

Appendix

Some Remarks concerning Tilting Modules and Tilted Algebras.

Origin. Relevance. Future.

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The project to produce a Handbook of Tilting Theory was discussed during the Fraueninsel Conference *20 Years of Tilting Theory*, in November 2002. A need was felt to make available surveys on the basic properties of tilting modules, tilting complexes and tilting functors, to collect outlines of the relationship to similar constructions in algebra and geometry, as well as reports on the growing number of generalizations. At the time the Handbook was conceived, there was a general consensus about the overall frame of tilting theory, with the tilted algebra as the core, surrounded by a lot of additional considerations and with many applications in algebra and geometry. One was still looking forward to further generalizations (say something like “pre-semi-tilting procedures for near-rings”), but the core of tilting theory seemed to be in a final shape. The Handbook was supposed to provide a full account of the theory as it was known at that time. The editors of this Handbook have to be highly praised for what they have achieved. But the omissions which were necessary in order to bound the size of the volume clearly indicate that there should be a second volume.

Part 1 will provide an outline of this core of tilting theory. Part 2 will then be devoted to topics where tilting modules and tilted algebras have shown to be relevant. I have to apologize that these parts will repeat some of the considerations of various chapters of the Handbook, but such a condensed version may be helpful as a sort of guideline. Both Parts 1 and 2 contain historical annotations and reminiscences. The final Part 3 will be a short report on some striking recent developments which are motivated by the cluster theory of Fomin and Zelevinsky. In particular, we will guide the reader to the basic properties of cluster tilted algebras,

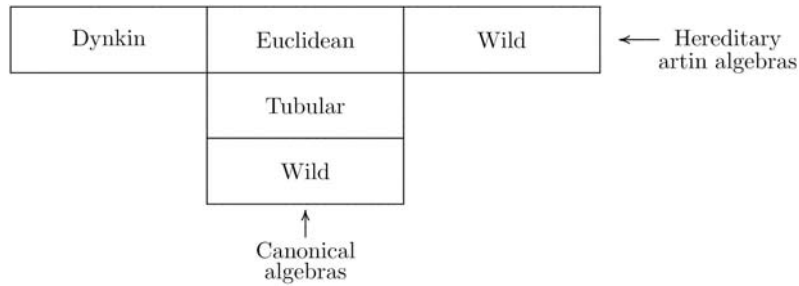
to the relationship between tilted algebras and cluster tilted algebras, but also to the cluster categories which provide a universal setting for all the related tilted and cluster tilted algebras. In addition, we will focus the attention to the complex of cluster tilting objects and exhibit a quite elementary description of this complex. In Part 1 some problems concerning tilting modules and tilted algebras are raised and one may jump directly to Part 3 in order to see in which way these questions have been answered by the cluster tilting theory. We stress that it should be possible to look at the Parts 2 and 3 independently.

1 Basic Setting

The setting to be exhibited is the following: We start with a hereditary artin algebra A and a tilting A -module T . It is the endomorphism ring $B = \text{End}(T)$, called a *tilted algebra*, which attracts the attention. The main interest lies in the comparison of the categories $\text{mod } A$ and $\text{mod } B$ (for any ring R , let us denote by $\text{mod } R$ the category of all R -modules of finite length).

1.1 The representation types of the hereditary categories one starts with

We may assume that A is connected (this means that 0 and 1 are the only central idempotents), and we may distinguish whether A is representation-finite, tame, or wild; for hereditary algebras, this distinction is well understood: the corresponding quiver (or better species) is a Dynkin diagram, a Euclidean diagram, or a wild diagram, respectively. There is a parallel class of algebras: if we start with a canonical algebra A instead of a finite-dimensional hereditary algebra (or, equivalently, with a weighted projective line, or a so called “exceptional curve” in the species case), there is a corresponding tilting procedure. Again the representation theory distinguishes three different cases: A may be domestic, tubular, or wild. Now two of the six cases coincide: the algebras obtained from the domestic canonical algebras via tilting are precisely those which can be obtained from a Euclidean algebra via tilting. Thus, there are 5 possibilities which are best displayed as the following “T”: the upper horizontal line refers to the hereditary artin algebras, the middle vertical line to the canonical algebras.



There is a common frame for the five different classes: start with an artin algebra A , such that the bounded derived category $D^b(\text{mod } A)$ is equivalent to the bounded derived category $D^b(\mathcal{H})$ of a hereditary abelian category \mathcal{H} . Let T be a tilting object in $D^b(\text{mod } A)$ and B its endomorphism ring. Then B has been called a *quasi-tilted* algebra by Happel-Reiten-Smalø, and according to Happel and Happel-Reiten these categories $D^b(\text{mod } A)$ are just the derived categories of artin algebras which are hereditary or canonical. In the “T” displayed above, the upper horizontal line concerns the derived categories with a slice, the middle vertical line those with a separating tubular family. More information can be found in Chapter 6 by Lenzing.

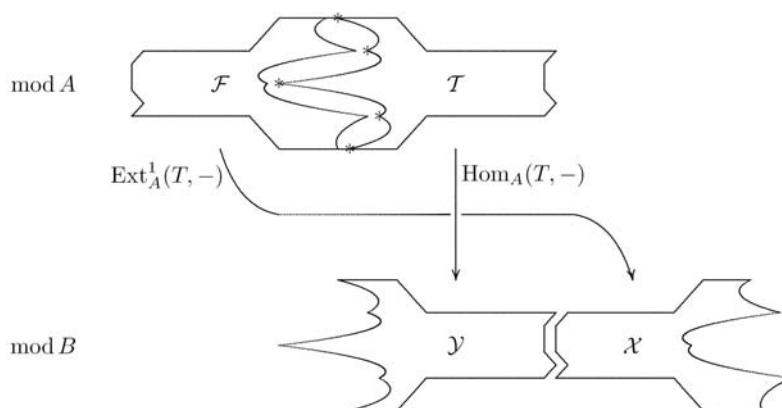
Most of the further considerations will be formulated for tilted algebras only. However usually there do exist corresponding results for all the quasi-tilted algebras. To restrict the attention to the tilted algebras has to be seen as an expression just of laziness, and does not correspond to the high esteem which I have for the remaining algebras (and the class of quasi-tilted algebras in general).

1.2 The functors $\text{Hom}_A(T, -)$ and $\text{Ext}_A^1(T, -)$

Thus, let us fix again a hereditary artin algebra A and let D be the standard duality of $\text{mod } A$ (if k is the center of A , then $D = \text{Hom}_k(-, k)$; note that k is semisimple). Thus DA is an injective cogenerator in $\text{mod } A$. We consider a tilting A -module T , and let $B = \text{End}(T)$. The first feature which comes to mind and which was the observation by Brenner and Butler which started the game¹, is the following: the func-

¹ Of course, we are aware that examples of tilting modules and tilting functors had been studied before. Examples to be mentioned are first the Coxeter functors introduced by Gelfand and Ponomarev in their paper on the four-subspace-problem (1970), then the BGP-reflection functors (Bernstein, Gelfand and Ponomarev,

tor $\text{Hom}_A(T, -)$ yields an equivalence between the category² \mathcal{T} of all A -modules generated by T and the category \mathcal{Y} of all B -modules cogenerated by the B -module $\text{Hom}_A(T, DA)$. Now the dimension vectors of the indecomposable A -modules in \mathcal{T} generate the Grothendieck group $K_0(A)$. If one tries to use $\text{Hom}_A(T, -)$ in order to identify the Grothendieck groups $K_0(A)$ and $K_0(B)$, one observes that the positivity cones overlap, but differ: the new axes which define the positive cone for B are “tilted” against those for A . This was the reason for Brenner and Butler to call it a tilting procedure. But there is a second “tilting” phenomenon which concerns the corresponding torsion pairs³. In order to introduce these torsion pairs, we have to look not only at the functor $\text{Hom}_A(T, -)$, but also at $\text{Ext}_A^1(T, -)$. The latter functor yields an equivalence between the category \mathcal{F} of all A -modules M with $\text{Hom}_A(T, M) = 0$ and the category \mathcal{X} of all B -modules N with $T \otimes_B N = 0$. Now the pair $(\mathcal{F}, \mathcal{T})$ is a torsion pair in the category of A -modules, and the pair $(\mathcal{Y}, \mathcal{X})$ is a torsion pair in the category of B -modules:



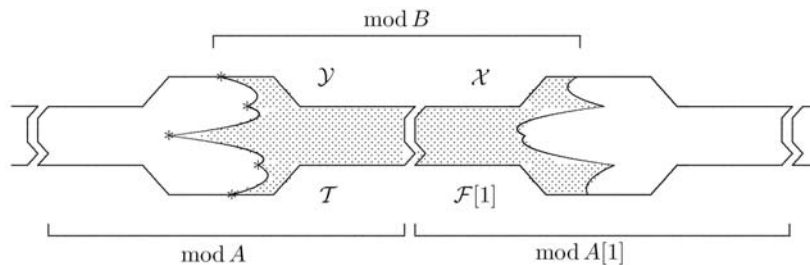
1973), their generalisation by Auslander, Platzeck and Reiten (1979), now called the APR-tilting functors, and also a lot of additional ad-hoc constructions used around the globe, all of which turn out to be special tilting functors. But the proper start of tilting theory is clearly the Brenner-Butler paper (1980). The axiomatic approach of Brenner and Butler was considered at that time as quite unusual and surprising in a theory which still was in an experimental stage. But it soon turned out to be a milestone in the development of representation theory.

² Subcategories like \mathcal{T} and \mathcal{F} will play a role everywhere in this appendix. In case we want to stress that they are defined using the tilting module T , we will write $\mathcal{T}(T)$ instead of \mathcal{T} , and so on.

³ In contrast to the usual convention in dealing with a torsion pair or a “torsion theory”, we name first the torsion-free class, then the torsion class: this fits to the rule that in a rough thought, maps go from left to right, and a torsion pair concerns regions with “no maps backwards”.

and one encounters the amazing fact that under the pair of functors $\text{Hom}_A(T, -)$ and $\text{Ext}_A^1(T, -)$ the torsion-free class of a torsion pair is flipped over the torsion class in order to form a new torsion pair in reversed order⁴. The stars * indicate a possible distribution of the indecomposable direct summands T_i of T , and one should keep in mind that for any i , the Auslander-Reiten translate τT_i of T_i belongs to \mathcal{T} (though it may be zero). We have said that the modules in \mathcal{T} are those generated by T , but similarly the modules in \mathcal{F} are those cogenerated by τT .

According to Happel⁵, the category $\text{mod } B$ should be seen as being embedded into the derived category $D^b(\text{mod } A)$



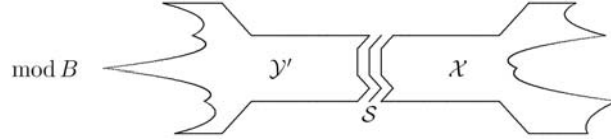
Under this embedding, $\mathcal{Y} = \mathcal{T}$ is the intersection of $\text{mod } B$ with $\text{mod } A$, whereas $\mathcal{X} = \mathcal{F}[1]$ is the intersection of $\text{mod } B$ with $\text{mod } A[1]$ (the shift functor in a triangulated category will always be denoted by $[1]$). This embedding functor $\text{mod } B \longrightarrow D^b(\text{mod } A)$ extends to an equivalence of $D^b(\text{mod } B)$ and $D^b(\text{mod } A)$, and this equivalence is one of the essential features of tilting theory.

Looking at the torsion pair $(\mathcal{Y}, \mathcal{X})$, there is a sort of asymmetry due to the fact that \mathcal{Y} is always sincere (this means that every simple module

⁴ The discovery of this phenomenon was based on a detailed examination of many examples (and contributions by Dieter Vossieck, then a student at Bielefeld, should be acknowledged). At that time only the equivalence of $\mathcal{T}(T)$ and $\mathcal{Y}(T)$ was well understood. The obvious question was to relate the remaining indecomposable B -modules (those in $\mathcal{X}(T)$) to suitable A -modules. As Dieter Happel recalls, the first examples leading to a full understanding of the whole tilting process were tilting modules for the \mathbb{E}_6 -quiver with subspace orientation.

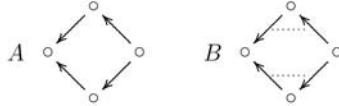
⁵ When he propagated this in 1984, it was the first clue that the use of derived categories may be of interest when dealing with questions in the representation theory of finite dimensional algebras. The derived categories had been introduced by Grothendieck in order to construct derived functors when dealing with abelian categories which have neither sufficiently many projective nor sufficiently many injective objects, and at that time they were considered as useless in case there are enough projectives and enough injectives, as in the cases $\text{mod } A$ and $\text{mod } B$.

occurs as a composition factor of some module in \mathcal{Y}), whereas \mathcal{X} does not have to be sincere (this happens if \mathcal{Y} contains an indecomposable injective module). As a remedy, one should divide \mathcal{Y} further as follows: \mathcal{Y} contains the slice module $S = \text{Hom}_A(T, DA)$, let $\mathcal{S} = \text{add } S$, and denote by \mathcal{Y}' the class of all B -modules in \mathcal{Y} without an indecomposable direct summand in \mathcal{S} . It is the triple $(\mathcal{Y}', \mathcal{S}, \mathcal{X})$, which really should be kept in mind:

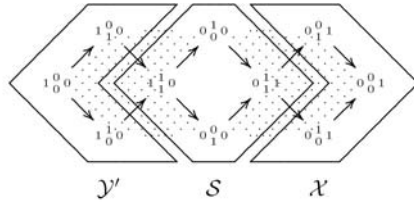


with all the indecomposables lying in one of the classes $\mathcal{Y}', \mathcal{S}, \mathcal{X}$ and with no maps backwards (the only maps from \mathcal{S} to \mathcal{Y}' , from \mathcal{X} to \mathcal{S} , as well as from \mathcal{X} to \mathcal{Y}' are the zero maps). Also note that any indecomposable projective B -module belongs to \mathcal{Y}' or \mathcal{S} , any indecomposable injective module to \mathcal{S} or \mathcal{X} . The module class \mathcal{S} is a slice (as explained in Chapter 3 by Brüstle) and any slice is obtained in this way. The modules in \mathcal{Y}' are those cogenerated by $\tau\mathcal{S}$, the modules in \mathcal{X} are those generated by $\tau^{-1}\mathcal{S}$.

Here is an example. Start with the path algebra A of a quiver of Euclidean type \tilde{A}_{22} having one sink and one source. Let $B = \text{End}(T)$, where T is the direct sum of the simple projective, the simple injective and the two indecomposable regular modules of length 3 (this is a tilting module), then the quiver of B is the same as the quiver of A , but B is an algebra with radical square zero. Thus B is given by a square with two zero relations.



The category $\text{mod } B$ looks as follows:



The separation of $\text{mod } B$ into the three classes $\mathcal{Y}', \mathcal{S}, \mathcal{X}$ can be phrased in the language of **cotorsion pairs**. Cotorsion pairs are very well related to tilting theory (see the Chapters 8 and 11 by Reiten and Trlifaj), but still have to be rated more as a sort of insider tip. We recall the definition: the pair $(\mathcal{V}, \mathcal{W})$ of full subcategories of $\text{mod } A$ is said to be a *cotorsion pair* provided \mathcal{V} is the class of all A -modules V with $\text{Ext}_A^1(V, W) = 0$ for all W in \mathcal{W} , and \mathcal{W} is the class of all A -modules W such that $\text{Ext}_A^1(V, W) = 0$ for all V in \mathcal{V} . The cotorsion pair is said to be *split*, provided every indecomposable A -module belongs to \mathcal{V} or \mathcal{W} . Usually some indecomposables will belong to both classes, they are said to form the *heart*. In our case the following holds: *The pair $(\text{add}(\mathcal{Y}', \mathcal{S}), \text{add}(\mathcal{S}, \mathcal{X}))$ forms a split cotorsion pair with heart \mathcal{S} .*

We also see that the modules in \mathcal{Y}' and in \mathcal{S} have projective dimension at most 1, those in \mathcal{S} and in \mathcal{X} have injective dimension at most 1. As a consequence, if X, Y are indecomposable modules with $\text{Ext}_B^2(X, Y) \neq 0$, then X belongs to \mathcal{X} and Y belongs to \mathcal{Y}' .

Let me add a remark even if it may be considered to be superfluous — its relevance should become clear in the last part of this appendix. If we feel that the subcategory \mathcal{Y}' has the same importance as \mathcal{X} (thus that it is of interest), then we should specify an equivalent subcategory, say \mathcal{T}' of $\text{mod } A$ and an equivalence $\mathcal{T}' \longrightarrow \mathcal{Y}'$. Such an equivalence is given by the functor

$$\text{Hom}_A(\tau^{-1}T, -): \text{mod } A \longrightarrow \text{mod } B$$

or, equivalently, by $\text{Hom}_A(T, \tau-)$, since τ^{-1} is left adjoint to τ . This functor vanishes on \mathcal{F} as well as on T , and it yields an equivalence between the subcategory \mathcal{T}' of all A -modules generated by $\tau^{-1}T$ and the subcategory \mathcal{Y}' of $\text{mod } B$. Note that the functor can also be written in the form $D\text{Ext}_A^1(-, T)$, due to the Auslander-Reiten formula $D\text{Ext}_A^1(M, T) \simeq \text{Hom}_A(T, \tau M)$. In this way, we see that we deal with equivalences

$$D\text{Ext}_A^1(-, T): \mathcal{T}' \longrightarrow \mathcal{Y}' \quad \text{and} \quad \text{Ext}_A^1(T, -): \mathcal{F} \longrightarrow \mathcal{X},$$

which are sort of dual to each other.

It seems to be worthwhile to have a short look at the rather trivial case when no modules are lost, so that the tilting procedure is a kind of rearrangement of module classes. *The following assertions are equivalent:*

- (i) *The tilting module T is a slice module.*
- (ii) *The endomorphism ring B is hereditary.*
- (iii) *The torsion pair $(\mathcal{F}, \mathcal{T})$ splits.*
- (iv) $\text{Ext}_A^1(\tau T, T) = 0$.

The equivalence of these assertions are well-known, but not too easy to trace. Some implications are quite obvious, for example that (ii) and (iii) are implied by (i). Let us show that (ii) implies (i): Since T is a tilting module, the B -module $T' = \text{Hom}_A(T, DA) \simeq \text{Hom}_k(T, k)$ is a slice module in $\text{mod } B$. Since the B -module T' is a tilting module and $A = \text{End}_B(T')$, we can use this tilting module in order to tilt from $\text{mod } B$ to $\text{mod } A$. Since B is hereditary, we obtain in $\text{mod } A$ the slice module $\text{Hom}_B(T', DB) \simeq \text{Hom}_k(T', k) \simeq T$. This shows (i). The equivalences of (ii) and (iv), as well as of (iii) and (iv), can be seen as consequences of more general considerations which will be presented later.

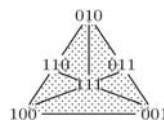
If T is not a slice module, so that B has global dimension equal to 2, then the algebras A and B play quite a different role: the first difference is of course the fact that A is hereditary, whereas B is not. Second, there are the two torsion pairs $(\mathcal{F}, \mathcal{T})$ in $\text{mod } A$ and $(\mathcal{Y}, \mathcal{X})$ in $\text{mod } B$ - the second one is a split torsion pair, the first one not. This means that we loose modules going from $\text{mod } A$ to $\text{mod } B$ via tilting. Apparently, no one cared about the missing modules, at least until quite recently. There are two reasons: First of all, we know (see Chapter 3 by Brüstle), that the study of indecomposable modules over a representation-finite algebra is reduced via covering theory to the study of representation-finite tilted algebras. Such an algebra B may be of the form $B = \text{End}_A(T)$, where T is a tilting A -module, with A representation-infinite. Here we describe the B -modules in terms of A and we are only interested in the finitely many indecomposable A -modules which belong to \mathcal{F} or \mathcal{T} , the remaining A -modules seem to be of no interest, we do not miss them. But there is a second reason: the fashionable reference to derived categories is used to appease anyone, who still mourns about the missing modules. They are lost indeed as modules, but they survive as complexes: since the derived categories of A and B are equivalent, corresponding to any indecomposable A -module, there is an object in the derived category which is given by a complex of B -modules. However, I have to admit that I prefer modules to complexes, whenever possible — thus I was

delighted, when the lost modules were actually found, as described in Part 3 of this appendix.

1.3 The simplicial complex Σ_A

We will always denote by $n = n(A)$ the number of isomorphism classes of simple A -modules. The interest in tilting A -modules directly leads to a corresponding interest in their direct summands. These are the modules without self-extensions and are called *partial tilting modules*. In particular, one may consider the indecomposable ones: an indecomposable A -module without self-extensions is said to be *exceptional* (or a “stone”, or a “brick without self-extensions”, or a “Schurian module without self-extensions”). But there is also an interest in the partial tilting modules with precisely $n-1$ isomorphism classes of indecomposable direct summands, the so-called *almost complete* partial tilting modules. If \bar{T} is an almost complete partial tilting module and X is indecomposable with $\bar{T} \oplus X$ a tilting module, then X (or its isomorphism class) is called a *complement* for \bar{T} . It is of interest that any almost complete partial tilting module \bar{T} has either 1 or 2 complements, and it has 2 if and only if \bar{T} is sincere. Recall that a module is said to be *basic*, provided it is a direct sum of pairwise non-isomorphic indecomposable modules. The isomorphism classes of basic partial tilting modules form a simplicial complex Σ_A , with vertex set the set of isomorphism classes of exceptional modules (the vertices of a simplex being its indecomposable direct summands), see Chapter 10 by Unger. Note that this simplicial complex is of pure dimension $n-1$. The assertion concerning the complements shows that it is a pseudomanifold with boundary. The boundary consists of all the non-sincere almost complete partial tilting modules.

As an example, consider the path algebra A of the quiver $\circ \leftarrow \circ \leftarrow \circ$. The simplicial complex Σ_A has the following shape:



Some questions concerning the simplicial complex Σ_A remained open: What happens under a change of orientation? What happens under a tilting functor? Is there a way to get rid of the boundary? Here we are

again in a situation where a remedy is provided by the derived categories: If we construct the analogous simplicial complex of tilting complexes in $D^b(\text{mod } A)$, then one obtains a pseudo manifold without boundary, but this is quite a large simplicial complex!

If \bar{T} is an almost complete partial tilting module, and X and Y are non-isomorphic complements for \bar{T} , then either $\text{Ext}_A^1(Y, X) \neq 0$ or $\text{Ext}_A^1(X, Y) \neq 0$ (but not both). If $\text{Ext}_A^1(Y, X) \neq 0$ (what we may assume), then there exists an exact sequence $0 \longrightarrow X \longrightarrow T' \longrightarrow Y \longrightarrow 0$ with $T' \in \text{add } \bar{T}$, and one may write $\bar{T} \oplus X < \bar{T} \oplus Y$. In this way, one gets a partial ordering on the set of isomorphism classes of basic tilting modules. One may consider the switch between the tilting modules $\bar{T} \oplus X$ and $\bar{T} \oplus Y$ as an exchange process which stops at the boundary. We will see in Part 3 that it is possible to define an exchange procedure across the boundary of Σ_A , and that this can be arranged in such a way that one obtains an interesting small extension of the simplicial complex Σ_A .

1.4 The BGP-reflection functors and the structure of tilted algebras

Bernstein, Gelfand and Ponomarev have defined reflection functors in order to be able to compare representations of quivers with different orientation, but also in order to construct inductively indecomposable representations. A BGP-reflection functor furnishes a quite small change of the given module category. Let us consider this in more detail. Let Q be a quiver and i a sink of Q . We denote by $\sigma_i Q$ the quiver obtained from Q by changing the orientation of all arrows ending in i , thus i becomes a source in $\sigma_i Q$. Let $S(i)$ be the simple kQ -module corresponding to the vertex i , and $S'(i)$ the simple $k\sigma_i Q$ -module corresponding to the vertex i . The BGP-reflection functor $\sigma_i: \text{mod } kQ \longrightarrow \text{mod } k\sigma_i Q$ provides an equivalence between the categories

$$\text{mod } kQ / \langle \text{add } S(i) \rangle \longrightarrow \text{mod } k\sigma_i Q / \langle \text{add } S'(i) \rangle.$$

In general, given rings R, R' one may look for a simple R -module S and a simple R' -module S' such that the categories

$$\text{mod } R / \langle \text{add } S \rangle \longrightarrow \text{mod } R' / \langle \text{add } S' \rangle$$

are equivalent. In this case, let us say that R, R' are *nearly Morita-equivalent*. As we have seen, for Q a quiver with a sink i , the path algebras kQ and $k\sigma_i Q$ are nearly Morita-equivalent. I am not aware that other pairs of nearly Morita-equivalent rings have been considered until very recently, but Part 3 will provide a wealth of examples.

Unfortunately, the BGP-reflection functors are defined only for sinks and for sources of the given quiver. This has to be considered as a real deficiency, since there is no similar restriction in Lie theory. Indeed, in Lie theory the use of reflections for all the vertices is an important tool. A lot of efforts have been made in representation theory in order to overcome this deficiency, see for example the work of Kac on the dimension vectors of the indecomposable representations of a quiver.

A final question should be raised here. There is a very nice homological characterization of the quasi-tilted algebras by Happel-Reiten-Smalø [28]: these are the artin algebras of global dimension at most 2, such that any indecomposable module has projective dimension at most 1 or injective dimension at most 1. But it seems that a corresponding characterization of the subclass of tilted algebras is still missing. Also, in case we consider tilted k -algebras, where k is an algebraically closed field, the possible quivers and their relations are not known.

2 Connections

The relevance of tilting theory relies on the many different connections it has not only to other areas of representation theory, but also to algebra and geometry in general. Let me give some indications. If nothing else is said, A will denote a hereditary artin algebra, T a tilting A -module and B its endomorphism ring.

2.1 Homology

Already the definition (the vanishing of Ext^1) refers to homology. We have formulated above that the first feature which comes to mind is the functor $\text{Hom}_A(T, -)$. But actually all the tilting theory concerns the study of the corresponding derived functors $\text{Ext}_A^i(T, -)$, or better, of the right derived functor $R\text{Hom}_A(T, -)$.

The best setting to deal with these functors are the corresponding derived categories $D^b(\text{mod } A)$ and $D^b(\text{mod } B)$, they combine to the right derived functor $R\text{Hom}_A(T, -)$, and this functor is an equivalence, as Happel has shown. Tilting modules T in general were defined in such a way that $R\text{Hom}_A(T, -)$ is still an equivalence. The culmination of this development was Rickard's characterization of rings with equivalent derived categories: such equivalences are always given by "tilting complexes". A detailed account can be found in Chapter 5 by Keller.

Tilting theory can be exhibited well by using spectral sequences. In Bongartz's presentation of tilting theory one finds the formulation: *Well-read mathematicians tend to understand tilting theory using spectral sequences* (which is usually interpreted as a critical comment about the earlier papers). But it seems that the first general account of this approach is only now available: the contribution of Brenner and Butler (see Chapter 4) in this volume. A much earlier one by Vossieck should have been his Bielefeld Ph.D. thesis in 1984, but he never handed it in.

2.2 Geometry and invariant theory.

The Bielefeld interest in tilting modules was first not motivated by homological, but by geometrical questions. Happel's Ph.D. thesis had focused the attention to quiver representations with an open orbit (thus to all the partial tilting modules). In particular, he showed that the number $s(V)$ of isomorphism classes of indecomposable direct summands of a representation V with open orbit is bounded by the number of simple modules. In this way, the study of open orbits in quiver varieties was a (later hidden) step in the development of tilting theory. When studying open orbits, we are in the setting of what Sato and Kimura [45] call **prehomogeneous vector spaces**. On the one hand, the geometry of the complement of the open orbit is of interest, on the other hand one is interested in the structure of the ring of semi-invariants.

Let k be an algebraically closed field and Q a finite quiver (with vertex set Q_0 and arrow set Q_1), and we may assume that Q has no oriented cyclic path, thus the path algebra kQ is just a basic hereditary finite-dimensional k -algebra. For any arrow α in Q_1 , denote by $t\alpha$ its tail and by $h\alpha$ its head, and fix some dimension vector \mathbf{d} . Let us consider representations V of Q with a fixed dimension vector \mathbf{d} , we may assume

$V(x) = k^{\mathbf{d}(x)}$; thus the set of these representations forms the affine space

$$\mathcal{R}(Q, \mathbf{d}) = \bigoplus_{\alpha \in Q_1} \operatorname{Hom}_k(k^{\mathbf{d}(t\alpha)}, k^{\mathbf{d}(h\alpha)}).$$

The group $\operatorname{GL}(Q, \mathbf{d}) = \prod_{x \in Q_0} \operatorname{GL}(\mathbf{d}(x))$ operates on this space via a sort of conjugation, and the orbits under this action are just the isomorphism classes of representations. One of the results of Happel [26] asserts that given a sincere representation V with open orbit, then $|Q_0| - s(V)$ is the number of isomorphism classes of representations W with $\mathbf{dim} V = \mathbf{dim} W$ and $\dim \operatorname{Ext}_A^1(W, W) = 1$ (in particular, there are only finitely many such isomorphism classes; we also see that $|Q_0| \geq s(V)$).

Consider now the ring $\operatorname{SI}(Q, \mathbf{d})$ of semi-invariants on $\mathcal{R}(Q, \mathbf{d})$; by definition these are the invariants of the subgroup $\operatorname{SL}(Q, \mathbf{d}) = \prod_{x \in Q_0} \operatorname{SL}(\mathbf{d}(x))$ of $\operatorname{GL}(Q, \mathbf{d})$. Given two representations V, W of Q , one may look at the map:

$$d_W^V: \bigoplus_{x \in Q_0} \operatorname{Hom}_k(V(x), W(x)) \longrightarrow \bigoplus_{\alpha \in Q_1} \operatorname{Hom}_k(V(t\alpha), W(h\alpha)),$$

sending $(f(x))_x$ to $(f(h\alpha)V(\alpha) - W(\alpha)f(t\alpha))_\alpha$. Its kernel is just $\operatorname{Hom}_{kQ}(V, W)$, its cokernel $\operatorname{Ext}_{kQ}^1(V, W)$. In case d_W^V is a square matrix, one may consider its determinant. According to Schofield [46], this is a way of producing semi-invariants. Namely, the Grothendieck group $K_0(kQ)$ carries a (usually non-symmetric) bilinear form $\langle -, - \rangle$ with $\langle \mathbf{dim} V, \mathbf{dim} W \rangle = \dim_k \operatorname{Hom}_{kQ}(V, W) - \dim_k \operatorname{Ext}_{kQ}^1(V, W)$, thus d_W^V is a square matrix if and only if $\langle \mathbf{dim} V, \mathbf{dim} W \rangle = 0$. So, if $\mathbf{d} \in \mathbb{N}_0^{Q_0}$ and if we select a representation W such that $\langle \mathbf{d}, \mathbf{dim} W \rangle = 0$, then $c_W(V) = \det d_W^V$ yields a semi-invariant c_W in $\operatorname{SI}(Q, \alpha)$. Derksen and Weyman (and also Schofield and Van den Bergh) have shown that these semi-invariants form a generating set for $\operatorname{SI}(Q, \mathbf{d})$. In fact, it is sufficient to consider only indecomposable representations W , thus exceptional kQ -modules.

2.3 Lie theory

It is a well-accepted fact that the representation theory of hereditary artinian rings has a strong relation to Lie algebras and quantum groups (actually one should say: a strong relation to Lie algebras via quantum groups). Such a relationship was first observed by Gabriel when he showed that the representation-finite connected quivers are just those

with underlying graph being of the form A_n, D_n, E_6, E_7, E_8 and that in these cases the indecomposables correspond bijectively to the positive roots. According to Kac, this extends to arbitrary finite quivers without oriented cycles: the dimension vectors of the indecomposable representations are just the positive roots of the corresponding Kac-Moody Lie-algebra. It is now known that it is even possible to reconstruct this Lie algebra using the representation theory of hereditary artin algebras, via Hall algebras. Here one encounters the problem of specifying the subring of a Hall algebra, generated by the simple modules. Schofield induction (to be discussed below) shows that all the exceptional modules belong to this subring.

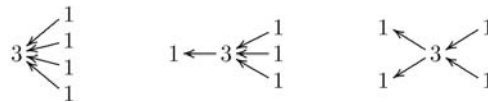
It seems to be appropriate to discuss the role of the necessary choices. Let me start with a semisimple finite-dimensional complex Lie-algebra. First, there is the choice of a Cartan subalgebra, it yields the root system of the Lie-algebra. Second, the choice of a root basis yields a triangular decomposition (and the set of positive roots), this is needed in order to define a Borel subalgebra and the corresponding category \mathcal{O} . Finally, the choice of a total ordering of the root basis (or, better, of an orientation of the edges of the Dynkin diagram) allows to work with a Coxeter element in the Weyl group. Of course, one knows that all these choices are inessential, when dealing with a finite dimensional Lie-algebra. The situation is more subtle if we deal with arbitrary Kac-Moody Lie-algebras: different orderings of the root basis may yield Coxeter elements which are not conjugate – the first case is \tilde{A}_3 , where one has to distinguish between $\tilde{A}_{3,1}$ and $\tilde{A}_{2,2}$.

On the other hand, when we start with a representation-finite hereditary artin algebra A , no choice at all is needed in order to write down its Dynkin diagram: it is intrinsically given as the Ext-quiver of the simple A -modules, and we obtain in this way a Dynkin diagram with orientation. A change of orientation corresponds to module categories with quite distinct properties (as already the algebras of type A_3 show). This difference is still preserved when one looks at the corresponding Hall algebras, and it comes as a big surprise that only a small twist of its multiplication is needed in order to get an algebra which is independent of the orientation.

2.4 The combinatorics of root systems

It is necessary to dig deeper into root systems since they play an important role for dealing with A -modules. Of interest is the corresponding quadratic form, and the reflections which preserve the root system (but not necessarily the positivity of roots), and their compositions, in particular the Coxeter transformations and the BGP-reflection functors. We will return to the reflection functors when we deal with generalizations of Morita equivalences, but also in Part 3.

As we have mentioned the relationship between the representation theory of a hereditary artin algebra A and root systems is furnished by the dimension vectors: We consider the Grothendieck group $K_0(A)$ (of all finite length A -modules modulo exact sequences). Given an A -module M , we denote by $\mathbf{dim} M$ the corresponding element in $K_0(A)$; this is what is called its *dimension vector*. The dimension vectors of the indecomposable A -modules are the positive roots of the root system in question. A positive root \mathbf{d} is said to be a *Schur root* provided there exists an indecomposable A -module M with $\mathbf{dim} M = \mathbf{d}$ and $\text{End}_A(M)$ a division ring. The dimension vectors of the exceptional modules are Schur roots, they are just the real (or Weyl) Schur roots. In case A is representation-finite then all the positive roots are Schur roots, also for $n(A) = 2$ all the real roots are Schur roots. But in all other cases, the set of real Schur roots depends on the choice of orientation. For example, consider the following three orientations of $\tilde{\mathbb{D}}_4$:



The two dimension vectors on the left are Schur roots, whereas the right one is not a Schur root.

In order to present the dimension vectors of the indecomposable A -modules, one may depict the Grothendieck group $K_0(A)$; a very convenient way seems to be to work with homogeneous coordinates, say with the projective space of $K_0(A) \otimes_{\mathbb{Z}} \mathbb{R}$. It is the merit of Derksen and Weyman [22] of having popularized this presentation well: they managed to get it to a cover of the Notices of the American Mathematical Society [21]. One such example has been shown in Part 1, when we presented the simplicial complex Σ_A , with A the path algebra of the linearly oriented quiver of type \mathbb{A}_3 . In general, dealing

with hereditary A , one is interested in the position of the Schur roots. Our main concern are the real Schur roots as the dimension vectors $\mathbf{dim} E$ of the exceptional modules E . They are best presented by marking the corresponding *exceptional lines*: look for orthogonal exceptional pairs E_1, E_2 (this means: E_1, E_2 are exceptional modules with $\mathrm{Hom}_A(E_1, E_2) = \mathrm{Hom}_A(E_2, E_1) = \mathrm{Ext}_A^1(E_2, E_1) = 0$) and draw the line segment from $\mathbf{dim} E_1$ to $\mathbf{dim} E_2$. The discussion of Schofield induction below will explain the importance of these exceptional lines.

As we know, a tilting A -module T has precisely $n = n(A)$ isomorphism classes of indecomposable direct summands, say T_1, \dots, T_n and the dimension vectors $\mathbf{dim} T_1, \dots, \mathbf{dim} T_n$ are linearly independent. We may consider the cone $C(T)$ in $K_0(A) \otimes \mathbb{R}$ generated by $\mathbf{dim} T_1, \dots, \mathbf{dim} T_n$. These cones are of special interest. Namely, if a dimension vector \mathbf{d} belongs to $C(T)$, then there is a unique isomorphism class of modules M with $\mathbf{dim} M = \mathbf{d}$ such that $\mathrm{End}_A(M)$ is of minimal dimension, and such a module M has no self-extensions. On the other hand, the dimension vector of any module M without self-extensions lies in such a cone (since these modules, the partial tilting modules, are the direct summands of tilting modules). The set of these cones forms a fan as they are considered in toric geometry (see for example the books of Fulton and Oda).

As Hille [29] has pointed out, one should use the geometry of these cones in order to introduce the following notion: If $T = \bigoplus_{i=1}^n T_i$ is a basic tilting module with indecomposable modules T_i of length $|T_i|$, he calls $\prod_{i=1}^n |T_i|^{-1}$ the *volume* of T . It follows that

$$\sum_T v(T) \leq 1$$

(where the summation extends over all isomorphism classes of basic tilting A -modules), with equality if and only if A is representation-finite or tame. This yields interesting equalities: For example, the preprojective tilting modules of the Kronecker quiver yield

$$\frac{1}{1} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{5} + \frac{1}{5} \cdot \frac{1}{7} + \dots = \frac{1}{2}.$$

One may refine these considerations by replacing the length $|T_i|$ by the k -dimension of T_i , at the same time replacing A by all the Morita equivalent algebras. In this way one produces power series identities in $n(A)$ variables which should be of general interest, for example

$$\frac{1}{x} \cdot \frac{1}{2x+y} + \frac{1}{2x+y} \cdot \frac{1}{3x+2y} + \frac{1}{3x+2y} \cdot \frac{1}{4x+3y} + \dots = \frac{1}{2xy}.$$

Namely, let x and y denote the k -dimension of the simple projective, or simple injective A -module, respectively. The indecomposable preprojective A -module P_t of length $2t-1$ is of dimension $tx + (t-1)y$. This means that the tilting module $P_t \oplus P_{t+1}$ contributes the summand $\frac{1}{tx+(t-1)y} \cdot \frac{1}{(t+1)x+ty}$. The sequence of tilting modules $P_1 \oplus P_2, P_2 \oplus P_3, \dots$ yields the various summands on the left side.

2.5 Combinatorial structure of modules

A lot of tilting theory is devoted to combinatorial considerations. The combinatorial invariants just discussed concern the Grothendieck group. But also the exceptional modules themselves have a combinatorial flavor: they are “tree modules” [42]. As we have mentioned, the orbit of a tilting module is open in the corresponding module variety, and this holds true with respect to all the usual topologies, in particular, the Zariski topology, but also the usual real topology in case the base field is \mathbb{R} or \mathbb{C} . This means that a slight change of the coefficients in any realization of T using matrices will not change the isomorphism class. Now in general to be able to change the coefficients slightly, will not allow to prescribe a finite set (for example $\{0, 1\}$) of coefficients which one may like to use: the corresponding matrices may just belong to the complement of the orbit. However, in case we deal with the path algebra of a quiver, the exceptional modules have this nice property: there always exists a realization of E using matrices with coefficients only 0 and 1. A stronger statement holds true: If E has dimension d , then there is a matrix realization which uses precisely $d-1$ coefficients equal to 1, and all the remaining ones are 0 (note that in order to be indecomposable, we need at least $d-1$ non-zero coefficients; thus we assert that really the minimal possible number of non-zero coefficients can be achieved).

2.6 Numerical linear algebra

Here we refer to the previous consideration: The relevance of 0-1-matrices in numerical linear algebra is well-known. Thus linear algebra problems, which can be rewritten as dealing with partial tilting modules, are very suitable for numerical algorithms, because of two reasons: one

can restrict to 0-1-matrices and the matrices to be considered involve only very few non-zero entries.

2.7 Module theory

Of course, tilting theory is part of module theory. It provides a very useful collection of non-trivial examples for many central notions in ring and module theory. The importance of modules without self-extensions has been realized a long time ago, for example one may refer to the lecture notes of Tachikawa from 1973. Different names are in use for such modules such as “splitters”.

It seems that the tilting theory exhibited for the first time a wide range of torsion pairs, with many different features: there are the splitting torsion pairs, which one finds in the module category of any tilted algebra, as well as the various non-split torsion pairs in the category $\text{mod } A$ itself. As we have mentioned in Part 1, tilting theory also gives rise to non-trivial examples of cotorsion pairs. And there are corresponding approximations, but also filtrations with prescribed factors. Questions concerning subcategories of module categories are considered in many of the contributions in this Handbook, in particular in Chapter 8 by Reiten, but also in the Chapters 9, 11 and 12 by Donkin, Trlifaj and Solberg, respectively.

We also should mention the use of **perpendicular categories**. Starting with an exceptional A -module E , the category E^\perp of all A -modules M with $\text{Hom}_A(E, M) = 0 = \text{Ext}_A^1(E, M)$ is again a module category, say $E^\perp \simeq \text{mod } A'$, where A' is a hereditary artin algebra with $n(A') = n(A) - 1$. These perpendicular categories are an important tool for inductive arguments and they can be considered as a kind of localisation.

Another notion should be illuminated here: recall that a left R -module M is said to have the **double centralizer property** (or to be balanced), provided the following holds: If we denote by S the endomorphism ring of ${}_R M$, say operating on the right on M , we obtain a right S -module M_S , and we may now look at the endomorphism ring R' of M_S . Clearly, there is a canonical ring homomorphism $R \longrightarrow R'$ (sending $r \in R$ to the left multiplication by r on M), and now we require that this map is surjective (in case M is a faithful R -module, so that the map $R \longrightarrow R'$ is injective,

this means that we can identify R and R' : the ring R is determined by the categorical properties of M , namely its endomorphism ring S , and the operation of S on the underlying abelian group of M). Modules with the double centralizer property are very important in ring and module theory. Tilting modules satisfy the double centralizer property and this is used in many different ways.

Of special interest is also the following **subquotient realization** of $\text{mod } A$. All the modules in \mathcal{T} are generated by T , all the modules in \mathcal{F} are cogenerated by τT . It follows that *for any A -module M , there exists an A -module X with submodules $X''' \subseteq X'' \subseteq X'$ such that X'' is a direct sum of copies of T , whereas X/X'' is a direct sum of copies of τT and such that $M = X'/X'''$* (it then follows that X''/X''' is the torsion submodule of M and X'/X'' its torsion-free factor module). In particular, we see that (\mathcal{F}, T) is a split torsion pair if and only if $\text{Ext}_A^1(\tau T, T) = 0$ (the equivalence of conditions (iii) and (iv) in Part 1). This is one of the results which stresses the importance of the bimodule $\text{Ext}_A^1(\tau T, T)$. Note that the extensions considered when we look at $\text{Ext}_A^1(\tau T, T)$ are opposite to those the Auslander-Reiten translation τ is famous for (namely the Auslander-Reiten sequences, they correspond to elements of $\text{Ext}_A^1(T_i, \tau T_i)$, where T_i is a non-projective indecomposable direct summand of T). We will return to the bimodule $\text{Ext}_A^1(\tau T, T)$ in Part 3.

2.8 Morita equivalence

Tilting theory is a powerful generalization of Morita equivalence. This can already be demonstrated very well by the reflection functors. When Gabriel showed that the representations of a Dynkin quiver correspond to the positive roots and thus only in an inessential way on the given orientation, this was considered as a big surprise. The BGP-reflection functors explain in which way the representation theory of a quiver is independent of the orientation: one can change the orientation of all the arrows in a sink or a source, and use reflection functors in order to obtain a bijection between the indecomposables. Already for the quivers of type \mathbb{A}_n with $n \geq 3$, we get interesting examples, relating say a serial algebra (using one of the two orientations with just one sink and one source) to a non-serial one.

The reflection functors are still near to classical Morita theory, since no

modules are really lost: here, we only deal with a kind of rearrangement of the categories in question. We deal with split torsion pairs $(\mathcal{F}, \mathcal{T})$ in $\text{mod } A$ and $(\mathcal{Y}, \mathcal{X})$ in $\text{mod } B$, with \mathcal{F} equivalent to \mathcal{X} and \mathcal{T} equivalent to \mathcal{Y} . Let us call two hereditary artin algebras *similar* provided they can be obtained from each other by a sequence of reflection functors. In case we consider the path algebra of a quiver which is a tree, then any change of orientation leads to a similar algebra. But already for the cycle with 4 vertices and 4 arrows, there are two similarity classes, namely the quiver $\tilde{A}_{3,1}$ with a path of length 3, and the quivers of type $\tilde{A}_{2,2}$.

One property of the reflection functors should be mentioned (since it will be used in Part 3). Assume that i is a sink for A (this means that the corresponding simple A -module $S(i)$ is projective). Let $S'(i)$ be the corresponding simple $\sigma_i A$ -module (it is injective). If M is any A -module, then $S(i)$ is not a composition factor of M if and only if $\text{Ext}_{\sigma_i A}^1(S'(i), \sigma_i M) = 0$. This is a situation, where the reflection functor yields a universal extension; for similar situations, let me refer to [39].

The general tilting process is further away from classical Morita theory, due to the fact that the torsion pair $(\mathcal{F}, \mathcal{T})$ in $\text{mod } A$ is no longer split.

2.9 Duality theory

Tilting theory is usually formulated as dealing with equivalences of subcategories (for example, that $\text{Hom}_A(T, -): \mathcal{T} \longrightarrow \mathcal{Y}$ is an equivalence). However, one may also consider it as a duality theory, by composing the equivalences obtained with the duality functor D , thus obtaining a duality between subcategories of the category $\text{mod } A$ and subcategories of the category $\text{mod } B^{\text{op}}$. The new formulations obtained in this way actually look more symmetrical, thus may be preferable. Of course, as long as we deal with finitely generated modules, there is no mathematical difference. This changes, as soon as one takes into consideration also modules which are not of finite length.

But the interpretation of tilting processes as dualities is always of interest, also when dealing with modules of finite length: In [2], Auslander considers (for R a left and right noetherian ring) the class $\mathcal{W}(R)$ of all left R -modules of the form $\text{Ext}_R^1(N_R, R_R)$, where N_R is a finitely generated right R -module, and he asserts that it would be of interest to know whether this class is always closed under submodules. A first example

of a ring R with $\mathcal{W}(R)$ not being closed under submodules has been exhibited by Huang [30], namely the path algebra $R = kQ$ of the quiver Q of type A_3 with 2 sources. Let us consider the general case when $R = A$ is a hereditary artin algebra. The canonical injective cogenerator $T = DA$ is a tilting module, thus $\text{Ext}_A^1(T, -)$ is a full and dense functor from $\text{mod } A$ onto $\mathcal{X}(T)$. The composition of functors

$$\text{mod } A^{\text{op}} \xrightarrow{D} \text{mod } A \xrightarrow{\text{Ext}_A^1(T, -)} \text{mod } A$$

is the functor $\text{Ext}_A^1(-, A_A)$, thus we see that $\mathcal{W}(A) = \mathcal{X}(T)$. On the other hand, $\mathcal{T}(T)$ are the injective A -modules, they are mapped under $\text{Hom}_A(T, -)$ to the class $\mathcal{Y}(T)$, and these are the projective A -modules. It follows that $\mathcal{X}(T)$ is the class of all A -modules without an indecomposable projective direct summand. As a consequence, $\mathcal{W}(A) = \mathcal{X}(T)$ is closed under submodules if and only if A is a Nakayama algebra. (It is an easy exercise to show that $\mathcal{X}(T)$ is closed under submodules if and only if the injective envelope of any simple projective module is projective, thus if and only if A is a Nakayama algebra).

It should be stressed that Morita himself seemed to be more interested in dualities than in equivalences. What is called Morita theory was popularized by P.M. Cohn and H. Bass, but apparently was considered by Morita as a minor addition to his duality theory. When Gabriel heard about tilting theory, he immediately interpreted it as a non-commutative analog of Roos duality.

The use of general tilting modules as a source for dualities has been shown to be very fruitful in the representation theory of algebraic groups, of Lie algebras and of quantum groups. This is explained in detail in Chapter 9 by Donkin. As a typical special case one should have the classical **Schur-Weyl duality** in mind, which relates the representation theory of the general linear groups and that of the symmetric groups, see Chapter 9 by Donkin, but also [36].

In the realm of commutative complete local noetherian rings, Auslander and Reiten [4] considered **Cohen-Macaulay rings** with dualizing module W . They showed that W is the only basic cotilting module. On the basis of this result, they introduced the notion of a dualizing module for arbitrary artin algebras.

2.10 Schofield induction

This is an inductive procedure for constructing all exceptional modules starting with the simple ones, by forming exact sequences of the following kind: Assume we deal with a hereditary k -algebra, where k is algebraically closed, and let E_1, E_2 be orthogonal exceptional modules with $\dim \operatorname{Ext}_A^1(E_1, E_2) = t$ and $\operatorname{Ext}_A^1(E_2, E_1) = 0$. Then, for every pair (a_1, a_2) of positive natural numbers satisfying $a_1^2 + a_2^2 - ta_1a_2 = 1$, there exists (up to equivalence) a unique non-split exact sequence of the form

$$0 \longrightarrow E_2^{a_2} \longrightarrow E \longrightarrow E_1^{a_1} \longrightarrow 0$$

(call it a *Schofield sequence*). Note that the middle term of such a Schofield sequence is exceptional again, and it is an amazing fact that starting with the simple A -modules without self-extension, all the exceptional A -modules are obtained in this way. Even a stronger assertion is true: If E is an exceptional module with support of cardinality s (this means that E has precisely s isomorphism classes of composition factors), then there are precisely $s - 1$ Schofield sequences with E as middle term. What is the relation to tilting theory? Starting with E one obtains the Schofield sequences by using the various indecomposable direct summands of its Bongartz complement as an A/I -module, where I is the annihilator of E : the $s - 1$ summands yield the $s - 1$ sequences [41].

2.11 Exceptional sequences, mutations

Note that a tilted algebra is always directed: the indecomposable summands of a tilting module E_1, \dots, E_m can be ordered in such a way that $\operatorname{Hom}_A(E_i, E_j) = 0$ for $i > j$. We may call such a sequence (E_1, \dots, E_m) a tilting sequence, and there is the following generalization which is of interest in its own (and which was considered by the Rudakov school [44]): Call (E_1, \dots, E_m) an *exceptional sequence* provided all the modules E_i are exceptional A -modules and $\operatorname{Hom}_A(E_i, E_j) = 0$ and $\operatorname{Ext}_A^1(E_i, E_j) = 0$ for $i > j$. There are many obvious examples of exceptional sequences which are not tilting sequences, the most important one being sequences of simple modules in case the Ext-quiver of the simple modules is directed. Now one may be afraid that this generalization could yield too many additional sequences, but this is not the case. In general most of the exceptional sequences are tilting sequences! An exceptional sequence

(E_1, \dots, E_m) is said to be complete provided $m = n(A)$ (the number of simple A -modules). There is a braid group action on the set of complete exceptional sequences, and this action is transitive [18, 40]. This means that all the exceptional sequences can be obtained from each other by what one calls “mutations”. As a consequence, one obtains the following: If (E_1, \dots, E_n) is a complete exceptional sequence, then there is a permutation π such that $\text{End}_A(E_i) = \text{End}_A(S_{\pi(i)})$, where S_1, \dots, S_n are the simple A -modules. In particular, this means that for any tilted algebra B , the radical factor algebras of A and of B are Morita equivalent.

An exceptional module E defines also **partial reflection functors** [39] as follows: consider the following full subcategories of $\text{mod } A$. Let \mathcal{M}^E be given by all modules M with $\text{Ext}_A^1(E, M) = 0$ such that no non-zero direct summand of M is cogenerated by E ; dually, let \mathcal{M}_E be given by all modules M with $\text{Ext}_A^1(M, E) = 0$ such that no non-zero direct summand of M is generated by E ; also, let \mathcal{M}^{-E} be given by all M with $\text{Hom}_A(M, E) = 0$ and \mathcal{M}_{-E} by all M with $\text{Hom}_A(E, M) = 0$. For any module M , let $\sigma^{-E}(M)$ be the intersection of the kernels of maps $M \rightarrow E$ and $\sigma_{-E}(M) = M/t_E M$, where $t_E M$ is the sum of the images of maps $M \rightarrow E$. In this way, we obtain equivalences

$$\sigma^{-E}: \mathcal{M}^E/\langle E \rangle \rightarrow \mathcal{M}^{-E}, \quad \text{and} \quad \sigma_{-E}: \mathcal{M}_E/\langle E \rangle \rightarrow \mathcal{M}_{-E}.$$

Here $\langle E \rangle$ is the ideal of all maps which factor through $\text{add } E$. The reverse functors σ^E and σ_E are provided by forming universal extensions by copies of E (from above or below, respectively). Note that on the level of dimension vectors these partial functors $\sigma = \sigma^E, \sigma^{-E}, \sigma_E, \sigma_{-E}$ yield the usual reflection formula:

$$\dim \sigma(M) = \dim M - \frac{2\langle \dim M, \dim E \rangle}{\langle \dim E, \dim E \rangle} \dim E.$$

2.12 Slices

An artin algebra B is a tilted algebra if and only if $\text{mod } B$ has a slice. Thus the existence of slices characterizes the tilted algebras. The necessity to explain the importance of slices has to be mentioned as a (further) impetus for the development of tilting theory. In my 1979 Ottawa lectures, I tried to describe several module categories explicitly. At that time, the knitting of preprojective components was one of the main tools, and I used slices in such components in order to guess what later turned

out to be tilting functors, namely functorial constructions using pushouts and pullbacks. The obvious question about a possible theoretical foundation was raised by several participants, but it could be answered only a year later at the Puebla conference. Under minor restrictions (for example, the existence of a sincere indecomposable module) preprojective components will contain slice modules and these are tilting modules with a hereditary endomorphism ring! This concerns the concealed algebras to be mentioned below, but also all the representation-directed algebras. Namely, using covering theory, the problem of describing the structure of the indecomposable modules over a representation-finite algebra is reduced to the representation-directed algebras with a sincere indecomposable module, and such an algebra is a tilted algebra, since it obviously has a slice module.

In dealing with an artin algebra of finite representation type, and looking at its Auslander-Reiten quiver, one may ask for sectional subquivers say of Euclidean types. Given such a subquiver Γ , applying several times τ or τ^{-1} (and obtaining in this way “parallel” subquivers), one has to reach a projective, or an injective vertex, respectively. Actually, Bautista and Brenner have shown that the number of parallel subquivers is bounded, the bound is called the **replication number**. If one is interested in algebras with optimal replication numbers, one only has to look at representation-finite tilted algebras of Euclidean type. Note that given a hereditary algebra A of Euclidean type and a tilting A -module T , then $B = \text{End}_A(T)$ is representation-finite if and only if T has both preprojective and preinjective indecomposable direct summands.

It is natural to look inside preprojective and preinjective components for slices. In 1979 one did not envision that there could exist even regular components with a slice module. But any connected wild hereditary algebra with at least three simple modules has a regular tilting module T , and then the **connecting component** of $B = \text{End}_A(T)$ is regular. One should be aware that the category $\text{mod } B$ looks quite amazing: the connecting component (which is a regular component in this case) connects two wild subcategories, like a tunnel between two busy regions. Inside the tunnel, there are well-defined paths for the traffic, and the traffic goes in just one direction.

* * *

Tilting modules can be used to study **specific classes of artin algebras**. Some examples have been mentioned already. We have noted that

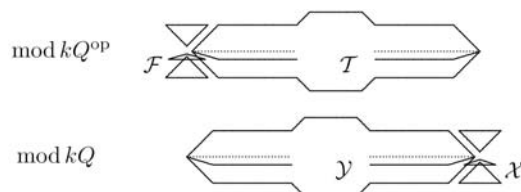
all the representation finite k -algebras, with k algebraically closed, can be described using tilted algebras (the condition on k is needed in order to be able to use covering theory). We obtain in this way very detailed information on the structure of the indecomposables. One of the first uses of tilting theory concerned the representation-finite tree algebras, see Chapter 3 by Brüstle.

2.13 Concealed algebras

By definition, B is a *concealed* algebra, provided $B = \text{End}_A(T)$, where T is a preprojective A -module with A hereditary. The tame concealed k -algebras B where k is algebraically closed, have been classified by Happel and Vossieck, and Bongartz has shown in which way they can be used in order to determine whether a k -algebra is representation-finite.

2.14 Representations of posets

The representation theory of posets always has been considered as an important tool when studying questions in representation theory in general: there are quite a lot of reduction techniques which lead to a vector space with a bunch of subspaces, but the study of a vector space with a bunch of subspaces with some inclusions prescribed, really concerns the representation theory of the corresponding poset. On the other hand, the representation theory of finite posets is very similar to the representation theory of some quite well-behaved algebras, and the relationship is often given by tilting modules. For example, when dealing with a disjoint union of chains, then we deal with the subspace representations of a star quiver Q (the quiver Q is obtained from a finite set of linearly oriented quivers of type \mathbb{A} , with all the sinks identified to one vertex, the center of the star). If c is the center of the star quiver Q , then the subspace representations are the torsion-free modules of the (split) torsion pair $(\mathcal{V}, \mathcal{X})$, with \mathcal{X} being the representations V of Q such that $V_c = 0$. We also may consider the opposite quiver Q^{op} and the (again split) torsion pair $(\mathcal{F}, \mathcal{T})$, where now \mathcal{F} are the representations V of Q^{op} with $V_c = 0$. The two orientations used here are obtained by a sequence of reflections, and the two split torsion pairs $(\mathcal{F}, \mathcal{G}), (\mathcal{V}, \mathcal{X})$ are given by a tilting module which is a slice module:



2.15 The Crawley-Boevey-Kerner functors

If R is an artin algebra and W an R -module, let us write $\langle \tau^\bullet W \rangle$ for the ideal of $\text{mod } R$ of all maps which factor through a direct sum of modules of the form $\tau^z W$ with $z \in \mathbb{Z}$. We say that the module categories $\text{mod } R$ and $\text{mod } R'$ are *almost equivalent* provided there is an R -module W and an R' -module W' such that the categories $\text{mod } R / \langle \tau^\bullet W \rangle$ and $\text{mod } R' / \langle \tau^\bullet W' \rangle$ are equivalent. The Crawley-Boevey-Kerner functors were introduced in order to show the following: *If k is a field and Q and Q' are connected wild quivers, then the categories $\text{mod } kQ$ and $\text{mod } kQ'$ are almost equivalent.* The proof uses tilting modules, and the result may be rated as one of the most spectacular applications of tilting theory. Thus it is worthwhile to outline the essential ingredients. This will be done below.

Here are some remarks concerning almost equivalent categories. It is trivial that the module categories of all representation-finite artin algebras are almost equivalent. If k is a field, and Q, Q' are tame connected quivers, then $\text{mod } kQ$ and $\text{mod } kQ'$ are almost equivalent only if Q and Q' have the same type $(\tilde{A}_{pq}, \tilde{D}_n, \tilde{E}_6, \tilde{E}_7, \tilde{E}_8)$. Let us return to wild quivers Q, Q' and a Crawley-Boevey-Kerner equivalence

$$\eta: \text{mod } kQ / \langle \tau^\bullet W \rangle \longrightarrow \text{mod } kQ' / \langle \tau^\bullet W' \rangle,$$

with finite length modules W, W' . Consider the case of an uncountable base field k , so that there are uncountably many isomorphism classes of indecomposable modules for $R = kQ$ as well as for $R' = kQ'$. The ideals $\langle \tau^\bullet W \rangle$ and $\langle \tau^\bullet W' \rangle$ are given by the maps which factor through a countable set of objects, thus nearly all the indecomposable modules remain indecomposable in $\text{mod } kQ / \langle \tau^\bullet W \rangle$ and $\text{mod } kQ' / \langle \tau^\bullet W' \rangle$, and non-isomorphic ones (which are not sent to zero) remain non-isomorphic. In addition, one should note that the equivalence η is really constructive (not set-theoretical rubbish), with no unfair choices whatsoever. This will be clear from the further discussion.

Nearly all quivers are wild. For example, if we consider the m -subspace quivers $Q(m)$, then one knows that $Q(m)$ is wild provided $m \geq 5$. Let us concentrate on a comparison of the wild quivers $Q(6)$ and $Q(5)$. To assert that Q, Q' are wild quivers means that there are full embeddings $\text{mod } kQ \longrightarrow \text{mod } kQ'$ and $\text{mod } kQ' \longrightarrow \text{mod } kQ$. But the Crawley-Boevey-Kerner theorem provides a completely new interpretation of what “wildness” is about. The definition of “wildness” itself is considered as quite odd, since it means in particular that there is a full embedding of $\text{mod } kQ(6)$ into $\text{mod } kQ(5)$. One may reformulate the wildness assertion as follows: any complication which occurs for 6 subspaces can be achieved (in some sense) already for 5 subspaces. But similar results are known in mathematics, since one is aware of other categories which allow to realize all kinds of categories as a subcategory. Also, “wildness” may be interpreted as a kind of fractal behaviour: inside the category $\text{mod } kQ(5)$ we find proper full subcategories which are equivalent to $\text{mod } kQ(5)$, again a quite frequent behaviour. These realization results are concerned with small parts of say $\text{mod } kQ(5)$; one looks at full subcategories of the category $\text{mod } kQ$ which have desired properties, but one does not try to control the complement. This is in sharp contrast to the Crawley-Boevey-Kerner property which provides a **global** relation between $\text{mod } kQ(5)$ and $\text{mod } kQ(6)$, actually, between the module categories of any two wild connected quivers. In this way we see that there is a kind of homogeneity property of wild abelian length categories which had not been anticipated before.

The Crawley-Boevey-Kerner result may be considered as a sort of Schröder-Bernstein property for abelian length categories. Recall that the Schröder-Bernstein theorem asserts that if two sets S, S' can be embedded into each other, then there is a bijection $S \longrightarrow S'$. For any kind of mathematical structure with a notion of embedding, one may ask whether two objects are isomorphic in case they can be embedded into each other. Such a property is very rare, even if we replace the isomorphism requirement by some weaker requirement. But this is what is asserted by the Crawley-Boevey-Kerner property.

Let us outline the construction of η . We start with a connected wild hereditary artin algebra A , and a regular exceptional module E which is quasi-simple (this means that the Auslander-Reiten sequence ending in E has indecomposable middle term, call it $\mu(E)$), such a module exists provided $n(A) \geq 3$. Denote by E^\perp the category of all A -modules M such that $\text{Hom}_A(E, M) = 0 = \text{Ext}_A^1(E, M)$. One knows (Geigle-Lenzing,

Strausemi simple) that E^\perp is equivalent to the category $\text{mod } C$, where C is a connected wild hereditary algebra C and $n(C) = n(A) - 1$. The aim is to compare the categories $\text{mod } C$ and $\text{mod } A$, they are shown to be almost equivalent.

It is easy to see that the module $\mu(E)$ belongs to E^\perp , thus it can be regarded as a C -module. Since $E^\perp = \text{mod } C$, there is a projective generator T' in E^\perp with $\text{End}_A(T') = C$. Claim: $T' \oplus E$ is a tilting module. For the proof we only have to check that $\text{Ext}_A^1(T', E) = 0$. Since T' is projective in E^\perp , it follows that $\text{Ext}_A^1(T', \mu(E)) = 0$. However, there is a surjective map $\mu(E) \rightarrow E$ and this induces a surjective map $\text{Ext}_A^1(T', \mu(E)) \rightarrow \text{Ext}_A^1(T', E)$.

As we know, the tilting module $T = \bar{T} \oplus E$ defines a torsion pair $(\mathcal{F}, \mathcal{T})$, with \mathcal{T} the A -modules generated by T . Let us denote by $\tau_{\mathcal{T}} M = t\tau_A M$ the torsion submodule⁶ of $\tau_A M$. The functor η is now defined as follows:

$$\eta(M) = \lim_{t \rightarrow \infty} \tau_A^t \tau_{\mathcal{T}}^{-2t} \tau_C^t(M).$$

One has to observe that the limit actually stabilizes: for large t , there is no difference whether we consider t or $t + 1$. The functor η is full, the image is just the full subcategory of all regular A -modules. There is a non-trivial kernel: a map is sent to zero if and only if it belongs to $\langle \tau^\bullet W \rangle$, where $W = C \oplus \mu(E) \oplus DC$. Also, let $W' = A \oplus DA$. Then η is an equivalence

$$\eta: \text{mod } C / \langle \tau^\bullet W \rangle \rightarrow \text{mod } A / \langle \tau^\bullet W' \rangle.$$

One may wonder how special the assumptions on A and C are. Let us say that A dominates C provided there exists a regular exceptional module E which is quasi-simple with $\text{mod } C$ equivalent to E^\perp . Given any two wild connected quivers Q, Q' , there is a sequence of wild connected quivers $Q = Q_0, \dots, Q_t = Q'$ such that kQ_i either dominates or is dominated by kQ_{i-1} , for $1 \leq i \leq t$. This implies that the module categories of all wild path algebras are almost equivalent.

The equivalence η can be constructed also in a different way [35], using

⁶ The notation shall indicate that this functor $\tau_{\mathcal{T}}$ has to be considered as an Auslander-Reiten translation: it is the relative Auslander-Reiten translation in the subcategory \mathcal{T} . And there is the equivalence $\mathcal{T} \simeq \mathcal{Y}$, where \mathcal{Y} is a full subcategory of $\text{mod } B$, with $B = \text{End}_A(T)$. Since \mathcal{Y} is closed under τ in $\text{mod } B$, the functor $\tau_{\mathcal{T}}$ corresponds to the Auslander-Reiten translation τ_B in $\text{mod } B$.

partial reflection functors. Let $E(i) = \tau^i E$, for all $i \in \mathbb{Z}$. Note that for any regular A -module M , one knows that

$$\mathrm{Hom}_A(M, E(-t)) = 0 = \mathrm{Hom}_A(E(t), M) \quad \text{for} \quad t \gg 0,$$

according to Baer and Kerner. Thus, if we choose t sufficiently large, we can apply the partial reflection functors $\sigma^{E(-t)}$ and $\sigma_{E(t)}$ to M . The module obtained from M has the form

$$\frac{\frac{E(-t) \ E(-t) \ \cdots \ E(-t)}{M}}{E(t) \ E(t) \ \cdots \ E(t)}$$

and belongs to

$$\mathcal{M}^{E(-t)} \cap \mathcal{M}_{E(t)} \subseteq \mathcal{M}^{-E(-t+1)} \cap \mathcal{M}_{-E(t-1)}.$$

Thus we can proceed, applying now $\sigma^{E(-t+1)}$ and $\sigma_{E(t-1)}$. We use induction, the last partial reflection functors to be applied are the functors $\sigma^{E(-1)}$, $\sigma_{E(1)}$, and then finally σ^E . In this way we obtain a module in

$$\mathcal{M}_{E(1)} \cap \mathcal{M}^E = E^\perp$$

as required. It has the following structure:

$$\frac{\frac{\frac{\frac{E \ \cdots \ E}{E(-1) \ \cdots \ E(-1)}}{\vdots}}{\frac{E(-t) \ \cdots \ E(-t)}{M}}}{\frac{E(t) \ \cdots \ E(t)}{E(1) \ \cdots \ E(1)}}$$

2.16 The shrinking functors for the tubular algebras

Again these are tilting functors (here, A no longer is a hereditary artin algebra, but say a canonical algebra - we are still in the realm of the “T” displayed in Part 1, now even in its center), and such functors belong to the origin of the development. If one looks at the Brenner-Butler tilting paper, the main examples considered there were of this kind. So one of the first applications of tilting theory was to show the similarity of

the module categories of various tubular algebras. And this is also the setting which later helped to describe in detail the module category of a tubular algebra: one uses the shrinking functors in order to construct all the regular tubular families, as soon as one is known to exist.

2.17 Self-injective algebras

Up to coverings and (in characteristic 2) deformations, the trivial extensions of the tilted algebras of Dynkin type (those related to the left arm of the “T” displayed in Part 1) yield all the representation-finite self-injective algebras (recall that the trivial extension of an algebra R is the semi-direct product $R \ltimes DR$ of R with the dual module DR). In private conversation, such a result was conjectured by Tachikawa already in 1978, and it was the main force for the investigations of him and Wakamatsu, which he presented at the Ottawa conference in 1979. There he also dealt with the trivial extension of a tilted algebra of Euclidean type (the module category has two tubular families). This motivated Hughes-Waschbüsch to introduce the concept of a repetitive algebra. But it is also part of one of the typical quarrels between Zürich and the rest of the world: with Gabriel hiding the Hughes-Waschbüsch manuscript from Bretscher-Läser-Riedtmann (asking a secretary to seal the envelope with the manuscript and to open it only several months later...), so that they could proceed “independently”.

The representation theory of artin algebras came into limelight when Dynkin diagrams popped up for representation-finite algebras. And this occurred twice, first for hereditary artin algebras in the work of Gabriel (as the Ext-quiver), but then also for self-injective algebras in the work of Riedtmann (as the tree class of the stable Auslander-Reiten quiver). The link between these two classes of rings is furnished by tilted algebras and their trivial extensions. As far as I know, it is Tachikawa who deserves the credit for this important insight.

The reference to trivial extensions of tilted algebras actually closes a circle in our considerations, due to another famous theorem of Happel. We have started with the fact that tilting functors provide derived equivalences. Thus the derived category of a tilted algebra can be identified with the derived category of a hereditary artin algebra. For all artin algebras R of finite global dimension (in particular our algebras A and B), there is an equivalence between $D^b(\text{mod } R)$ and the stable module

category of the repetitive algebra \hat{R} . But \hat{R} is just a \mathbb{Z} -covering of the trivial extension of R .

2.18 Artin algebras with Gorenstein dimension at most 1

We have mentioned that the two classes of algebras: the selfinjective ones and the hereditary ones, look very different, but nevertheless they have some common behaviour. Auslander and Reiten [4] have singled out an important property which they share, they are Gorenstein algebras of Gorenstein dimension at most 1. An artin algebra A is called *Gorenstein*⁷ provided ${}_A DA$ has finite projective dimension and ${}_A A$ has finite injective dimension. For Gorenstein algebras, $\text{proj-dim } {}_A DA = \text{inj-dim } {}_A A$, and this number is called the *Gorenstein dimension* of A . It is not known whether the finiteness of the projective dimension of ${}_A DA$ implies the finiteness of the injective dimension of ${}_A A$. It is conjectured that this is the case: this is the Gorenstein symmetry conjecture, and this conjecture is equivalent to the conjecture that the small finitistic dimension of A is finite. The artin algebras of Gorenstein dimension 0 are the selfinjective algebras. An artin algebra has Gorenstein dimension at most 1 if and only if DA is a tilting module (of projective dimension at most 1).

If A is a Gorenstein algebra of Gorenstein dimension at most 1, then there is a strict separation of the indecomposable modules: an A -module M of finite projective dimension or finite injective dimension satisfies both $\text{proj-dim } M \leq 1$, $\text{inj-dim } M \leq 1$. (The proof is easy: Assume $\text{proj-dim } M \leq m$, thus the m -th syzygy module $\Omega_m(M)$ is projective. Now for any short exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$, it is clear that $\text{inj-dim } X \leq 1, \text{inj-dim } Y \leq 1$ imply $\text{inj-dim } Z \leq 1$. One applies this inductively to the exact sequences $\Omega_i(M) \rightarrow P_i \rightarrow \Omega_{i-1}(M)$, where P_i is projective, starting with $i = m$ and ending with $i = 0$. This shows that $\text{inj-dim}(M) \leq 1$. The dual argument shows that a module of finite injective dimension has projective dimension at most 1.) As a consequence, if A is not hereditary, then the global dimension of A

⁷ This definition is one of the many possibilities to generalize the notion of a commutative Gorenstein ring to a non-commutative setting. Note that a commutative artin algebra R is a Gorenstein ring if and only if R is selfinjective. Of course, a commutative connected artin algebra R is a local ring, and a local ring has a non-zero module of finite projective dimension only in case R is selfinjective.

is infinite. Also, if P is an indecomposable projective A -module, then either its radical is projective or else the top of P is a simple module which has infinite projective and infinite injective dimension.

Until very recently, the interest in artin algebras of Gorenstein dimension at most 1 has been quite moderate, the main reason being a lack of tempting examples: of algebras which are neither selfinjective nor hereditary. But now there is a wealth of such examples, as we will see in Part 3.

* * *

We hope that we have convinced the reader that the use of tilting modules and tilted algebras lies at the heart of nearly all the major developments in the representation theory of artin algebras in the last 25 years. In this report we usually restrict to tilting modules in the narrow sense (as being finite length modules of projective dimension at most 1). In fact, most of the topics mentioned are related to tilting A -modules T , where A is a hereditary artin algebra (so that there is no need to stress the condition $\text{proj-dim } T \leq 1$). However, the following two sections will widen the viewpoint, taking into account also various generalizations.

2.19 Representations of semisimple complex Lie algebras and algebraic groups

The highest weight categories which arise in the representation theory of semisimple complex Lie algebras and algebraic groups can be analyzed very well using quasi-hereditary artin algebras as introduced by Cline-Parshall-Scott. One of the main features of such a quasi-hereditary artin algebra is its characteristic module, this is a tilting module (of finite projective dimension). Actually, the experts use a different convention, calling its indecomposable direct summands “tilting modules”, see Chapter 9 by Donkin. If T is the characteristic module, then $\text{add } T$ consists of the A -modules which have both a standard filtration and a costandard filtration, and it leads to a duality theory which seems to be of great interest.

2.20 The homological conjectures

The homological conjectures are one of the central themes of module theory, so clearly they deserve special interest. They go back to mathematicians like Nakayama, Eilenberg, Auslander, Bass, but also Rosenberg, Zelinsky, Buchsbaum and Nunke should be mentioned, and were formulated between 1940 and 1960. Unfortunately, there are no written accounts about the origin, but we may refer to surveys by Happel, Smalø and Zimmermann-Huisgen. The modern development in representation theory of artin algebras was directed towards a solution of the Brauer-Thrall conjectures, and there was for a long time a reluctance to work on the homological conjectures. The investigations concerning the various representation types have produced a lot of information on special classes of algebras, but for these algebras the homological conjectures are usually true for trivial reasons. As Happel has pointed out, the lack of knowledge of non-trivial examples may very well mean that counter-examples could exist. Here is a short discussion of this topic, in as far as modules without self-extensions are concerned.

Let me start with the Nakayama conjecture which according to B. Müller can be phrased as follows: If R is an artin algebra and M is a generator and cogenerator for $\text{mod } R$ with $\text{Ext}_R^i(M, M) = 0$ for all $i \geq 1$, then M has to be projective. Auslander and Reiten [3] proposed in 1975 that the same conclusion should hold even if M is not necessarily a cogenerator (this is called the “generalized Nakayama conjecture”). This incorporates a conjecture due to Tachikawa (1973): If R is self-injective and M is an R -module with $\text{Ext}_R^i(M, M) = 0$ for all $i > 0$, then M is projective. The relationship of the generalized Nakayama conjecture with tilting theory was noted by Auslander and Reiten [3, 4]. Then there is the conjecture on the finiteness of the number of complements of an almost complete partial tilting module, due to Happel and Unger. And there is a conjecture made by Beligiannis and Reiten [5], called the Wakamatsu tilting conjecture (because it deals with Wakamatsu tilting modules, see Chapter 8 by Reiten): If T is a Wakamatsu tilting module of finite projective dimension, then T is a tilting module. The Wakamatsu tilting conjecture implies the generalized Nakayama conjecture (apparently, this was first observed by Buan) and also the Gorenstein symmetry conjecture, see [5]. In a joint paper, Mantese and Reiten [37] showed that it is implied by the finitistic dimension conjecture, and that it implies the conjecture on a finite number of complements, which according to Buan

and Solberg is known to imply the generalized Nakayama conjecture. There is also the equivalence of the generalized Nakayama conjecture with projective almost complete partial tilting modules having only a finite number of complements (Happel-Unger, Buan-Solberg, both papers are in the Geiranger proceedings). Coelho, Happel and Unger proved that the finitistic dimension conjecture implies the conjecture on a finite number of complements.

Further relationship of tilting theory with the finitistic dimension conjectures is discussed in detail in Chapter 11 by Trlifaj and in Chapter 12 by Solberg. But also other results presented in the Handbook have to be seen in this light. We know from Auslander and Reiten, that the finitistic dimension of an artin algebra R is finite, in case the subcategory of all modules of finite projective dimension is contravariantly finite in $\text{mod } R$. This has been the motivation to look at the latter condition carefully (see for example Chapter 10 by Unger).

With respect to applications outside of ring and module theory, many more topics could be mentioned. We have tried to stay on a basic level, whereas there are a lot of mathematical objects which are derived from representation theoretical data and this leads to a fruitful interplay (dealing with questions on quantum groups, with the shellability of simplicial complexes, or with continued fraction expansions of real numbers): There are many unexpected connections to analysis, to number theory, to combinatorics — and again, it is usually the tilting theory which plays an important role.

3 The new cluster tilting approach

Let me repeat: at the time the Handbook was conceived, there was a common feeling that the tilted algebras (as the core of tilting theory) were understood well and that this part of the theory had reached a sort of final shape. But in the meantime this has turned out to be wrong: the tilted algebras have to be seen as factor algebras of the so called cluster tilted algebras, and it may very well be, that in future the cluster tilted algebras and the cluster categories will topple the tilted algebras. The impetus for introducing and studying cluster tilted algebras came from outside, in a completely unexpected way. We will mention below some

of the main steps of this development. But first let me jump directly to the relevant construction.

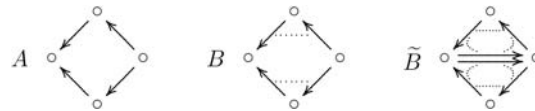
3.1 The cluster tilted algebras

We return to the basic setting, the hereditary artin algebra A , the tilting A -module T and its endomorphism ring B . Consider the semi-direct ring extension

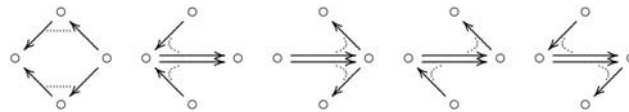
$$\tilde{B} = B \ltimes \text{Ext}_B^2(DB, B).$$

This is called the *cluster tilted algebra* corresponding to B . Since this is the relevant definition, let me say a little more about this construction⁸ : \tilde{B} has B as a subring, and there is an ideal J of \tilde{B} with $J^2 = 0$, such that $\tilde{B} = B \oplus J$ as additive groups and J is as a B - B -bimodule isomorphic to $\text{Ext}_B^2(DB, B)$; in order to construct \tilde{B} one may take $B \oplus \text{Ext}_B^2(DB, B)$, with componentwise addition, and one uses $(b, x)(b', x') = (bb', bx' + xb')$, for $b, b' \in B$ and $x, x' \in \text{Ext}_B^2(DB, B)$ as multiplication. The definition shows that \tilde{B} can be considered as a \mathbb{Z} -graded (or also $\mathbb{Z}/2$ -graded) algebra, with $(\tilde{B})_0 = B$ and $(\tilde{B})_1 = J$.

We consider again the example of B given by a square with two zero relations. Here $\text{Ext}_B^2(DB, B)$ is 8-dimensional and \tilde{B} is a 16-dimensional algebra:



Non-isomorphic tilted algebras B may yield isomorphic cluster tilted algebras \tilde{B} . Here are all the tilted algebras which lead to the cluster tilted algebra just considered:



It is quite easy to write down the **quiver of a cluster-tilted algebra**. Here, we assume that we deal with k -algebras, where k is an algebraically

⁸ One may wonder what properties the semi-direct product $R \ltimes \text{Ext}_R^2(DR, R)$ for any artin algebra R has in general (at least in case R has global dimension at most 2); it seems that this question has not yet been studied.

closed field. We get the quiver of \tilde{B} from the quiver with relations of B by just replacing the dotted arrows⁹ by solid arrows in opposite direction [1]. The reason is the following: Let us denote by N the radical of B . Then $N \oplus J$ is the radical of $\tilde{B} = B \ltimes J$, and $N^2 \oplus (NJ + JN)$ is equal to the square of the radical of \tilde{B} . This shows that the additional arrows for \tilde{B} correspond to $J/(NJ + JN)$. Note that $J/(NJ + JN)$ is the top of the B - B -bimodule J . Now the top of the bimodule $\text{Ext}_B^2(DB, B)$ is $\text{Ext}_B^2(\text{soc}_B DB, \text{top}_B B)$, since B has global dimension at most 2. It is well-known that $\text{Ext}_B^2(\text{soc}_B DB, \text{top}_B B)$ describes the relations of the algebra B , and we see in this way that relations for B correspond to the additional arrows for \tilde{B} . Since $\text{rad } \tilde{B} = \text{rad } B \oplus J$ and J is an ideal of \tilde{B} with $J^2 = 0$, we also see: *If $(\text{rad } B)^t = 0$, then $(\text{rad } \tilde{B})^{2t} = 0$.* The quiver of any tilted algebra is directed, thus $(\text{rad } B)^{n(B)} = 0$, therefore $(\text{rad } \tilde{B})^{2n(B)} = 0$.

The recipe for obtaining the quiver of \tilde{B} shows that there are always oriented cyclic paths (unless B is hereditary). However, such a path is always of length at least 3. Namely, since the quiver of B has no loops, there cannot be any relation for B starting and ending at the same vertex. Thus, the quiver of \tilde{B} cannot have a loop [8]. Also, Happel ([27], Lemma IV.1.11) has shown that for simple B -modules S, S' with $\text{Ext}_B^1(S, S') \neq 0$ one has $\text{Ext}_B^2(S, S') = 0$. This means that the quiver of \tilde{B} cannot have a pair of arrows in opposite direction [11].

It should be of interest whether knowledge about the quiver with relations of a cluster tilted algebra \tilde{B} can provide new insight into the structure of the tilted algebras themselves. There is a lot of ongoing research on cluster tilted algebras, let us single out just one result. Assume that we deal with k -algebras, where k is algebraically closed. Then: *Any cluster tilted k -algebra of finite representation type is uniquely determined by its quiver* [12]. This means: in the case of finite representation type, the quiver determines the relations! What happens in general is still under investigation.

If A is a hereditary artin algebra and T a tilting A -module with endomorphism ring B , we have introduced the corresponding cluster tilted algebras as the algebra $\tilde{B} = B \ltimes J$, with B - B -bimodule $J = \text{Ext}_B^2(DB, B)$.

⁹ Actually, the usual convention for indicating relations is to draw dotted lines, not dotted arrows. However, these dotted lines are to be seen as being directed, since the corresponding relations are linear combinations of paths with fixed starting point and fixed end point.

The original definition of \tilde{B} by Buan, Marsh and Reiten [10] used another description of J , namely $J = \text{Ext}_A^1(T, \tau^{-1}T)$, and it was observed by Assem, Brüstle and Schiffler [1] that the bimodules $\text{Ext}_A^1(T, \tau^{-1}T)$ and $\text{Ext}_B^2(DB, B)$ are isomorphic¹⁰ (using this Ext^2 -bimodule has the advantage that it refers only to the algebra B itself, and not to T). It was Zhu Bin [51] who stressed that cluster tilted algebras should be explored as semi-direct ring extensions.

Since this isomorphism is quite essential, let me sketch an elementary proof, without reference to derived categories. Let V be the universal extension of τT by copies of T from above, thus there is an exact sequence

$$(*) \quad 0 \longrightarrow \tau T \longrightarrow V \longrightarrow T^m \longrightarrow 0$$

for some m , and $\text{Ext}_A^1(T, V) = 0$. Applying $\text{Hom}_A(-, T)$ to $(*)$ shows that $\text{Ext}_A^1(V, T) \simeq \text{Ext}_A^1(\tau T, T)$. Applying $\text{Hom}_A(T, -)$ to $(*)$ yields the exact sequence

$$0 \longrightarrow \text{Hom}_A(T, V) \longrightarrow \text{Hom}_A(T, T^m) \longrightarrow \text{Ext}_A^1(T, \tau T) \longrightarrow 0.$$

This is an exact sequence of B -modules and $\text{Hom}_A(T, T^m)$ is a free B -module, thus we see that $\text{Hom}_A(T, V)$ is a syzygy module for the B -module $\text{Ext}_A^1(T, \tau T)$. But the latter means that

$$\text{Ext}_B^2(\text{Ext}_A^1(T, \tau T), {}_B B) \simeq \text{Ext}_B^1(\text{Hom}_A(T, V), {}_B B).$$

The left hand side is nothing else than $\text{Ext}_B^2(DB, B)$, since the B -module DB and $\text{Ext}_A^1(T, \tau T)$ differ only by projective-injective direct summands. The right hand side $\text{Ext}_B^1(\text{Hom}_A(T, V), \text{Hom}_A(T, T))$ is the image of $\text{Ext}_A^1(V, T)$ under the (exact) equivalence $\text{Hom}_A(T, -): \mathcal{T} \longrightarrow \mathcal{Y}$ (here we use that V belongs to \mathcal{T}). This completes the proof¹¹.

Now let us deal with the **representations** of \tilde{B} . The \tilde{B} -modules can

¹⁰ In addition, we should remark that $\text{Ext}_A^1(T, \tau^{-1}T)$ can be identified with $\text{Ext}_A^1(\tau T, T)$ (as B - B -bimodules). The reason is the fact that the functor τ^{-1} is left adjoint to τ , for A hereditary, thus $\text{Ext}_A^1(T, \tau^{-1}T) \simeq D \text{Hom}_A(\tau^{-1}T, \tau T) \simeq D \text{Hom}_A(T, \tau^2 T) \simeq \text{Ext}_A^1(\tau T, T)$. The importance of the bimodule $\text{Ext}_A^1(\tau T, T)$ has been stressed already in section 2.7; I like to call it the “magic” bimodule for such a tilting process. All the bimodule isomorphisms mentioned here should be of interest when dealing with the magic bimodule J . In particular, when working with injective \tilde{B} -modules, it seems to be convenient to know that $DJ \simeq \text{Hom}_A(T, \tau^2 T)$.

¹¹ Note that the isomorphy of $\text{Ext}_B^2(DB, B)$ and $\text{Ext}_A^1(\tau T, T)$ yields a proof for the implication (ii) \Rightarrow (iv) mentioned in Part 1. Since we know that B has global dimension at most 2, the vanishing of $\text{Ext}_B^2(DB, B)$ implies that $\text{Ext}_B^2(X, Y) = 0$ for all B -modules X, Y , thus we also see that (iv) \Rightarrow (ii).

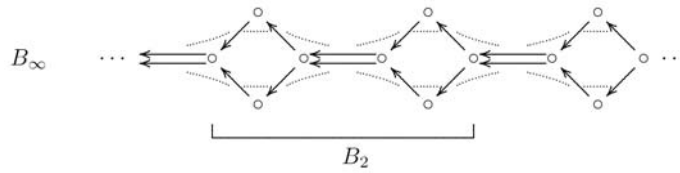
be described as follows: they are pairs of the form (M, γ) , where M is a B -module, and $\gamma: J \otimes_B M \longrightarrow M$ is a B -linear map. As we know, in $\text{mod } B$ there is the splitting torsion pair $(\mathcal{Y}, \mathcal{X})$ and it turns out that $J \otimes_B X = 0$ for $X \in \mathcal{X}$, and that $J \otimes_B Y$ belongs to \mathcal{X} for all $Y \in \mathcal{Y}$ (for the definition of the module classes \mathcal{X}, \mathcal{Y} , but also for \mathcal{Y}' and \mathcal{S} we refer to section 1.2). Let us consider a pair (M, γ) in $\text{mod } \tilde{B}$ and write $M = Y \oplus S \oplus X$, with $Y \in \mathcal{Y}'$, $S \in \mathcal{S}$, and $X \in \mathcal{X}$. Then the image of γ is contained in \mathcal{Y}' and $Y \oplus S$ is contained in the kernel of γ (in particular, $(S, 0)$ is a direct summand of (M, γ)).

Note that $(\mathcal{Y}, \mathcal{X})$ still is a torsion pair in $\text{mod } \tilde{B}$ (a module $(X \oplus Y, \gamma)$ with $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$ has $(X, 0)$ as torsion submodule, has $(Y, 0)$ as its torsion-free factor module, and the map γ is the obstruction for the torsion submodule to split off). Let us draw the attention to a special feature of this torsion pair $(\mathcal{Y}, \mathcal{X})$ in $\text{mod } \tilde{B}$: there exists an ideal, namely J , such that the modules annihilated by J are just the modules in $\text{add}(\mathcal{X}, \mathcal{Y})$.

Buan, Marsh and Reiten [10] have shown that the category $\text{mod } \tilde{B}$ can be described in terms of $\text{mod } A$ (via the corresponding cluster category). Let us present such a description in detail. We will use that $J = \text{Ext}_A^1(T, \tau^{-1}T)$ (as explained above). The algebra \tilde{B} has as \mathbb{Z} -covering the following (infinite dimensional) matrix algebra:

$$B_\infty = \begin{bmatrix} & & & & \\ & B & J & & \\ & & B & J & \\ & & & B & J \\ & & & & B \end{bmatrix}$$

with B on the main diagonal, J directly above the main diagonal, and zeros elsewhere (note that this algebra has no unit element in case $B \neq 0$). It turns out that it is sufficient to determine the representations of the convex subalgebras of the form $B_2 = \begin{bmatrix} B & J \\ 0 & B \end{bmatrix}$. We can write B_2 -modules as columns $\begin{bmatrix} N \\ N' \end{bmatrix}$ and use matrix multiplication, provided we have specified a map $\gamma: J \otimes N' \longrightarrow N$. In the example considered (B a square, with two zero relations), the algebras B_∞ and B_2 are as follows:



In order to exhibit all the B_2 -modules, we use the functor $\Phi: \text{mod } A \longrightarrow \text{mod } B_2$ given by

$$\Phi(M) = \begin{bmatrix} \text{Ext}_A^1(T, M) \\ \text{Hom}_A(\tau^{-1}T, M) \end{bmatrix},$$

with $\gamma: \text{Ext}_A^1(T, \tau^{-1}T) \otimes \text{Hom}_A(\tau^{-1}T, M) \longrightarrow \text{Ext}_A^1(T, M)$ being the canonical map of forming induced exact sequences (this is just the Yoneda multiplication)¹². Now Φ itself is not faithful, since obviously T is sent to zero¹³. However, it induces a fully faithful functor (which again will be denoted by Φ):

$$\Phi: \text{mod } A/\langle T \rangle \longrightarrow \text{mod } B_2,$$

where $\text{mod } A/\langle T \rangle$ denotes the factor category of $\text{mod } A$ modulo the ideal of all maps which factor through $\text{add } T$. The image of the functor Φ is given by

$$\begin{bmatrix} \mathcal{X} \\ 0 \end{bmatrix} \left[\begin{bmatrix} 0 \\ \mathcal{Y}' \end{bmatrix} \right].$$

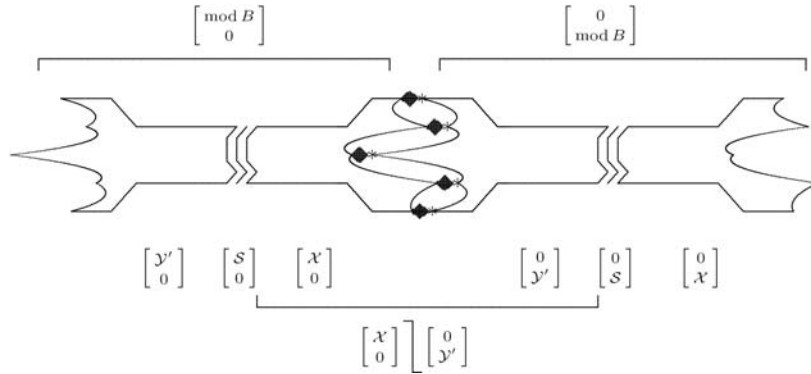
In general, given module classes \mathcal{K}, \mathcal{L} in $\text{mod } R$, we write $\mathcal{K} \left[\mathcal{L} \right.$ for the class of all R -modules M with a submodule K in \mathcal{K} such that

¹² The reader should recall that the functors $\text{Ext}_A^1(T, -)$ and $\text{Hom}_A(\tau^{-1}T, -)$ have been mentioned already in Part 1. These are the functors which provide the equivalences $\mathcal{F} \simeq \mathcal{X}$ and $\mathcal{T}' \simeq \mathcal{Y}'$, respectively.

¹³ The comparison with the Buan-Marsh-Reiten paper [10] shows a slight deviation: The functor they use vanishes on the modules τT and not on T (and if we denote by T_i an indecomposable direct summand of T , then the image of T_i becomes an indecomposable projective \tilde{B} -module). Instead of looking at the functor Φ , we could have worked with $\Phi'(M) = \begin{bmatrix} \text{Ext}_A^1(\tau T, M) \\ \text{Hom}_A(T, M) \end{bmatrix}$, again taking for γ the canonical map. This functor Φ' vanishes on τT . On the level of cluster categories, the constructions corresponding to Φ and Φ' differ only by the Auslander-Reiten translation in the cluster category, and this is an auto-equivalence of the cluster category. But as functors $\text{mod } A \longrightarrow \text{mod } B_2$, the two functors Φ, Φ' are quite different. Our preference for the functor Φ has the following reason: the functor Φ kills precisely $n = n(A)$ indecomposable A -modules, thus the number of indecomposable \tilde{B} -modules which are not contained in the image of Φ is also n , and these modules form a slice. This looks quite pretty: the category $\text{mod } \tilde{B}$ is divided into the image of the functor Φ and one additional slice.

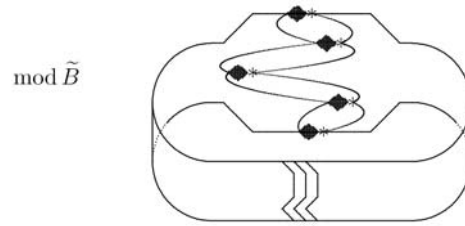
M/K belongs to \mathcal{L} . Thus, we assert that the image of Φ is the class of the \tilde{B} -modules $\begin{bmatrix} N \\ N' \end{bmatrix}$ with $N \in \mathcal{X}$ and $N' \in \mathcal{Y}'$. (In order to see that $\text{Hom}_A(\tau^{-1}T, M) \in \mathcal{Y}'$, first note that $\text{Hom}_A(\tau^{-1}T, M) = \text{Hom}_A(T, \tau M)$, thus this is a B -module in \mathcal{Y} . We further have $\text{Hom}_A(T, \tau M) = \text{Hom}_A(T, t\tau M)$, where $t\tau M$ is the torsion submodule of τM . If we assume that $\text{Hom}_A(T, t\tau M)$ has an indecomposable submodule in \mathcal{S} , say $\text{Hom}_A(T, Q)$, where Q is an indecomposable injective A -module, then we obtain a non-zero map $Q \longrightarrow t\tau M \subseteq \tau M$, since $\text{Hom}_A(T, -)$ is fully faithful on \mathcal{T} . However, the image of this map is injective (since A is hereditary) and τM is indecomposable, thus τM is injective, which is impossible).

We want to draw a rough sketch of the shape of $\text{mod } B_2$, in the same spirit as we have drawn a picture of $\text{mod } B$ in Part 1:



As we have mentioned, the middle part $\begin{bmatrix} X \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ Y' \end{bmatrix}$ (starting with $\begin{bmatrix} X \\ 0 \end{bmatrix}$ and ending with $\begin{bmatrix} 0 \\ Y' \end{bmatrix}$) is the image of the functor Φ , thus this part of the category $\text{mod } B_2$ is equivalent to $\text{mod } A/\langle T \rangle$. Note that this means that there are some small “holes” in this part, they are indicated by black lozenges; these holes correspond to the position in the Auslander-Reiten quiver of A which are given by the indecomposable direct summands T_i of T (and are directly to the left of the small stars).

It follows that $\text{mod } \tilde{B}$ has the form:



Here, we have used the covering functor $\Pi: \text{mod } B_\infty \longrightarrow \text{mod } \tilde{B}$ (or better its restriction to $\text{mod } B_2$): under this functor the subcategories $[\text{mod } B]_0$ and $[\text{mod } B]_0^0$ are canonically identified. In particular, a fundamental domain for the covering functor is given by the module classes $[\mathcal{X}] \bigsqcup [\mathcal{Y}']$ and $[\mathcal{S}]$.

This shows that $\text{mod } \tilde{B}$ decomposes into the modules in $\mathcal{X} \bigsqcup \mathcal{Y}'$ (these are the \tilde{B} -modules N with a submodule $X \subseteq N$ in \mathcal{X} , such that N/X belongs to \mathcal{Y}') on the one hand, and the modules in \mathcal{S} on the other hand. Under the functor Φ , $\text{mod } A/\langle T \rangle$ is embedded into $\text{mod } \tilde{B}$ with image the module class $\mathcal{X} \bigsqcup \mathcal{Y}'$. This is a controlled embedding (as defined in [43]), with control class \mathcal{S} .

The functor

$$\text{mod } A \xrightarrow{\Phi} \text{mod } B_2 \xrightarrow{\Pi} \text{mod } \tilde{B}$$

has the following interesting property: only finitely many indecomposables are killed by the functor (the indecomposable direct summands of T) and there are only finitely many indecomposables (actually, the same number) which are not in the image of the functor (the indecomposable modules in \mathcal{S}). Otherwise, it yields a bijection between indecomposables.

It should be noted that some of the strange phenomena of tilted algebras disappear when passing to cluster tilted algebras. For example, the tunnel effect mentioned above changes as follows: there still is the tunnel, but no longer does it connect two separate regions; it now is a sort of by-pass for a single region. On the other hand, we should stress that the pictures which we have presented and which emphasise the existence of cyclic paths in $\text{mod } \tilde{B}$ are misleading in the special case when T is a slice module: in this case, $J = \text{Ext}_B^2(DB, B) = 0$, thus $\tilde{B} = B$ is again hereditary.

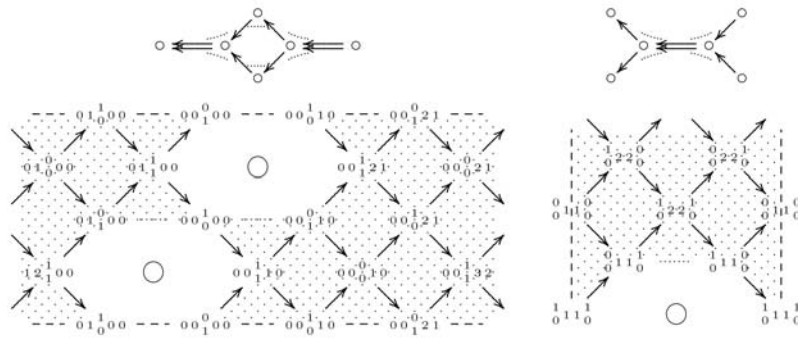
The cluster tilting theory has produced a lot of surprising results —

it even answered some question which one did not dare to ask. For example, dealing with certain classes of algebras such as special biserial ones, one observes that sometimes there do exist indecomposable direct summands X of the radical of an indecomposable projective module P , such that the Auslander-Reiten translate τX is a direct summand of the socle factor module of an indecomposable injective module I . Thus, in the Auslander-Reiten quiver of \tilde{B} , there are non-sectional paths of length 4 from I to P

$$I \longrightarrow \tau X \longrightarrow X' \longrightarrow X \longrightarrow P.$$

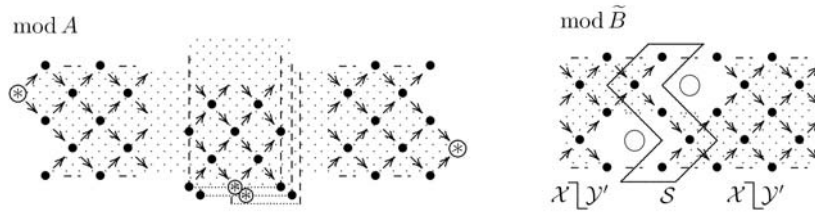
Is this configuration of interest? I did not think so before I was introduced to cluster tilting theory, but according to [10], this configuration is a very typical one when dealing with cluster tilted algebras.

As an illustration, we show what happens in the non-regular components of our example B_∞ (where B is the square with two zero relations). The upper line exhibits the part of the quiver of B_∞ which is needed as support for the modules shown below:

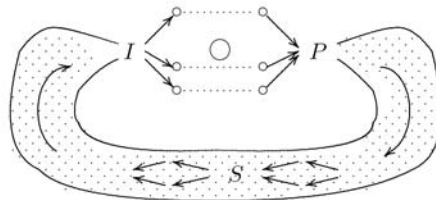


For both components, the dashed boundary lines have to be identified. In this way, the right picture with the vertical identification yields what is called a tube, the left picture gives a kind of horizontal hose. In contrast to the tube with its mouth, the hose extends in both directions indefinitely. The big circles indicate the position of the modules T_i in the corresponding components of $\text{mod } A$, these are the modules which are killed by the functor Φ . In both components we find non-sectional paths of length 4 from an indecomposable injective B_∞ -module I to an indecomposable projective B_∞ -module P such that the simple modules $\text{soc } I$ and $\text{top } P$ are identified under the covering functor Π .

We also want to use this example in order to illustrate the fact that the image of Φ in $\text{mod } \tilde{B}$ is complemented by a slice \mathcal{S} :



When looking at the non-sectional paths from I to P of length 4, where I is an indecomposable injective \tilde{B} -module, P an indecomposable projective \tilde{B} -module such that $S = \text{soc } I \simeq \text{top } P$, one should be aware that the usual interest lies in paths from P to I . Namely, there is the so called “hammock” for the simple module S , dealing with pairs of maps of the form $P \longrightarrow M \longrightarrow I$ with composition having image S (and M indecomposable).



Taking into account not only the hammock, but also the non-sectional paths of length 4 from I to P leads to a kind of organized round trip. Since the simple module S has no self-extension, it is the only indecomposable module M such that $\text{Hom}_{\tilde{B}}(P', M) = 0$, for any indecomposable projective \tilde{B} -module $P' \neq P$. We will return to this hammock configuration (P, S, I) later.

Readers familiar with the literature will agree that despite of the large number of papers devoted to questions in the representation theory of artin algebras, only few classes of artin algebras are known where there is a clear description of the module categories¹⁴. The new developments

¹⁴ Say in the same way as the module categories of hereditary artin algebras are described. We consider here algebras which may be wild, thus we have to be cautious of what to expect from a “clear description”.

outlined here show that the cluster tilted algebras are such a class: As for the hereditary artin algebras, the description of the module category is again given by the root system of a Kac-Moody Lie algebra.

Keller and Reiten [34] have shown, that *cluster tilted algebras are Gorenstein algebras of Gorenstein dimension at most 1*. This is a very remarkable assertion! The proof uses in an essential way cluster categories, and provides further classes of Gorenstein algebras of Gorenstein dimension at most 1. Note that the cluster tilted algebra \tilde{B} is hereditary only in case $\tilde{B} = B$, thus only for T a slice module. There are examples where \tilde{B} is self-injective (for example for $B = kQ/\langle\rho\rangle$ with Q the linearly directed A_3 -quiver and ρ the path of length 2). In general, \tilde{B} will be neither hereditary nor self-injective.

3.2 The complex Σ'_A

We have mentioned in Part 1 that the simplicial complex Σ_A of tilting modules always has a non-empty boundary (for $n(A) \geq 2$). Now the cluster theory provides a recipe for embedding this simplicial complex in a slightly larger one without boundary. Let me introduce here this complex Σ'_A directly in terms of $\text{mod } A$, using a variation of the work of Marsh, Reineke und Zelevinsky [38]¹⁵. It is obtained from Σ_A by just adding $n = n(A)$ vertices, and of course further simplices. Recall that a *Serre subcategory*¹⁶ \mathcal{U} of an abelian category is a subcategory which is closed under submodules, factor modules and extensions; thus in case we deal with a length category such as $\text{mod } A$, then \mathcal{U} is specified by the simple modules contained in \mathcal{U} (an object belongs to \mathcal{U} if and only if its composition factors lie in \mathcal{U}). In particular, for a simple A -module S , let us denote by $(-S)$ the subcategory of all A -modules which do not have S as a composition factor. Any Serre subcategory is the intersection of such subcategories.

Here is the definition of Σ'_A : As simplices take the pairs (M, \mathcal{U}) where \mathcal{U} is a Serre subcategory of $\text{mod } A$ and M is (the isomorphism class of) a basic module in \mathcal{U} without self-extensions; write $(M', \mathcal{U}') \leq (M, \mathcal{U})$

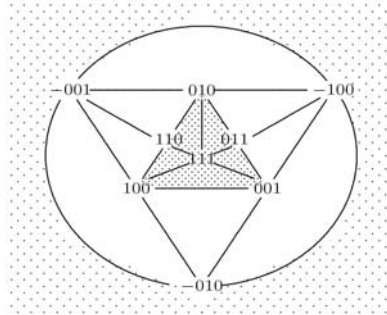
¹⁵ The title of the paper refers to “associahedra”: in the case of the path algebra of a quiver of type A_n , the dual of the simplicial complex Σ'_A is an associahedron (or Stasheff polytope). For quivers of type B_n and C_n one obtains a Bott-Taubes cyclohedron.

¹⁶ The Serre subcategories are nothing else then the subcategories of the form $\text{mod } A/AeA$, where e is an idempotent of A .

provided M' is a direct summand of M and $\mathcal{U}' \supseteq \mathcal{U}$ (note the reversed order!). Clearly¹⁷, Σ_A can be considered as a subcomplex of Σ'_A , namely as the set of all pairs $(M, \text{mod } A)$.

There are two kinds of vertices of Σ'_A , namely those of the form $(E, \text{mod } A)$ with E an exceptional A -module (these are the vertices belonging to Σ_A), and those of the form $(0, (-S))$ with S simple. It is fair to say that the latter ones are indexed by the “negative simple roots”; of course these are the vertices which do not belong to Σ_A . Given a simplex (M, \mathcal{U}) , its vertices are the elements $(E, \text{mod } A)$, where E is an indecomposable direct summand of M , and the elements $(0, (-S))$, where $\mathcal{U} \subseteq (-S)$. The $(n-1)$ -simplices are those of the form (M, \mathcal{U}) , where M is a basic tilting module in \mathcal{U} . The vertices outside Σ_A belong to one $(n-1)$ -simplex, namely to $(0, \{0\})$. The $(n-2)$ -simplices are of the form (M, \mathcal{U}) , where M is an almost complete partial tilting module for \mathcal{U} . If it is sincere in \mathcal{U} , there are precisely two complements in \mathcal{U} . If it is not sincere in \mathcal{U} , then there is only one complement in \mathcal{U} , but there also is a simple module S such that X belongs to $(-S)$, thus X is a tilting module for $\mathcal{U} \cap (-S)$. This shows that any $(n-2)$ -simplex belongs to precisely two $(n-1)$ -simplices.

As an example, we consider again the path algebra A of the quiver $\circ \leftarrow \circ \leftarrow \circ$. The simplicial complex Σ'_A is a 2-sphere and looks as follows (considering the 2-sphere as the 1-point compactification of the real plane):



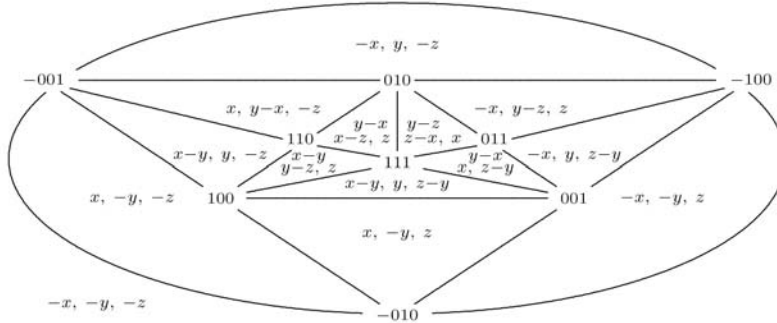
Here, the vertex $(0, (-S))$ is labeled as $-\dim S$. We have shaded the

¹⁷ In the same way, we may identify the set of simplices of the form (M, \mathcal{U}) with \mathcal{U} fixed, as $\Sigma_{A/AeA}$, where $\mathcal{U} = \text{mod } A/AeA$. In this way, we see that Σ'_A can be considered as a union of all the simplicial complexes $\Sigma_{A/AeA}$.

subcomplex Σ_A (the triangle in the middle) as well as the $(n-1)$ -simplex $(0, \{0\})$ (the outside).

Consider now a reflection functor σ_i , where i is a sink, say. We obtain an embedding of $\Sigma_{\sigma_i A}$ into Σ'_A as follows: There are the exceptional $\sigma_i A$ -modules of the form $\sigma_i E$ with E an exceptional A -module, different from the simple A -module $S(i)$ concentrated at the vertex i , and in between these modules $\sigma_i E$ the simplex structure is the same as in between the modules E . In addition, there is the simple $\sigma_i A$ -module $S'(i)$ again concentrated at i . Now we know that E has no composition factor $S(i)$ if and only if $\text{Ext}_{\sigma_i A}^1(S'(i), \sigma_i M) = 0$. This shows that the simplex structure of Σ_A involving $(0, (-S(i)))$ and vertices of the form $(E, \text{mod } A)$ is the same as the simplex structure of $\Sigma_{\sigma_i A}$ in the vicinity of $(S'(i), \text{mod } \sigma_i A)$.

We may consider the simplicial complex Σ'_A as a subset of the real n -dimensional space $K_0(A) \otimes \mathbb{R}$, where $n = n(A)$, namely as a part of the corresponding unit $(n-1)$ -sphere, with all the $(n-1)$ -simplices defined by n linear inequalities. In case A is representation-finite, we deal with the $(n-1)$ -sphere itself, otherwise with a proper subset. For example, in the case of the path algebra A of the quiver $\circ \leftarrow \circ \leftarrow \circ$, the inequalities are $\varphi_1 \geq 0$, $\varphi_2 \geq 0$, $\varphi_3 \geq 0$, where $\varphi_1, \varphi_2, \varphi_3$ are the linear forms inserted in the corresponding triangle:



In general, any $(n-1)$ -simplex (M, \mathcal{U}) is equipped with n linear forms $\varphi_1, \dots, \varphi_n$ on $K_0(A)$ such that the following holds: an A -module N without self-extensions belongs to add M if and only if $\varphi_i(\dim N) \geq 0$.

In the same way as Σ_A , also Σ'_A can be identified with a fan in $K_0(A) \otimes \mathbb{R}$. For any simplex (M, \mathcal{U}) with vertices $(E, \text{mod } A)$ and $(0, (-S))$, where

E are indecomposable direct summands of M , and S simple modules which do not belong to \mathcal{U} , take the cone $C(M, \mathcal{U})$ generated by the vectors $\mathbf{dim} E$ and $-\mathbf{dim} S$.

3.3 The cluster categories

We have exhibited the cluster tilted algebras without reference to cluster categories, in order to show the elementary nature of these concepts. But a genuine understanding of cluster tilted algebras as well as of Σ'_A is not possible in this way. Starting with a hereditary artin algebra A , let us introduce now the corresponding cluster category \mathcal{C}_A . We have to stress that this procedure reverses the historical development¹⁸ : the cluster categories were introduced first, and the cluster tilted algebras only later. The aim of the definition of cluster categories was to illuminate the combinatorics behind the so called cluster algebras, in particular the combinatorics of the cluster complex.

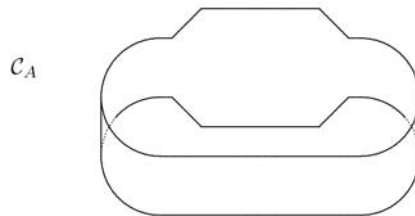
Let me say a little how cluster tilted algebras were found. Everything started with the introduction of “cluster algebras” by Fomin and Zelevinsky [23]: these are certain subrings of rational function fields, thus commutative integral domains. At first sight, one would not guess any substantial relationship to non-commutative artin algebras. But it turned out that the Dynkin diagrams, as well as the general Cartan data, play an important role for cluster algebras too. As it holds true for the hereditary artin algebras, it is the corresponding root system, which is of interest. This is a parallel situation, although not completely. For the cluster algebras one needs to understand not only the positive roots, but the *almost positive* roots: this set includes besides the positive roots also the negative simple roots. As far as we know, the set of almost positive roots had not been considered before¹⁹ . The first link between cluster

¹⁸ In the words of Fomin and Zelevinsky [25], this Part 3 altogether is completely revisionistic.

¹⁹ Lie theory is based on the existence of perfect symmetries — partial structures (such as the set of positive roots) which allow only broken symmetries tend to be accepted just as necessary working tools. The set of almost positive roots seems to be as odd as that of the positive ones: it depends on the same choices, but does not even enjoy the plus-minus merit of being half of a neat entity. This must have been the mental reasons that the intrinsic beauty of the cluster complex was realized only very recently. But let me stress here that the cluster complex seems to depend not only on the choice of a root basis, but on the ordering of the basis (or better, on the similarity class): with a difference already for the types $\tilde{A}_{2,2}$ and $\tilde{A}_{3,1}$.

theory and tilting theory was given by Marsh, Reineke, Zelevinsky in [38] when they constructed the complex Σ'_A . Buan, Marsh, Reineke, Reiten, Todorov [8] have shown in which way the representation theory of hereditary artin algebras can be used in order to construct a category \mathcal{C}_A (the cluster category) which is related to the set of almost positive roots²⁰ in the same way as the module category of a hereditary artin algebra is related to the corresponding set of positive roots.

As we have seen, a tilted algebra B should be regarded as the factor algebra of its cluster tilted algebra \tilde{B} , if we want to take into account also the missing modules. But $\text{mod } \tilde{B}$ has to be considered as the factor category of some triangulated category \mathcal{C}_A , the corresponding cluster category. Looking at \mathcal{C}_A , we obtain a common ancestor of all the algebras tilted from algebras in the similarity class of A . In the setting of the pictures shown, the corresponding cluster category has the form



The cluster category \mathcal{C}_A should be considered as a universal kind of category belonging to the similarity class of the hereditary artin algebra A in order to obtain all the module categories $\text{mod } \tilde{B}$, where \tilde{B} is a cluster tilted algebra of type similar to A .

What one does is the following: start with the derived category $D^b(\text{mod } A)$ of the hereditary artin algebra A , with shift functor $[1]$,

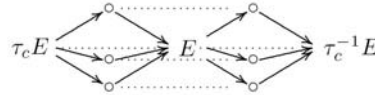
²⁰ A slight unease should be mentioned: as we will see, there is an embedding of $\text{mod } A$ into the cluster category which preserves indecomposability and reflects isomorphy (but it is not a full embedding), thus this part of the cluster category corresponds to the positive roots. There are precisely $n = n(A)$ additional indecomposable objects: they should correspond to the negative simple roots, but actually the construction relates them to the negative of the dimension vectors of the indecomposable projectives. Thus, the number of additional objects is correct, and there is even a natural bijection between the additional indecomposable objects and the simple modules, thus the simple roots. But in this interpretation one may hesitate to say that “one has added the negative simple roots” (except in case any vertex is a sink or a source). On the other hand, in our presentation of the cluster complex we have used as additional vertices the elements $(0, (-S))$, and they really look like “negative simple roots”. Thus, we hope that this provides a better feeling.

and take as \mathcal{C}_A the orbit category with respect to the functor $\tau_D^{-1}[1]$ (we write τ_D for the Auslander-Reiten translation in the derived category, and τ_C for the Auslander-Reiten translation in \mathcal{C}_A). As a fundamental domain for the action of this functor one can take the disjoint union of $\text{mod } A$ (this yields all the positive roots) and the shifts of the projective A -modules by $[1]$ (this yields $n = n(A)$ additional indecomposable objects). It should be mentioned that Keller [33] has shown that \mathcal{C}_A is a triangulated category; this is now the basis of many considerations dealing with cluster categories and cluster tilted algebras. Now if we take a tilting module T in $\text{mod } A$, we may look at the endomorphism ring \tilde{B} of T in \mathcal{C}_A (or better: the endomorphism ring of the image of T under the canonical functors $\text{mod } A \subseteq D^b(\text{mod } A) \longrightarrow \mathcal{C}_A$), and obtain a cluster tilted algebra²¹ as considered above. The definition immediately yields that $\hat{B} = B \ltimes J$, where $J = \text{Hom}_{D^b(\text{mod } A)}(T, \tau_D^{-1}T[1]) = \text{Ext}_A^1(T, \tau^{-1}T)$. The decisive property is that there is a canonical equivalence of categories²²

$$\mathcal{C}_A/\langle T \rangle \longrightarrow \text{mod } \tilde{B}.$$

In particular, we see that the triangulated category \mathcal{C}_A has many factor categories which are abelian²³.

What happens when we form the factor category $\mathcal{C}_A/\langle T \rangle$? Consider an indecomposable direct summand E of the tilting A -module T as an object in the cluster category \mathcal{C}_A and the meshes starting and ending in E :



²¹ This is the way, the cluster tilted algebras were introduced and studied by Buan, Marsh and Reiten [10].

²² Instead of $\mathcal{C}_A/\langle T \rangle$, one may also take the equivalent category $\mathcal{C}_A/\langle \tau_C T \rangle$. The latter is of interest if one wants the indecomposable summands of T in \mathcal{C}_A to become indecomposable projective objects.

²³ We have mentioned that the cluster theory brought many surprises. Here is another one: One knows for a long time many examples of abelian categories \mathcal{A} with an object M such that the category $\mathcal{A}/\langle M \rangle$ (obtained by setting zero all maps which factor through M) becomes a triangulated category: just take $\mathcal{A} = \text{mod } R$, where R is a self-injective artin algebra and $M = {}_R R$. The category $\text{mod } R/\langle {}_R R \rangle = \underline{\text{mod}} R$ is the stable module category of R . But we are not aware that non-trivial examples were known of a triangulated category \mathcal{D} with an object N such that $\mathcal{D}/\langle N \rangle$ becomes abelian. Cluster tilting theory is just about this!

In the category $\mathcal{C}_A/\langle T \rangle$, the object E becomes zero, whereas both $\tau_{\mathcal{C}}E$ and $\tau_{\mathcal{C}}^{-1}E$ remain non-zero. In fact, $\tau_{\mathcal{C}}^{-1}E$ becomes a projective object and $\tau_{\mathcal{C}}E$ becomes an injective object: We obtain in this way in $\text{mod } \tilde{B} = \mathcal{C}_A/\langle T \rangle$ an indecomposable projective module $P = \tau_{\mathcal{C}}^{-1}E$ and an indecomposable injective module $I = \tau_{\mathcal{C}}E$, such that $\text{top } P \simeq \text{soc } I$. This explains the round trip phenomenon for \tilde{B} mentioned above: there is the hammock corresponding to the simple \tilde{B} -module $\text{top } P \simeq \text{soc } I$, starting from $I = \tau_{\mathcal{C}}E$, and ending in $P = \tau_{\mathcal{C}}^{-1}E$. And either $\text{rad } P$ is projective (and $I/\text{soc } I$ injective) or else there are non-sectional paths of length 4 from I to P .

There is a decisive symmetry condition²⁴ in the cluster category $\mathcal{C} = \mathcal{C}_A$:

$$\text{Hom}_{\mathcal{C}}(X, Y[1]) \simeq D \text{Hom}_{\mathcal{C}}(Y, X[1]).$$

This is easy to see: since we form the orbit category with respect to $\tau_D^{-1}[1]$, this functor becomes the identity functor in \mathcal{C} , and therefore the Auslander-Reiten functor $\tau_{\mathcal{C}}$ and the shift functor $[1]$ in \mathcal{C} coincide. On the other hand, the Auslander-Reiten (or Serre duality) formula for \mathcal{C} asserts that $\text{Hom}_{\mathcal{C}}(X, Y[1]) \simeq D \text{Hom}_{\mathcal{C}}(Y, \tau_{\mathcal{C}}X)$. A triangulated category is said to be *d-Calabi-Yau* provided the shift functor $[d]$ is a Serre (or Nakayama) functor, thus provided there is a functorial isomorphism

$$\text{Hom}(X, -) \simeq D \text{Hom}(-, X[d])$$

(for a discussion of this property, see for example [33]). As we see, *the cluster category is 2-Calabi-Yau*.

The cluster category has Auslander-Reiten sequences. One component Γ_0 of the Auslander-Reiten quiver of \mathcal{C}_A has only finitely many $\tau_{\mathcal{C}}$ -orbits, namely the component containing the indecomposable projective (as well as the indecomposable injective) A -modules. The remaining components of the Auslander-Reiten quiver of \mathcal{C}_A have tree class \mathbb{A}_{∞} .

In a cluster category $\mathcal{C} = \mathcal{C}_A$, an object is said to be a *cluster-tilting object*²⁵ provided first $\text{Hom}_{\mathcal{C}}(T, T[1]) = 0$, and second, that T is maximal

²⁴ If we write $\text{Ext}^1(X, Y) = \text{Hom}_{\mathcal{C}}(X, Y[1])$, then this symmetry condition reads that $\text{Ext}^1(X, Y)$ and $\text{Ext}^1(Y, X)$ are dual to each other, in particular they have the same dimension.

²⁵ It has to be stressed that the notion of a “cluster-tilting object” in a cluster category does not conform to the tilting notions used otherwise in this Handbook! If T is such a cluster-tilting object, then it may be that $\text{Hom}_{\mathcal{C}}(T, T[i]) \neq 0$ in $\mathcal{C} = \mathcal{C}_A$ for some $i \geq 2$. Observe that in a 2-Calabi-Yau category such as \mathcal{C}_A , we have $\text{Hom}_{\mathcal{C}}(X, X[2]) \neq 0$, for any non-zero object X .

with this property in the following sense: if $\text{Hom}_{\mathcal{C}}(T \oplus X, (T \oplus X)[1]) = 0$, then X is in $\text{add } T$. If T is a tilting A -module, then one can show quite easily that T , considered as an object of \mathcal{C}_A is a cluster-tilting object.

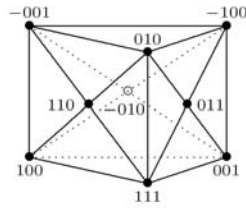
Let us consider the hereditary artin algebras in one similarity class and the reflection functors between them. One may identify the corresponding cluster categories using the reflection functors, as was pointed out by Bin Zhu [52]. In this way, one can compare the tilting modules of all the hereditary artin algebras in one similarity class. It turns out that *the cluster-tilting objects in \mathcal{C}_A are just the tilting modules for the various artin algebras obtained from A by using reflection functors* [8]. In order to see this, let T be a cluster-tilting object in \mathcal{C}_A . Let Γ_0 be the component of the Auslander-Reiten quiver of \mathcal{C}_A which contains the indecomposable projective A -modules. If no indecomposable direct summand of T belongs to Γ_0 , then T can be considered as an A -module, and it is a regular tilting A -module. On the other hand, if there is an indecomposable direct summand of T , say T_1 , which belongs to Γ_0 , then let \mathcal{S} be the class of all indecomposable objects X in Γ_0 with a path from X to T_1 in Γ_0 , and such that any path from X to T_1 in Γ_0 is sectional. Then no indecomposable direct summand of T belongs to $\tau_{\mathcal{C}}\mathcal{S}$. We may identify the factor category $\mathcal{C}_A / \langle \tau_{\mathcal{C}}\mathcal{S} \rangle$ with $\text{mod } A'$ for some hereditary artin algebra A' , and consider T as an A' -module (the object T_1 , considered as an A' -module, is projective and faithful). Clearly, A' is obtained from A by a sequence of BGP-reflection functors.

Also, the usual procedure of going from a tilting module to another one by exchanging just one indecomposable direct summand gets more regular. Of course, there is the notion of an almost complete partial cluster-tilting object and of a complement, parallel to the corresponding notions of an almost complete partial tilting module and its complements. Here we get: *Any almost complete partial cluster-tilting object \overline{T} has precisely two complements* [8]. We indicate the proof: We can assume that \overline{T} is an A -module. If \overline{T} is sincere, then we know that there are two complements for \overline{T} considered as an almost complete partial tilting A -module. If \overline{T} is not sincere, then there is only one complement for \overline{T} considered as an almost complete partial tilting A -module. But there is also one (and obviously only one) indecomposable projective module P with $\text{Hom}_A(P, \overline{T}) = 0$, and the $\tau_{\mathcal{C}}$ -shift of P in the cluster category is the second complement we are looking for!

An important point seems to be the following: *The simplicial complex*

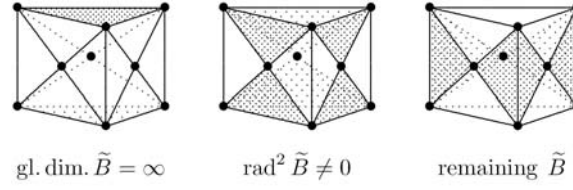
of partial cluster-tilting objects in the cluster category \mathcal{C}_A is nothing else than Σ'_A , with the following identification: If T is a basic partial cluster-tilting object in \mathcal{C}_A , we can write T as the direct sum of a module M in $\text{mod } A$ and objects of the form $\tau_{\mathcal{C}}P(i)$, with $P(i)$ indecomposable projective in $\text{mod } A$, and i in some index set Θ . Then M corresponds in Σ'_A to the pair (M, \mathcal{U}) , where $\mathcal{U} = \bigcap_{i \in \Theta} (-S(i))$. The reason is very simple: $\text{Hom}_{\mathcal{C}}(\tau_{\mathcal{C}}P(i), M[1]) \simeq \text{Hom}_{\mathcal{C}}(P(i), M) = \text{Hom}_A(P(i), M)$, with $\mathcal{C} = \mathcal{C}_A$.

The complex Σ'_A should be viewed as a convenient index scheme²⁶ for the set of cluster tilted algebras obtained from the hereditary artin algebras in the similarity class of A . Any maximal simplex of σ'_A is a cluster-tilting object in \mathcal{C}_A , and thus we can attach to it its endomorphism ring. Let us redraw the complex Σ'_A for the path algebra A of the quiver $\circ \leftarrow \circ \leftarrow \circ$, so that the different vertices and triangles are better seen:



There are two kinds of vertices, having either 4 or 5 neighbours. The vertices with 5 neighbours form two triangles (the bottom and the top triangle), and these are the cluster-tilting objects with endomorphism ring of infinite global dimension. The remaining triangles yield hereditary endomorphism rings and again, there are two kinds: The quiver of the endomorphism ring may have one sink and one source, these rings are given by the six triangles which have an edge in common with the bottom or the top triangle. Else, the endomorphism ring is hereditary and the radical square is zero: these rings correspond to the remaining six triangles:

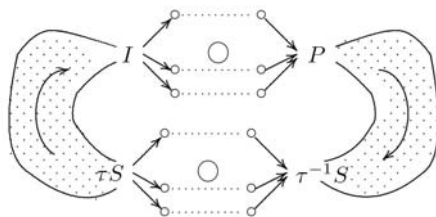
²⁶ But we should also mention the following: The set of isomorphism classes of basic cluster-tilting objects in \mathcal{C}_A is no longer partially ordered. In fact, given an almost complete partial cluster-tilting object \bar{T} and its two complements X and Y , there are triangles $X \rightarrow T' \rightarrow Y \rightarrow$ and $Y \rightarrow T'' \rightarrow X \rightarrow$ with $T', T'' \in \text{add } \bar{T}$.



Consider an almost complete partial cluster-tilting object \overline{T} in $\mathcal{C} = \mathcal{C}_A$. As we have mentioned, there are precisely two complements for \overline{T} , say E and E' . Let $T = \overline{T} \oplus E$, and $T' = \overline{T} \oplus E'$. Thus, there are given two cluster-tilted algebras $\tilde{B} = \text{End}_{\mathcal{C}}(T)$, and $\tilde{B}' = \text{End}_{\mathcal{C}}(T')$, we may call them *adjacent*, this corresponds to the position of T and T' in the complex Σ'_A . We can identify $\mathcal{C}_A/\langle T \rangle$ with $\text{mod } \tilde{B}$, and $\mathcal{C}_A/\langle T' \rangle$ with $\text{mod } \tilde{B}'$. We saw that E as an indecomposable direct summand of T yields an indecomposable projective \tilde{B} -module $P = \tau_{\mathcal{C}}^{-1}E$ and an indecomposable injective \tilde{B} -module $I = \tau_{\mathcal{C}}E$, such that $\text{soc } I \simeq \text{top } P$. Since $\overline{T} \oplus E'$ is a cluster-tilting object, it is not difficult to show that $\text{Hom}_{\tilde{B}}(P', E') = 0$ for any indecomposable projective \tilde{B} -module $P' \neq P$. But this implies that E' is identified under the equivalence of $\mathcal{C}_A/\langle T \rangle$ and $\text{mod } \tilde{B}$ with the simple \tilde{B} -module which is the socle of I and the top of P . In the same way, we see that E is a simple \tilde{B}' -module, namely the top of the \tilde{B}' -module $P = \tau_{\mathcal{C}}^{-1}E'$ and the socle of the indecomposable injective \tilde{B}' -module $I = \tau_{\mathcal{C}}E'$. Thus, there is the following sequence of identifications:

$$\begin{aligned} \text{mod } \tilde{B}/\langle \text{add } E' \rangle &\simeq \mathcal{C}_A/\langle \text{add}(T \oplus E') \rangle \\ &= \mathcal{C}_A/\langle \text{add}(T' \oplus E) \rangle \simeq \text{mod } \tilde{B}'/\langle \text{add } E \rangle. \end{aligned}$$

Altogether this means that *artin algebras \tilde{B} and \tilde{B}' which are adjacent, are nearly Morita equivalent* [10]. We had promised to the reader, that we will return to the hammock configuration (P, S, I) , where S is a simple \tilde{B} -module, $P = P(S)$ its projective cover, and $I = I(S)$ its injective envelope: but this is the present setting. Using the cluster category notation, we can write $P = \tau_{\mathcal{C}}^{-1}E$, $I = \tau_{\mathcal{C}}E$, and then $S = E'$, where E, E' are complements to an almost complete partial cluster-tilting object \overline{T} . When we form the category $\text{mod } \tilde{B}/\langle \text{add } S \rangle$, the killing of the simple \tilde{B} -module S creates a hole in $\text{mod } \tilde{B}$. From the hammock $\text{Hom}(P, -)$ in $\text{mod } \tilde{B}$ the following parts survive:



Note that the new hole is of the same nature as the hole between I and P (which was created when we started from the cluster category \mathcal{C} , killing the object E). Indeed, one may fill alternatively one of the two holes and obtains $\text{mod } \tilde{B}$, or $\text{mod } \tilde{B}'$, respectively.

Altogether, we see: A cluster category $\mathcal{C} = \mathcal{C}_A$ has a lot of nice factor categories which are abelian (the module categories $\text{mod } \tilde{B}$), and one should regard \mathcal{C} as being obtained from patching together the various factor categories in the same way as manifolds are built up from open subsets by specifying the identification maps of two such subsets along what will become their intersection. The patching process for the categories $\text{mod } \tilde{B}$ is done via the nearly Morita equivalences for adjacent tilting objects²⁷.

The reader will have noticed that this exchange process for adjacent algebras generalizes the BGP-reflection functors (and the APR-tilting functors) to vertices which are not sinks or sources. Indeed, for $\tilde{B} = \text{End}_{\mathcal{C}}(\overline{T} \oplus E)$, and $\tilde{B}' = \text{End}_{\mathcal{C}}(\overline{T} \oplus E')$, the indecomposable direct summand E of $\overline{T} \oplus E$ corresponds to a vertex of the quiver of \tilde{B} , and similarly, E' corresponds to a vertex of the quiver of \tilde{B}' . In the BGP and the APR setting, one of the modules E, E' is simple projective, the other one is simple injective — here now E and E' are arbitrary exceptional modules²⁸.

This concludes our attempt to report about some of the new results in tilting theory which are based on cluster categories. Let us summarize the importance of this development. First of all, the cluster tilted algebras provide a nice depository for storing the modules which are lost when we pass from hereditary artin algebras to tilted algebras; there is

²⁷ It seems that there is not yet any kind of axiomatic approach to this new patching process.

²⁸ A direct description of this reflection process seems to be still missing. It will require a proper understanding of all the cluster tilted algebras \tilde{B} with $n(\tilde{B}) = 3$. A lot is already known about such algebras, see [11].

a magic bimodule which controls the situation. We obtain in this way a wealth of algebras whose module categories are described by the root system of a Kac-Moody Lie-algebra. These new algebras are no longer hereditary, but are still of Gorenstein dimension at most 1. For the class of cluster tilted algebras, there is a reflection process at any vertex of the quiver, not only at sinks and sources. This is a powerful generalization of the APR-tilting functors (thus also of the BGP-reflection functors), and adjacent cluster tilted algebras are nearly Morita equivalent. The index set for this reflection process is the simplicial complex Σ'_A and the introduction of this simplicial complex solved also another riddle of tilting theory: it provides a neat way of enlarging the simplicial complex of tilting A -modules in order to get rid of its boundary. We have mentioned in Part 1 that both the missing modules problem as well as the boundary problem concern the module category, but disappear on the level of derived categories. Thus it is not too surprising that derived categories play a role: as it has turned out, the cluster categories, as suitable orbit categories of the corresponding derived categories, are the decisive new objects. These are again triangulated categories, and are to be considered as the universal structure behind all the tilted and cluster tilted algebras obtained from a single hereditary artin algebra A (and the hereditary artin algebras similar to A).

3.4 Appendix: Cluster algebras

Finally we should speak about the source of all these developments, the introduction of cluster algebras by Fomin and Zelevinsky. But we are hesitant, for two reasons: first, there is our complete lack of proper expertise, but also it means that we leave the playground of tilting theory. Thus this will be just an appendix to the appendix. The relationship between cluster algebras on the one hand, and the representation theory of hereditary artin algebras and cluster tilted algebras on the other hand is fascinating, but also very subtle²⁹. At first, one observed certain analogies and coincidences. Then there was an experimental period, with

²⁹ Since this report is written for the Handbook of Tilting Theory, we are only concerned with the relationship of the cluster algebras to tilting theory. There is a second relationship to the representation theory of artin algebras, namely to Hall algebras, as found by Caldero and Chapoton [15], and Caldero-Keller [16, 17], see also Hubery [31]. And there are numerous interactions between cluster theory and many different parts of mathematics. But all this lies beyond the scope of this volume.

many surprising findings (for example, that the Happel-Vossieck list of tame concealed algebras corresponds perfectly to the Seven list of minimal infinite cluster algebras [47], as explained in [14]). In the meantime, many applications of cluster-tilted algebras to cluster algebras have been found [13, 9], and the use of Hall algebra methods provides a conceptual understanding of this relationship [15, 16, 17, 31].

Here is at least a short indication what cluster algebras are. As we said already, the cluster algebras are (commutative) integral domains. The cluster algebras we are interested in (those related to hereditary artin algebras)³⁰ are finitely generated (this means finitely generated “over nothing”, say over \mathbb{Z}), thus they can be considered as subrings of a finitely generated function field $\mathbb{Q}(x_1, \dots, x_n)$ over the rational numbers \mathbb{Q} . This is the way they usually are presented in the literature (but the finite generation is often not stressed). In fact, one of the main theorems of cluster theory asserts that we deal with subrings of the ring of Laurent polynomials $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ (this is the subring of all elements of the form $\frac{p}{q}$ where p is in the polynomial ring $\mathbb{Z}[x_1, \dots, x_n]$ and q is a monomial in the variables x_1, \dots, x_n).

Since we deal with a noetherian integral domain, the reader may expect to be confronted with problems in algebraic geometry, or, since we work over \mathbb{Z} with those of arithmetical geometry. But this was not the primary interest. Instead, the cluster theory belongs in some sense to algebraic combinatorics, and the starting question concerns the existence of a nice \mathbb{Z} -basis of such a cluster algebra, say similar to all the assertions about canonical bases in Lie theory.

What are clusters? Recall that a cluster algebra is a subring of $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. What one is looking for is a convenient \mathbb{Z} -basis of the cluster algebra. One may assume that the elements of the basis are written in the form $\frac{p}{q}$, where $p \in \mathbb{Z}[x_1, \dots, x_n]$ is not divisible by the variables x_1, \dots, x_n and $q = x_1^{d_1} \cdots x_n^{d_n}$ with exponents $d_i \in \mathbb{Z}$; the Laurent monomial q is said to be the *denominator*³¹ of $\frac{p}{q}$ and one may call $\mathbf{dim} q = (d_1, \dots, d_n)$ its dimension vector. There seems to be an inductive procedure to produce at least a part of a \mathbb{Z} -basis by first obtaining the “cluster variables”, and then forming monomials of the cluster variables belonging to a fixed cluster. At least, this works for the cluster

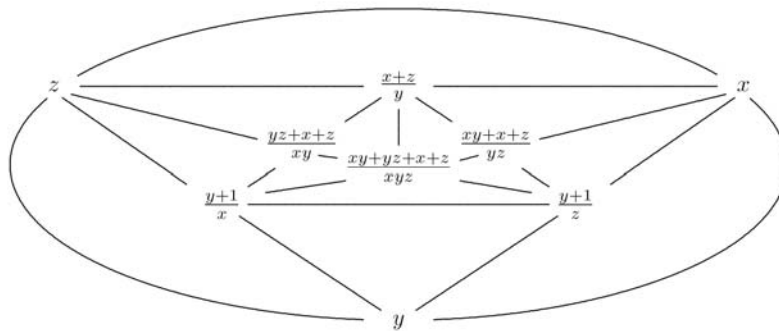
³⁰ these are the so-called acyclic cluster algebras [6].

³¹ Note that this means that the variable x_i itself will be rewritten in the form $1/(x_i^{-1})$; its denominator is $q = x_i^{-1}$.

algebras of finite type and in this case one actually obtains a complete \mathbb{Z} -basis. One of the main topics discussed in cluster theory concerns the shape of the cluster variables in general.

Consider the case of the path algebra of a finite quiver Q without oriented cycles. According to Caldero-Keller [17], the simplicial complex Σ'_A with $A = kQ$ can be identified with the cluster complex corresponding to Q . Under this correspondence, the cluster variables correspond to the exceptional A -modules and the elements of the form $(0, (-S))$. When we introduced the simplicial complex Σ'_A , the maximal simplices were labeled (M, \mathcal{U}) with M a basic tilting module in a Serre subcategory \mathcal{U} of $\text{mod } A$. Recall that such an $(n-1)$ -simplex (M, \mathcal{U}) in Σ'_A is equipped with n linear forms p_1, \dots, p_n on $K_0(A)$ such that an A -module N without self-extensions belongs to \mathcal{U} if and only if $\varphi_i(\dim N) \geq 0$, for $1 \leq i \leq n$. And there is the parallel assertion: A cluster monomial with denominator q belongs to the cluster corresponding to (M, \mathcal{U}) if and only if $\varphi_i(\dim q) \geq 0$, for $1 \leq i \leq n$.

Here are the cluster variables for the cluster algebra of type \mathbb{A}_3 , inserted as the vertices of the cluster complex Σ'_A :



Acknowledgment. We were able to provide only a small glimpse of what cluster tilting theory is about — it is a theory in fast progress, but also with many problems still open. The report would not have been possible without the constant support of many experts. In particular, we have to mention Idun Reiten who went through numerous raw versions. The author has also found a lot of help by Thomas Brüstle, Philippe Caldero, Dieter Happel, Otto Kerner and Robert Marsh, who answered all kinds of questions. And he is indebted to Aslak Buan, Philipp Fahr,

Rolf Farnsteiner, Lutz Hille, Bernhard Keller and Markus Reineke for helpful comments concerning an early draft of this report.

References. In order to avoid a too long list of references, we tried to restrict to the new developments. We hope that further papers mentioned throughout our presentation can be identified well using the appropriate chapters of this Handbook as well as standard lists of references. But also concerning the cluster approach, there are many more papers of interest and most are still preprints (see the arXiv). Of special interest should be the survey by Buan and Marsh [7].

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