrow \cdots \leftarrow N_n'$$

which is conical with respect to b. In particular, the map $U_1 \oplus U_2 \to N_3'$ has to be surjective. This means that $N_3' = \mu_1(U_1) + \mu_2(U_2)$, thus N_3' is just the subspace $U_1 + U_2$ of N_3.

It follows that N is the direct sum $N = N' \oplus N''$ of

$$N'' \;=\; \left(\begin{tikzcd} 0 \arrow[dr] & & & & \\ & N_3'' & N_4'' \arrow[l] & \cdots \arrow[l] & N_n'' \arrow[l] \\ 0 \arrow[ur] & & & & \end{tikzcd}\right).$$

and

$$N' \;=\; \left(\begin{tikzcd} U_1 \arrow[dr, "\mu_1"] & & & & \\ & U_1+U_2 & N_4' \arrow[l] & \cdots \arrow[l] & N_n' \arrow[l] \\ U_2 \arrow[ur, "\mu_2"'] & & & & \end{tikzcd}\right)$$

with $\mu_i \colon U_i \to U_1 + U_2 = N_3'$ being the inclusion maps, and all the maps $N_i' \leftarrow N_{i+1}'$ for $3 \le i < n$ being injective.

Given a representation N in $\mathcal{N}$, we denote by $\sigma^- N$ the representation

$$\sigma^- N \;=\; \left(\begin{tikzcd} N_3/U_1 & & & & \\ & N_3 \arrow[ul, "\pi_1"'] \arrow[dl, "\pi_2"] & N_4 \arrow[l] & \cdots \arrow[l] & N_n \arrow[l] \\ N_3/U_2 & & & & \end{tikzcd}\right)$$

such that the maps π_1, π_2 are the canonical projections.

Now, let us start with a representation M of $Q(n)$.

(1) We split off copies of $S(1)$ and $S(2)$ so that we can assume that π_1 and π_2 are surjective.

(2) We look at the representation

$$M_1 \oplus M_2 \xleftarrow{\left[\begin{smallmatrix}\pi_1\\ \pi_2\end{smallmatrix}\right]} M_3 \leftarrow M_4 \leftarrow \cdots \leftarrow M_n$$

of the quiver Q' of type $\mathbb{A}_{n-1}$

$$Q' \qquad\qquad a \leftarrow 3 \leftarrow 4 \leftarrow \cdots \leftarrow n.$$

We use the conification procedure of Corollary 1.3 in order to write this representation as the direct sum of a representation $(0 \leftarrow M_3'' \leftarrow \cdots \leftarrow M_n'')$ and a conical representation $(M_1 \oplus M_2 \leftarrow M_3' \leftarrow \cdots \leftarrow M_n')$ of (Q', a).

It follows that M is the direct sum $M = M' \oplus M''$, where

$$M'' \;=\; \left(\begin{tikzcd}[row sep=small, column sep=small] 0 & & & & \\ & M_3'' \arrow[ul] \arrow[dl] & M_4'' \arrow[l] & \cdots \arrow[l] & M_n'' \arrow[l] \\ 0 & & & & \end{tikzcd}\right).$$

and

$$M' = \left(\begin{tikzcd}[row sep=small, column sep=small] M_1 & & & & \\ & M_3' \arrow[ul, "\pi_1"'] \arrow[dl, "\pi_2"] & M_4' \arrow[l] & \cdots \arrow[l] & M_n' \arrow[l] \\ M_2 & & & & \end{tikzcd}\right)$$

with the map $\left[\begin{smallmatrix}\pi_1\\ \pi_2\end{smallmatrix}\right] : M_3' \to M_1 \oplus M_2$ and all the maps $M_{i+1}' \to M_i'$ for $3 \leq i < n$ being injective. Of course, the representation M'' is a direct sum of thin representations, thus it remains to deal with M'.

(3) We assume now that we deal with a representation M

$$\begin{tikzcd}[row sep=small, column sep=small] M_1 & & & & \\ & M_3 \arrow[ul, "\pi_1"'] \arrow[dl, "\pi_2"] & M_4 \arrow[l] & \cdots \arrow[l] & M_n \arrow[l] \\ M_2 & & & & \end{tikzcd}$$

which is conical and that in addition the map $\left[\begin{smallmatrix}\pi_1\\ \pi_2\end{smallmatrix}\right] : M_3 \to M_1 \oplus M_2$ is injective.

Let U_i be the kernel of π_i with inclusion map $\mu_i \colon U_i \to M_3$. Since the map $\left[\begin{smallmatrix}\pi_1\\ \pi_2\end{smallmatrix}\right] : M_3 \to M_1 \oplus M_2$ is injective, we have $U_1 \cap U_2 = 0$.

Since π_i is surjective, we may assume that $M_i = M_3/U_i$ and that π_i is the canonical projection map. The representation

$$N \;=\; \left(\begin{tikzcd}[row sep=small, column sep=small] U_1 \arrow[dr, "\mu_1"] & & & & \\ & M_3 & M_4 \arrow[l] & \cdots \arrow[l] & M_n \arrow[l] \\ U_2 \arrow[ur, "\mu_2"'] & & & & \end{tikzcd}\right)$$

of $\sigma Q(n)$ belongs to $\mathcal{N}$ and $\sigma^- N = M$. Since M is conical, also N is conical.

We use (0) in order to write $N = N' \oplus N''$ with

$$N'' \;=\; \left(\begin{tikzcd}[row sep=small, column sep=small] 0 \arrow[dr] & & & & \\ & N_3'' & N_4'' \arrow[l] & \cdots \arrow[l] & N_n'' \arrow[l] \\ 0 \arrow[ur] & & & & \end{tikzcd}\right).$$

and

$$N' \;=\; \left(\begin{tikzcd}[row sep=small, column sep=small] U_1 \arrow[dr, "\mu_1"] & & & & \\ & U_1+U_2 & N_4' \arrow[l] & \cdots \arrow[l] & N_n' \arrow[l] \\ U_2 \arrow[ur, "\mu_2"'] & & & & \end{tikzcd}\right),$$

where again the maps $\mu_i \colon U_i \to U_1 + U_2$ are the inclusion maps. We have $M = \sigma^- N = \sigma^- N' \oplus \sigma^- N''$. Since M is conical, also the direct summands $\sigma^- N', \sigma^- N''$ of M are conical.

Since N'' is conical and $N_1'' = N_2'' = 0$, we know that N'' is the direct sum of indecomposable representations of $\sigma Q(n)$ with dimension vectors of the form $\begin{smallmatrix} 0 & & & & & & \\ & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 0 & & & & & & \end{smallmatrix}$, thus $\sigma^- N''$ is the direct sum of indecomposable representations W of $Q(n)$ with dimension vectors of the form $\begin{smallmatrix} 1 & & & & & & \\ & 1 & \cdots & 1 & 0 & \cdots & 0 \\ 1 & & & & & & \end{smallmatrix}$. In this way, we see that $\sigma^- N''$ is a direct sum of thin representations.

It remains to look at N' and $\sigma^- N'$. Now N' is a subrepresentation of N, with $U_1 = N'_1 = N_1$, $U_2 = N'_2 = N_2$ and $U_1 + U_2 = N'_3 \subseteq N_3$. Since we have $U_1 \cap U_2 = 0$, we see that $N'_3 = U_1 \oplus U_2$ (and that the map $[\,\mu_1 \quad \mu_2\,] : N'_1 \oplus N'_2 \to N'_3$ is the identity map). It follows that

$$\sigma^- N' \;=\; \left(\begin{tikzcd}[column sep=small, row sep=small] (U_1{+}U_2)/U_1 & & & & \\ & U_1{+}U_2 \arrow[ul, "\pi_1"'] \arrow[dl, "\pi_2"] & N'_4 \arrow[l] & \cdots \arrow[l] & N'_n \arrow[l] \\ (U_1{+}U_2)/U_2 & & & & \end{tikzcd}\right),$$

belongs to $\check{\mathcal{B}}$. This completes the proof of Theorem 2.1.

Remark. We have denoted by $\mathcal{N}$ the category of representations N of $\sigma Q(n)$ such that the maps μ_1, μ_2 are inclusion maps. Similarly, we may denote by $\mathcal{M}$ the category of representations M of $Q(n)$ such that the maps π_1, π_2 are surjective. The functor σ^- maps $\mathcal{N}$ to $\mathcal{M}$. Similarly, we may consider a functor $\sigma^+ \colon \mathcal{M} \to \mathcal{N}$ by using the kernels of the maps π_1, π_2. Clearly, σ^- is an equivalence of categories with (quasi-)inverse σ^+.

Let us now investigate the subcategory $\check{\mathcal{B}}$.

2.2. Theorem. *The category $\check{\mathcal{B}}$ is equivalent to* $\operatorname{Rep} Q(n{-}1)$*; it is the image of the fully faithful functor*

$$\eta\colon\ \operatorname{Rep} Q(n{-}1) \to \operatorname{Rep} Q(n)$$

defined as follows:

$$\left(\begin{tikzcd}[column sep=small, row sep=small] M_1 & & & & \\ & M_3 \arrow[ul, "\pi_1"'] \arrow[dl, "\pi_2"] & M_4 \arrow[l] & \cdots \arrow[l] & M_{n-1} \arrow[l] \\ M_2 & & & & \end{tikzcd}\right) \mapsto \left(\begin{tikzcd}[column sep=small, row sep=small] M_1 & & & & & \\ & M_1 \oplus M_2 \arrow[ul, "\epsilon_1"'] \arrow[dl, "\epsilon_2"] & M_3 \arrow[l, "\left[\begin{smallmatrix}\pi_1\\ \pi_2\end{smallmatrix}\right]"'] & M_4 \arrow[l] & \cdots \arrow[l] & M_{n-1} \arrow[l] \\ M_2 & & & & & \end{tikzcd}\right);$$

here, for $i = 1, 2$, the map ϵ_i is the canonical projection $M_1 \oplus M_2 \to M_i$.

Proof. Clearly, the image of η is just $\check{\mathcal{B}}$. There is a (quasi-)inverse functor from $\operatorname{Rep} Q(n)$ to $\operatorname{Rep} Q(n{-}1)$ which deletes the vector space at the branching vertex:

$$\left(\begin{tikzcd}[column sep=small, row sep=small] M_1 & & & & \\ & M_3 \arrow[ul, "\pi_1"'] \arrow[dl, "\pi_2"] & M_4 \arrow[l, "\gamma"'] & \cdots \arrow[l] & M_n \arrow[l] \\ M_2 & & & & \end{tikzcd}\right) \mapsto \left(\begin{tikzcd}[column sep=small, row sep=small] M_1 & & & \\ & M_4 \arrow[ul, "\pi_1\gamma"'] \arrow[dl, "\pi_2\gamma"] & \cdots \arrow[l] & M_n \arrow[l] \\ M_2 & & & \end{tikzcd}\right).$$

This shows that η is fully faithful.

2.3. Corollary. *Any representation of $Q(n)$ is the direct sum of finite-dimensional indecomposable representations.*

Proof, by induction. For $n = 3$ we deal with a quiver of type $\mathbb{A}_3$, thus the assertion has been shown in Part 1. Thus, assume that $n \geq 4$. According to Theorem 2.1, any representation M of $Q(n)$ is the direct sum of thin representations and a representation M' in $\check{\mathcal{B}}$. According to Theorem 2.2, the functor η provides an equivalence between $\check{\mathcal{B}}$ and $\operatorname{Rep} Q(n-1)$, and the representations in $\mathcal{B}$ correspond under η to the finite-dimensional representations of $Q(n-1)$. By induction, $\eta^{-1}(M')$ is a direct sum of finite-dimensional representations of $Q(n-1)$, thus M' is the direct sum of finite-dimensional representations of $Q(n)$.

2.4. Twin representations. We call an indecomposable representation M of $Q(n)$ a *twin* representation provided $M_1 \neq 0, M_2 \neq 0$.

Here are typical examples of twin representations: First of all, we will call the thin twin representations 0-*twin* representations; they are the indecomposable representations with dimension vector of the form $\begin{smallmatrix} 1 & & \\ & 1 \cdots 1 \; 0 \cdots 0 \\ 1 & & \end{smallmatrix}$ (and dimension at least 3). Second, the following representations

$$\begin{tikzcd}
k & & & & & & & & & & \\
& k^2 \arrow[ul, "\pi_1"'] \arrow[dl, "\pi_2"] & \cdots \arrow[l, "1"'] & k^2 \arrow[l, "1"'] & k \arrow[l, "\delta"'] & \cdots \arrow[l, "1"'] & k \arrow[l, "1"'] & 0 \arrow[l] & \cdots \arrow[l] & 0 \arrow[l] \\
k & & & & & & & & & &
\end{tikzcd}$$

with $r \geq 1$ vector spaces of the form k^2, and at least three vector spaces of the form k will be called the r-*twin* representations; here $\delta \colon k \to k^2$ denotes the diagonal embedding, and $\pi_i \colon k^2 \to k$ with $i = 1, 2$ are the two canonical projections. In order to see that these r-twin representations are twin representations, we need to know that they are indecomposable. But this is easy to check (it is also a direct consequence of the following Lemma).

Lemma. (a) *Let $n \geq 4$ and $1 \leq r \leq n-3$. The functor η provides a bijection between the $(r-1)$-twin representations of $Q(n-1)$ and the r-twin representations of $Q(n)$.*

(b) *A twin representation of $Q(n)$ is an r-twin representation for some $r \geq 0$.*

Proof. (a) is straightforward. (b) Let M be a twin representation of $Q(n)$. If M is thin, then it is a 0-twin representation, by definition. Thus, we many assume that M is not thin. According to Theorem 2.1, M belongs to $\mathcal{B}(n)$, thus it is of the form $\eta(N)$ for some indecomposable representation of $Q(n-1)$. By the definition of η, we have $N_i = M_i \neq 0$, for $i = 1, 2$, thus N is a twin representation. By induction, N is an s-twin representation for some $s \geq 0$. According to (a), $M = \eta(N)$ is an $(s+1)$-twin representation.

Corollary. *An indecomposable representation of $Q(n)$ is either thin or an r-twin representation with $1 \leq r \leq n-3$. Any representation of $Q(n)$ is the direct sum of thin and twin representations.*

In particular, $Q(n)$ has a uniquely determined maximal indecomposable representation, namely the $(n-3)$-twin representation (here, *maximal* means that the composition length is maximal).

From now on, we mainly will deal with finite-dimensional representations. Thus, vector spaces and representations will be assumed to be finite-dimensional unless otherwise stated.

2.5. The position of some subcategories in the Auslander-Reiten quiver of $Q(n)$. Given a class $\mathcal{U}$ of representations of a quiver, we denote by $\operatorname{add}\mathcal{U}$ the class of all direct summands of finite direct sums of representations in $\mathcal{U}$. If $\mathcal{U}', \mathcal{U}''$ are subcategories, we write $\mathcal{U}' \vee \mathcal{U}''$ for the class of representations of the form $C' \oplus C''$ with $C' \in \mathcal{U}'$, $C'' \in \mathcal{U}''$.

Here are the subcategories which we are interested in:

$$\mathcal{V} = \operatorname{add} S(1) \vee \operatorname{add} S(2)$$

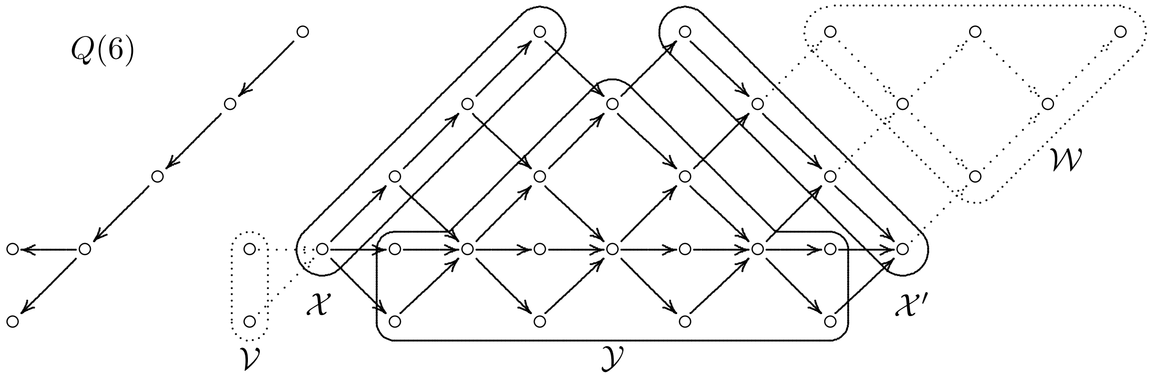
$$\mathcal{X} = \operatorname{add}\{M \mid M \text{ is a 0-twin representation}\}$$

$$\mathcal{Y} = \{M \mid \begin{bmatrix} \pi_1 \\ \pi_2 \end{bmatrix} : M_3 \to M_1 \oplus M_2 \text{ bijective, and } M_{i+1} \to M_i \text{ for } 3 \le i < n \text{ injective}\}$$

$$\mathcal{X}' = \operatorname{add}\{M \mid M \text{ indecomposable thin representation with } M_1 = 0 = M_2, M_3 \neq 0\}$$

$$\mathcal{W} = \{\text{representations with support in } [4, n]\}$$

For the notion of an Auslander-Reiten quiver, we refer to [R3, ARS]. For $n = 6$, let us draw the Auslander-Reiten quiver of $Q(n)$. We have encircled by solid lines the parts $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{X}'$, and by dotted lines the parts $\mathcal{V}$ and $\mathcal{W}$.



These are the building blocks of $\operatorname{rep} Q(n)$. The dotted part on the left are the two simple representations $S(1)$ and $S(2)$, this is $\mathcal{V}$. The dotted part on the right shows the indecomposable representations with support in $[4, n]$, this is $\mathcal{W}$. We should stress that $\mathcal{X} \vee \mathcal{Y} \vee \mathcal{X}'$ is just the category $\operatorname{Conic} Q(n)$.

Let us introduce the following notation:

$$\mathcal{A} = \mathcal{X} \vee \mathcal{W}, \quad \mathcal{B} = \mathcal{Y} \vee \mathcal{W}, \quad \mathcal{A}' = \mathcal{X}' \vee \mathcal{W}.$$

(thus, $\mathcal{W}$ belongs to $\mathcal{A}, \mathcal{B}, \mathcal{A}'$).

The category $\mathcal{A}'$ are the representations with support in $[3, n]$, thus it is the category of representations of a quiver of type $\mathbb{A}_{n-2}$.

The category $\mathcal{A}$ is equivalent to the category of representations of a quiver of type $\mathbb{A}_{n-2}$, since the functor σ^+ (introduced in the proof of Theorem 2.1) provides an equivalence between $\mathcal{A}$ and $\mathcal{A}'$.

The shape of the category $\mathcal{B} = \mathcal{Y} \vee \mathcal{W}$ illustrates nicely that $\mathcal{B} = \mathcal{B}(n)$ may be identified with the category of representations of a quiver of type $\mathbb{D}_{n-1}$, as we have shown in Theorem 2.2.

Analysis of the proof of Theorem 2.1. Let us use the Auslander-Reiten quiver shown above to describe the steps in the proof of Theorem 2.1. In step (1), we have split off representations in $\mathcal{V}$. Step (2) was devoted to splitting off representations in $\mathcal{X}' \vee \mathcal{W}$. In the final step (3), we have split off representations in $\mathcal{X}$. What remains turns out to be a direct sum of representations in $\mathcal{Y}$ (and $\mathcal{Y} \subset \mathcal{B}$).

2.6. Antichains in posets and additive categories. Given a poset P, an *antichain* in P is a subset A of P such that $a \le a'$ for a, a' in A implies $a = a'$. Given a k-category $\mathcal{C}$, an *antichain* in $\mathcal{C}$ a is a set of pairwise (Hom-)orthogonal objects with endomorphism rings being division rings. (Starting with a poset P, one may consider its linearization kP, see for example [R5, N1.8], this is an additive category and the antichains in the poset P are just the antichains in the additive category kP.) An antichain set of cardinality 2 or 3 will be called an *antichain pair* or an *antichain triple,* respectively.

Simplification. Let us assume now that $\mathcal{C}$ is an abelian category. Starting with an antichain A in $\mathcal{C}$, we can consider the full subcategory $\mathcal{E}(A)$ of all objects with a filtration with factors in the antichain: this is a thick subcategory (an exact abelian subcategory which is closed under extensions). Starting with an antichain $A = \{A_1, \dots, A_t\}$ in an abelian k-category $\mathcal{C}$ such that the endomorphism ring of any object A_i is equal to k, its Ext-*quiver* $Q(A)$ has t vertices, and the number of arrows $j \to i$ is the k-dimension of $\operatorname{Ext}^1(A_j, A_i)$. In case the Ext-quiver is finite and acyclic, the category $\mathcal{E}(A)$ is equivalent to the category $\operatorname{mod}\Lambda$ of all finite-dimensional Λ-modules, where Λ is a finite-dimensional k-algebra, see [R1]. Note that the simple objects in the category $\mathcal{E}(A)$ are just the elements of A, thus to focus the attention to $\mathcal{E}(A)$ is called the process of *simplification.*

Let us return to the quiver $Q(n)$. It is easy to see that the representations

$$M(\langle 1,3\rangle), M(\langle 2,3\rangle), S(4), \dots, S(n)$$

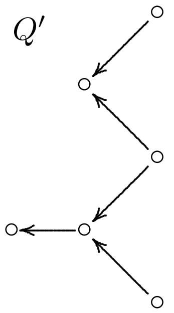
form an antichain and that $\mathcal{E}(M(\langle 1,3\rangle), M(\langle 2,3\rangle), S(4), \dots, S(n))$ are the finite-dimensional representations in $\mathcal{B}(n)$.

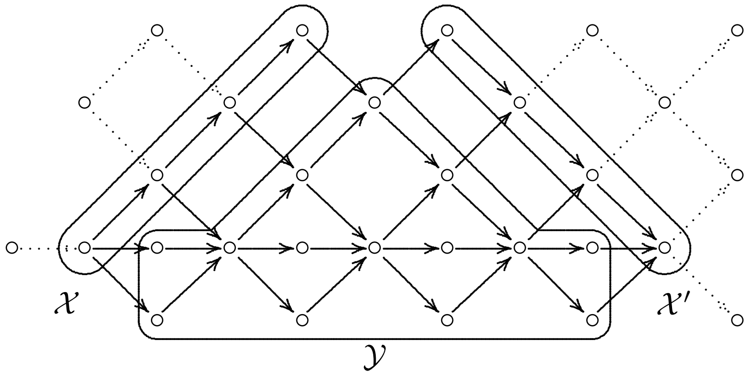
The Ext-quiver of this antichain looks as follows:

$$\begin{tikzcd}
M(\langle 1,3\rangle) & & & \\
& S(4) \arrow[ul] \arrow[dl] & \cdots \arrow[l] & S(n) \arrow[l] \\
M(\langle 2,3\rangle) & & &
\end{tikzcd}$$

thus, it is the quiver $Q(n-1)$. This shows (again) that the category of finite-dimensional representations in $\mathcal{B}(n)$ is equivalent to $\operatorname{rep} Q(n-1)$.

2.7. The conical representations. Let us recall that the category Conic $Q(n)$ is the intrinsic object to look at, we have Conic $Q(n) = \mathcal{X} \vee \mathcal{Y} \vee \mathcal{X}'$. As we have mentioned in Proposition 1.6, the category of conical representations does not change, if we change the orientation of the quiver. Here is the Auslander-Reiten quiver of a second quiver Q' of type $\mathbb{D}_6$, it should facilitate the reader to compare the embeddings of the category of conical representations into the representation categories.





Change of orientation: Antichains. For the quiver $Q(n)$, we have exhibited above an antichain A such that the objects of $\mathcal{E}(A)$ are the finite-dimensional representations inside $\mathcal{B}(n)$. Similar antichains do exist for any orientation, but we have to distinguish two cases, namely whether the two short arms have the same orientation or not. In case they have the same orientation, the antichain to be considered consists (as in the case $Q(n)$) of the representations:

$$M(\langle 1,3\rangle), M(\langle 2,3\rangle), S(4), \ldots, S(n).$$

Now assume that the short arms have different orientation. Let us look at the quiver $Q'(n)$ given as follows:

$$Q'(n) \qquad \begin{tikzcd}[row sep=small] 1 \arrow[dr, "\mu"] & & & & \\ & 3 \arrow[dl, "\pi"] & 4 \arrow[l, "\gamma"'] & \cdots \arrow[l] & n \arrow[l] \\ 2 & & & & \end{tikzcd}$$

Then the antichain to be considered is the following:

$$M(\langle 1,2,3\rangle), S(3), S(4), \ldots, S(n).$$

Of course, its Ext-quiver

$$\begin{tikzcd}[row sep=small] M(\langle 1,2,3\rangle) & & & \\ & S(4) \arrow[ul] \arrow[dl] & \cdots \arrow[l] & S(n) \arrow[l] \\ S(3) & & & \end{tikzcd}$$

is again of type $\mathbb{D}_{n-1}$ (here, the orientation of the two short arms is determined by the orientation of γ). Note that $\mathcal{E}(M(\langle 1,2,3\rangle), S(3), S(4), \ldots, S(n))$ is the full subcategory of all representations of $Q'(n)$ such that $\pi\mu$ is bijective.

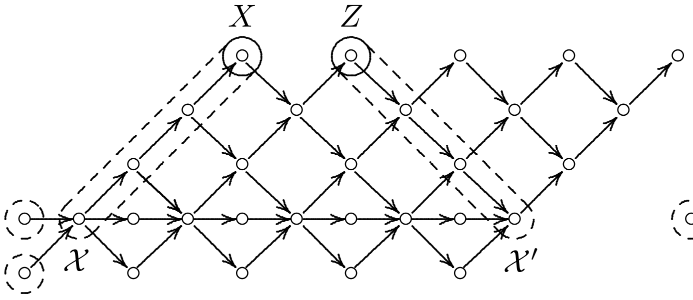
2.8. Perpendicular categories. Given a class $\mathcal{U}$ of representation of a quiver Q, we denote by $\mathcal{U}^\perp$ the full subcategory of $\operatorname{rep} Q$ given by all representations M with $\operatorname{Hom}(C, M) = 0 = \operatorname{Ext}^1(C, M)$ for all C in $\mathcal{U}$. Similarly, we denote by ${}^\perp\mathcal{U}$ the full subcategory of $\operatorname{rep} Q$ given by all representations M with $\operatorname{Hom}(M, C) = 0 = \operatorname{Ext}^1(M, C)$ for all C in $\mathcal{U}$.

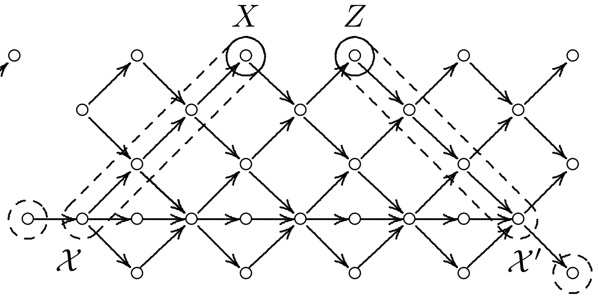
The reduction from $\mathbb{D}_n$-quivers to $\mathbb{D}_{n-1}$-quivers can be described very well using such perpendicular categories: Let X be the indecomposable representation in $\mathcal{X}$ which is a sink in $\mathcal{X}$ (thus any non-zero homomorphism $X \to X'$ in $\mathcal{X}$ is a split monomorphism). Let Z be the indecomposable representation in $\mathcal{X}'$ which is a source in $\mathcal{X}'$ (thus any non-zero homomorphism $Z' \to Z$ in $\mathcal{X}'$ is a split epimorphism). Then $\tau Z = X$. For any orientation, we may look at

$$\mathcal{B}(n) = {}^\perp X = Z^\perp = \{M \mid \operatorname{Hom}(M, X) = 0 = \operatorname{Hom}(Z, M)\}$$

Thus, in order to find $\mathcal{B}(n)$, we have to delete on the one hand the indecomposable representations M with $\operatorname{Hom}(M, X) \neq 0$, these are the indecomposable representations in $\mathcal{X}$ as well as the simple projective modules of the form $S(1), S(2)$, and on the other hand the indecomposable representations M with $\operatorname{Hom}(Z, M) \neq 0$, these are the indecomposable representations in $\mathcal{X}'$ as well as the simple injective modules of the form $S(1), S(2)$. Thus $\mathcal{B}(n)$ is obtained from $\operatorname{rep} Q(n)$ by deleting $\mathcal{X}, \mathcal{X}'$ as well as $S(1)$ and $S(2)$.

Here are the Auslander-Reiten quivers of $Q(6)$ and Q' with X, Z encircled (the subcategories $\mathcal{X}$ and $\mathcal{X}'$ as well as the two simple representations in $\mathcal{V}$ are indicated by dotted lines):





The subcategory $\mathcal{B}(n)$ can be characterized as follows.

2.9. Theorem. *Let Q be a quiver of type $\mathbb{D}_n$ with $n \geq 4$. There exists a smallest thick subcategory $\mathcal{T}$ of* $\operatorname{rep} Q$ *which contains all non-thin indecomposable representations. If $n \geq 5$, we have $\mathcal{T} = \mathcal{B}(n)$, thus $\mathcal{T}$ is equivalent to* $\operatorname{rep} Q'$ *for some quiver Q' of type $\mathbb{D}_{n-1}$. If $n = 4$, we have $\mathcal{T} = \operatorname{add} M$, where M is the maximal indecomposable representation, and thus $\mathcal{T}$ is of type $\mathbb{A}_1$.*

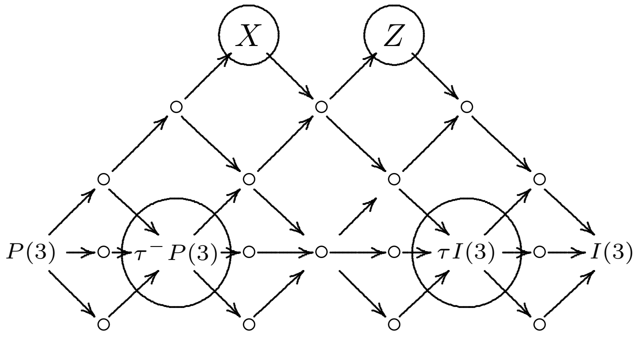
Proof. For $n = 4$, there is just one non-thin indecomposable representation M. Thus $\operatorname{add} M$ is the smallest thick subcategory which contains all non-thin indecomposable representations.

Thus, let us assume that $n \geq 5$. We know that $\mathcal{B} = \mathcal{B}(n)$ is a proper thick subcategory which contains all non-thin indecomposable representations. Conversely, let $\mathcal{T}$ be a proper thick subcategory of $\operatorname{rep} Q$ which contains all non-thin indecomposable representations. We have to show that $\mathcal{T} = \mathcal{B}$. Since $\mathcal{T}$ is a proper subcategory, $\mathcal{T}^{\perp}$ is not zero (see for example [R5]). Let X' be an indecomposable representation in $\mathcal{T}^{\perp}$, and $Z' = \tau^{-}X'$. We want to show that $X' = X$ (and $Z' = Z$). Since X' belongs to $\mathcal{T}^{\perp}$, we know the following: If Y is an indecomposable representation which is not thin, then $\operatorname{Hom}(Y, X') = 0$ and $\operatorname{Hom}(Z', Y) = D\operatorname{Ext}(Y, X') = 0$ (since Y belongs to $\mathcal{T}$).

Let us look at the successors of $\tau^{-}P(3)$ and at the predecessors of $\tau I(3)$. If N is indecomposable and a successor of $\tau^{-}P(3)$, then one easily sees that there is a non-thin indecomposable representation Y with $\operatorname{Hom}(Y, N) \neq 0$. Dually, if N is indecomposable and a predecessor of $\tau I(3)$, then there is a non-thin indecomposable representation Y with $\operatorname{Hom}(N, Y) \neq 0$.

It follows that X' is not a successor of $\tau^{-}P(3)$, and Z' is not a predecessor of $\tau I(3)$. But the only such indecomposable representation is $X' = X$ (and then $Z' = Z$).

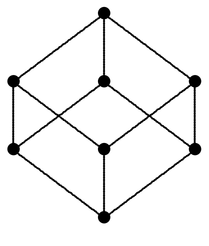
Let us indicate the position of the representations $\tau^{-}P(3)$ and $\tau I(3)$ as well as X and Z inside the category $\operatorname{Conic} Q(n) = \mathcal{X} \vee \mathcal{Y} \vee \mathcal{X}'$:



Altogether, we have shown that $\mathcal{T}^{\perp} = \operatorname{add} X$ and therefore $\mathcal{T} = {}^{\perp}X = \mathcal{B}(n)$. This completes the proof.

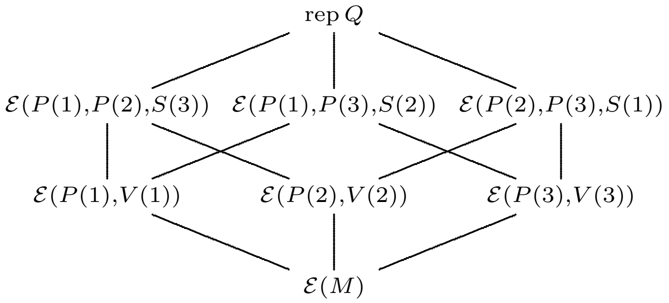
If Q is a quiver of type $\mathbb{D}_n$ with $n \geq 5$, then $\mathcal{B}$ itself and $\operatorname{rep} Q$ are the only thick subcategories which contain $\mathcal{B}$.

If Q is a quiver of type $\mathbb{D}_4$, then the thick subcategories of $\operatorname{rep} Q$ which contain the non-thin indecomposable representation M form a lattice L of the form



Namely, let Q' be obtained from Q by deleting the center, thus Q' is a quiver of type $\mathbb{A}_1 \sqcup \mathbb{A}_1 \sqcup \mathbb{A}_1$. The lattice of the thick subcategories of $\operatorname{rep} Q$ which contain M is isomorphic to the lattice of thick subcategories of $\operatorname{rep} Q'$ (see for example [R5]).

For example, let Q be the 3-subspace quiver with simple injective representations $S(1), S(2), S(3)$. As we have mentioned, $P(i)$ denotes the projective cover of $S(i)$, and we let $V(i) = \tau S(i) = \tau^{-1}P(i)$. Then L looks as follows:



We may proceed inductively: Let Q be a quiver of type $\mathbb{D}_n$. For $0 \le t \le n-3$, let $\mathcal{B}^{(t)}$ be the smallest thick subcategory which contains all s-twin representations with $s \ge t$. Then

$$\operatorname{rep} Q = \mathcal{B}^{(0)} \supset \mathcal{B} = \mathcal{B}^{(1)} \supset \cdots \supset \mathcal{B}^{(n-3)},$$

and $\mathcal{B}^{(t)}$ is of type $\mathbb{D}_{n-t}$ for $0 \le t \le n-4$ and of type $\mathbb{A}_1$ for $t = n-3$.

2.10. Final remark for Part 2. The aim of Part 2 was to collect the relevant indecomposable representations as a subcategory $\mathcal{B}$ which contains all the non-thin indecomposable representations and only few thin representations.

For a long time there was the strong belief that one of the first themes of representation theory should be to provide lists of all the indecomposable representations, whenever this is possible, and, only then, as a second step, to determine homomorphisms and extensions between the indecomposable objects. Of course, such a listing will be possible in case we deal with a representation-finite artin algebras, but it turns out that usually the list of indecomposables is quite uninteresting: it is the internal categorical structure and the interplay between indecomposable representations which should be described. In order to do so, one may look at sets of indecomposables which are related either by small changes of parameters or by the existence of irreducible maps.

This procedure is well accepted in case one deals with tame artin algebras, where the one-parameter families always are considered as units. There are three classes of artin algebras with good descriptions of all the indecomposable modules: the tame concealed algebras, the tubular algebras and the special biserial algebras. For the special biserial algebras, the string modules may be considered individually, but the band modules for a primitive cyclic word always are considered as a unit. For the tame concealed and the tubular algebras, one looks at the preprojective and the preinjective components, as well as the tubular families.

Our discussion of the quivers of type $\mathbb{D}$ uses a similar principle. The full subcategory of the conical representations is a unit which is independent of the orientation and which draws the attention to the representations which are not thin. As we have seen, we can decompose this unit into smaller blocks $\mathcal{X}, \mathcal{Y}, \mathcal{X}'$ whose internal structure is important, as well. Clearly, these blocks are the data of interest, whereas the individual indecomposables gain their relevance by their position inside these blocks.

Part 3. The quivers of type $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$ (and again $\mathbb{D}_n$).

Recall that any Dynkin quiver has exceptional vertices, those of type $\mathbb{D}_n$ and $\mathbb{E}_m$ have just one exceptional vertex. If Δ is a Dynkin quiver of type $\mathbb{D}_n$ or $\mathbb{E}_m$, and y is its exceptional vertex, we denote by Δ' the quiver obtained from Δ by deleting y, and by Δ'' the quiver obtained from Δ be deleting y and the neighbors of y.

A main result of Part 3 is the following theorem.

3.1. Theorem. *Let Δ be a Dynkin quiver of type $\mathbb{D}_n$ or $\mathbb{E}_m$. There is a unique antichain triple $A(1), A(2), A(3)$ in $\operatorname{rep}\Delta$ such that the support of any $A(i)$ is in Δ', but not in Δ''.*

In addition, we have: $\operatorname{Ext}^1(A(i), A(j)) = 0$ for all i, j. For any i, there is only one neighbor x of the exceptional vertex with $A(i)_x \neq 0$, and $\dim A(i)_x = 1$ for this vertex x.

Finally, if M is the maximal indecomposable representation of Δ, then

$$M|\Delta' = A(1) \oplus A(2) \oplus A(3)$$

(and $\dim M_y = 2$, for the exceptional vertex y).

The triple $A(1), A(2), A(3)$ will be called the *special antichain triple* in $\operatorname{rep}\Delta$. The proof will be based on a careful study of the quiver Δ'. It will be completed in section 3.6. The further considerations in part 3 will outline some consequences. In addition, we will draw the attention also to $M|\Delta''$.

We add two remarks, using the following definition. Given a dimension vector $\mathbf{d}$, let $\mathrm{n}_a(\mathbf{d})$ be the sum of the numbers $\mathbf{d}_b$ with b a neighbor of a.

Remark 1. If Δ is a Dynkin quiver of type $\mathbb{D}_n$ or $\mathbb{E}_m$ and M is its maximal indecomposable representation, then it is easy to see that $M|\Delta'$ is the direct sum of an antichain triple, since $\mathrm{n}_y(\mathbf{dim}\, M) = 3$.

Namely, there is the following general fact: *Let Δ be a Dynkin quiver and $a \in \Delta_0$. If N is an indecomposable representation of Δ, with $N_a \neq 0$, and $\mathrm{n}_a(\mathbf{dim}\, N) \leq 3$, then $N|\Delta \setminus \{a\}$ is the direct sum of an antichain of cardinality $\mathrm{n}_a(\mathbf{dim}\, N)$.*

Proof. If X is an indecomposable direct summand of $N|\Delta \setminus \{a\}$, then $\mathrm{n}_a(\mathbf{dim}\, X) = 1$, since $\mathrm{n}_a(\mathbf{dim}\, X) \geq 2$ would imply that $\mathcal{E}(X, S(a))$ is representation-infinite. This shows that $N|\Delta \setminus \{a\}$ is the direct sum of at most 3 indecomposable representations. In order to see that these representations form an antichain, one may, for example, refer to Kleiner's list of posets of finite type [R3].

Remark 2. *If M is the maximal indecomposable representation of the Dynkin quiver Δ and $a \in \Delta_0$, then $\mathrm{n}_a(\mathbf{dim}\, M) = 3$ if and only if Δ is of type $\mathbb{D}_n$ with $n \geq 4$ or of type $\mathbb{E}_m$ and a is the exceptional vertex.*

For the proof, we just have to calculate the numbers $\mathrm{n}_a(\mathbf{dim}\, M)$. The following table shows on the left the dimension vector $\mathbf{dim}\, M$; on the right, we display in the same way the numbers $\mathrm{n}_a(\mathbf{dim}\, M)$.

Δ	$\mathbf{dim}\, M$	$\mathrm{n}(\mathbf{dim}\, M)$
$\mathbb{A}_n$	$1 - 1 - 1 - \cdots - 1 - 1$	$1 - 2 - 2 - \cdots - 2 - 1$
$\mathbb{D}_n$	$\begin{smallmatrix}1 & \diagdown & \\ & & 2\\ 1 & \diagup & \end{smallmatrix} - \cdots - 2 - 2 - 1$	$\begin{smallmatrix}2 & \diagdown & \\ & & 4\\ 2 & \diagup & \end{smallmatrix} - \cdots - 4 - 3 - 2$
$\mathbb{E}_6$	$1 - 2 - \overset{2}{\overset{\vert}{3}} - 2 - 1$	$2 - 4 - \overset{3}{\overset{\vert}{6}} - 4 - 2$
$\mathbb{E}_7$	$2 - 3 - \overset{2}{\overset{\vert}{4}} - 3 - 2 - 1$	$3 - 6 - \overset{4}{\overset{\vert}{8}} - 6 - 4 - 2$
$\mathbb{E}_8$	$2 - 4 - \overset{3}{\overset{\vert}{6}} - 5 - 4 - 3 - 2$	$4 - 8 - \overset{6}{\overset{\vert}{12}} - 10 - 8 - 6 - 3$

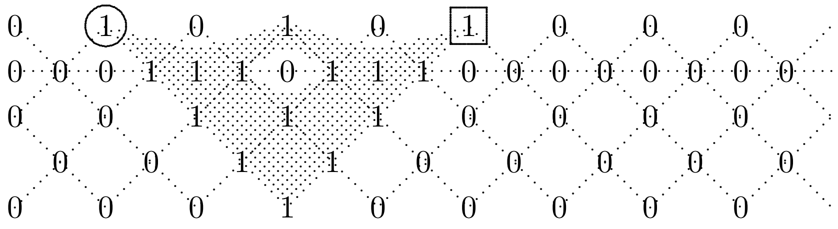
We see that the value 3 occurs in the right column only in the cases $\mathbb{D}_n$ and $\mathbb{E}_m$, and only at the exceptional vertex. Actually, for $\mathbb{D}_n$ and $\mathbb{E}_m$ the value 3 is the only odd value which occurs (and it is the only value $\mathrm{n}_a(\mathbf{dim}\, M)$ which is different from $2 \cdot \mathbf{dim}\, M_a$).

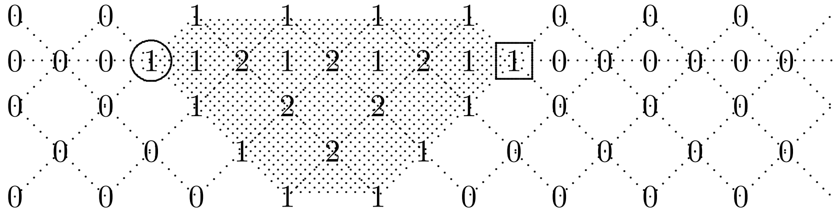
In view of Remark 1, we can say: *If Δ is of type $\mathbb{D}_n$ or $\mathbb{E}_m$, there is a unique vertex a of Δ such that* $\mathrm{n}(M|\Delta \setminus \{a\}) = 3$, *namely the exceptional vertex of* Δ.

3.2. Hammocks. Hammocks have been introduced by Brenner [Br] (for a general theory see [RV], the special case of dealing with quiver representations was already considered by Gabriel [G2]). If Q is a Dynkin quiver, and x a vertex of Q, the hammock $H(Q, x)$ is a subset of $\mathbb{Z}Q$ which is obtained using the knitting algorithm in the same way as one constructs a preprojective component (however, not using representations or dimension vectors, but just single numbers; actually, each number can be interpreted as the dimension of the vector space M_x, where M is an indecomposable preprojective representation). Here is the definition:

Knitting algorithm for the hammock function $h_{(Q,x)}$, where Q is a finite connected acyclic quiver and $x \in Q_0$. The hammock function $h = h_{(Q,x)}$ is a function $(\mathbb{Z}Q)_0 \to \mathbb{N}_0$ defined as follows: Start: Let $h(a) = 0$ for any vertex a in $\mathbb{Z}Q = Q_0 \times \mathbb{Z}$ with a proper path from a to $(x, 0)$; and let $h(x, 0) = 1$. Inductive procedure: Assume that c is a vertex of $(\mathbb{Z}Q)_0$ such that $h(\tau c)$ as well as $h(b)$ for any arrow $b \to c$ in $\mathbb{Z}Q$ are defined. Then either $-h(\tau(c)) + \sum_{\alpha:\ t(\alpha)\to c} h(s(\alpha))$ is non-negative, then this should be the value $h(c)$. Otherwise put $h(c) = 0$. By definition, the *hammock* $H(Q, x)$ is the support of the hammock function $h_{(Q,x)}$, this is a translation subquiver of $\mathbb{Z}Q$. Note that $H(Q, x)$ depends only on the underlying graph $\overline{Q}$ of the quiver Q (and not on the orientation of the edges).

Let us exhibit two hammock functions for a quiver of type $\mathbb{D}_6$:





In both examples, the corresponding hammock is shaded; the vertex $(x, 0)$ (this is the source of the hammock) is encircled, the last vertex c obtained with non-zero value $h(c)$ is marked by a square (this is the sink of the hammock).

In case $h_{(Q,x)}(y) \leq 1$ for all vertices y, we define on the set $H(Q, x)$ a relation $\leq$ as follows: $y \leq z$ provided there is a path $y = y_0 \to y_1 \to \cdots y_t = z$ in $\mathbb{Z}Q$ such that all vertices y_i belong to $H(Q, x)$. It is well-known (and easy to see) that in this way $H(Q, x)$ becomes a poset. By abuse of language, we will say that $H(Q, x)$ *is a poset* provided $h_{(Q,x)}(y) \leq 1$ for all vertices y.

If we look at the two hammock functions for a quiver of type $\mathbb{D}_6$ exhibited above, we see that the first hammock satisfies the poset condition $h_{(Q,x)}(a) \leq 1$ for all vertices of $\mathbb{Z}Q$; the shape of this poset will be shown below. The second hammock function does not satisfy the poset condition.

Hammock sets. Let $Q(1), \ldots, Q(t)$ be Dynkin quivers and let Q be the disjoint union of the quivers $Q(i)$. Let $x(i)$ be a vertex of $Q(i)$, for $1 \leq i \leq t$. The disjoint union

$$H(Q, x(1), \ldots, x(t)) = \bigsqcup\nolimits_i H(Q(i), x(i))$$

of the hammocks $H(Q(i), x(i))$ will be called a *hammock set.* In case all the hammocks $H(Q(i), x(i))$ are posets, we consider also $H(Q, x(1), \ldots, x(t))$ as a poset and we will say that $H(Q, x(1), \ldots, x(t))$ *is a poset.*

Hammock categories. Given a representation M of a quiver Q, we denote by $\mathcal{H}(M)$ the following category: its objects are the indecomposable representations X of Q and $\operatorname{Hom}_{\mathcal{H}(M)}(X, X')$ is the set of equivalence classes of maps $f \colon X \to X'$ where maps $f, f' \colon X \to X'$ are equivalent provided $\operatorname{Hom}(M, f - f') = 0$. Note that the isomorphism classes of indecomposable objects in $\mathcal{H}(M)$ are just the isomorphism classes of indecomposable representations X of Q such that $\operatorname{Hom}(M, X) \neq 0$. If $M = P(x)$, then $\mathcal{H}(M)$ is the image of the forgetful functor from rep Q to the category of k-spaces which send a representation X to X_x and a map f to f_x.

Let us consider the case $M = P(x)$, where x is a vertex of a Dynkin quiver Q. Since $\dim \operatorname{Hom}(P(x), M) = \dim M_x$, for any representation M of Q, the hammock function $h = h_{(Q,x)}$ describes $\mathcal{H}(P(x))$: *The hammock $H(Q,x)$ is the quiver of the hammock category $\mathcal{H}(P(x))$;* the vertices a of the hammock $H(Q,x)$ may be considered as the isomorphism classes $[M]$ of the indecomposable representations N of Q with $N_x \neq 0$ *and $h_{(Q,x))}(a)$ is just the dimension of the vector space N_x.*

Proof. The knitting algorithm for the hammock function is nothing else than the knitting algorithm for one of the coordinates of the dimension vector $\mathbf{dim}\, N$ of the indecomposable representations N, namely for $(\mathbf{dim}\, N)_x = \dim N_x$.

Always, the hammock $H(Q,x)$ has a unique source a, namely the isomorphism class of the representation $P(x)$. If Q is a Dynkin quiver, then $H(Q,x)$ also has a (unique) sink z, namely the isomorphism class of the indecomposable injective representation $M(z) = I(x)$. In the two $\mathbb{D}_6$-examples above, the encircled vertex a is the position of $P(x)$, the vertex z marked by a square is the position of $I(x)$.

If Q is a star quiver with center c and representation-finite, then the hammock category $\mathcal{H}(Q,c)$ is just the full subcategory of all conical representations. The second example above is of this kind.

More generally, assume that we deal with a hammock set

$$H(Q, x(1), \ldots, x(t)) = \bigsqcup\nolimits_i H(Q(i), x(i)),$$

thus $Q(1), \ldots, Q(t)$ are the connected components of a quiver Q and $x(i)$ is a vertex of the Dynkin quiver $Q(i)$, for $1 \leq i \leq t$. Then we have: *The hammock set $H(Q, x(1), \ldots, x(t))$ is the quiver of the hammock category $\mathcal{H}(\bigoplus_i P(x(i)))$.*

3.3. Quivers with a special vertex set. Let $Q(1), \ldots, Q(t)$ be Dynkin quivers and let Q be the disjoint union of the quivers $Q(i)$. Let $x(i)$ be a vertex of $Q(i)$, for $1 \leq i \leq t$. We say that $x(1), \ldots, x(t)$ is a *special vertex set of Q* provided $H(Q, x(1), \ldots, x(t))$ is a poset with precisely one antichain triple.

Proposition. *Let $Q(1), \ldots, Q(t)$ be Dynkin quivers and let Q be the disjoint union of the quivers $Q(i)$. Let $x(i)$ be a vertex of $Q(i)$, for $1 \leq i \leq t$. Then $x(1), \ldots, x(t)$ is s special vertex set if and only if one of the following cases holds:*

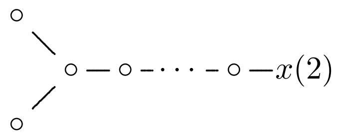
The first possibility $\mathbb{A}_1 \sqcup \mathbb{A}_1 \sqcup \mathbb{A}_1$ is

$$Q = Q(1) \sqcup Q(2) \sqcup Q(3) \quad \textit{with} \quad Q(i) = \{x(i)\} \quad \textit{for} \;\; 1 \leq i \leq 3,$$

thus all the connected components $Q(i)$ of Q are of type $\mathbb{A}_1$.

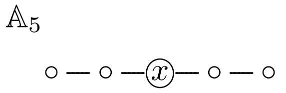
The second possibility $\mathbb{A}_1 \sqcup \mathbb{D}_m$ is

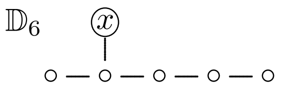
$$Q = Q(1) \sqcup Q(2) \;\; \textit{with} \;\; Q(1) = \{x(1)\} \;\; \textit{and} \;\; Q(2) =$$

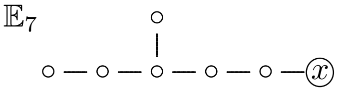


here, the component $Q(1)$ *of* Q *is of type* $\mathbb{A}_1$*, the component* $Q(2)$ *is of type* $\mathbb{D}_m$ *with* $m \geq 3$ (recall that by definition $\mathbb{D}_3 = \mathbb{A}_3$ and, in this case, $x(2)$ is the central vertex).

And finally, there are the possibilities that Q *is of type* $\mathbb{A}_5, \mathbb{D}_6, \mathbb{E}_7$ *and* $x = x(1)$ *is the following encircled vertex:*





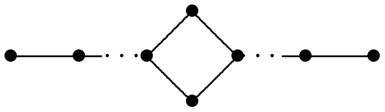


Note that *the vertices* $x(1), \ldots, x(t)$ *are determined by* Q *up to symmetry.*

Sketch of the proof. We assume that $H = H(Q, x(1), \ldots, x(t))$ is a poset with precisely one antichain triple. Since $H(Q, x(1), \ldots, x(t))$ is the disjoint union of the hammocks $H(Q(i), x(i))$, and H has width 3, we see that $t \leq 3$.

First assume that $t = 3$. We want to show that any $Q(i)$ is of type $\mathbb{A}_1$. If say $Q(t)$ is not of type $\mathbb{A}_1$, then $H(Q(t), x(t))$ has at least two elements. This shows that there are at least 2 antichain triples (using one element from $H(Q(1), x(1))$ and one from $H(Q(2), x(2))$).

Second, let $t = 2$, then we may assume that $H(Q(1), x(1))$ is a chain, whereas $H(Q(2), x(2))$ has an antichain pair. As in the case $t = 3$, we see that $H(Q(1), x(1))$ has to consist of a single vertex, thus $Q(1)$ has to be of type $\mathbb{A}_1$. In addition, $H(Q(2), x(2))$ can have only one antichain pair. Since a hammock has a unique source and a unique sink, we see that $H(Q(2), x(2))$ must have the form

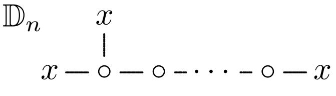


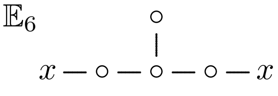
But then there are only the following possibilities: $Q(2)$ is of type $\mathbb{A}_3$ and x is the central vertex, or $Q(2)$ is of type $\mathbb{D}_4$ and x is a leaf vertex (a vertex of a quiver is a *leaf* provided it has at most one neighbor), or $Q(2)$ is of type $\mathbb{D}_n$ with $n \geq 5$ and x is the leaf vertex of the long arm. In the formulation of Proposition 3.3, these cases have been collected under the label $\mathbb{D}_m$ with $m \geq 3$.

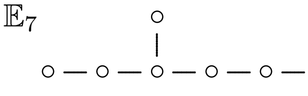
It remains to consider the case that Q is connected and $x = x(1)$ is a vertex such that the hammock $H(Q, x)$ is a poset with precisely one antichain triple. There are only few cases such that a hammock $H(Q, x)$ is a poset, namely the following cases.

First of all, Q can be of type $\mathbb{A}$ and x arbitrary. If x is a leaf vertex, then $H(Q, x)$ has width 1, if x is the neighbor of a leaf vertex (and not a leaf vertex), then $H(Q, x)$ has width 2. Otherwise, the width of $H(Q, x)$ is greater or equal to 3, as we want, however usually there are several antichain triples, the only exception is the $\mathbb{A}_5$ case.

The remaining possibilities for x are the leaf vertices of $\mathbb{D}_n$, two leaf vertices for $\mathbb{E}_6$ and one for $\mathbb{E}_7$.





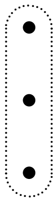


For $\mathbb{D}_4$ and x any leaf vertex, for $\mathbb{D}_5$ and x the leaf vertex of a short arm, as well as for $\mathbb{D}_n$ with $n \geq 5$ and x the leaf vertex of the long arm, the hammock $H(Q, x)$ has width 2.

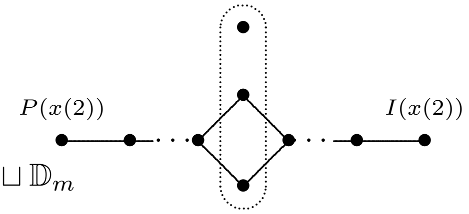
For $n > 6$ and x the leaf vertex on a short arm, the hammock $H(Q,x)$ contains several antichain triples. Finally, for $\mathbb{E}_6$, and x the leaf vertex of a long arm, the hammock $H(Q,x)$ has width 2. This shows that only the cases mentioned in the theorem remain.

Here are the posets $H(Q, x_1, \dots, x_t)$ for the various special vertex sets $x(1), \dots, x(t)$ of a quiver Q. We draw the Hasse diagram of each poset, with increase from left to right; we also may interpret the pictures as describing the quiver of $\mathcal{H}(\bigoplus P(x(i)))$, but deleting the arrow heads (all arrows point to the right, thus are northeast, east or southeast arrows).

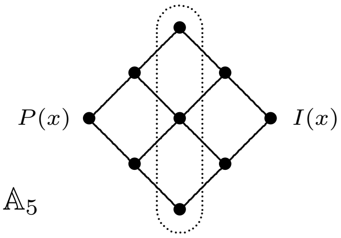
$\mathbb{A}_1 \sqcup \mathbb{A}_1 \sqcup \mathbb{A}_1$



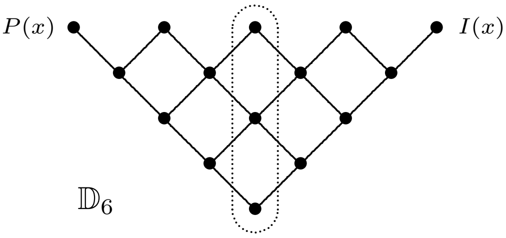
$\mathbb{A}_1 \sqcup \mathbb{D}_m$



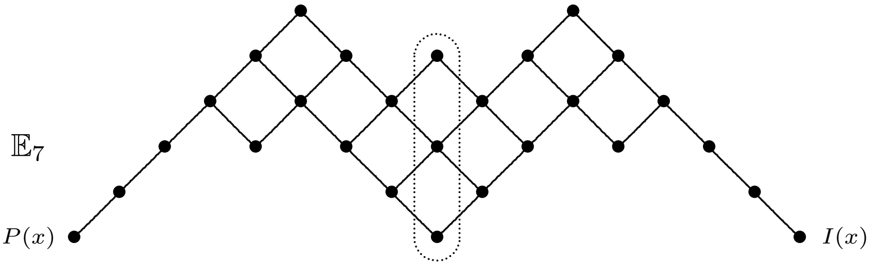
$\mathbb{A}_5$



$\mathbb{D}_6$



$\mathbb{E}_7$

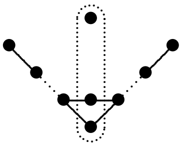


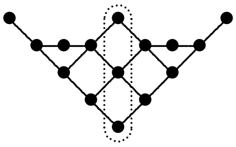
the dotted lines show the unique antichain triple. In the case $\mathbb{A}_1 \sqcup \mathbb{D}_m$, the number of elements of $H(Q, x(1), x(2))$ is $2m - 2$, and the poset is self-dual. We also should remark that we have poset embeddings

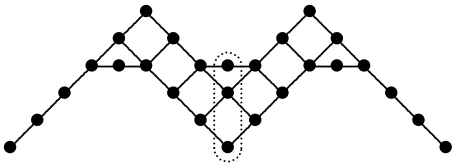
$$H(\mathbb{A}_5, x) \subset H(\mathbb{D}_6, x) \subset H(\mathbb{E}_7, x).$$

(Note that the hammock $H(\mathbb{E}_7, x)$ was used already in [R4], Example 7, with a similar aim, namely to describe the largest indecomposable representation of a quiver of type $\mathbb{E}_8$.)

Actually, if we want to stress that these posets arise as subquivers of a translation quiver of the form $\mathbb{Z}Q$, we should draw them in the cases $\mathbb{A}_1 \sqcup \mathbb{D}_m$, $\mathbb{D}_6$, $\mathbb{E}_7$ differently, namely as follows:





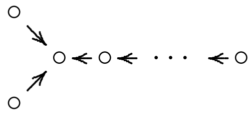


If $x(1), \dots, x(t)$ is a special vertex set of the quiver Q, the three representations $A(1), A(2), A(3)$ in $\operatorname{rep} Q$ which form an antichain triple in $H(Q, x(1), \dots, x(t))$ will be called the *special triple* of $(Q, x(1), \dots, x(t))$. The hammocks $H(Q(i), x(i))$ show:

If $A(1), A(2), A(3)$ *is the special triple of* $(Q, x(1), \dots, x(t))$*, then for* $i \neq j$*, there is no path in* $\operatorname{rep} Q$ *from* $A(i)$ *to* $A(j)$. It follows that $A(1), A(2), A(3)$ is not only an antichain triple in $H(Q, x(1), \dots, x(t))$, but we even have $\operatorname{Hom}(A(i), A(j)) = 0$ for $i \neq j$ (it is an antichain triple in $\operatorname{rep} Q$), as well as $\operatorname{Ext}^1(A(i), A(j)) = 0$ (by the previous argument for $i \neq j$, but, of course, also for $i = j$).

Let us exhibit the dimension vectors of such an antichain triple for some fixed orientation of the quiver Q. In case $Q = \mathbb{A}_1 \sqcup \mathbb{A}_1 \sqcup \mathbb{A}_1$, the special triple is given by the simple representations. In case $Q = Q(1) \sqcup Q(2)$, where $Q(1) = \mathbb{A}_1$ and $Q(2)$ is the quiver of type $\mathbb{D}_m$ with subspace orientation, the first representation $A(1)$ is the simple representation of the component of type $\mathbb{A}_1$, the remaining two representations $A(2), A(3)$ are as follows:

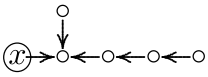
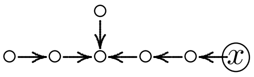
$\mathbb{D}_m$



$A(2)$ $\begin{smallmatrix}1 & \\ & 1\ 1\ 1 \cdots 1\\ 0 & \end{smallmatrix}$

$A(3)$ $\begin{smallmatrix}0 & \\ & 1\ 1\ 1 \cdots 1\\ 1 & \end{smallmatrix}$

Finally we look at quivers of type $\mathbb{A}_5, \mathbb{D}_6, \mathbb{E}_7$, again dealing with a subspace orientation:

	$\mathbb{A}_5$	$\mathbb{D}_6$	$\mathbb{E}_7$
			
$A(1)$	1 1 1 0 0	0 1 1 1 1 0	1 1 1 2 2 1 1
$A(2)$	0 1 1 1 0	1 1 2 2 1 1	2 1 2 3 2 2 1
$A(3)$	0 0 1 1 1	1 1 1 0 0 0	0 0 1 1 1 1 1

3.4. The one-point extension for a special vertex set.

The one-point extension for a special vertex set. *Let* $x(1), \dots, x(t)$ *be a special vertex set of the acyclic quiver* Q. *Then* $\Delta = Q[x(1), \dots, x(t)]$ *is a Dynkin quiver of type* $\mathbb{D}_n$ *or* $\mathbb{E}_m$ *and the extension vertex of* Δ *is its exceptional vertex and is a source.*

Conversely, let Δ *be a Dynkin quiver of type* $\mathbb{D}_n$ *or* $\mathbb{E}_m$ *and* y *its exceptional vertex. Recall that* Δ' *is obtained from* Δ *by deleting* y *and let* $x(1), \dots, x(t)$ *be the neighbors of* y *in* Δ. *Then* $x(1), \dots, x(t)$ *is a special vertex set of the quiver* Δ'.

Proof: Direct verification.

Corollary. *The Dynkin quivers of type* $\mathbb{D}_n$ *or* $\mathbb{E}_m$ *are representation-finite.*

Proof. Let Δ be of a quiver of type $\mathbb{D}_n, \mathbb{E}_6, \mathbb{E}_7$ or $\mathbb{E}_8$. Changing if necessary the orientation, we may assume that the vertex y is a source. Thus, Δ is the one-point extension

$\Delta'[x(1), \ldots, x(t)]$ of Δ', where $x(1), \ldots, x(t)$ is a special vertex set. Its representation type is determined by the poset $H(\Delta', x(1), \ldots, x(t))$, see for example [R2] or [R3], sections 2.5 and 2.6. Since $H(\Delta', x(1), \ldots, x(t))$ has a unique antichain triple, $\Delta'[x(1), \ldots, x(t)]$ has to be representation-finite.

3.5. The indecomposable representations of a Dynkin quiver of type $\mathbb{D}_n$ and $\mathbb{E}_m$. As we will see, the poset $H(\Delta', x(1), \ldots, x(t))$ provides an easy way to construct explicitly all the indecomposable representations of Δ. Let us distinguish 6 different kinds:

(1) The indecomposable representations Y of Δ with $N_y = 0$ and $N_{x(i)} = 0$ for all $1 \le i \le t$, these are just the indecomposable representations of the quiver Δ''.
(2) The indecomposable representations Y of Δ with $N_y = 0$ and $N_{x(i)} \neq 0$ for some $1 \le i \le t$; these are the representations which belong to the hammocks $H(\Delta', x(i))$ with $1 \le i \le t$.
(3) For any representation N in $H(\Delta', x(1), \ldots, x(t))$, there is an indecomposable representation Y of Δ with $Y|\Delta' = N$ and $\dim \overline{Y}_y = 1$.
(4) For any antichain pair $B(1), B(2)$ in $H(\Delta', x(1), \ldots, x(t))$, there is an indecomposable representation Y of Δ with $Y|\Delta' = B(1) \oplus B(2)$ and $\dim Y_y = 1$.
(5) Two representations Y of Δ with $Y|\Delta' = A(1) \oplus A(2) \oplus A(3)$ such that $\dim Y_y$ is equal to 1 or 2.
(6) The simple representation $S(y)$.

Here is the table which lists the number of indecomposable representations Y of Δ of the various kinds. Given a representation N of any quiver, we denote by $\mathrm{m}(N)$ the number of direct summands when N is written as a direct sum of indecomposable representations. The upper three rows name the types of Δ, of $\Delta' = \Delta \setminus \{y\}$ and of $\Delta'' = \Delta \setminus \{x, y\}$.

	Δ	$\mathbb{D}_n$ $(n \ge 4)$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$
	Δ'	$\mathbb{D}_{n-2}$	$\mathbb{A}_5$	$\mathbb{D}_6$	$\mathbb{E}_7$
	Δ''	$\mathbb{D}_{n-3}$	$\mathbb{A}_2 \sqcup \mathbb{A}_2$	$\mathbb{A}_5$	$\mathbb{E}_6$
(1)	$Y \in \operatorname{rep} \Delta''$	$(n-4)(n-3)$	6	15	36
(2)	$Y \in \operatorname{rep} \Delta' \setminus \operatorname{rep} \Delta''$	$2n-5$	9	15	27
(3)	$\mathrm{m}(Y\|\Delta') = 1,\ Y_y \neq 0$	$2n-5$	9	15	27
(4)	$\mathrm{m}(Y\|\Delta') = 2$	$2n-5$	9	15	27
(5)	$\mathrm{m}(Y\|\Delta') = 3$	2	2	2	2
(6)	$Y\|\Delta' = 0$	1	1	1	1
sum		$(n-1)n$	36	63	120

The Dynkin graphs $\mathbb{D}_n$ are usually considered only for $n \ge 4$. But our induction procedure requires to define also $\mathbb{D}_n$ with $n = 1, 2, 3$: we set $\mathbb{D}_1 = \emptyset$, $\mathbb{D}_2 = \mathbb{A}_1 \sqcup \mathbb{A}_1$ and $\mathbb{D}_3 = \mathbb{A}_3$.

It is clear that the numbers in the rows (2) and (3) coincide: this number is just the cardinality of the vertex set of the poset $H(\Delta', x(1), \dots, x(t))$. However, it seems to be a curious coincidence that in all cases the number of vertices of the poset $H(\Delta', x(1), \dots, x(t))$ is equal to the number of antichain pairs in $H(\Delta', x(1), \dots, x(t))$ (thus, that the number of the representations in row (4) is equal to the number in row (2) and (3)).

The indecomposable representation Y in (5) with $Y|\Delta' = A(1) \oplus A(2) \oplus A(3)$ and $\dim Y_y = 3$ is the maximal indecomposable representation of Δ, we usually label it M.

3.6. The special antichain triple of a Dynkin quiver of type $\mathbb{D}_n$ and $\mathbb{E}_m$. Let Δ be a Dynkin quiver of type $\mathbb{D}_n$ or $\mathbb{E}_m$. Let y be its exceptional vertex. By definition, Δ' is obtained from Δ by deleting y and we denote by $x(1), \dots, x(t)$ the neighbors of y in Δ. Then $x(1), \dots, x(t)$ is a special vertex set of the quiver Δ', thus there is a unique antichain triple $A(1), A(2), A(3)$ in $H(\Delta', x(1), \dots, x(t))$. We call $A(1), A(2), A(3)$ the *special antichain triple* of Δ. The previous considerations show that the special antichain triple is the unique antichain triple $A(1), A(2), A(3)$ in $\operatorname{rep}\Delta'$ such that for $1 \le i \le 3$, there is at least one vertex x in the set $\{x(1), \dots, x(t)\}$ with $A(i)_x \neq 0$. To say that $A(i)$ is a representation of Δ' is the same as to say that it is a representation of Δ with $A(i)_y = 0$. In this way, we have shown the main assertion of Theorem 3.1.

Of course, we know that for any i, there is just one vertex x in the set $\{x(1), \dots, x(t)\}$ with $A(i)_x \neq 0$ and for this vertex x, we have $\dim A(i)_x = 1$. Also, we know that for $i \neq j$ there is no path in $\operatorname{rep}\Delta'$ from $A(j)$ to $A(i)$. In particular, we have $\operatorname{Ext}^1(A(i), A(j)) = 0$. Since also $\operatorname{Ext}^1(X, X) = 0$ for any indecomposable representation of Δ', we see that $\operatorname{Ext}^1(A(i), A(j)) = 0$ for all i, j.

In order to complete the proof of Theorem 3.1, it remains to be seen that $M|\Delta' = A(1) \oplus A(2) \oplus A(3)$. This will be shown now.

Let us consider the special antichain triple in more detail.

Theorem. *Let Δ be a Dynkin quiver of type $\mathbb{D}_n$ or $\mathbb{E}_m$ with exceptional vertex y and special antichain triple $A(1), A(2), A(3)$. The* Ext*-quiver $\mathcal{C}(\Delta)$ of*

$$\mathcal{C}(\Delta) = \mathcal{E}(A(1), A(2), A(3), S(y))$$

has the following form

$$\begin{tikzcd}
A(1) \arrow[rd, dash] & \\
A(2) \arrow[r, dash] & S(y) \\
A(3) \arrow[ru, dash] &
\end{tikzcd}$$

It is of type $\mathbb{D}_4$ and $\mathcal{C}(\Delta)$ contains the maximal indecomposable representation M of Δ.

As an element of $\mathcal{C}$, the representation M has dimension vector

$$\mathbf{dim}_{\mathcal{C}}\, M = \begin{smallmatrix} 1 & \\ 1 & 2 \\ 1 & \end{smallmatrix} \; ,$$

thus $M|\Delta' = A(1) \oplus A(2) \oplus A(3)$ and $\dim M_y = 2$.

Proof. In order to determine the quiver $C(\Delta)$, we first note that the indecomposable representations of a Dynkin quiver have no self-extensions, thus there is no loop in $C(\Delta)$. Second, since the representations $A(1), A(2), A(3)$ are Ext-orthogonal, there is no arrow between these representations. Finally, for any i, there is a unique neighbor x of y with $A(i)_x \neq 0$, and one has $\dim A(i)_x = 1$. In case the arrow between x and y points to y, we have $\dim \operatorname{Ext}^1(A(i), S(y)) = 1$ and $\operatorname{Ext}^1(S(y), A(i)) = 0$. Otherwise we have $\dim \operatorname{Ext}^1(S(y), A(i)) = 1$ and $\operatorname{Ext}^1(A(i), S(y)) = 0$.

It remains to look at the maximal indecomposable representation M of Δ. It is the only indecomposable representation of Δ with $\dim M_y = 2$ (since y is the exceptional vertex of Δ). Since $\mathcal{C}(\Delta)$ is of type $\mathbb{D}_4$, there is an indecomposable object X in $\mathcal{C}(\Delta)$ with dimension vector $\mathbf{dim}_{\mathcal{C}}\, X = \begin{smallmatrix} 1 & \\ 1 & 2 \\ 1 & \end{smallmatrix}$. In particular, X is an indecomposable representation with $\dim X_y = 2$, thus X is isomorphic to M.

As we have seen in the proof, there is the following **recipe** for obtaining the orientation of the quiver $C(\Delta)$: For $1 \leq i \leq 3$, the orientation of the edge between $[A(i)]$ and $[S(y)]$ is the same as the orientation between x and y in Δ, where $A(i)_x \neq 0$.

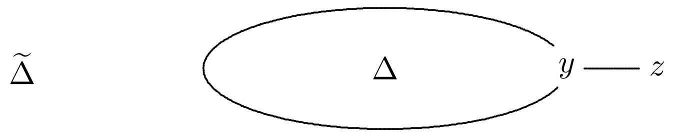
We call $\mathcal{C}(\Delta)$ the *core* of $\operatorname{rep}\Delta$.

Corollary. *If Δ is a Dynkin quiver of type $\mathbb{E}_m$, then the core of* $\operatorname{rep}\Delta$ *is the smallest thick subcategory of* $\operatorname{rep}\Delta$ *which contains M and $S(y)$.*

Proof. Of course, the core $\mathcal{C}(\Delta)$ contains both M and $S(y)$. Conversely, assume that $\mathcal{T}$ is a thick subcategory which contains M and $S(y)$. If y is a source, then $A(1) \oplus A(2) \oplus A(3)$ is the kernel of a map $M \to S(y)^2$, thus $A(1), A(2), A(3)$ belong to $\mathcal{T}$. A similar agument works if y is a sink.

3.7. The corresponding Euclidean quivers.

We consider now pairs $\Delta \subset \widetilde{\Delta}$, where Δ is a Dynkin quiver of type $\mathbb{D}_n$ or $\mathbb{E}_m$ with exceptional vertex y, and $\widetilde{\Delta}$ is a corresponding Euclidean quiver, thus there is a unique vertex z of $\widetilde{\Delta}$ outside of Δ and z is a neighbor of y



(the orientation of the arrow between y and z is not fixed).

Theorem. *Let $A(1), A(2), A(3)$ be the special antichain triple for Δ and let*

$$\mathcal{C}_z(\widetilde{\Delta}) = \mathcal{E}(A(1), A(2), A(3), S(y), S(z)).$$

Then, the Ext-*quiver of* $\mathcal{C}_z(\widetilde{\Delta})$ *has the following form*

$$\begin{tikzcd}
A(1) \arrow[dr, dash] & & \\
A(2) \arrow[r, dash] & S(y) \arrow[r, dash] & S(z) \\
A(3) \arrow[ur, dash] & &
\end{tikzcd}$$

It is of type $\widetilde{\mathbb{D}}_4$ *and the subcategory* $\mathcal{C}_z(\widetilde{\Delta})$ *contains an element from any* τ*-orbit of simple regular representations of* $\widetilde{\Delta}$*.*

We should remark that actually $\mathcal{C} = \mathcal{C}_z(\widetilde{\Delta})$ only depends on y, not on z itself: For quivers of type $\widetilde{\mathbb{D}}_4$, there is just one subcategory $\mathcal{C}$, namely $\operatorname{rep}\Delta$ itself (but four different choices of z). For quivers of type $\widetilde{\mathbb{D}}_n$, with $n \geq 5$, there are two subcategories of the form $\mathcal{C}$ (but again four different choices of z). For the quivers of type $\widetilde{\mathbb{E}}_m$, the possible vertices z correspond bijectively to the vertices y (there are three choices for $\widetilde{\mathbb{E}}_6$, two choices for $\widetilde{\mathbb{E}}_7$, and only one choice for $\widetilde{\mathbb{E}}_8$).

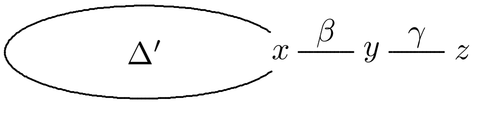
Proof of the Theorem. We know from Theorem 3.6 the shape of the Ext-quiver $C(\Delta)$. Clearly, the Ext-quiver for $\mathcal{C}_z(\widetilde{\Delta})$ is obtained from $C(\Delta)$ by adding the vertex $[S(z)]$ and an arrow between $[S(y)]$ and $[S(z)]$. The orientation of the edge between $[S(y)]$ and $[S(z)]$ is the same as the orientation of the edge between y and z in the quiver Δ.

It remains to show that any τ-orbit of a simple regular representation N of $\widetilde{\Delta}$ contains a representation in $\mathcal{C}_z(\widetilde{\Delta})$. If N is homogeneous (that means $\tau N = N$), then $N|\Delta$ is a direct sum of copies of the maximal indecomposable representation M. Since M belongs to $\mathcal{C}(\Delta)$, we see that N belongs to $\mathcal{C}_z(\widetilde{\Delta})$. Thus, we now may assume that N is exceptional. We distinguish according to the number t of connected components of Δ'.

The types $\widetilde{\mathbb{D}}_4$ (the case $t = 3$). In case $\widetilde{\Delta}$ is of type $\widetilde{\mathbb{D}}_4$, we have $\mathcal{C}_z(\widetilde{\Delta}) = \operatorname{rep}\widetilde{\Delta}$ (since $\operatorname{rep}\widetilde{\Delta}$ is its only thick subcategory of rank 5, see for example [R5]), thus nothing has to be shown in this case.

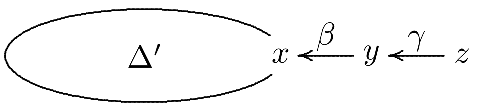
The types $\widetilde{\mathbb{E}}_m$ (the cases $t = 1$). Here, $\widetilde{\Delta}$ has the following shape:

$\widetilde{\Delta}$



with $\{x\}$ a special vertex set of Δ'. Using BGP reflection functors at vertices $a \in \widetilde{\Delta}_0 \setminus \{x, y\}$, we may change the orientation of all the arrows different from β. Under such a change of orientation, the image of the special antichain triple of Δ will again be a special antichain triple. In addition, using duality, we may replace the direction of β by the opposite direction. Altogether, we see that it is sufficient to look at a single orientation, say at the subspace orientation as shown below (with Δ' the full subquiver with vertex set $\widetilde{\Delta}_0 \setminus \{y, z\}$). In particular, the edges β and γ are oriented as follows:

$\widetilde{\Delta}$



For such an orientation, the Ext-quiver R of $\mathcal{C} = \mathcal{C}_z(\widetilde{\Delta})$ is as follows:

R

$$\begin{tikzcd} A(1) & & \\ A(2) & S(y) \arrow[l] \arrow[ul] \arrow[dl] & S(z) \arrow[l] \\ A(3) & & \end{tikzcd}$$

Given a representation X of Δ', we denote by $\overline{X}$ the representation of $\widetilde{\Delta}$ with $\overline{X}|\Delta' = X$, $\overline{X}_\beta \colon \overline{X}_y \to \overline{X}_x = X_x$ the identity map, and $\overline{X}_z = 0$. There is the following interesting observation: *Let X be an indecomposable representation of Δ'. Then $\overline{X}$ is a regular representation of $\widetilde{\Delta}$ if and only if X is one of the representations $A(1), A(2), A(3)$.*

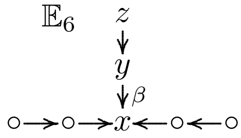
As elements of the subcategory $\mathcal{C}$, the representations $\overline{A(i)}$ have the dimension vectors

$$\mathbf{dim}_{\mathcal{C}}\, \overline{A(1)} = \begin{smallmatrix} & 1 & \\ 0 & 1 & 0 \\ & 0 & \end{smallmatrix}, \quad \mathbf{dim}_{\mathcal{C}}\, \overline{A(2)} = \begin{smallmatrix} & 0 & \\ 1 & 1 & 0 \\ & 0 & \end{smallmatrix}, \quad \mathbf{dim}_{\mathcal{C}}\, \overline{A(3)} = \begin{smallmatrix} & 0 & \\ 0 & 1 & 0 \\ & 1 & \end{smallmatrix};$$

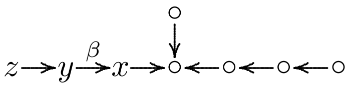
these are simple regular objects of $\mathcal{C}$.

At the end of section 3.3, the antichain triple $A(1), A(2), A(3)$ has been exhibited for our quivers Δ'. Thus, the representations $\overline{A(i)}$ are as follows:

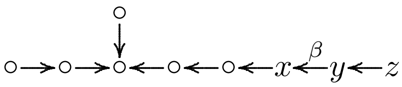
$\widetilde{\mathbb{E}}_6$



$\widetilde{\mathbb{E}}_7$



$\widetilde{\mathbb{E}}_8$

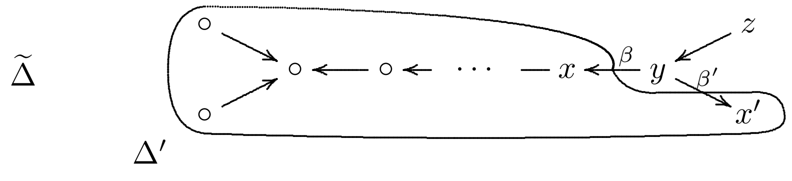


	$\widetilde{\mathbb{E}}_6$	$\widetilde{\mathbb{E}}_7$	$\widetilde{\mathbb{E}}_8$
$\overline{A(1)}$	$\begin{array}{c} 0 \\ 1 \\ 1\ 1\ 1\ 0\ 0 \end{array}$	$\begin{array}{c} 0 \\ 0\ 1\ 1\ 1\ 1\ 1\ 0 \end{array}$	$\begin{array}{c} 1 \\ 1\ 1\ 2\ 2\ 1\ 1\ 1\ 0 \end{array}$
$\overline{A(2)}$	$\begin{array}{c} 0 \\ 1 \\ 0\ 1\ 1\ 1\ 0 \end{array}$	$\begin{array}{c} 1 \\ 0\ 1\ 1\ 2\ 2\ 1\ 1 \end{array}$	$\begin{array}{c} 2 \\ 1\ 2\ 3\ 2\ 2\ 1\ 1\ 0 \end{array}$
$\overline{A(3)}$	$\begin{array}{c} 0 \\ 1 \\ 0\ 0\ 1\ 1\ 1 \end{array}$	$\begin{array}{c} 1 \\ 0\ 1\ 1\ 1\ 0\ 0\ 0 \end{array}$	$\begin{array}{c} 0 \\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 0 \end{array}$

The essential claim is the following: *The representations $\overline{A(i)}$ of $\widetilde{\Delta}$ are simple regular and belong to pairwise different τ-orbits. For $i = 1$, the τ-period of $\overline{A(i)}$ in* rep $\widetilde{\Delta}$ *is 3, for $i = 2$, it is 2, and for $i = 3$ it is $3, 4, 5$ in the cases $\widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$, respectively.*

For a proof, one should write down the corresponding τ-orbits. Actually, in case we deal with a star quiver (as we do), an indecomposable regular representation which is thin has to be simple regular (and all but three of the listed representations are thin).

The types $\widetilde{\mathbb{D}}_n$ with $n \geq 5$ (the cases $t = 2$). We can use a similar procedure as in the cases $\widetilde{\mathbb{E}}_m$. We consider the quiver $\widetilde{\Delta}$ with the following orientation:



(since it is sufficient to work up to duality, we may deal with a fixed orientation of the arrow β between x and y and we may use BGP reflection functors at vertices $a \in \widetilde{\Delta}_0 \setminus \{x, y\}$, in order to change the orientation of all the arrows different from β).

The special antichain triple of Δ' is given by the following representations of $\widetilde{\Delta}$:

$$\mathbf{dim}\, A(1) = \begin{smallmatrix} 1 & & & 0 \\ & 1 \cdots 1 & 0 & \\ 0 & & & 0 \end{smallmatrix}, \quad \mathbf{dim}\, A(2) = \begin{smallmatrix} 0 & & & 0 \\ & 1 \cdots 1 & 0 & \\ 1 & & & 0 \end{smallmatrix}, \quad \mathbf{dim}\, A(3) = \begin{smallmatrix} 0 & & & 0 \\ & 0 \cdots 0 & 0 & \\ 0 & & & 1 \end{smallmatrix}.$$

Thus, it turns out that the Ext-quiver of $\mathcal{C} = \mathcal{C}_z(\Delta)$ is again R and we consider again the representations $\overline{A(i)}$ of $\widetilde{\Delta}$ which coincide on Δ' with $A(i)$, with $\overline{A(1)}_\beta$, $\overline{A(2)}_\beta$, $\overline{A(3)}_{\beta'}$ the identity map and with $\overline{A(i)}_z = 0$:

$$\mathbf{dim}\, \overline{A(1)} = \begin{smallmatrix} 1 & & & 0 \\ & 1 \cdots 1 & 1 & \\ 0 & & & 0 \end{smallmatrix}, \quad \mathbf{dim}\, \overline{A(2)} = \begin{smallmatrix} 0 & & & 0 \\ & 1 \cdots 1 & 1 & \\ 1 & & & 0 \end{smallmatrix}, \quad \mathbf{dim}\, \overline{A(3)} = \begin{smallmatrix} 0 & & & 0 \\ & 0 \cdots 0 & 1 & \\ 0 & & & 1 \end{smallmatrix}$$

Again, we see: *The representations $\overline{A(i)}$ of $\widetilde{\Delta}$ are simple regular and belong to pairwise different τ-orbits. The τ-period of $\overline{A(1)}$ and $\overline{A(2)}$ in* $\operatorname{rep} \widetilde{\Delta}$ *is* 2, *the τ-period of $\overline{A(3)}$ in* $\operatorname{rep} \widetilde{\Delta}$ *is $n - 2$.*

We call $\mathcal{C}_y(\widetilde{\Delta})$ the *y-core* of $\operatorname{rep} \widetilde{\Delta}$.

Corollary. *If Δ is a Dynkin quiver of type $\mathbb{E}_m$, then the y-core of* $\operatorname{rep} \widetilde{\Delta}$ *is the smallest thick subcategory of* $\operatorname{rep} \widetilde{\Delta}$ *which contains $M, S(y), S(z)$.*

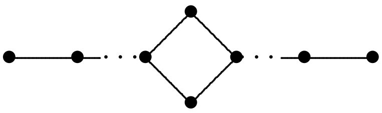
3.8. The quivers Δ'' for Δ a Dynkin quiver of type $\mathbb{D}_n$ and $\mathbb{E}_m$.

Theorem. *Let Δ be a Dynkin quiver and M its maximal indecomposable representation. If Δ is of type $\mathbb{D}_4$, then $M|\Delta'' = 0$. If Δ is of type $\mathbb{D}_n$ with $n \geq 5$, then $M|\Delta''$ is the direct sum of an antichain pair. If Δ is of type $\mathbb{E}_m$, then $M|\Delta''$ is the direct sum of four indecomposable representations U, V, U', V' with* $\operatorname{Hom}(U, V) \neq 0$, $\operatorname{Hom}(U', V') \neq 0$ *and* $\dim \operatorname{End}(M|\Delta'') = 6$.

The proof is similar to the proof of Theorem 3.1. We indicate the main steps. Details are left to the reader.

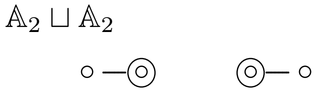
Proposition 1. *Let $Q(1), \ldots, Q(t)$ be Dynkin quivers and let Q be the disjoint union of the quivers $Q(i)$. Let $x(i)$ be a vertex of $Q(i)$, for $1 \le i \le s$. Then $H(Q, w(1), \ldots, w(s)$ has precisely one antichain pair if and only if one of the following cases holds: $Q = \mathbb{A}_1 \sqcup \mathbb{A}_1$ and $w(1), w(2)$ are the two vertices, or $Q = \mathbb{A}_3$ and $w(1), w(2)$ the two leaves, or $Q = \mathbb{D}_n$ with $n \ge 5$ and $w = w(1)$ is the leaf of the long arm.*

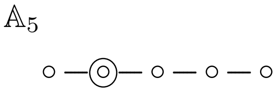
In all these cases, we have $Q = \mathbb{D}_n$ with $n \ge 2$, where $\mathbb{D}_2 = \mathbb{A}_1 \sqcup \mathbb{A}_1$ and $\mathbb{D}_3 = \mathbb{A}_3$. The hammock set for $\mathbb{D}_n$ mentioned in Proposition 1 is the following self-dual poset with $2n - 2$ vertices:

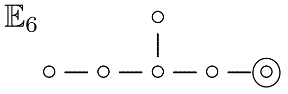


Given a poset P, a $(2,2)$-*set* in P is by definition a full subposet which is the disjoint union of two chains of cardinality 2. (More generally, an $(n_1, \ldots, n_t)$-set in P is a full subposet of P which is the disjoint union of chains of cardinalities $n_1, \ldots, n_t$. In particular, an antichain in P is a $(1, 1, \ldots, 1)$-set in P.)

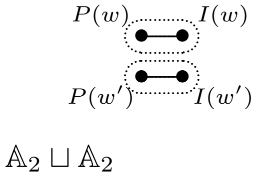
Proposition 2. *Let $Q(1), \ldots, Q(t)$ be Dynkin quivers and let Q be the disjoint union of the quivers $Q(i)$. Let $w(i)$ be a vertex of $Q(i)$, for $1 \le i \le s$. Then $H(Q, w(1), \ldots, w(s))$ has width 2 and contains a unique $(2,2)$-set if and only Q is a quiver of the following form, and $w(1), \ldots, w(s)$ are the encircled vertices:*

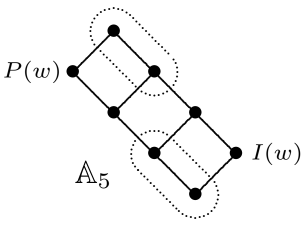


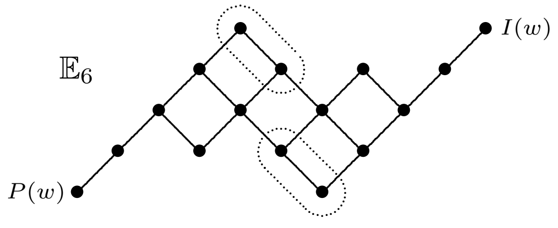




Here are the hammock sets $H(Q, w(1), \ldots, w(s))$ of width 2 with precisely one $(2,2)$-set. Of course, $s \le 2$ and we write $w = w(1)$ and $w' = w(2)$.



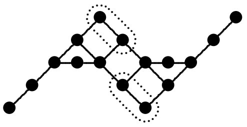




The unique $(2,2)$-subsets are encircled by dotted lines. Again, we have poset embeddings:

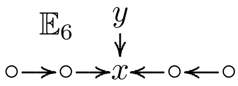
$$H(\mathbb{A}_2 \sqcup \mathbb{A}_2, w, w') \subset H(\mathbb{A}_5, w) \subset H(\mathbb{E}_6, w).$$

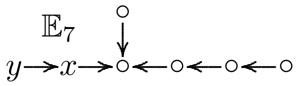
If we want to visualize that the poset $H(\mathbb{E}_6, w)$ is a subquiver of $\mathbb{Z}\mathbb{E}_6$, we should draw it slightly different, namely as follows:

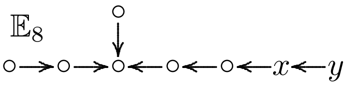


Given one of these hammock sets, the representations belonging to the $(2,2)$-subset will be denoted by U, V, U', V' with $\operatorname{Hom}(U,V) \neq 0$, $\operatorname{Hom}(U',V') \neq 0$. Of course, in the cases $\mathbb{A}_2 \sqcup \mathbb{A}_2$ and $\mathbb{A}_5$, the representations U, V, U', V' are thin (as all indecomposable representations of a quiver of type $\mathbb{A}$). For Q of type $\mathbb{E}_6$, two of the representations U, V, U', V' are thin, the remaining two are representations N with $\dim N_c = 2$, where c is the central vertex of Q.

If we consider the quivers of type $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ with subspace orientation, the representations U, V, U', V' are as follows:







U	0 0 1 0 0 0	1 0 0 1 0 0 0	0 0 1 1 1 1 0 0
V	0 1 1 0 0 0	1 0 0 1 1 0 0	1 1 2 2 1 1 0 0
U'	0 0 0 0 1 0	0 0 0 1 1 1 0	1 1 1 2 2 1 0 0
V'	0 0 0 0 1 1	0 0 0 1 1 1 1	1 0 0 1 1 1 0 0

Let us complete the proof of Theorem 3.8. There are no problems if Δ is of type $\mathbb{D}_n$, thus we assume that Δ is of type $\mathbb{E}_m$. We are going to discuss the relationship between the representations of Δ' and of Δ''. The quiver Δ'' is of the form $\mathbb{A}_2 \sqcup \mathbb{A}_2$, $\mathbb{A}_5$, $\mathbb{E}_6$ and the neighbors in Δ'' of the vertex $x \in \Delta_0$ are just the vertices $w(1), \dots, w(s)$ mentioned in Proposition 2. We write $H(\Delta'')$ instead of $H(\Delta'', w(1), \dots, w(s))$, thus, for Δ'' of type $\mathbb{A}_2 \sqcup \mathbb{A}_2$, we have $H(\Delta'') = H(\Delta'', w, w')$, and for Δ'' of type $\mathbb{A}_5$ and $\mathbb{E}_6$, we have $H(\Delta'') = H(\Delta'', w)$.

Given a representation N in $H(\Delta'')$, we denote by $\overline{N}$ the indecomposable representation of Δ' with $\overline{N}|\Delta'' = N$ and $\overline{N}_x \neq 0$ (thus $\dim \overline{N}_x = 1$). Given an antichain pair N, N' in $H(\Delta'')$, we denote by $\overline{NN'}$ the indecomposable representation of Δ' with $\overline{NN'}|\Delta'' = N \oplus N'$ (and $\dim \overline{NN'}_x = 1$). The representations of Δ' of the form $\overline{N}$ and $\overline{NN'}$ with $N, N' \in H(\Delta'')$ together with $S(x)$ form the hammock $H(\Delta', x)$.

Let us describe inside the hammock $H(\Delta', x)$ the subset $\mathcal{X}$ given by the indecomposable representations X of Δ' with $X|\Delta'' \in \operatorname{add}\{U, V, U', V'\}$. This depends on the orientation of the arrows outside of Δ''. Up to duality, we can assume that the arrow between x and w points to w. In the case of Δ'' being of type $\mathbb{A}_2 \sqcup \mathbb{A}_2$ we have to distinguish the two cases $x \to w'$ (thus x is a source) and $x \leftarrow w'$.

First, let us assume that x is a source. Then $\mathcal{X}$ looks as follows:

$$\begin{tikzcd}
 & & \overline{U} \arrow[dr] & & \\
 & \overline{UV'} \arrow[ur] \arrow[dr] & & \overline{V} \arrow[dr] & \\
\overline{UU'} \arrow[ur] \arrow[dr] & & \overline{VV'} \arrow[ur] \arrow[dr] & & S(x) \\
 & \overline{VU'} \arrow[ur] \arrow[dr] & & \overline{V'} \arrow[ur] & \\
 & & \overline{U'} \arrow[ur] & &
\end{tikzcd}$$

Second, we assume that Δ'' is of type $\mathbb{A}_2 \sqcup \mathbb{A}_2$ and that there is the subquiver $w \leftarrow x \leftarrow w'$ in Δ'. Then $\mathcal{X}$ looks as follows:

$$\begin{tikzcd}
 & & S(x) \arrow[dr] & & \\
 & \overline{V} \arrow[ur] \arrow[dr] & & \overline{U'} \arrow[dr] & \\
\overline{U} \arrow[ur] \arrow[dr] & & \overline{VU'} \arrow[ur] \arrow[dr] & & \overline{V'} \\
 & \overline{UU'} \arrow[ur] \arrow[dr] & & \overline{VV'} \arrow[ur] & \\
 & & \overline{UV'} \arrow[ur] & &
\end{tikzcd}$$

Always we see that the subset $\mathcal{X}$ of $H(\Delta', x)$ contains an antichain triple. But as we know $H(\Delta', x)$ has a unique antichain triple, namely the representations $A(1), A(2), A(3)$. This shows that

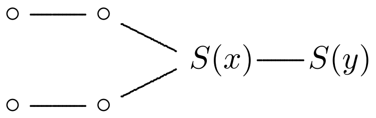
$$\{A(1), A(2), A(3)\} = \begin{cases} \{\overline{U}, \overline{VV'}, \overline{U'}\}, & \text{if } x \text{ is a source,} \\ \{S(x), \overline{U'V}, \overline{UV'}\} & \text{for } w \leftarrow x \leftarrow w'. \end{cases}$$

Of course, in both cases we have

$$A(1) \oplus A(2) \oplus A(3) \mid \Delta'' = U \oplus V \oplus U' \oplus V',$$

this completes the proof of Theorem 3.8.

Proposition 3. *Let Δ be a quiver of type $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ with exceptional vertex y and x the neighbor of y. Then there is a unique thick subcategory of* rep Δ *which contains $S(x)$ and $S(y)$ such that its* Ext*-quiver is of the following form*



Remark. In case Δ is of type $\mathbb{E}_6$, there is only one thick subcategory of $\operatorname{rep}\Delta$ with 6 simple objects, namely $\operatorname{rep}\Delta$ itself, thus nothing has to be shown.

Proof. It is sufficient to observe that the smallest thick subcategories containing U, V of U', V', respectively, both are of type $\mathbb{A}_2$.

Let us add a table which provides for any quiver Δ of Dynkin type $\mathbb{D}_n$ or $\mathbb{E}_m$ the number of indecomposable representations X of Δ' of the various kinds (similar to the table in section 3.5).

Δ'	$\mathbb{D}_{n-2}$	$\mathbb{A}_5$	$\mathbb{D}_6$	$\mathbb{E}_7$
Δ''	$\mathbb{D}_{n-3}$	$\mathbb{A}_2 \sqcup \mathbb{A}_2$	$\mathbb{A}_5$	$\mathbb{E}_6$
$X \in \operatorname{rep}\Delta''$	$(n-4)(n-3)$	6	15	36
$\mathrm{m}(X\|\Delta'') = 1,\ X_x \neq 0$	$2n-8$	4	8	16
$\mathrm{m}(X\|\Delta'') = 2$	1	4	6	10
$X\|\Delta'' = 0$	1	1	1	1
sum	$(n-3)(n-2)$	15	30	63

Of course, the number of indecomposable representations X of Δ' with $\mathrm{m}(X|\Delta'') = 1$ and $X_x \neq 0$ is just the number of vertices of the hammock set $H(\Delta'')$, whereas the number of indecomposable representations X of Δ' with $\mathrm{m}(X|\Delta'') = 2$ is the number of antichain pairs in $H(\Delta'')$.

3.9. Invariants with value 2, 4, 8**.** Let Δ be a Dynkin quiver of type $\mathbb{E}_m$ with exceptional vertex y. Let x be the neighbor of y and $\Delta' = \Delta \setminus \{y\}, \Delta' = \Delta \setminus \{x, y\}$.

Let $P'(x)$ be the indecomposable projective representation of Δ' corresponding to the vertex x, let $I'(x)$ be the indecomposable injective representation of Δ' corresponding to the vertex x. Let τ' be the Auslander-Reiten translation in $\operatorname{rep}\Delta'$. *Then $P'(x)$ and $I'(x)$ belong to the same τ'-orbit and there is the following* $2-4-8$ *assertion:*

$$P'(x) = \begin{cases} (\tau')^2 I'(x) \\ (\tau')^4 I'(x) \\ (\tau')^8 I'(x) \end{cases} \text{for } \Delta' \text{ of type} \begin{cases} \mathbb{A}_5\,, \\ \mathbb{D}_6\,, \\ \mathbb{E}_7\,, \end{cases} \text{thus for } \Delta \text{ of type} \begin{cases} \mathbb{E}_6\,, \\ \mathbb{E}_7\,, \\ \mathbb{E}_8\,. \end{cases}$$

Actually, also the number of vertices of the hammock $H(\Delta', x)$ is related to this number $r = 2, 4, 8$, namely it is $3(r+1)$ (and, as we have seen, the number of vertices of the hammock coincides with the number of antichain pairs in the hammock).

Already the quiver Δ'' with its vertices $w(1), \dots, w(s)$ (the neighbors of x in Δ') provides a $2-4-8$ assertion. Namely, the number of vertices of the hammock set $H(\Delta'')$ is equal to $2r$, with $r = 2, 4, 8$ for Δ of type $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$, respectively.

3.10. Final Remarks for Part 3. It is a well-known procedure to use one-point extensions (and vector space categories) in order to determine the representation type of quivers, in particular to deal with the quivers of type $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$. A usual way seems to be to use a sequence of one-point extensions starting with $\mathbb{D}_5$, and adding successively vertices to obtain $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$. Our method presented is slightly different: whereas as usual $\mathbb{E}_8$ is obtained as a one-point extension from $\mathbb{E}_7$, we draw the attention to a parallel way for obtaining $\mathbb{E}_7$ from $\mathbb{D}_6$ and $\mathbb{E}_6$ from $\mathbb{A}_5$. In this way, the three exceptional cases $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ are considered as being on equal footing.

Let us stress that any Dynkin diagram Γ should be seen as being accompanied by the corresponding Euclidean diagram $\widetilde{\Gamma}$; many properties of Γ can be read off very nicely by looking at $\widetilde{\Gamma}$. In contrast to the inclusion sequence of the Dynkin diagrams $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$, the corresponding Euclidean diagrams $\widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8$ are pairwise incomparable. Of course, our equal footing approach to $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ is motivated by a look at the further extensions from Γ to $\widetilde{\Gamma}$.

Henning Krause has drawn my attention to a famous letter included in the paper [Ad] by F. Adams, and purportedly written by $\mathbb{E}_8$ itself: it regrets the prevailing opinion that the case $\mathbb{E}_8$ is *remote and unapproachable* and needs *an arduous course of preparation with* $\mathbb{E}_6$ *and* $\mathbb{E}_7$. The letter adds the following comment: *Any right-thinking mathematician who wishes to construct the root-system of* $\mathbb{E}_6$ *does so as follows: first he constructs the root-system of* $\mathbb{E}_8$*, and then inside it he locates the root-system of* $\mathbb{E}_6$. *In this way he benefits from the great symmetry of the root-system of* $\mathbb{E}_8$ *and its perspicuous nature. If this good precedent is not followed in other researches, one should consider whether to infer a lack of boldness in the investigator rather than a lack of cooperation from the subject-matter.*

In contrast to Adams suggestion to start with $\mathbb{E}_8$ and to consider the root systems of $\mathbb{E}_6$ and $\mathbb{E}_7$ just as subsystems of $\mathbb{E}_8$, we have the feeling that our attempt to provide a unified treatment of $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$ is also supported by the subject-matter.

Let us add a remark concerning our use of the representations of a quiver R of type $\widetilde{\mathbb{D}}_4$ in order to deal with the representations of an arbitrary Euclidean quiver. To invoke R means a reduction to the 4-subspace quiver. In the paper [GP], Gelfand and Ponomarev have drawn the attention to the 4-subspace quiver and have provided a classification of all the indecomposable representation. In the paper, they have stressed the importance of the 4-subspace quiver: *Many tame problems of Linear Algebra can be reduced to the classification* of representations of the 4-subspace quiver ([GP] p. 163). The results presented in Part 3 confirm this observation. Our approach provides for the Euclidean quivers $\widetilde{\Delta}$ of type $\widetilde{\mathbb{D}}_n, \widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8$ a thick subcategory of type $\widetilde{\mathbb{D}}_4$. It recovers not only the 1-parameter families of indecomposable representations, but also one representative from any τ-orbit of simple regular representations. Of course, if one is only interested in the 1-parameter families of indecomposable representations, one will be content with a thick subcategory which is equivalent to the category of Kronecker modules, say with $\mathcal{E}(M, S(z))$ where M is the maximal indecomposable representation of a corresponding Dynkin subquiver Δ and z is the vertex in $\widetilde{\Delta} \setminus \Delta$.

The $2-4-8$ assertion should remind the reader on mathematical settings which rely on an invariant r being equal to 1, 2, 4, or 8. The prototype is the Hurwitz theorem which asserts that the only Euclidean Hurwitz algebras are the real numbers $\mathbb{R}$, the complex numbers $\mathbb{C}$, the quaternions $\mathbb{H}$ and the octonions $\mathbb{O}$ (a *Euclidean Hurwitz algebra* is a finite-dimensional (not necessarily associative) $\mathbb{R}$-algebra A with identity endowed with a positive quadratic form q such that $q(ab) = q(a)q(b)$). Is there a direct relationship between the Dynkin quivers $\mathbb{E}_6$, $\mathbb{E}_7, \mathbb{E}_8$ and the division algebras $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$? We do not know. But we should draw the attention of the reader to the magic Freudenthal-Tits square, see [F, Ti, V]. It is a symmetric (4×4)-matrix whose entries D_{AB} are Dynkin types, with column index A and row index B being one $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, $\mathbb{O}$:

	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathbb{A}_1$	$\mathbb{A}_2$	$\mathbb{C}_3$	$\mathbb{F}_4$
$\mathbb{C}$	$\mathbb{A}_2$	$\mathbb{A}_2 \sqcup \mathbb{A}_2$	$\mathbb{A}_5$	$\mathbb{E}_6$
$\mathbb{H}$	$\mathbb{C}_3$	$\mathbb{A}_5$	$\mathbb{D}_6$	$\mathbb{E}_7$
$\mathbb{O}$	$\mathbb{F}_4$	$\mathbb{E}_6$	$\mathbb{E}_7$	$\mathbb{E}_8$

The magic square concerns constructions of the Lie groups or Lie algebras of type D_{AB} starting with A and B. Since all possible A and B are subalgebras of $\mathbb{O}$, the magic square asserts that all exceptional Lie groups and Lie algebras can be constructed starting with the octonions $\mathbb{O}$ (note that the exceptional Lie type $\mathbb{G}_2$ is missing in the magic square, but the Lie group of type $\mathbb{G}_2$ is just the automorphism group of $\mathbb{O}$).

We are interested in the right lower corner of the magic square, it is the following matrix:

$$\begin{array}{ccc} \mathbb{A}_2 \sqcup \mathbb{A}_2 & \mathbb{A}_5 & \mathbb{E}_6 \\ \mathbb{A}_5 & \mathbb{D}_6 & \mathbb{E}_7 \\ \mathbb{E}_6 & \mathbb{E}_7 & \mathbb{E}_8 \end{array}$$

Here, any row (or column) lists the Dynkin types of $\Delta'', \Delta', \Delta$, where Δ is a Dynkin quiver of type $\mathbb{E}_m$ with exceptional vertex y and where x is the neighbor of y, such that $\Delta' = \Delta \setminus \{y\}$ and $\Delta'' = \Delta \setminus \{x, y\}$.

What we have recovered in our presentation are invariants r of Δ which have the value $2, 4, 8$ respectively.

One may wonder about the case $r = 1$. According to the magic square, this concerns the Dynkin type $\mathbb{F}_4$. Now, there is no quiver of type $\mathbb{F}_4$. There are the species of type $\mathbb{F}_4$, but they are outside of the scope of the present investigation. Actually, it turns out that for the species of type $\mathbb{F}_4$, the table in 3.5 can be completed by a column concerning $\mathbb{F}_4$ with entries $\mathbb{F}_4, \mathbb{C}_3, \mathbb{A}_2, 3, 6, 6, 6, 2, 1, 24$, and $6 = 3 \cdot (r + 1)$ with $r = 1$. This confirms the magic square philosophy. However, if we look at the hammock in question, we have $(\tau')^t I'(x) = P'(x)$ with $t = 2$ and not $t = 1$.

Panoramic View.

The Dynkin graphs have been exhibited in the Preliminaries. There, at the end, one also finds the Euclidean graphs $\widetilde{\mathbb{A}}_n$. The remaining Euclidean graphs $\widetilde{\mathbb{D}}_n$, $\widetilde{\mathbb{E}}_6, \widetilde{\mathbb{E}}_7, \widetilde{\mathbb{E}}_8$ have been discussed in Part 3.

(1) *A finite connected quiver Q without a Euclidean subquiver is a Dynkin quiver.* Proof: Since Q does not contain a subquiver of type $\widetilde{\mathbb{A}}_n$, we see that Q is a tree quiver Since Q has no subquiver of type $\widetilde{\mathbb{D}}_4$, any vertex of Q has at most 3 neighbors. Vertices with 3 neighbors are called branching vertices. Since Q has no subquiver of type $\widetilde{\mathbb{D}}_n$, with $n \geq 5$, there is at most one branching vertex. If there is no branching vertex, then Q is of type $\mathbb{A}_n$, thus a Dynkin quiver. If Q has a branching vertex, then Q is a star quiver of type $(t(1), t(2), t(3))$ with $t(1) \geq t(2) \geq t(3) \geq 2$. If $t(3) \geq 3$, then Q contains a subquiver of type $\widetilde{\mathbb{E}}_6$. Thus $t(3) = 2$. If $t(2) \geq 4$, then Q contains a subquiver of type $\widetilde{\mathbb{E}}_7$. Thus $t(2)$ is equal to 2 or 3. If $t(2) = 2$, then Q is of Dynkin type $\mathbb{D}_n$. It remains that $t(2) = 3$. If $t(1) \geq 6$, then Q contains a subquiver of type $\widetilde{\mathbb{E}}_8$. Thus, the only remaining cases are $t(1) \in \{3, 4, 5\}$, these are the Dynkin cases $\mathbb{E}_6, \mathbb{E}_7, \mathbb{E}_8$.

(2) *Dynkin quivers are representation-finite.* This has been shown above.

(3) *Euclidean quivers are representation-infinite.*

Proof. The quivers of type $\widetilde{\mathbb{A}}_n$ have been discussed at the end of the Preliminaries in section 0.3. Let Q be such a quiver and let us fix one of the arrows, say α. For any $\lambda \in k$, we define $M(\lambda)$ as follows: $M(\lambda)_x = k$ for all vertices x, let $M(\alpha)_\alpha = \lambda$ and $M(\lambda)_\beta = 1$ for the remaining arrows β. Then the representations $M(\lambda)$ with $\lambda \in k$, are indecomposable and pairwise non-isomorphic. This yields infinitely many isomorphism classes of indecomposable representations, provided k is an infinite field. More generally, for any natural number n, we define $M(\lambda, n)$ as follows: $M(\lambda, n)_x = k^n$ for all vertices x, let $M(\lambda, n)_\alpha$ be the Jordan $(n \times n)$-matrix with eigenvalue $\lambda \in k$ and $M(\lambda, n)_\beta$ the identity map, for the remaining arrows β. Then the representations $M(\lambda, n)$ with $\lambda \in k$, are indecomposable and pairwise non-isomorphic.

Next, let us consider the 4-subspace quiver Q (its type is $\widetilde{\mathbb{D}}_4$). We consider the quiver $\widetilde{\mathbb{A}}_0$ (one vertex, one loop) and define a full exact embedding $\zeta\colon \operatorname{rep}\widetilde{\mathbb{A}}_0 \to \operatorname{rep} Q$ as follows: a representation of $\widetilde{\mathbb{A}}_0$ is a pair (V, ϕ), where V is a vector space and $\phi\colon V \to V$ a linear endomorphism (note that $\operatorname{rep}\widetilde{\mathbb{A}}_0$ can also be considered as the category of $k[T]$-modules, where $k[T]$ is the polynomial ring in one variable, since a $k[T]$-module is also just a vector space with a linear endomorphism, namely given by the multiplication with T). If (V, ϕ) is a representation of $\widetilde{\mathbb{A}}_0$, let $\zeta(V, \phi)$ be the following representation of Q

$$\begin{tikzcd}
 & V \times 0 \arrow[ddl] \\
 & 0 \times V \arrow[dl] \\
V \times V & \\
 & \Gamma(1) \arrow[ul] \\
 & \Gamma(\phi) \arrow[uul]
\end{tikzcd}$$

where $\Gamma(\phi) = \{(v, \phi(v)) \mid v \in V\}$ (and thus $\Gamma(1) = \{(v, v) \mid v \in V\}$). Since $\widetilde{\mathbb{A}}_0$ is representation-infinite, also Q is representation-infinite. Of course, almost all the indecomposable representations of Q are conical, thus any quiver of type $\widetilde{\mathbb{D}}_4$ is representation-infinite, by Proposition 1.6.

Consider now a Euclidean quiver Q of type $\widetilde{\mathbb{E}}_n$ (thus $n = 6, 7, 8$), but also those of type $\widetilde{\mathbb{D}}_n$ with $n \geq 5$. As we have seen in Part 3, there is a full exact embedding $\operatorname{rep} R \to \operatorname{rep} Q$, where R is of type $\widetilde{\mathbb{D}}_4$. This shows that the quivers of type $\widetilde{\mathbb{E}}_6$, $\widetilde{\mathbb{E}}_7$, $\widetilde{\mathbb{E}}_8$ are representation-infinite.

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