INFINITE LENGTH MODULES.
SOME EXAMPLES AS INTRODUCTION.

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The aim of this introduction is to outline the general setting and to exhibit some examples, in order to show interesting features of infinite length modules, but also to point out the relevance of these features in representation theory. We try to present examples as explicit as possible, in contrast to the quite common attitude of being satisfied with the mere existence, an attitude which indicates the desire to rate such features as unpleasant and to avoid them. In contrast, the phenomena we deal with should be considered as typical and as exciting.

The main references to be quoted are the books by Jensen and Lenzing [JL] and Prest [P1], but also volume 2 of Faith [Fa2]. The reader will realize that we follow closely the path of the Trondheim lectures of Crawley-Boevey [CB4] and we have to admit that we are strongly indebted to his mathematical insight. The text is written to be accessible even for a neophyte. We hope that some of the considerations, in particular several examples, are of interest for a wider audience, as we try to present as easy as possible some gems which seem to be hidden in the literature. Part of the text may be rated as "descriptive mathematics", definitely not fitting into the usual pattern of "definition – theorem – proof", but we hope that the topics presented in this way will be illuminating and will provide a better understanding of some problems in representation theory.

General conventions: Always, k will be a field. We denote by $\subseteq$ an inclusion of sets, and we write $\subset$ in case the inclusion is proper. Recall that a quiver $Q = (Q_0, Q_1, s, t)$ consists of a set Q_0 (its elements are called vertices) and a set Q_1 (of arrows); for every arrow $\alpha \in Q_1$, $s(\alpha)$ is its starting point, $t(\alpha)$ its end point.

If R is a ring, we consider mainly left modules and we denote by $\operatorname{Mod} R$ the category of all (left) R-modules, by $\operatorname{mod} R$ the full subcategory of all finitely presented ones. The module ${}_RR$ will be called the regular representation of R.

We denote by $\mathbb{N}$ the set of natural numbers $\{0, 1, 2, \dots\}$, by $\mathbb{Z}$ the integers, and by $\mathbb{Q}$ the rational numbers. For any natural number $n \geq 2$, let $\mathbb{Z}_{(n)}$ be the set of rational numbers $\frac{a}{b}$ such that 1 is the only positive integer which divides both b and n.

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Contents.

Finite length modules

Composition series. Let R be a ring and M an R-module. A *composition series* of M is a finite sequence of submodules

$$(*) \qquad 0 = M_0 \subset M_1 \subset \cdots \subset M_{n-1} \subset M_n = M$$

which cannot be refined (this means that if N is a submodule of M with $M_{i-1} \subseteq N \subseteq M_i$, then $M_{i-1} = N$ or $N = M_i$). Of course, a sequence $(*)$ of submodules is a composition series if and only of all the factors M_i/M_{i-1}, for $1 \le i \le n$ are simple R-modules. Given a composition series $(*)$ of M, then the modules M_i/M_{i-1} are called the *composition factors* of M and $n = |M|$ is said to be its *length.* If M has a composition series, then M is said to be *of finite length.* The **Theorem of Jordan-Hölder** asserts that *the length of a finite length module is uniquely determined,* and that also the composition factors are unique in the following sense: *Given a second composition series*

$$0 = M'_0 \subset M'_1 \subset \cdots \subset M'_{n-1} \subset M'_n = M,$$

there is a permutation σ of $\{1, 2, \ldots, n\}$ such that $M'_i/M'_{i-1} \simeq M_{\sigma(i)}/M_{\sigma(i)-1}$. We can phrase this uniqueness assertion also as follows: *Given a simple R-module S and a composition series $(*)$, then the number of indices i with $M_i/M_{i-1} \simeq S$ only depends on M and S and not on the chosen composition series $(*)$;* this number is called the *Jordan-Hölder multiplicity of S in M.* For further reference, we have given in detail the definition of a finite length module, even though the book exhibited here — as the title makes clear — is devoted to modules which are just **not** of finite length. But the aim of the book is, on the one hand, the comparison of properties of finite length modules with those of general ones, and, on the other hand, to stress the need for using infinite length modules in situations which seem to be confined to finite length modules.

The term *finite length module* obviously refers to the existence of a composition series and to its length. But it is worthwhile to note that there exists a characterization of these modules which does not refer to the actual length and which does not give any hint at all what the length of such a module M could be: *The module M has finite length if and only if M is both artinian and noetherian.* (Recall that M is said to be *noetherian* - or to satisfy the ascending chain condition for submodules - provided for every chain of submodules $M_1 \subseteq M_2 \subseteq \cdots \subseteq M_i \subseteq M_{i+1} \subseteq \cdots$ there exists some number n with $M_n = M_{n+1}$; similarly, M is said to be *artinian* - or to satisfy the descending chain condition for submodules - provided for any chain of submodules $M_1 \supseteq M_2 \supseteq \cdots \supseteq M_i \supseteq M_{i+1} \supseteq \cdots$ there exists some number n with $M_n = M_{n+1}$. The properties of being artinian and noetherian are dual to each other, but there is a characterization of the noetherian modules for which no neat dual property can be named: a module M is noetherian if and only if all submodules are finitely generated.)

Let us consider the interrelation between the various composition series of a finite length module M. Starting with a composition series $(*)$, one looks at two consecutive composition factors M_i/M_{i-1}, M_{i+1}/M_i and the corresponding length 2 module M_{i+1}/M_{i-1}. In case the exact sequence

$$0 \to M_i/M_{i-1} \to M_{i+1}/M_{i-1} \to M_{i+1}/M_i \to 0$$

splits, we find some submodule $M_i' \neq M_i$ with $M_{i-1} \subset M_i' \subset M_{i+1}$, and we obtain a different composition series

$$0 = M_0 \subset \cdots \subset M_{i-1} \subset M_i' \subset M_{i+1} \subset \cdots \subset M_n = M.$$

By a sequence of such changes, we obtain from the given composition series any other composition series (Proof: Given two composition sequences $(M_i)_{0\le i\le n}$ and $(M_i')_{0\le i\le n}$ of a module M, we may assume that $M_{n-1} \neq M_{n-1}'$, otherwise use induction. Take any composition series $(M_i'')_{0\le i\le n-2}$ of $M_{n-1} \cap M_{n-1}'$. The submodules M_i and M_i'' provide two composition series for M_{n-1}, and similarly, the submodules M_i'' and M_i' provide two composition series of M_{n-1}'. It remains to use induction.)

Direct decompositions. Any finite length module can be written as the direct sum of a finite number of indecomposable[1] modules. Now, it is of great importance that such a decomposition is essentially unique, this follows from the following fact: *The endomorphism ring of an indecomposable module of finite length is a local ring* (recall that a ring R is local provided (it is non-zero and) the set of non-invertible elements of R is an ideal). The only idempotents of a local ring are 0 and 1; thus a module with a local endomorphism ring necessarily is indecomposable. The uniqueness statement is known as the **Theorem of Krull-Remak-Schmidt**[2]**:** *Let $M = \bigoplus_{i=1}^m M_i = \bigoplus_{j=1}^n N_j$, such that all the modules M_i have local endomorphism rings, $1 \le i \le m$, and the modules N_j are indecomposable, $1 \le j \le n$.*

[1] A module M is said to be *indecomposable* provided M is non-zero and there is no direct sum decomposition $M = M_1 \oplus M_2$ with both M_1, M_2 non-zero.

[2] This theorem sometimes is referred just as the Krull-Schmidt Theorem, and

Then $n = m$ and there is a permutation σ with $N_j \simeq M_{\sigma(j)}$ for all $1 \le j \le m$. Actually, the corresponding result for arbitrary (not necessarily finite) direct sums is also true, this is the **Theorem of Azumaya:** *Let $M = \bigoplus_{i\in I} M_i = \bigoplus_{j\in J} N_j$, such that all the module M_i have local endomorphism rings, $i \in I$, and the modules N_j are indecomposable, $j \in J$. Then there is a bijection $\sigma\colon J \to I$ such that $N_j \simeq M_{\sigma(j)}$ for all $j \in J$.*

The Krull-Remak-Schmidt theorem yields a reduction of the study of finite length modules to the indecomposable ones, and one of the main tasks of present representation theory is to describe the shape of indecomposable modules of finite length, to look for invariants which allow to read off their properties. In addition to the knowledge of the simple R-modules S_i, one needs to know the possible extensions of these modules, thus the extension groups $\operatorname{Ext}^1_R(S_i, S_j)$, for all simple R-modules S_i, S_j. This knowledge is collected in the *quiver* of R, its vertices are the isomorphism classes $[S_i]$ of the simple R-modules, and one draws an arrow $[S_i] \to [S_j]$ provided $\operatorname{Ext}^1_R(S_i, S_j) \neq 0$. Actually, in case we deal with a k-algebra and $\operatorname{End}(S_i) = k = \operatorname{End}(S_j)$, one will draw not just one arrow, but d arrows $[S_i] \to [S_j]$, where d is the k-dimension of $\operatorname{Ext}^1_R(S_i, S_j)$; in general one may endow the arrow $[S_i] \to [S_j]$ with the $\operatorname{End}(S_i)$-$\operatorname{End}(S_j)$-bimodule $\operatorname{Ext}^1_R(S_i, S_j)$.

Serial modules. Since ancient times, the modules whose submodule lattice is a chain, the so called *serial* (or uniserial) modules, have attracted a lot of interest, these are those modules M such that any pair N_1, N_2 of submodules is comparable: we have $N_1 \subseteq N_2$ or $N_2 \subset N_1$.

First of all, for some quite important rings, all the indecomposable finite length modules are serial: this includes the two most prominent commutative rings, the integers and the polynomial ring in one variable (and the necessity to use their finite length modules should be out of doubt), more generally, all Dedekind rings, but also, in the non-commutative realm, the full rings of upper triangular matrices over some field. A second reason is a very trivial one: any indecomposable module of length 2 is serial, and the R-modules of length 2 provide the information exhibited in the quiver of R. However, a general ring, even a general finite-dimensional k-algebra may have few serial modules: for example, consider the exterior algebra $\Lambda_n(k)$ (with generators $X_1, \dots, X_n$ and relations $X_iX_j + X_jX_i = 0$ and $X_i^2 = 0$ for all i, j). Then the only serial modules are the indecomposable modules of length 1 and 2. In particular, if J is the radical of $\Lambda_n(k)$, then J^2 annihilates all the serial modules. Thus, the structure of the algebra is not determined by the serial modules.

Uniform and couniform modules. Recall that a module M is said to be *uniform* provided M is non-zero, and the intersection of any two non-zero sub-

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we also did so in earlier publications. But one should note that already in 1911 Remak was aware of this kind of results, whereas the relevant papers by Krull and Schmidt are from 1925 and 1928, respectively. The Germans took his life (he died in Auschwitz in 1942), we should not take also his mathematical insight.

modules of M is non-zero again. Thus, M is uniform if and only if M is non-zero and any non-zero submodule of M is indecomposable, again. An injective indecomposable module always has to be uniform, but usually there will exist many indecomposable modules which are not uniform. A module is uniform if and only if its injective envelope is indecomposable. We obtain all uniform R-modules for a ring R as follows: start with all cyclic uniform R-modules U_i, and take all non-zero submodules of their injective envelopes; in this way, we see that the isomorphism classes of uniform modules form a set. There is the dual notion of a couniform module: M is said to be *couniform* provided the following condition is satisfied: if M_1, M_2 are submodules with $M = M_1 + M_2$, then $M = M_1$ or $M = M_2$. Serial modules are both uniform and couniform. Recall that a submodule N of the module M is said to be *essential* provided the only submodule $U \subseteq M$ with $N \cap U = 0$ is the zero module $U = 0$. Dually, N is said to be *small* in M provided the only submodule $V \subseteq M$ with $N + V = M$ is $V = M$ itself. A non-zero module is uniform if and only if every non-zero submodule is essential; it is couniform if any only if every proper submodule is small.

A module M with an essential simple submodule is uniform. In this case, there is a non-zero submodule M_1 which is contained in any non-zero submodule. More generally, one may be interested in modules which have a chain of non-zero submodules $M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$ indexed by the natural numbers such that any non-zero submodule U of M contains at least one of the modules M_i (and then almost all). We say that such a module is $\mathbb{N}$*-uniform.* If R is a Dedekind ring with only countably many prime ideals (for example $R = \mathbb{Z}$, or $R = k[T]$ where k is a countable field), then the module ${}_RR$ is $\mathbb{N}$-uniform. (Proof: Let $P_1, P_2, \ldots$ be all the prime ideals and form $M_i = P_1^{i-1}P_2^{i-2}\cdots P_{i-1}^1 = \prod_{1\le j<i} P_j^{i-j}$. Then these ideals form a chain, and any non-zero ideal contains one of them.)

Diamonds. An important class of modules which are both uniform and couniform should be mentioned: We call a module D a *diamond* provided it has a simple essential submodule and also a small maximal submodule (such modules look really quite like diamonds, and their value in ring and module theory is that of diamonds). Let us show that for a semiperfect ring, the direct sum of all diamonds, one from each isomorphism class, is a faithful module: *For any ideal I of a semiperfect ring R, let $\mathcal{D}(I)$ be the set of diamonds annihilated by I. Then the intersection of the annihilators of the diamonds in $\mathcal{D}(I)$ is I.* (Proof: Let I' be the intersection of the annihilators of all the modules in $\mathcal{D}(I)$. Of course, $I \subseteq I'$. Let a be an element of $R \setminus I$, we show that it does not belong to I'. Choose a left ideal L with $I \subseteq L \subset R$ such that $a \notin L$, and maximal with this property, here we use the axiom of choice. It is well-known and easy to see that R/L has a unique minimal submodule, namely the left module generated by the residue class of a, and this submodule is contained in any non-zero submodule. Since R is semiperfect, any module is a sum of local modules, in particular this is true for R/L. Let $R/L = \sum M_i$, with local modules M_i. All these modules M_i are diamonds. Since $a \cdot (1 + L) = a + L \neq L$, we see that R/L is not annihilated by a. It follows that there exists some i such that M_i is not annihilated by a. On the other hand, R/L and therefore all modules M_i are annihilated by I. This shows that a does not belong to I'.)

Let us assume for a moment that R is a finite-dimensional algebra (or, more generally, an artin algebra). Note that all indecomposable modules of length 2 are diamonds, thus it is quite usual to have infinitely many isomorphism classes of diamonds. In case there are only finitely many isomorphism classes of indecomposable R-modules of finite length, the algebra R is said to be *representation-finite.* For a representation-finite algebra the shape of the possible diamonds is very restricted[2]: For example, any semisimple subfactor of a diamond is of length at most 3. Also, for a representation-finite algebra R, diamonds D, D' with the same Loewy length, and with isomorphic top and isomorphic socle are isomorphic[3].

1. Modules in general.

Let us consider now an arbitrary R-module M, also without any restriction on the ring R.

Composition factors. Note that any module M has sufficiently many simple subfactors, in the following sense: given a non-zero element $m \in M$, there are submodules $M'' \subseteq M' \subseteq M$ with M'/M'' simple, such that $m \in M' \setminus M''$.

Generalizing the concept of a composition series, one may look for chains of submodules M_i of M indexed by $i \in I$, where I is one of the sets $\mathbb{N}, -\mathbb{N}, \mathbb{Z}$, with $M_{i-1} \subset M_i$, such that the factors M_i/M_{i-1} are simple, for all i, with $\bigcap_{i\in I} M_i = 0$ and $\bigcup_{i\in I} M_i = M$.

First, consider the case of an ascending chain, thus $I = \mathbb{N}$: in this case, the corresponding factors are again uniquely determined, as in the Jordan-Hölder-Theorem: Assume that there is given a second chain $M'_i,\ i \in \mathbb{N}$, with the corresponding properties. Consider the index $j \in \mathbb{N}$. The submodule M_j is contained

[2] Let us assume that R is a k-algebra, where k is an algebraically closed field. Using covering theory [GR], we can assume that R is representation-directed. In addition, we may assume that we deal with a faithful diamond D. But then D is both projective and injective and R is the incidence algebra of a finite poset with one minimal and one maximal element. The possible cases are well-known: For example, if there are more than 13 simple R-modules, then there are just seven possibilities labelled (Bo1), (Bo15), (Bo16), (Bo17), (Bo19), (Bo20), (Bo21) in [R4]. Note that all the Jordan-Hölder multiplicities of such a module D are equal to 1 and one obtains in this way a maximal positive root of the corresponding quadratic form. Conversely, Dräxler [Dl] has observed that any representation-directed k-algebra which has a maximal positive root with all coefficients equal to 1 is obtained from an algebra with a faithful indecomposable projective-injective module by a change of orientation.

[3] In case there do exist non-isomorphic diamonds D, D' with the same Loewy length, isomorphic top and isomorphic socle, then it is easy to show that R is even of "strongly unbounded representation type".

in $M = \bigcup M'_j$, and since M_j is finitely generated, it is contained already in some $M'_{j'}$; but $M'_{j'}$ is a finite length module, thus the composition factors of M_j occur as composition factors of $M'_{j'}$. It follows that for any simple module S, the number of indices i with $M_i/M_{i-1} \simeq S$ is smaller or equal to the number of factors $M'_i/M'_{i-1} \simeq S$. By symmetry, these numbers actually are equal, and this is what we wanted to show.

In contrast, for descending chains ($I = -\mathbb{N}$) we cannot expect a corresponding assertion, as already the case $R = \mathbb{Z}$ shows: It is easy to write down all possible descending chains with simple factors: just pick any sequence $p_1, p_2, \dots$ of prime numbers and take $M_{-i} = p_1 \cdots p_i\mathbb{Z}$. Then $M_{-i}/M_{-i-1} \simeq \mathbb{Z}/\mathbb{Z}p_{i+1}$. For example, we can take the constant sequences $2, 2, \dots$ or $3, 3, \dots$; they will not have any common factor. We also should note another fact: both $\mathbb{Z}$ and its p-adic completion $\hat{\mathbb{Z}}_p$ have descending chains M_i, $i \in -\mathbb{N}$ with all the factors $M_i/M_{i-1} \simeq \mathbb{Z}/\mathbb{Z}p$; but the sets $\mathbb{Z}$ and its p-adic completion $\hat{\mathbb{Z}}_p$ have different cardinalities. Of course, similar features occur for the case $I = \mathbb{Z}$. Here another example: again, take $R = \mathbb{Z}$ and consider the $\mathbb{Z}$-module $\mathbb{Q}$. Given a chain M_i $i \in \mathbb{Z}$ of submodules with simple factors such that $\bigcap_i M_i = 0$ and $\bigcup_i M_i = \mathbb{Q}$, we obtain via $M_i/M_{i-1} \simeq \mathbb{Z}/\mathbb{Z}p_i$ a family p_i of prime numbers indexed by $i \in \mathbb{Z}$ such that for any prime number p and any natural number n, there is an index $i \geq n$ with $p_i = p$. And conversely, any such indexed family of prime numbers occurs in this way.

Direct decompositions. Concerning direct decompositions of modules, quite surprising phenomena are possible:

- **Failure of cancelation.** The existence of indecomposable modules M_0, M_1, M_2 such that $M_0 \oplus M_1 \simeq M_0 \oplus M_2$, but $M_1 \not\simeq M_2$.
- **Failure of the Schröder-Bernstein property.** The existence of non-isomorphic modules M, N with monomorphisms $M \to N$ and $N \to M$.
- **Non-uniqueness of roots.** The existence of non-isomorphic modules M, N such that with $M^s \simeq N^s$ for some $s \geq 2$.
- **Isomorphism of specified powers.** The existence of a module M such that $M^s \simeq M^t$ for natural numbers s, t if and only if $s \cong t$ modulo some fixed number n.
- **Decompositions into an arbitrary finite number of indecomposables.** The existence of a module M which can be written as the direct sum of t indecomposable modules for any $t \geq 2$, but not as the direct sum of infinitely many non-zero modules.
- **Superdecomposability.** The existence of a non-zero module M such that no non-zero direct summand of M is indecomposable (thus M itself is not indecomposable, and if we take any non-trivial direct decomposition $M = M_1 \oplus M_2$, then both direct summands M_1, M_2 can be further decomposed).
- **Finiteness of direct decompositions, but any direct decomposition involves decomposable modules.** The existence of a module M with the following property: If $M = \bigoplus_{i \in I} M_i$ is a direct decomposition with non-zero

modules M_i, then the index set I is finite, but at least one of the modules M_i is decomposable.

• **Existence of large indecomposables.** The existence of indecomposable modules which have arbitrarily large cardinality.

The free algebra $k\langle X, Y\rangle$ in two variables has such modules, but many other algebras also. Sometimes, such modules are called "pathological", but this term is misleading: what is called "pathological" seems to be the general behaviour of modules, and why should we think of a general module to be pathological? Let us exhibit some examples which are easy to comprehend and to remember. In addition, we want to formulate some consequences.

Modules and their endomorphism rings. Properties concerning the possible direct decompositions of a module M are reflected in its endomorphism ring $\mathrm{End}(M)$, more precisely in the set of idempotents in $\mathrm{End}(M)$. Namely, the direct sum decompositions $M = M_1 \oplus M_2$ correspond bijectively to the idempotents $e \in \mathrm{End}(M)$ (given such an idempotent e, let $M_1 = eM$ and $M_2 = (1-e)M$). Thus, the study of modules with prescribed behaviour with respect to direct decompositions is part of the question what kind of rings can be realized as endomorphism rings of modules. This topic will be discussed in detail by Eklof in his contribution [E].

Indecomposable modules. Let us start with the indecomposable modules themselves. Of course, a non-zero module M is indecomposable if and only if the only idempotents in $\mathrm{End}(M)$ are the elements 0 and 1.

One of the assumptions in the theorem of Krull-Remak-Schmidt is that one deals with modules M_i with local endomorphism rings. It should be kept in mind that the conclusion of the theorem may hold also in case some of the endomorphism rings are not local. For example, the finitely generated abelian groups are direct sums of indecomposable torsion groups (they are of finite length, thus have local endomorphism rings) and copies of $\mathbb{Z}$ (here the endomorphism ring is $\mathbb{Z}$ again, thus a ring which is **not** local). Given a module M, the full force of $\mathrm{End}(M)$ being local is expressed in the exchange property [CJ,Wa]: If there is a split embedding $M \subset N$ and $N = \bigoplus_{i\in I} N_i$ with arbitrary modules N_i, then there exist submodules $N_i' \subseteq N_i$ such that $N = M \oplus \left(\bigoplus_{i\in I} N_i'\right)$ (and these submodules N_i' are necessarily direct summands of N_i).

There do exist indecomposable modules whose endomorphism rings are not local, but still semilocal; such modules are discussed by Facchini [Fc3] and Pimenov and Yakovlev [PY]; they share some of the properties of modules with local endomorphism rings. An artinian indecomposable module always has a semilocal endomorphism ring [CD]. Let us quote from [PY] an example of an artinian modules which is cyclic, but not of finite length[4]: Take the ring $R = \begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{bmatrix}$ and

[4] Observe that for R being commutative, cyclic artinian modules always are

consider the indecomposable projective module P and its submodules U and V, where

$$P = \begin{bmatrix} \mathbb{Q} \\ \mathbb{Q} \end{bmatrix} \quad \supset \quad V = \begin{bmatrix} 0 \\ \mathbb{Q} \end{bmatrix} \quad \supset \quad U = \begin{bmatrix} 0 \\ \mathbb{Z}_{(p)} \end{bmatrix},$$

where p is a prime number. We are interested in the factor module P/U. First of all, as a factor module of P, this is a cyclic module. Second, it is artinian and of infinite length: consider the filtration $0 \subset V/U \subset P/U$. Here, $(P/U)/(V/U) = P/V$ is a simple R-module, whereas V/U is annihilated by P (this is a twosided ideal), thus an R/P-module and $R/P = \mathbb{Z}$, and actually, V/U is just a Prüfer group. Also, it is easy to check that $\operatorname{End}(P/U)$ is equal to $\mathbb{Z}_{(p)}$. — We may do the same construction, replacing p by a product pq of two different prime numbers, thus we consider

$$U' = \begin{bmatrix} 0 \\ \mathbb{Z}_{(pq)} \end{bmatrix}.$$

Again, P/U' is cyclic and artinian (V/U' is the direct sum of two Prüfer groups), and this time $\operatorname{End}(P/U') = \mathbb{Z}_{(pq)}$, thus we obtain a semilocal ring which is not local[5]. Note that the semilocal ring $\operatorname{End}(P/U')$ has precisely two maximal ideals and is a domain; in particular, it is not semiperfect. In the same way, replacing p by a product of n pairwise different prime numbers, we obtain a cyclic artinian module whose endomorphism ring is a semilocal domain with precisely n maximal ideals.

Large indecomposables. Using transfinite induction, Fuchs (1959) has constructed large indecomposable abelian groups, namely groups whose cardinality is any cardinal number less than the first strongly inaccessible cardinal number (a cardinal number λ is said to be strongly inaccessible, provided first: $\lambda > \aleph_0$, second: $2^\mu < \lambda$ for every cardinal number $\mu < \lambda$, and third: $\sum_{i\in I} \mu_i < \lambda$, whenever I is an index set of cardinality smaller than λ and also all $\mu_i < \lambda$). These strongly inaccessible cardinal numbers are huge, and their existence is independent of the usual axioms of set theory. On the other hand, in 1973, Shelah was able to remove this cardinality restriction. Shelah's methods will be presented in this volume by Göbel [Gö]. To quote from his introduction: these are "simple, but

of finite length. The reason is that in the commutative case, any cyclic module is really a factor ring, and artinian rings always are noetherian. Let us stress the importance of understanding cyclic artinian modules in case one is interested in artinian modules in general: after all, every module is the sum of its cyclic submodules, thus an artinian one is the sum of artinian cyclic modules. The usual predominance of commutative ring theory is here a clear source for misdirection: the possible existence of non-trivial artinian modules is one of the intrinsic features of non-commutative algebra.

[5] In [Wi], 31.14, Wisbauer claims that the endomorphism ring of an indecomposable artinian module is local.

clever counting arguments"; they are put together in "Shelah's Black Box" and designed for applications in different areas of mathematics.

Formatted modules. We suggest to call a module M *formatted* provided it contains an essential submodule which is a direct sum of uniform modules. Note that a module M is formatted if and only if every non-zero submodule of M contains a uniform submodule (for the proof, use the lemma of Zorn). As a consequence, the class of formatted modules is closed under submodules.

Let M be a formatted module, thus there is an essential submodule $\bigoplus_{i\in I} U_i$ with uniform modules U_i. Note that the cardinality of I is an invariant of M, which may be called its *uniform dimension* (or *Goldie dimension*). (Proof: Consider the set $\mathcal{U}$ of uniform submodules of an module. We call $Q \in \mathcal{U}$ dependent on $P_i (i \in I)$ provided Q intersects $\sum P_i$ non trivially. A family in $\mathcal{U}$ is said to be independent, if no element is dependent on the rest. With these definitions, the set $\mathcal{U}$ satisfies the axioms of an abstract dependence relation [Cn]: First, any element of a set is dependent on the set. Second: If Z is dependent on the independent set Y and each element of Y is dependent on X, then Z is dependent on X. And finally, the exchange property: If Y is dependent on the set $\{X_1, \dots, X_n\}$, but not on $\{X_2, \dots, X_n\}$, then X_1 is dependent on $\{Y, X_2, \dots, X_n\}$. For such a dependence relation $\mathcal{U}$, the maximal independent subsets of $\mathcal{U}$ are precisely the minimal spanning sets [Cn, 1.4.1] (a spanning set is a subset of $\mathcal{U}$ such that every element of $\mathcal{U}$ is dependent on it) — such a set is called a basis, and it is a general assertion that the cardinality of a basis is an invariant.)

It is easy to construct indecomposable modules of infinite uniform dimension. Consider the subring $R = \begin{bmatrix} k & 0 \\ k[T] & k \end{bmatrix}$ of the ring of 2×2 matrices over $k[T]$. There are up to isomorphism two indecomposable projective R-modules: one is simple, the other one is a local module of infinite uniform dimension.

In noetherian ring theory, the modules of finite uniform dimension play a decisive role, these are just those modules M which do not contain an infinite direct sum of non-zero submodules, as one easily verifies. As a trivial consequence, one sees that any noetherian module has finite uniform dimension, thus is an essential extension of a finite direct sum of uniform modules. More generally, any module M which is generated by noetherian modules is formatted[6]. (Proof: Assume that M is generated by noetherian submodules M_i with $i \in I$. Note that the sum of two noetherian submodules is again noetherian, thus we may assume that the submodules M_i form a directed family. If U is any non-zero submodule of M, then $U = U \cap M = U \cap \sum M_i = \sum (U \cap M_i)$ yields some $i \in I$ with $U \cap M_i$ non-zero. Now, $U \cap M_i$ is noetherian, since it is a submodule of M_i, thus $U \cap M_i$ contains some uniform submodule. This shows that U has a uniform submodule.)

As a consequence, given a left noetherian ring R, then any R-module is formatted. The same is true in case R is semi-artinian, since the semi-artinian

[6] Warning: In general, the sum of two formatted submodules does not have to be formatted, an example will be given below.

rings are characterized by the property that every non-zero module has a simple submodule.

Discrete modules. Next, we want to discuss the question of the existence of indecomposable direct summands. A module M is said to be *discrete* provided every non-zero direct summand of M has an indecomposable direct summand. As we will see below, for injective modules the formatted ones are just the discrete ones. For non-injective modules, being formatted and being discrete are completely different properties: For an artinian ring T, all the R-modules are formatted, but any strictly wild finite-dimensional algebra has many superdecomposable modules. This shows that a formatted module does not have to be discrete. Conversely, it is easy to construct local modules which do not contain uniform submodules (this provides an indecomposable, thus discrete module which is not formatted): consider the matrix algebra $R = \begin{bmatrix} k & 0 \\ k\langle X,Y\rangle & k\langle X,Y\rangle \end{bmatrix}$, where $k\langle X,Y\rangle$ is the free algebra in two generators. The indecomposable projective R-module given by the first column is local and has no uniform submodule. Namely, observe that for any element $a \in k\langle X,Y\rangle$, the two elements Xa and Ya of $k\langle X,Y\rangle$ generate submodules with zero intersection:

$$k\langle X,Y\rangle Xa \,\cap\, k\langle X,Y\rangle Ya \;=\; 0.$$

Superdecomposable modules. A module M is said to be *superdecomposable* provided no direct summand of M is indecomposable. Note that the class of superdecomposable modules is closed under direct summands.

Superdecomposable modules may look surprising at the first sight, but there are at least two natural examples to be mentioned. The first is the injective envelope I of the regular representation ${}_{k\langle X,Y\rangle}k\langle X,Y\rangle$ of the free algebra $k\langle X,Y\rangle$ in two variables. Given a non-zero direct summand N of I, the intersection $N \cap k\langle X,Y\rangle$ is non-zero. Take an element a in this intersection, then the injective hull of $k\langle X,Y\rangle Xa$ is a proper non-zero direct summand of N. This shows that I is superdecomposable.

Another example of a superdecomposable module is the regular representation of the ring $\mathcal{C}(X)$ of all continuous functions $X \to \mathbb{R}$, where X is the Cantor discontinuum (or any totally disconnected topological space X without isolated points). As for any ring R, the direct summands of the regular representation ${}_RR$ are of the form Re, where e is an idempotent in R. The idempotents e of R correspond bijectively to the subsets of X which are both open and closed. Note that we assume that X is totally disconnected and that there are no isolated points. Thus, if X' is any non-empty subset of X which is open and closed, then X' can be written (in many ways) as the disjoint union of subsets which again are open and closed.

Finiteness of direct decompositions, but any direct decomposition involves decomposable modules. Let us stress that there do exist superdecomposable modules which cannot be written as infinite direct sums of non-zero modules. Actually, both examples presented above can be quoted. First, consider the injective envelope I of the regular representation ${}_RR$ of the free algebra $R = k\langle X, Y\rangle$ in two variables, and assume $I = \bigoplus_{j\in J} I_j$. The unit element 1_R belongs to a finite direct sum $\bigoplus_{j\in J'} I_j$, where J' is a finite subset of J, but then also $R = R\cdot 1_R$ is contained in $\bigoplus_{j\in J'} I_j$, thus also I. Similarly, consider $R = \mathcal{C}(X)$, where X is totally disconnected without isolated points. If we assume in addition that X is compact, then again we see that ${}_RR$ cannot be written as the direct sum of infinitely many non-zero submodules.

But there are also discrete modules with only finite direct decompositions, but such that any direct decomposition involves decomposable modules. Consider again the subring $R = \begin{bmatrix} k & 0 \\ k[T] & k \end{bmatrix}$ of the ring of 2×2 matrices over $k[T]$ and let $P(1)$ be the indecomposable projective R-module which is not simple. We denote by I the injective envelope of $P(1)$. Since the socle U of $P(1)$ is an essential submodule, it follows that U is essential in I. Also, $\operatorname{Hom}(I/U, I) = 0$, thus the intersection with U yields a bijection between the direct decompositions $I = I'\oplus I''$ and the direct decompositions of U. In particular, we see that I is a discrete module. On the other hand, assume there is given a countable direct decomposition $I = \bigoplus_i I_i$ with non-zero submodules I_i. Note that $I_i\cap U \neq 0$, thus $I_i\cap U$ contains a simple submodule S_i. Of course, all these submodules S_i are isomorphic to the simple projective module $P(2)$, thus $\bigoplus_i S_i$ is isomorphic to U. In this way, we obtain an injective map $U \to \bigoplus_i S_i \subset \bigoplus_i I_i = I$. Using the injectivity of I, we obtain an extension of f to a homomorphism $f'\colon P(1) \to \bigoplus_i I_i$. But $P(1)$ is cyclic, thus the image of f' is contained in a finite direct sum, a contradiction. This shows that any direct decomposition of I has to be finite.

Comparison of different direct decompositions. Let us return to the theorem of Krull-Remak-Schmidt: If M is a module whose endomorphism ring is local, one has the following nice situation: if M occurs as a direct summand of a module M' and M' can be decomposed in the form $M' = M_1' \oplus M_2'$, then M is isomorphic to a direct summand of M_1 or M_2. This means that such a module M behaves like a prime element when we consider the relation of being a direct summand in analogy to the relation that one element divides another element in a commutative ring.

We are going to exhibit some typical ways for obtaining modules M_1, M_2, N_1, N_2 with an isomorphism $M_1 \oplus M_2 \simeq N_1 \oplus N_2$. The aim is to provide examples of indecomposable modules M_1, N_1, N_2 such that M_1 is a direct summand of $N_1 \oplus N_2$, but not isomorphic to any one of the modules N_1, N_2.

(1) Let M be a projective module with submodules N_1, N_2 such that $N_1 + N_2 = M$. Let $M' = N_1 \cap N_2$. Then the inclusion maps ι give rise to an exact sequence

$$0 \to M' \xrightarrow{\begin{bmatrix} \iota \\ -\iota \end{bmatrix}} N_1 \oplus N_2 \xrightarrow{[\iota \ \ \iota]} M \to 0,$$

and, since M is projective, the sequence splits. Thus

$$M \oplus M' \simeq N_1 \oplus N_2.$$

We consider the special case where R is an integral domain and consider the regular representation $M = {}_RR$. Given non-zero ideals N_1, N_2, then also $M' = N_1 \cap N_2$ is non-zero (an integral domain is a uniform module), thus all the modules M, M', N_1, N_2 are indecomposable.

Here are some typical cases. We may take as N_1, N_2 any pair of two different maximal ideals of an integral domain R, then clearly $N_1 + N_2 = R$. For example, let $R = k[X,Y]$ be the polynomial ring in two variables, $N_1 = RX + RY$, and $N_2 = RX + R(Y-1)$. The submodules N_1, N_2 are not cyclic; in particular, they are not isomorphic to ${}_RR$. Thus we see that ${}_RR$ is an indecomposable direct summand of $N_1 \oplus N_2$, and not isomorphic to N_1 or N_2.

Second, an example suggested by Mazorchuk: Let R be the subring of the polynomial ring $k[X]$ generated by X^2 and X^3. Let $N_1 = R(X^2+1)$ and $N_2 = RX^2 + RX^3$. Again we see that $N_1 + N_2 = R$. Note that $N_1 \cap N_2 = R(X^4 + X^2)$, in particular, both N_1 and $N_1 \cap N_2$ are principal ideals, thus isomorphic, as R-modules, to ${}_RR$. The isomorphism $M \oplus M' \simeq N_1 \oplus N_2$ can be rewritten in the form

$${}_RR \oplus {}_RR \simeq {}_RR \oplus N_2,$$

and N_2 is not cyclic. A typical instance where cancelation fails.

(2) Consider a module with submodules $0 \subset U \subset M' \subset M$. There is the following exact sequence:

$$0 \to M' \xrightarrow{\begin{bmatrix} \iota \\ -\pi' \end{bmatrix}} M \oplus M'/U \xrightarrow{[\pi \quad \iota']} M/U \to 0,$$

again inclusion maps are denoted by ι, projection maps by π. Let us assume that the module M' is both uniform and couniform, and, in addition[7], that there is a monomorphism $f\colon M \to M'$ and an epimorphism $g\colon M'/U \to M'$. Under these assumptions, the sequence splits, thus

$$M \oplus M'/U \simeq M' \oplus M/U.$$

(Proof: Consider the endomorphism $h = f \circ \iota - g \circ \pi'$ of M'. First, let us show that h is a monomorphism. Note that $\mathrm{Ker}(h) \cap \mathrm{Ker}(g \circ \pi')$ is contained in the

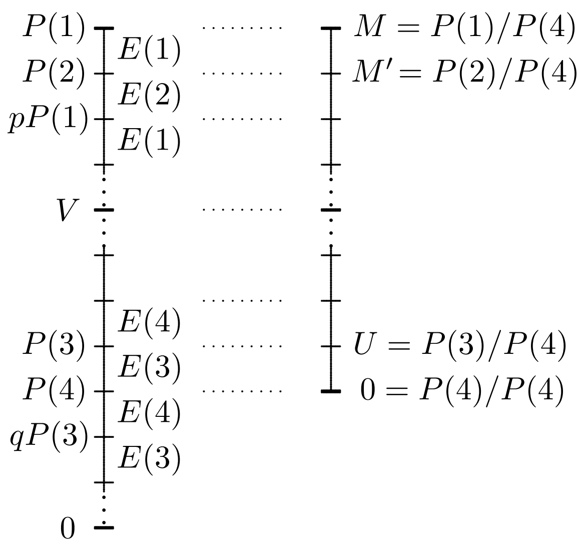
[7] These additional assumptions imply that the given module M is isomorphic to a proper submodule as well as a proper factor module of itself (note that an artinian module is never isomorphic to any of its proper submodules, a noetherian module is never isomorphic to any of its proper factor modules; thus, the modules we are dealing with are neither artinian nor noetherian).

kernel $\mathrm{Ker}(f\circ\iota)$ and $\mathrm{Ker}(f\circ\iota)=0$. Since M' is uniform, and $\mathrm{Ker}(g\circ\pi')$ is non-zero, it follows that $\mathrm{Ker}(h)=0$. Second, $\mathrm{Im}(f\circ\iota)$ is a proper submodule of M', since both f and ι are monomorphism and at least ι is a proper inclusion. Now, $\mathrm{Im}(f\circ\iota)+\mathrm{Im}(h)\supseteq \mathrm{Im}(g\circ\pi')=M'$. Since M' is couniform, it follows that $\mathrm{Im}(h)=M'$. This shows that $h=[\,f\quad g\,]\circ\begin{bmatrix}\iota\\ -\pi'\end{bmatrix}$ is an isomorphism, thus the sequence splits.)

The latter construction is the one used by Facchini in order to show that the Krull-Remak-Schmidt property does not hold for finitely presented modules over serial rings [Fc1,Fc2,Fc3], in this way solving a problem raised by Warfield. Consider the ring

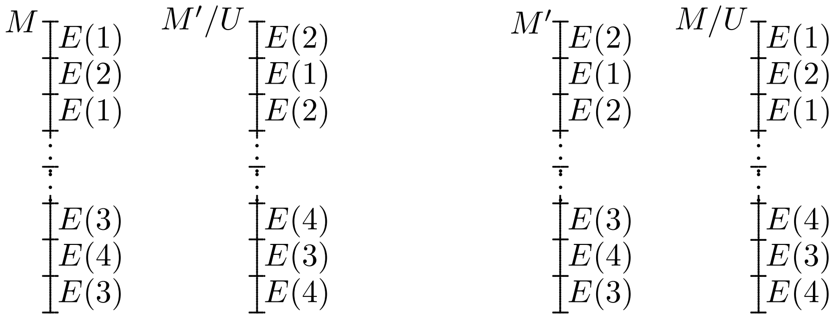
$$R=\begin{bmatrix}\mathbb{Z}_{(p)} & p\mathbb{Z}_{(p)} & 0 & 0\\ \mathbb{Z}_{(p)} & \mathbb{Z}_{(p)} & 0 & 0\\ \mathbb{Q} & \mathbb{Q} & \mathbb{Z}_{(q)} & q\mathbb{Z}_{(q)}\\ \mathbb{Q} & \mathbb{Q} & \mathbb{Z}_{(q)} & \mathbb{Z}_{(q)}\end{bmatrix}$$

where p,q are different primes. Denote by $P(i)$, for $1\le i\le 4$, the indecomposable projective R-module given by the ith column, let $V=\begin{bmatrix}0\\0\\ \mathbb{Q}\\ \mathbb{Q}\end{bmatrix}$ and note the various inclusions of these modules: all are submodules of $P(1)$. Here, on the left, is the complete submodule lattice of $P(1)$, this module $P(1)$ is a serial module. On the right, the module $M=P(1)/P(4)$ is depicted and we have marked its submodules $U=P(3)/P(4)$ and $M'=P(2)/P(4)$:



The following observation is of importance: The scalar multiplication by p maps $P(1)$ onto the submodule $pP(1)$, with $pP(1)\subset P(2)\subset P(1)$, and the factors $E(1)=P(1)/P(2)$ and $E(2)=P(2)/pP(1)$ are simple. However $pP(3)=P(3)$

and $pP(4) = P(4)$, since p is invertible in $\mathbb{Z}_{(q)}$. Similarly, $qP(1) = P(1)$, $qP(2) = P(2)$ and $qP(3) \subset P(4) \subset P(3)$ with simple factors $E(3) = P(3)/P(4)$ and $E(4) = P(4)/qP(3)$. As monomorphism $f\colon M \to M'$ we take the multiplication by p. As epimorphism $g\colon M'/U \to M'$ we compose the isomorphism $M'/U = P(2)/P(3) \to P(2)/qP(3)$ given by the multiplication with q and the canonical projection $P(2)/qP(3) \to P(2)/P(4)$. In this way, we see that all the requirements needed are fulfilled and $M \oplus M'/U \simeq M' \oplus M/U$. The modules occurring in these two direct decompositions have the following form:



What we see is a sort of partial exchange, we may describe it as follows: if we keep the upper (noetherian) parts of the modules, then the lower (artinian) parts are exchanged. Remember that for all these considerations, the essential ingredient is the existence of a proper monomorphism $M \to M$ and a proper epimorphism $M \to M$; here the scalar multiplications by p and by q serve this purpose[8]. It should also be remarked that the endomorphism ring of our module M is just $\mathbb{Z}_{(pq)}$. In general [Fc3], the endomorphism ring of a serial module is semilocal with at most two maximal ideals, one consists of all endomorphisms which are not injective, the other of all endomorphisms which are not surjective, and any proper one-sided ideal is contained in at least one of these ideals (these ideals may coincide, as one knows from the indecomposable modules of finite length, but in general they will not, a typical example is the module M considered above).

The isomorphism property for specified powers. Consider a module with the isomorphism property for specified powers, say for $n = 2$. In particular, this means that $M \not\simeq M \oplus M$, but $M \simeq M \oplus M \oplus M$. Two consequences should be mentioned: Consider the modules M and $N = M \oplus M$. *The modules M, N are non-isomorphic and have the following properties:* First, *M is a direct summand of N and N is a direct summand of M* (failure of the Schröder-Bernstein property). Second, *the modules $M \oplus M$ and $N \oplus N$ are isomorphic* (non-uniqueness

[8] Note that the existence of the scalar multiplications by p and by q has the consequence that all serial modules with top $E(1)$ and socle $E(3)$ are isomorphic. It follows that the ring R has only finitely many isomorphism classes of serial modules of infinite length. In contrast, if we deal with the similarly defined 4×4 matrix ring using only one prime number $p = q$, then there are infinitely many isomorphism classes of serial modules of infinite length.

of roots). In case we deal with abelian groups, thus with $\mathbb{Z}$-modules, the question whether non-isomorphic modules with one of these properties do exist, was raised by Kaplansky [Ka] in 1954 as his two famous *Test Problems;* the existence of non-isomorphic abelian groups M and N such that $M \oplus M$ and $N \oplus N$ are isomorphic was shown by Jónsson in 1957, that of non-isomorphic abelian groups M and N such that M is a direct summand of N and N is a direct summand of M was shown by Sąsiada in 1961. In 1964, Corner exhibited an abelian group M which is not isomorphic to $M \oplus M$, but isomorphic to $M \oplus M \oplus M$. We refer to Eklof's contribution [E] in this volume which focuses the attention precisely to this problem.

The non-uniqueness of roots is discussed in detail in the very nice booklet by Lam [La] which just has appeared. He starts with a detailed discussion of the isomorphism $P \oplus P \simeq R \oplus R$, where $R = \mathbb{Z}[\sqrt{-5}]$ and P is the ideal of R generated by 2 and $1 + \sqrt{-5}$. Lam presents several additional examples, but also many affirmative results valid under suitable conditions on the ring or the modules and he outlines the relationship to problems in number theory, K-theory and the theory of operator algebras. He also includes related results concerning the cancelation problem.

2. The categorical setting

For the problems to be discussed here, it seems to be appropriate to consider not only modules over a ring with 1, but more generally over a ring R which has sufficiently many idempotents: this means that $R = \bigoplus_{i,j \in I} e_i R e_j$ for some set of idempotents e_i, $i \in I$ (and here we should not be too fuzzy about the word "set"). Of course, this means that R is really just an additive category (preferably with a set of objects), and the category of R-modules is the category of all contravariant functors to abelian groups (this point of view is often expressed by Mitchell's formulation that we deal with a "ring with several objects"). The advantage of this slight generalization is the following: starting with such a ring with several objects, thus with an additive category $\mathcal{A}$, the functor category $\operatorname{Mod} \mathcal{A}$ of all controvariant functors from $\mathcal{A}$ to $\operatorname{Mod} \mathbb{Z}$ again is an additive category, and we may repeat this process. For set theoretical reasons, it will be necessary to restrict the size of the modules which we consider, or to work with increasing universes, but this should not lead to confusion.

Additive categories arise very frequently in mathematics and all the questions which usually are asked when dealing with a ring (a commutative one as it is considered in number theory or in algebraic geometry, or the non-commutative versions which are now very popular under the name of quantizations) can and have to be asked in this more general setting. One of the main questions concerns the description of such a ring or such a category, by generators and relations. If we consider a representation-finite k-algebra R, where k is an algebraically closed field of characteristic different from 2, it is the Auslander-Reiten quiver Γ_R which produces such a presentation: the generators are given by the vertices and the arrows of Γ_R, thus by the indecomposable R-modules and the irreducible maps between them, the relations are just the mesh relations. Also for a general, not necessarily representation-finite k-algebra R, the Auslander-Reiten quiver Γ_R describes at least part of the category of R-modules, the next section will be devoted to discuss some examples.

When dealing in this way with a module category as a sort of ring, the individual modules are just mere elements of this ring (note that a category is just a set or class of "maps", these are the elements of the category; in this interpretation, the objects of the category are the identity maps, thus special idempotents.) On the other hand, many of the properties of a module which are of interest are really categorical properties, they can be recovered from the module category. In particular, this holds true for the property of having finite or infinite length, at least as long as we work with the full module category.

Ideals. When dealing with a ring, one of the basic concepts is that of an ideal and the corresponding factor ring. Clearly, dealing with an additive category $\mathcal{A}$, there also is the notion of an ideal $\mathcal{I}$ in $\mathcal{A}$ and of the corresponding factor category[9] $\mathcal{A}/\mathcal{I}$. In particular, given a class $\mathcal{U}$ of objects in $\mathcal{A}$ (or, equivalently, a full subcategory), we may consider the ideal $\langle\mathcal{U}\rangle$ given by all maps which factor through a finite direct sum of objects in $\mathcal{U}$. Such an ideal is idempotent since it is generated by idempotent elements (the identity maps 1_U, where U is a finite direct sum of objects in $\mathcal{U}$). But note that not all idempotent ideals have to be of this form, as we will see below.

As for a ring, we may speak of the Jacobson radical $\operatorname{rad}(\mathcal{A})$ of an additive category $\mathcal{A}$: it consists of all homomorphisms $f\colon A \to A'$ such that for any homomorphism $g\colon A' \to A$, the endomorphism $1 + g \circ f$ is invertible (this just means that $\operatorname{Hom}(A', A) \circ f$ is contained in the Jacobson radical of $\operatorname{End}(A)$).

If we consider idempotent ideals in an additive category $\mathcal{A}$, two different kinds can be distinguished. First of all, the ideal $\mathcal{I}$ may be generated by idempotent elements, thus "by objects", as mentioned above. But second, it is also easy to construct idempotent ideals which contain only nilpotent elements. We stress that the only idempotents contained in the Jacobson radical of $\mathcal{A}$ are the zero endomorphisms. In case every object of $\mathcal{A}$ is a finite direct sum of objects with local endomorphism rings, then any ideal of $\mathcal{A}$ which properly contains $\operatorname{rad}(\mathcal{A})$ will also contain non-trivial idempotents; thus, in his case $\operatorname{rad}(\mathcal{A})$ is the largest

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[9] When starting with an abelian category $\mathcal{A}$, the symbol / sometimes also is used in order to denote quotient categories with respect to a Serre subcategory; this is a completely different construction and should not be confused. Well-known situations of forming the factor categories $\mathcal{A}/\mathcal{J}$ are the categories $\underline{\operatorname{mod}}R = \operatorname{mod} R/\langle\mathcal{P}\rangle$ and $\overline{\operatorname{mod}}R = \operatorname{mod} R/\langle\mathcal{I}\rangle$ where $\langle\mathcal{P}\rangle$ is the ideal of all homomorphisms which factor through a projective module, and $\langle\mathcal{I}\rangle$ is that of all homomorphisms which factor through an injective module. For R selfinjective, these categories coincide and are of special importance.

ideal without non-trivial idempotents. The Jacobson radical $\operatorname{rad}(\mathcal{A})$ never contains non-trivial idempotents, but it may contain non-trivial idempotent ideals. Actually, we may define by transfinite induction the power $\operatorname{rad}^\alpha(\mathcal{A})$ for any ordinal number α, and then the transfinite radical[10] $\operatorname{rad}^\infty(\mathcal{A}) = \bigcap_\alpha \operatorname{rad}^\alpha(\mathcal{A})$, where α runs through all ordinal numbers, see [P1,Sc1,Sc3]. This transfinite radical $\operatorname{rad}^\infty(\mathcal{A})$ is an idempotent ideal and it contains all the idempotent ideals of $\mathcal{A}$ which are contained in $\operatorname{rad}(\mathcal{A})$. An easy way for obtaining an idempotent ideal is the following: select a class $\mathcal{H}$ of homomorphisms such that any $h \in \mathcal{H}$ can be written in the form $h = g \circ h' \circ g' \circ h'' \circ g''$ with $h', h'' \in \mathcal{H}$ and arbitrary maps g, g', g'', then $\mathcal{H}$ generates an idempotent ideal.

Let us mention two typical situations where it is easy to specify such classes $\mathcal{H}$ inside $\operatorname{rad}(\mathcal{A})$. First of all, consider a tubular algebra [R6]. Recall that there are a preprojective component and a preinjective component, and that the remaining components form tubular families $\mathcal{T}_\gamma$ with γ a non-negative rational number or the symbol ∞. Let $\mathcal{H}$ be the set of all homomorphisms $h\colon A \to A'$, where A belongs to a family $\mathcal{T}_\gamma$, A' to $\mathcal{T}_{\gamma'}$ and $\gamma < \gamma'$. Choose γ'' with $\gamma < \gamma'' < \gamma'$ and note that $\mathcal{T}_{\gamma''}$ is a separating tubular family: this shows that h can be factored through a module in $\mathcal{T}_{\gamma''}$. Note that the idempotent ideal generated by $\mathcal{H}$ has the following property: any element of this ideal is the sum of elements f with square zero.

Second, consider a non-domestic string algebra, see [Sc3]. There are arrows $\alpha, \beta, \gamma, \delta$ and words of the form $u\gamma^{-1}v, u\delta w$ which both start with the letter α and end in the letter β^{-1}. We fix $\alpha, \beta, \gamma, \delta$ and v, w, and consider the set $\mathcal{U}$ of all words which start with α and such that both words $u' = u\gamma^{-1}v$ and $u'' = u\delta w$ exist. Note that with u also $u'u$ and $u'u''u$ belong to $\mathcal{U}$. Let $\mathcal{H}$ be the set of canonical maps $h_u\colon M(u') \to M(u'')$ with image $M(u)$, where $u \in \mathcal{U}$. Given $u \in \mathcal{U}$, we can factorize h_u as follows:

$$M(u') \xrightarrow{g''} M(u'u') \xrightarrow{h_{u'u}} M(u'u'') \xrightarrow{g'} M(u'u''u') \xrightarrow{h_{u'u''u}} M(u'u''u'') \xrightarrow{g} M(u''),$$

here g'', g' are the canonical inclusions, and g is the canonical map with image $M(u)$.

On the other hand, it has been conjectured (by Prest and others) that for a domestic algebra R, the transfinite radical $\operatorname{rad}(\operatorname{mod} R)$ is zero, thus that the only idempotent ideal inside $\operatorname{rad}(\operatorname{mod} R)$ is the zero ideal. This conjecture has been verified by Skowroński for strongly simply connected algebras [Sk] and by Schröer for string algebras.

Ideals and "ideal objects". Recall that the concept of an ideal in a ring R goes back to Kummer. Consider the ring of algebraic integers in a number field, it is what now is called a Dedekind ring, but in contrast to say the

[10] One should be careful to distinguish the transfinite radical $\operatorname{rad}^\infty$ and the "infinite" radical $\operatorname{rad}^\omega = \bigcap_{n\in\mathbb{N}} \operatorname{rad}^n$; often the latter also is denoted by $\operatorname{rad}^\infty$.

integers $\mathbb{Z}$, it need not be a principal ideal domain. In a principal ideal domain, every non-zero element can be written as a product of prime elements and such a factorization is essentially unique. Such a result is not valid for arbitrary Dedekind rings, already in $\mathbb{Z}[\sqrt{-5}]$ we have two completely different factorizations $6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$. To overcome this difficulty, Kummer introduced his "ideal numbers" and he obtained an essentially unique factorization of non-zero elements (even of arbitrary "ideal numbers") as products of "ideal numbers". In the terminology of Dedekind which is used today, instead of considering an element, we look at the corresponding principal ideal generated by the element, but we also take into account ideals which are not principal ideals. As it turns out, in a Dedekind ring, the set of non-zero ideals is a free commutative semigroup, the free generators are the non-zero prime ideals. As we have mentioned, ideals serve as a generalization of the principal ideals. One may try to associate to an ideal I of a ring R an "ideal element" s which generates I; of course, in case I is not a principal ideal of R, then s cannot be an element of R, but it may be an element of some overring R' of R. We may reverse these considerations: let $R \subset R'$ be commutative rings, and let $s \in R'$ be a non-zero element. The intersection of the principal ideal of R' generated by s with R yields an ideal of R which usually will not be principal.

In the context of categories, Krause has followed these ideas: starting with a category $\mathcal{C}'$ and a subcategory $\mathcal{C}$, we may intersect any ideal of $\mathcal{C}'$ with $\mathcal{C}$ and we will obtain an ideal of $\mathcal{C}$. In particular, we may start with an object S in $\mathcal{C}'$ and take the ideal of $\mathcal{C}'$ generated by 1_S, this is just the class of all homomorphisms which factor through a direct sum of copies of S; this is an idempotent ideal of $\mathcal{C}'$, however the intersection of this ideal with $\mathcal{C}$ does not have to be idempotent again: Actually, it may be nilpotent: take $\mathcal{C} = \operatorname{mod} R$, $\mathcal{C}' = \operatorname{Mod} R$, where R is a tame hereditary finite-dimensional algebra and let S be the generic module of infinite length. Of course, the ideal of $\operatorname{Mod} R$ generated by 1_S is idempotent, but its intersection $\mathcal{I}$ with $\operatorname{mod} R$ satisfies $(\mathcal{I})^2 = 0$. Krause [K6,K7] has introduced the concept of fp-idempotence for ideals, and it turns out that such ideals as $\mathcal{I}$ are fp-idempotent. We may say that $\mathcal{I}$ is a sort of shadow of the generic module S inside the category $\operatorname{mod} R$; the object S itself is not visible in $\operatorname{mod} R$, but its shadow is. Conversely, starting from the category $\mathcal{C}$, there may exist ideals $\mathcal{I}$ in $\mathcal{C}$ which can be considered as shadows of objects outside of the category, of "ideal objects": ideals which can be obtained as the intersection $\langle S\rangle_{\mathcal{C}'} \cap \mathcal{C}$ where S is an object of some category $\mathcal{C}' \supset \mathcal{C}$.

Categorical equivalences. Concerning the categorical approach, a further view point should be stressed: categorical equivalences show that a given category may be realized in different ways. The paper [PY] by Pimenov and Yakovlev provides a particularly nice example of two different realizations of a category. They consider abelian groups and maps between them. Let $\mathcal{T}$ be the category of all torsionfree groups and as second category take the category[11] $\mathcal{R}$ of surjective group homomorphisms $f\colon M \to N$, where M is torsionfree divisible and N is torsion (and also divisible, since N is a factor group of the divisible group M). It is easy to see that the categories $\mathcal{T}$ and $\mathcal{R}$ equivalent: given a torsionfree group F, let $I(F)$ be its injective hull, then the canonical projection $I(F) \to I(F)/F$ belongs to $\mathcal{R}$; conversely, given an object (f) in $\mathcal{R}$, then $\mathrm{Ker}(f)$ is a torsionfree group. In this way we obtain mutually inverse equivalences. Note that the functor $\mathcal{R} \to \mathcal{T}$ which attaches to (f) its kernel $\mathrm{Ker}(f)$ is faithful, since any homomorphism between torsionfree abelian groups has a unique extension to their injective envelopes.

The category $\mathcal{R}$ can be identified with the the category of those representations[12] $(f) = (M, N, f)$ of the bimodule ${}_{\mathbb{Z}}\mathbb{Q}_{\mathbb{Q}}$, for which f is surjective. Note that a general representation (M, N, f) of ${}_{\mathbb{Z}}\mathbb{Q}_{\mathbb{Q}}$ is the direct sum of $(M, f(M), f)$ (which belongs to $\mathcal{R}$) and of $(0, N/f(M), 0)$; here we use that with M also $f(M)$ is divisible, thus $f(M)$ is a direct summand of N. As we know, representations of the bimodule ${}_{\mathbb{Z}}\mathbb{Q}_{\mathbb{Q}}$ are just R-modules, where $R = \begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z} \end{bmatrix}$. In $\mathrm{Mod}\, R$, the full subcategory $\mathcal{R}$ is the class of all torsion modules for a split torsion pair, the torsionfree modules being those of the form $(0, G, 0)$, where G is an arbitrary abelian group.

Let $\mathcal{T}'$ be the full subcategory of all torsionfree groups F of finite rank such that $pF = F$ for almost all primes p. Let $\mathcal{R}'$ be the full subcategory of $\mathcal{R}$ consisting of all artinian R-modules. Observe that an R-module (M, N, f) is artinian if and only if both M and N are artinian abelian groups, thus if we assume that f is surjective, (M, N, f) is artinian in $\mathrm{Mod}\, R$ if and only if M is of finite rank and N is the direct sum of finitely many Prüfer groups. Under the equivalences described above, these subcategories $\mathcal{T}'$ and $\mathcal{R}'$ correspond to each other. There is nothing strange about the equivalence of the subcategory $\mathcal{T}$ of $\mathrm{Mod}\, \mathbb{Z}$ and the subcategory $\mathcal{R}$ of $\mathrm{Mod}\, R$, or also about the equivalence of the categories $\mathcal{T}'$ and $\mathcal{R}'$, but nevertheless it is worthwhile to contemplate. By definition, all the R-modules in $\mathcal{T}'$ are artinian, whereas the only artinian $\mathbb{Z}$-module in $\mathcal{R}'$ is the zero module. Also, the artinian property of the modules in $\mathcal{T}'$ does not seem to be related to a dual property for the modules in $\mathcal{R}'$ such as being noetherian: some of the $\mathbb{Z}$-modules in $\mathcal{R}'$ are noetherian, most of them not.

Recall that the category $\mathcal{T}'$ of all torsionfree abelian groups of finite rank was

[11] We write the objects in $\mathcal{R}$ in the form (f), where f is a homomorphism as indicated; given two homomorphisms $f\colon M \to N$ and $f'\colon M' \to N'$, the maps $(\alpha, \beta)\colon (f) \longrightarrow (f')$ in $\mathcal{R}$ are as usual pairs, with $\alpha : M \to M'$ and $\beta\colon N \to N'$ such that $\beta \circ f = f' \circ \alpha$.

[12] If R, S are rings and ${}_SX_R$ is a bimodule, a *representation* of ${}_SX_R$ is by definition of the form $(f) = ({}_RM, {}_SN, f\colon {}_SX \otimes_R M \to {}_SN)$, with f being S-linear, but this is nothing else than a C-module, where $C = \begin{bmatrix} R & 0 \\ X & S \end{bmatrix}$.

the main playing ground for considering the failure of the Krull-Remak-Schmidt property, since papers by Jónsson (1945, 1957) and Corner (1961). Many different constructions are known, and via the stated equivalence they carry over to the category $\mathcal{R}'$, thus one obtains in this way many examples of artinian modules which do not satisfy the Krull-Remak-Schmidt property. Note that Krull raised the question whether this property holds for artinian modules in 1932, and the first answer to this question is only from 1995, see the paper [FHLV] by Facchini, Herbera, Levy and Vámos. Let us repeat: If we assume the knowledge on $\mathcal{T}'$ established a long time ago, we obtain the corresponding assertions for $\mathcal{R}'$ using the equivalence of the categories. However, it may be reasonable to revert these considerations. If one analyzes the usual constructions made in $\mathcal{T}'$ for obtaining a torsionfree group F of finite rank, one observes that these are really constructions starting with a finite direct sum of copies of $\mathbb{Q}$, thus with $I(F)$, and prescribing a direct sum of Prüfer modules as factor module: the group F is given by a minimal injective resolution, thus by an object in $\mathcal{R}'$.

It may be sufficient to discuss just one example, the construction of a torsionfree group $F_1 \oplus F_2 \simeq F_3 \oplus F_4 \oplus F_5$, with indecomposable direct summands F_i such that F_1, F_2, F_3 have rank 2 (and F_4, F_5 rank 1), see Fuchs [Fu2] Theorem 90.1. We use the following notation: The factor group $\mathbb{Q}/\mathbb{Z}$ is the direct sum of all Prüfer groups P_p, p a prime number, each occurring with multiplicity one. Denote by π_p the composition of the canonical projections $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to P_p$. We also will need the map $p\pi_p\colon \mathbb{Q} \to P_p$; note that its kernel $\operatorname{Ker}(p\pi_p)$ contains $\operatorname{Ker}(\pi_p)$ with index p. For $1 \le i \le 3$, let F_i be the kernel of a map

$$f_i\colon \mathbb{Q}^2 \to P_2 \oplus P_3 \oplus P_5 \oplus P_5 \oplus P_7 \oplus P_7,$$

namely of

$$f_1 = \begin{bmatrix} \pi_2 & 0 \\ 0 & \pi_3 \\ \pi_5 & \pi_5 \\ 5\pi_5 & 0 \\ \pi_7 & 0 \\ 0 & \pi_7 \end{bmatrix}, \qquad f_2 = \begin{bmatrix} \pi_2 & 0 \\ 0 & \pi_3 \\ \pi_5 & 0 \\ 0 & \pi_5 \\ \pi_7 & \pi_7 \\ 7\pi_7 & 0 \end{bmatrix}, \qquad f_3 = \begin{bmatrix} \pi_2 & 0 \\ 0 & \pi_3 \\ \pi_5 & \pi_5 \\ 5\pi_5 & 0 \\ \pi_7 & \pi_7 \\ 7\pi_7 & 0 \end{bmatrix}.$$

The additional groups F_4 and F_5 are the kernels of the maps

$$f_4 = \begin{bmatrix} \pi_2 \\ \pi_5 \\ \pi_7 \end{bmatrix} : \mathbb{Q} \to P_2 \oplus P_5 \oplus P_7, \quad \text{and} \quad f_5 = \begin{bmatrix} \pi_3 \\ \pi_5 \\ \pi_7 \end{bmatrix} : \mathbb{Q} \to P_3 \oplus P_5 \oplus P_7,$$

respectively. Of course, the map f_i is the minimal injective resolution of F_i and belongs to $\mathcal{R}'$. Thus, we really deal with indecomposable artinian R-modules $(f_1), \dots, (f_5)$ and with an isomorphism[13] $(f_1) \oplus (f_2) \simeq (f_3) \oplus (f_4) \oplus (f_5)$.

[13] Matrices which yield an isomorphism can be calculated easily; or see [R13].

The example presented here uses four prime numbers, but actually it is sufficient to work with a single prime. Namely, as Butler has pointed out, for any prime $p \geq 5$, there do exist torsionfree abelian groups F of finite rank which do not satisfy the Krull-Remak-Schmidt property, such that $qF = F$ for all primes $q \neq p$, see [Ar] 2.15. Of course, such examples give rise to corresponding artinian modules over the ring $\begin{bmatrix} \mathbb{Q} & 0 \\ \mathbb{Q} & \mathbb{Z}_{(p)} \end{bmatrix}$. Note that this ring is a very well-behaved ring: all its indecomposable projective (left or right) modules are serial.

A slight modification of the Pimenov-Yakovlev construction allows to replace the rings considered by local rings [R13], and we obtain in this way examples of artinian modules even over a local ring which do not satisfy the Krull-Remak-Schmidt property.

Categorical equivalences, again. We have seen above an interesting example of realizing a category in two different ways, once as a full subcategory of the category of abelian groups, once as modules over some non-commutative ring. But the most important setting for using a categorical equivalence concerns a process which may be called projectification. It is the following quite trivial, but very effective procedure: Given any module M, its endomorphism ring $\mathrm{End}(M)$ is also the endomorphism ring of a projective module, just take the regular representation of E^{op}, where $E = \mathrm{End}(M)$; the categorical version of this statement is: Given an R-module M with endomorphism ring $E = \mathrm{End}(M)$, let $\mathrm{add}\, M$ be the additive closure, this is the full subcategory of $\mathrm{Mod}\, R$ consisting of all direct summands of finite direct sums of copies of M. Then this category is equivalent to the category $\mathrm{pro}\, E^{\mathrm{op}}$ of all finitely generated projective E^{op}-modules, an equivalence is given by the functor $\mathrm{Hom}_R(M, -)$.

A related result has to be mentioned here, a theorem due to Swan [Sw,Ba]: *Let X be a compact Hausdorff space and denote by $C_0(X)$ the ring of continuous functions $X \to \mathbb{R}$. Then the category of $\mathbb{R}$-vector bundles on X is equivalent to the category of finitely generated projective $C_0(X)$-modules,* an equivalence is given by sending the $\mathbb{R}$-vector bundle E to the $C_0(X)$-module $\Gamma(E)$ of all continuous sections of E. Note that this result provides a bridge to topology and differential geometry. Note that the most prominent vector bundles are the tangent bundles of differential manifolds, thus questions concerning vector fields (these are the sections of the tangent bundle) can be reformulated in terms of projective modules. The analogy between finitely generated projective modules and vector bundles can be an important source for inspiration, it provides a nice geometrical model for questions concerning projective modules.

There is a similar bridge to number theory: many problems about rings of integers in number fields concern the structure of their finitely generated projective modules. In particular, the ideal class group of a Dedekind ring can be interpreted as the set of isomorphism classes of projective modules of rank 1 with respect to the tensor product (for a general commutative ring, the latter group is called its Picard group).

We end this section with some general considerations concerning the role of simple objects and of finite length objects in abelian categories.

Abelian categories with no finite length modules. First of all, we have to stress that *an abelian category may not have simple objects at all,* a typical example $\mathcal{A}$ can be constructed as follows: As set of indecomposable objects take the set of open (non-empty) intervals $(a,b) = \{q \in \mathbb{Q} \mid a < q < b\}$ in $\mathbb{Q}$, let $\operatorname{Hom}_{\mathcal{A}}((a,b),(c,d)) = k$ if $a \le c < b \le d$ and 0 otherwise; as composition take the multiplication in k. By adding formal (finite) direct sums (see for example [GR], p.18), we obtain an additive category $\bigoplus \mathcal{A}$ which easily can be shown to be abelian: just observe that it is sufficient to consider a finite sequence $c_0 < c_1 < \cdots < c_m$ of rational numbers and the full subcategory of direct sums of objects of the form (c_i, c_j) with $i < j$. This subcategory is equivalent to the module category of the ring of upper triangular matrices over k.

Of course, a Grothendieck category[14], in particular the module category $\operatorname{Mod} R$ over a ring R, always has sufficiently many simple objects. There do exist non-trivial examples of rings R where all simple modules are injective. A ring R with all simple modules injective is called a *V-ring*. Cozzens [Cz,Fa2] has constructed rings of differential polynomials and also twisted polynomial rings which are V-rings, but have zero socle.

Objects of finite Loewy length. *Let $\mathcal{A}$ be an abelian category, let $S_1, \dots, S_n$ be pairwise non-isomorphic simple objects in $\mathcal{A}$. Let $\mathcal{S}(S_1, \dots, S_n)$ be the class of semisimple objects in $\mathcal{A}$ which are (finite or infinite) direct sums of copies of $S_1, \dots, S_n$. If t is a natural number, let $\mathcal{A}(S_1, \dots, S_n; t)$ be the set of objects in $\mathcal{A}$ which have a filtration of length t with factors in $\mathcal{S}(S_1, \dots, S_n)$. Then $\mathcal{A}(S_1, \dots, S_n; t)$ is equivalent to the module category of a semiprimary ring R. If J is the radical of R, then $J^t = 0$ and R/J is isomorphic to the endomorphism ring of $\bigoplus_{i=1}^n S_i$.* Proof: For every object S_i we can construct its relative projective cover P_i in $\mathcal{A}' = \mathcal{A}(S_1, \dots, S_n; t)$. In this way, we clearly obtain a progenerator $P = \bigoplus_{i=1}^n P_i$ for $\mathcal{A}'$. Let R be the endomorphism ring of P.

Note that all the finite length objects are recovered in this way: If M has length at most t and if all the composition factors of M belong to the set $\{S_1, \dots, S_n\}$, then M belongs to $\mathcal{A}(S_1, \dots, S_n; t)$.

Decomposing projective objects in an abelian category. Let us stress that for general abelian categories we cannot expect any structure theory. In case we consider a Grothendieck category, the assertions concerning projective objects and those concerning injective objects are very different. Of course, dealing with the dual of a Grothendieck category, we obtain examples with the opposite features.

Let $\mathcal{C}$ be a Grothendieck category. Note that $\mathcal{C}$ may not have enough projective modules (example: the category of all abelian p-groups); there may be

[14] For the definition, see for example [Fa1].

enough projectives, but not enough projective covers (example: $\operatorname{Mod}\mathbb{Z}$); an indecomposable projective object P does not have to be couniform (example again: ${}_RR$ for $R=\mathbb{Z}$), and this may happen even if the radical of P is superfluous in P (example: take ${}_RR$, where $R=\mathbb{Z}_{(pq)}$ is obtained from $\mathbb{Z}$ by localizing at the product of two different primes p,q). If $\mathcal{C}=\operatorname{Mod}R$ for some ring R, then there are enough projectives, but all the other anomalies mentioned occur already in module categories. On the other hand there is **Kaplansky's Theorem:** *Every projective module is a direct sum of countably generated projective modules.* Thus, we obtain a strong bound on the size of indecomposable projective modules.

For an integral domain R, the projective modules of rank 1 play an important role; as we have mentioned, the set of isomorphism classes with respect to the tensor product is called the Picard group $\operatorname{Pic}R$ of R. Of course, all these projective rank 1 modules are indecomposable. There do exist also indecomposable projective modules of rank greater than 1. The typical example of an integral domain R which has such a module is the coordinate algebra $\mathbb{R}[x_1,x_2,x_3]/\langle x_1^2+x_2^2+x_3^2-1\rangle$ of the sphere S^2, see [Sw]: Let P be the kernel of the homomorphism $\phi\colon R^3\to R$ defined by $\phi(r_1,r_2,r_3)=\sum r_ix_i$. Note that ϕ is surjective, thus it splits and therefore $R^3\simeq P\oplus R$. This shows that P is projective and has rank 2. In case P would be decomposable, one would have $P\simeq R^2$, since the ring R is known to be a principal ideal domain. But an isomorphism $P\simeq R^2$ would provide a continuous vector field on S^2 which nowhere vanishes, impossible.

Decomposing injective objects in an abelian category. Consider now the injectives in a Grothendieck category $\mathcal{C}$. There are always sufficiently many injective objects, even sufficiently many injective envelopes. Indecomposable injective objects are always uniform. In case $\mathcal{C}=\operatorname{Mod}R$ for some ring R, we obtain all indecomposable injective modules as injective envelopes of uniform cyclic modules. In contrast to the case of projective modules, we cannot expect to be able to write all the injective modules as direct sums of countably generated modules (example: the injective envelope of ${}_RR$, where R is the polynomial ring $k[T]$ in one variable is given by the field $k(T)$ of rational functions, and if k is an uncountable field, then ${}_{k[T]}k(T)$ is not countably generated), and not even of modules which are generated by λ elements, where λ is a fixed cardinal number (this is only possible for left noetherian rings, by the Faith-Walker theorem mentioned already). Any non-zero ring has indecomposable injective modules (since it has uniform modules, namely at least the simple modules), but, as we have noted already, there are rings R which also have non-zero superdecomposable injective modules, for example the free algebra $k\langle X,Y\rangle$ in two variables.

Let I be an injective module. Then

(i) *I is indecomposable if and only if I is uniform.*
(ii) *I is discrete if and only if I is formatted.*
(iii) *I is superdecomposable if and only if I has no uniform submodule.*

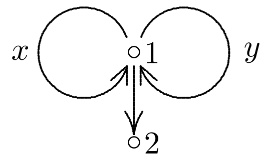
Here, the conditions mentioned left deal with the behaviour with respect to direct decompositions, the right ones with the submodule structure, namely the uniform submodules of the module. The direct sum conditions are properties which concern the endomorphism rings, thus they remain valid when we apply a functor which preserves endomorphism rings. On the other hand, the submodule conditions are preserved under submodules as follows: a non-zero submodule of a uniform module is uniform; any submodule of a formatted module is formatted; and finally, if a module has no uniform submodule, then the same is true for any of its submodules. (Let us sketch the proof of (ii). First, assume that the injective module I is formatted and consider a direct decomposition $I = I' \oplus I''$ with $I' \neq 0$. Since I is formatted, I' contains a uniform submodule, say U. But the injective envelope of U is a direct summand of I. This shows that I is discrete. Assume conversely that I is discrete and let N be a non-zero submodule. Then the injective envelope I' of N is a direct summand of I and by assumption, I' has an indecomposable direct summand, say J. Now $J \cap N \neq 0$, since J is a non-zero submodule of I' and N is essential in I'. Since J is indecomposable injective, its submodule $J \cap N$ is uniform. This shows that I is formatted.)

Theorem of Gabriel and Oberst. *Any injective module I is the direct sum of a discrete module I_1 and a superdecomposable module I_2. If $I = I_1' \oplus I_2'$ is a second decomposition with I_1' discrete and I_2' superdecomposable, then $I = I_1 \oplus I_2'$.* (In particular, the modules I_1 and I_1' are isomorphic, and similarly, the modules I_2 and I_2' are isomorphic.) The usual discussions of the Gabriel-Oberst-Theorem invoke so called spectral categories [GO] (they are obtained by factoring out from the category of all injective R-modules the ideal of all maps $f\colon I \to I'$ which vanish on an essential submodule). The use of spectral categories is illuminating, but it seems also misleading[15]. Let us sketch a direct and elementary proof. Let I be an injective module. Using the lemma of Zorn, choose a submodule I' of I which is a direct sum of uniform modules and such that there does not exist a uniform submodule U of I with $I' \cap U = 0$. Let I_1 be an injective envelope of I', thus I_1 is discrete. Since I is injective, we may assume that I_1 is a submodule of I, thus there is a direct decomposition $I = I_1 \oplus I_2$. Since I_2 has no uniform submodule, it is superdecomposable. Now assume that there is given a second decomposition $I = I_1' \oplus I_2'$ with I_1' discrete and I_2' superdecomposable. The intersection $I_1 \cap I_2'$ is a submodule of I_1, thus formatted, and a submodule of I_2, thus also superdecomposable, and therefore zero. It follows that $I_1 + I_2'$ is a direct sum; it is an injective module, thus there is a submodule C with $I = I_1 \oplus I_2' \oplus C$. Both I_2 and $I_2' \oplus C$ are direct complements for I_1, thus they are isomorphic and C is isomorphic to a submodule of I_2. This implies that C is superdecomposable. On the other hand, both I_1' and $I_1 \oplus C$ are direct complements for I_2', thus isomorphic and therefore C is isomorphic to a submodule of I_1', thus formatted. Since C is both formatted and superdecomposable, we see that $C = 0$. This

[15] For example, based on the use of spectral categories, the book by Jensen-Lenzing [JL,8.24] asserts that the maximal discrete direct summand I_1 of I is uniquely determined, Prest [P1, Corollary 4.A14] even claims that both summands I_1, I_2 are unique, in contrast to examples which we are going to present.

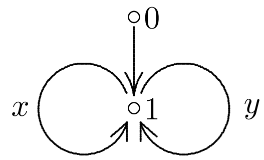
completes the proof.

In the decomposition $I = I_1 \oplus I_2$ of an injective module I, with I_1 discrete and I_2 superdecomposable, none of the direct summands I_1 and I_2 has to be unique, as the following examples show. It is sufficient to find a ring R, a discrete injective R-module I_1 and a superdecomposable injective R-module I_2 such that $\mathrm{Hom}(I_1, I_2) \neq 0$ or $\mathrm{Hom}(I_2, I_1) \neq 0$ (since the graph $G(f)$ of an R-linear map $f\colon M \to M'$ is an R-submodule of $M \oplus M'$ which satisfies $M \oplus M' = G(f) \oplus M'$). For an example with $\mathrm{Hom}(I_1, I_2) \neq 0$, consider the quiver with two vertices, say labeled 1 and 2, with two loops x, y at the vertex 1 and one arrow $1 \to 2$.



Consider the indecomposable projective representation $P(1)$ corresponding to the vertex 1, it has a k-basis given by the set of paths starting in 1. The socle N of $P(1)$ is an infinite direct sum of copies of the simple module $S(2)$ corresponding to the vertex 2. Since N is essential in $P(1)$, we see that $P(1)$ is formatted, whereas $P(1)/N$ is superdecomposable. Take for I_1 the injective envelope of $P(1)$ and for I_2 the injective envelope of $P(1)/N$. The canonical map $P(1) \to P(1)/N$ induces a non-zero map $I_1 \to I_2$. Thus we see[16] that in $I_1 \oplus I_2$, there are other maximal discrete submodules than I_1.

As an example with $\mathrm{Hom}(I_2, I_1) \neq 0$, we take the opposite quiver:



Also here, the injective envelope I_2 of $P(1)$ is superdecomposable, and now it maps onto the injective module $I_1 = S(0)$. This shows that in $I_1 \oplus I_2$, there are other maximal superdecomposable submodules than I_2. Actually, given an arbitrary ring R and I_2 a non-zero superdecomposable R-module, we obtain a corresponding example: consider a simple subfactor S of I_2 and its injective envelope I_1. Clearly, I_1 is discrete and $\mathrm{Hom}(I_2, I_1) \neq 0$. This shows that in $I_1 \oplus I_2$, there are several different maximal superdecomposable submodules.

[16] This ring R provides also an example of a sum of two formatted modules which is not formatted: Consider $M = P(1) \oplus P(1)/N$. Note that both modules $P(1)$ and $P(1)/N$ are cyclic, say generated by elements a and $b = a + N$, respectively. Then M is generated by the elements $(a, 0)$ and (a, b). The submodules generated by $(a, 0)$ and by (a, b) both are isomorphic to $P(1)$, thus formatted, but M is not formatted, since its submodule $P(1)/N$ has no uniform submodules.

One may ask under what conditions on a ring R every discrete injective R-module actually is a direct sum of uniform modules. In case all the injective modules are discrete, this has been answered by the **Matlis–Papp theorem**: *A ring R is left noetherian if and only if every left module is a direct sum of uniform modules.* We should mention also the following characterization of left noetherian rings. **Faith-Walker theorem.** *A ring R is left noetherian if and only if each injective module is a direct sum of modules, each being generated by λ elements, where λ is some fixed cardinal number.* For both results we may refer to [Fa2].

Diamond categories. Let us assume that $\mathcal{C}$ is an abelian category. We say that $\mathcal{C}$ is a *diamond category* provided first, every simple object has a projective cover and an injective envelope, and second every object C has an essential subobject C' which is semisimple and of finite length, and dually, a superfluous subobject C'' such that C/C'' is semisimple and of finite length. Of course, the subobject C' will be called the socle of C and denoted by $\operatorname{soc} C$, the subobject C'' is called the radical of C and denoted by $\operatorname{rad} C$. Let us state some properties of a diamond category $\mathcal{C}$. First of all, it follows easily that every object has a projective cover and an injective envelope. Next, *any object in $\mathcal{C}$ is a finite direct sum of indecomposable objects, and indecomposable objects have local endomorphism rings.* In particular, the Krull-Remak-Schmidt Theorem holds in $\mathcal{C}$. Also, a projective object is indecomposable if and only if its radical is a maximal subobject; an injective object is indecomposable if and only if its socle is simple.

A set of subfactors $C_i' \subset C_i$ of an object C is said to *cover* C provided for any subfactor $U' \subset U$, there exists an index i such that $U \cap C_i \not\subseteq U' + C_i'$. *Any object C in a diamond category has finitely many subfactors $C_i' \subset C_i \subseteq C$ which cover C, such that any object C_i/C_i' is a diamond* (this explains the name). (For the proof, take a projective cover $(p_a)_a \colon \bigoplus_{a=1}^n P_a \to C$ with indecomposable projective objects P_a, and an injective envelope $(u_b)_b \colon C \to \bigoplus_{b=1}^m I_b$ with indecomposable injective objects I_b. Consider a pair $i = (a,b)$ such that the composition $u_b \circ p_a$ is non-zero. Let C_i be the image of p_a in C, let C_i' be the intersection of C_i with the kernel of u_b. Then $C_i' \subset C_i$ and C_i/C_i' is a diamond, since it is isomorphic to a factor object of P_a and to a subobject of I_b. In order to see that these subfactors cover C, take a subfactor $U' \subset U$ in C.)

Given an object C in a diamond category $\mathcal{C}$, we may define its socle sequence $\operatorname{soc}_i C$ by $\operatorname{soc}_0 C = 0$ and $\operatorname{soc}_{i+1} C/\operatorname{soc}_i C = \operatorname{soc}(C/\operatorname{soc}_i C)$, for all natural numbers i. Similarly, let $\operatorname{rad}^0 C = C$ and $\operatorname{rad}^{i+1} C = \operatorname{rad}(\operatorname{rad}^i C)$, for all i. Note that all the objects $\operatorname{soc}_i C$ and $C/\operatorname{rad}^i C$ are of finite length. As a consequence, all objects in a diamond category which are of finite Loewy length are of finite length.

Starting with a diamond category $\mathcal{C}$, we may add formal direct limits in order to obtain a Grothendieck category $\varinjlim \mathcal{C}$. Note that an object of $\mathcal{C}$ may have in $\varinjlim \mathcal{C}$ subfactors $C'' \subset C' \subset C$ such that C'/C'' is an infinite direct sum of simple objects.

If C is an object of $\mathcal{C}$ which is not of finite length, then all the inclusions occurring in the socle sequence and in the radical sequence of C

$$0 \subset \operatorname{soc} C \subset \operatorname{soc}_2 C \subset \cdots \qquad \text{and} \qquad \cdots \subset \operatorname{rad}^2 C \subset \operatorname{rad} C \subset C$$

are proper. If we take the union $\bigcup_i \operatorname{soc}_i C$ in $\varinjlim \mathcal{C}$, we obtain an artinian object. Similarly, we may consider the intersection $C' = \bigcap_i \operatorname{rad}^i C$ in $\varinjlim \mathcal{C}$, the corresponding factor object C/C' will be a noetherian object. Note that for C not of finite length, C/C' as well as $\bigcup_i \operatorname{soc}_i C$ never will belong to $\mathcal{C}$ itself.

Functor categories. If $\mathcal{C}$ is an additive category, recall that $\operatorname{Mod} \mathcal{C}$ denotes the category of all additive contravariant functors from the category $\mathcal{C}$ into the category of all abelian groups, and $\operatorname{mod} \mathcal{C}$ the full subcategory of all finitely presented functors (a functor F is said to be finitely presented provided there is an exact sequence of the form $\operatorname{Hom}_{\mathcal{C}}(-, C) \to \operatorname{Hom}_{\mathcal{C}}(-, C') \to F \to 0$, with objects C, C' in $\mathcal{C}$). Of course, in case R is a ring and $\operatorname{pro} R$ is the category of all finitely generated projective R-modules, then $\operatorname{Mod} R = \operatorname{Mod} \operatorname{pro} R$ and $\operatorname{mod} R = \operatorname{mod} \operatorname{pro} R$.

Consider now the case of $\mathcal{C}$ being a *dualizing k-variety*, where k is a commutative artinian ring (for example a field), as studied by Auslander and Reiten [AR1]. This means that first of all $\mathcal{C}$ is a k-category with all Hom-sets being finitely generated k-modules, and second, that for every contravariant functor F in $\operatorname{mod} \mathcal{C}$, also the covariant functor $D(F)$ is finitely presented, and dually, that for every finitely presented covariant functor G on $\mathcal{C}$, also the contravariant functor $D(G)$ is finitely presented; here D is the duality with respect to the minimal injective cogenerator I for k, thus $D(F)(C) = \operatorname{Hom}_k(F(C), I)$ and $D(G)(C) = \operatorname{Hom}_k(G(C), I)$. For example, for any artin algebra R, the category $\operatorname{pro} R$ is such a dualizing k-variety. The first result to remember is: **Theorem (Auslander-Reiten).** *If $\mathcal{C}$ is a dualizing k-variety, then so is* $\operatorname{mod} \mathcal{C}$. ([AR1], 2.6). As a consequence, we may iterate the construction, and consider the sequence $\mathcal{C}, \operatorname{mod} \mathcal{C}, \operatorname{mod} \operatorname{mod} \mathcal{C}, \ldots$ and so on. We obtain a sequence of categories which are getting larger and larger, but in some sense more and more well behaved. Note the following: if we start with a representation-finite algebra R, then $\operatorname{mod} R = \operatorname{pro} A(R)$, where $A(R)$ is the Auslander algebra of R, and, $A(R)$ is well-behaved both with respect to global dimension and dominant dimension (it has global dimension at most 2 and dominant dimension at least 2). Also, the category $\mathcal{C}$ does not have to be abelian, however $\operatorname{mod} \mathcal{C}$ always will be. The second result to remember: **Theorem (Auslander-Reiten).** *If $\mathcal{C}$ is a dualizing k-variety, then* $\operatorname{mod} \mathcal{C}$ *is a diamond category.* ([AR1], 3.7).

An additive category which is equivalent to a category of the form $\operatorname{mod} \mathcal{C}'$ will be said to have *Auslander dimension* at least 1. Inductively, we may say that $\mathcal{C}$ has Auslander dimension at least $n+1$ provided $\mathcal{C}$ is equivalent to a category of the form $\operatorname{mod} \mathcal{C}'$ where $\mathcal{C}'$ has Auslander dimension at least n. Note that for any additive category $\mathcal{C}'$, the category $\operatorname{pro} \operatorname{mod} \mathcal{C}'$ is just the closure of $\mathcal{C}'$ under finite direct sums and direct summands. Thus, if $\mathcal{C}$ is equivalent to $\operatorname{mod} \mathcal{C}'$, and if $\mathcal{C}'$ has finite direct sums and split idempotents, then we can recover $\mathcal{C}'$ as the full subcategory $\operatorname{pro}\mathcal{C}'$ of all projective objects in $\mathcal{C}$. If the Auslander dimension of $\mathcal{C}$ is at least n, we can apply n times pro to $\mathcal{C}$. In particular, if $\mathcal{C} = \operatorname{mod}\operatorname{mod}\mathcal{B}$, and $\mathcal{B}$ has finite sums and split idempotents, then we get $\mathcal{B} = \operatorname{pro}\operatorname{pro}\mathcal{C}$.

Let us assume that $\mathcal{B}$ is a dualizing k-variety with finite direct sums and split idempotents, and let $\mathcal{C} = \operatorname{mod}\operatorname{mod}\mathcal{B}$. Then we can recover $\mathcal{B}$ from $\mathcal{C}$ also in a different way, namely as the full subcategory of $\mathcal{C}$ of all objects which are both projective and injective. The indecomposable objects in $\operatorname{mod}\operatorname{mod}\mathcal{C}$ which are both projective and injective may be compared with the "hammocks" as considered by S. Brenner: The name "hammock" was introduced by her when she considered $\mathcal{C} = \operatorname{mod}\operatorname{mod} R$ for a representation-finite algebra R aiming at a combinatorial characterization of these hammock functors $\mathcal{H}(S) = \operatorname{Hom}(-, I(S)) = D\operatorname{Hom}(P(S), -)$, here $I(S)$ is the injective envelope of a simple module S and $P(S)$ its projective cover. It seems to be of great importance to study these functors $\operatorname{Hom}(-, I(S)) = D\operatorname{Hom}(P(S), -)$ not only in the case of a representation-finite algebra, but in general. We also have to refer to Tachikawa [T] who emphasized the importance of the objects in $\operatorname{mod}\operatorname{mod} R$ which are projective as well as injective.

The dualizing k-varieties are the proper setting for the Auslander-Reiten theory. *If $\mathcal{C}$ is a dualizing k-variety, then the category* $\operatorname{mod}\mathcal{C}$ *has Auslander-Reiten sequences*[17], and if $0 \to X \to Y \to Z \to 0$ is such a sequence, then $X = \tau Z$ is calculated as usual as the "dual of the transpose".

The categories to be considered later are of the form $\operatorname{mod} R$ or $\operatorname{mod}\operatorname{mod} R$, where R is a finite-dimensional k-algebra with k a field, thus they are dualizing k-varieties and therefore diamond categories. The combinatorial flavour of the representation theory of such algebras R is due to this fact. In case R is not representation-finite, the assertion that $\operatorname{mod}\operatorname{mod} R$ is a diamond category has not yet found the appreciation which it deserves, even though many classical facts from Auslander-Reiten theory find a natural interpretation in this setting. Only in very special cases, the structure of finitely presented functors on $\operatorname{mod} R$, thus of objects in the category $\operatorname{mod}\operatorname{mod} R$, has been analyzed in detail. The first question to be raised concerns the possible serial objects. Finitely presented functors which are serial have been studied by Auslander and Reiten in [AR2]. Some typical examples of such functors will be seen below.

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[17] A general diamond category $\mathcal{C}$ may not have sufficiently many Auslander-Reiten sequences, even if every object in $\mathcal{C}$ has finite length. A typical example has been exhibited in [R2]: Let F be a field with a derivation δ and consider the F-F-bimodule ${}_FM_F$ where ${}_FM = {}_FF \oplus {}_FF$ and such that the action by F on the right is given by $(a,b)c = (ac + b\delta(c), bc)$ for all $a,b,c \in F$. The category of representations of ${}_FM_F$ of finite length is a diamond category. The finite direct sums of indecomposable representations of ${}_FM_F$ of even length form an abelian subcategory $\mathcal{R}_0$. If (F,δ) is differentially closed, there are just two indecomposable representations of length two. One of them is projective and injective in $\mathcal{R}_0$ and does not occur in any Auslander-Reiten sequence.

3. Well-behaved modules

We have seen that for most of the rings there is an abundance of different types of modules so that it will be a waste of time to try to deal with all of them. Even if we restrict to finite length modules, Maurice Auslander strongly argued against the usual classification procedures many mathematicians are fond of: he stressed that it should be of greater interest to deal with some classes of modules with specific and important properties, than to publish large lists of normal forms no-one is going to use. Of course, up to now such lists have been put forward only in tame cases (and some people believe that the notion of wildness just excludes the possibility of producing complete lists) and there it seems that really all the indecomposable modules have specific and presumably also important properties. Thus Auslander's argument could be rephrased in these cases better as follows: it may be of little interest to exhibit lists of modules unless one cannot figure out their properties. But the by now usual procedure of determining complete Auslander-Reiten components or even families of such components as well as their global behaviour aims at a reasonable description of such module categories and this seems to be an endeavor which has to be appreciated. Too little is known at the moment about any wild module category for arguing in real favour for a classification program, but there do exist dreams about a "very well-behaved wild world": the Kac conjecture [Kc] that the module varieties of wild hereditary algebras have a cellular decomposition with affine strata is part of it. What one may hope for are, on the one hand, discrete invariants which fit into some concise combinatorial picture, and, on the other hand, for any given set of such invariants, a nice, hopefully even rational variety which describes the modules with these data. In this way, it may turn out that all the finite length modules are considered to be of interest and of importance, at least when considered in the natural setting of their relatives, as part of such a family with a fixed set of combinatorial data. It is one of the main reason for the present book to stress that a description of the category of all finite length modules quite naturally has to rely on incorporating infinite length modules, but clearly only some of them. It should be out of question that we have to be very restrictive about the infinite length modules which we are going to involve.

The algebraically compact modules. There is a natural choice of the class of modules to be considered, a choice which can be justified both by usual algebraic arguments, but also by mathematical logic. Recall that there is an overlap between the interests of algebraist and logicians. There are many important questions handled in a different, but quite parallel way by algebraists and logicians. Many notions and constructions both in algebra and in logic stem from difficulties which have been encountered in the 19th century before the set theoretical foundation of modern mathematics was laid. To overcome such difficulties, different, and sometimes incompatible remedies have been found. For example, in first order logic, the notion of a logic "with equality" just tries to formalize a specific way of dealing with factor objects. The whole model theory should be considered as part of algebra, but the conflicting terminologies make it difficult for algebraists and logicians to communicate. Some attempts have been made by Ziegler, and especially by Prest and Herzog in order to overcome these difficulties and to provide some reasonable dictionary in order to translate questions and results from one language to the other. Unfortunately, as it is usual with dictionaries, the scope of notions in the languages to be compared often are not really compatible, so that convenient groupings in one language may look slightly artificial in the other, at least on the first sight.

One of the basic notions presented in this book is a concept which can easily be described in both languages, on the first sight in completely different ways (but a deeper understanding relates the two definitions quite clearly), namely that of a module which is *algebraically compact* or, what is the same, *pure injective.* The two denominations already point out the two different roles these modules play.

Algebraic compactness refers to solving systems of linear equations. Such a system consists of equations indexed by the elements i of a set I, the equations are of the form

$$(*) \qquad \sum\nolimits_{j\in J} r_{ij}x_j = a_i,$$

where a_i are elements of a given R-module M, the elements r_{ij} belong to the ring R, for $i \in I, j \in I$, and one assumes that for given $i \in I$, almost all r_{ij} are supposed to be zero (so that forming the sum $\sum_{j\in J} r_{ij}x_j$ makes sense in this algebraic context). In principle, the x_j, $j \in J$ are variables; a solution of this system of equations consists of elements $x_j \in M$ such that all the equations $(*)$ are satisfied. Such a system of equations is said to be finitely solvable provided for any finite subset $I' \subseteq I$, there exists a solution for the equations $(*)$ with $i \in I'$. The R-module M is said to be *algebraically compact* provided any system of linear equations which is finitely solvable has a solution.

On the other hand, pure injectivity means "relative injectivity with respect to pure embeddings". Recall that forming tensor products does not respect monomorphisms: a typical example is the map $f\colon \mathbb{Z} \to \mathbb{Z}$ given by the multiplication with 2, this is a monomorphism, but if we tensor f with $\mathbb{Z}/2$, we obtain the zero map (of course, the tensor product of $\mathbb{Z}$ with $\mathbb{Z}/2$ is just $\mathbb{Z}/2$ and $1_{\mathbb{Z}/2}\otimes f\colon \mathbb{Z}/2 \to \mathbb{Z}/2$ is still the multiplication by 2, but this is now the zero map). For any ring R, a homomorphism $f\colon M \to N$ of R-modules (as usual, this means left R-modules) is said to be a *pure monomorphism* provided $1_X \otimes_R f\colon X \otimes_R M \to X \otimes_R N$ is a monomorphism, for any right R-module X_R. This concept of purity will be discussed in detail by Huisgen-Zimmermann in [H]. The R-module M is said to be *pure injective* provided the only pure monomorphisms $M \to N$ (with arbitrary N) are the split monomorphisms.

The two names of "algebraic compactness" and "pure injectivity" show the main directions of interest, but actually, these modules can be characterized in many additional ways (and some of these could equally well serve as the source of naming these modules). For a very elegant and concise treatment of these modules including seven characterizations we refer to the book of Jensen and Lenzing [JL], Chapter 7. Chapter 8 of the same book provides a corresponding account dealing with the Σ-algebraically compact modules: these are those modules M, for which any direct sum of copies of M is again algebraically compact.

It seems to be of interest to quote at least the following characterization: *A module M is algebraically compact if and only if for any index set I the summation map $\bigoplus_I M \to M$ can be extended to a map $\prod_I M \to M$.* The importance of this characterization is due to the fact that it has the following consequence: **Corollary.** *Let F be an additive functor which commutes with direct sums and direct products. If M is algebraically compact, then also $F(M)$ is algebraically compact.* This is a convenient tool for showing that modules are algebraically compact. For example, given a ring homomorphism $\varphi\colon R \to S$, any S-module M may be considered as an R-module via $r \cdot m = \varphi(r)m$ for $m \in M$, $r \in R$, this yields a functor $F\colon \operatorname{Mod} S \to \operatorname{Mod} R$ which does not change the underlying sets, thus it commutes with direct sums and direct products. As a consequence, we see that any algebraically compact S-module is also algebraically compact when considered as R-module. In particular, if I is an ideal of R, and M is an R-module which is annihilated by I, then M is algebraically compact as an R-module if and only if it is algebraically compact when considered as an R/I-module. This shows that all injective R/I-modules are algebraically compact R-modules.

Algebraically compact modules as injective objects in a Grothendieck category. According to Gruson and Jensen [GJ], the category of algebraically compact modules is equivalent to the category of injective objects in some Grothendieck category – a Grothendieck category which is usually far away from module categories of rings with finiteness conditions. The equivalence is given as follows: Let R be a ring and $\operatorname{mod}(R^{\rm op})$ the category of finitely presented right R-modules. Any (left) R-module M gives rise to a functor $(-\otimes_R M)\colon \operatorname{mod}(R^{\rm op}) \to \operatorname{Mod}\mathbb{Z}$, and we obtain in this way a functor

$$\Phi\colon \operatorname{Mod} R \to \operatorname{Mod}\operatorname{mod}(R^{\rm op}), \qquad \Phi(M) = (-\otimes_R M).$$

It is easy to see that this functor Φ is a full embedding[17]. The R-module M is algebraically compact if and only if $\Phi(M)$ is an injective object in $\operatorname{Mod}\operatorname{mod}(R^{\rm op})$. As a consequence, the restriction of Φ to the subcategory of all algebraically compact R-modules yields an equivalence with the category of injective objects in $\operatorname{Mod}\operatorname{mod}(R^{\rm op})$.

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[17] In addition, it also commutes with direct sums, direct products and direct limits; the image consists of those additive functors which are right exact, and these are just the objects Q in $\operatorname{Mod}\operatorname{mod}(R^{\rm op})$ with $\operatorname{Ext}^1(F,Q)=0$ for all finitely presented functor F, see for example [JL, Theorems B15 and B.16]. A sequence $0 \to M' \to M \to M'' \to 0$ in $\operatorname{Mod} R$ is pure exact if and only if its image under Φ is exact.

For any ring R, the R-modules which are indecomposable and algebraically compact are those we are interested in, thus we have to deal with the indecomposable injective objects in $\operatorname{Mod}\operatorname{mod}(R^{\mathrm{op}})$. The first observation to be stressed is the fact that such an object always has a local endomorphism ring, thus the endomorphism ring of an indecomposable algebraically compact R-module is local. Next, let us note that indecomposable injective objects in a Grothendieck category always are uniform; they are the injective envelopes of uniform objects. Thus, in order to get hold of some indecomposable algebraically compact R-module, it is sufficient to find an R-module M such that the functor $\Phi(M) = (-\otimes M)$ is uniform: an injective envelope of $\Phi(M)$ will be of the form $\Phi(\mu)\colon \Phi(M) \to \Phi(\overline{M})$, and $\mu\colon M \to \overline{M}$ is the so called pure-injective envelope[18]. Let us reformulate what it means that the functor $\Phi(M)$ is $\mathbb{N}$-uniform: we need modules N_i and maps $g_i\colon M \to N_i$, $h_i\colon N_{i+1} \to N_i$, for all $i \in \mathbb{N}$, with the following properties: first, $g_i = h_i \circ g_{i+1}$; second, no g_i is a pure monomorphism; and third, given any map $f\colon M \to N$ which is not a pure monomorphism, then there exists an index i and $f'\colon N_i \to N$ such that $f = f' \circ g_i$. (Let us show that these conditions imply that $\Phi(M)$ is $\mathbb{N}$-uniform; the converse can be shown in the same way. Let U_i be the kernel of the transformation $\Phi(g_i) = (-\otimes g_i)$. This U_i is a subobject of $\Phi(M) = (-\otimes M)$. Also, since g_i is not a pure monomorphism, we see that $\Phi(g_i)$ is not a monomorphism, thus $U_i \neq 0$. It follows from $g_i = h_i \circ g_{i+1}$ that $U_{i+1} \subseteq U_i$. Now assume there is given any subobject U of $\Phi(M)$. The injective envelope of $\Phi(M)/U$ is of the form $\Phi(N)$ for some module N, thus U is the kernel of some map $\Phi(M) \to \Phi(N)$. But since Φ is full, such a map is of the form $\Phi(f)$ for some $f\colon M \to N$. As we require, there exists some map $f_i\colon N_i \to N$ such that $f = f' \circ g_i$. But this means that $U_i \subseteq U$.) Actually, the third condition has to be checked only in very special cases, namely in case $f\colon M \to N$ is a monomorphism whose cokernel is indecomposable and of finite length, as well as in case f is an epimorphism with simple kernel. (Namely, assume the third condition holds true in these special cases, and let $f\colon M \to N$ be any map which is not a pure monomorphism. If f is not a monomorphism, then there is a simple submodule S of M which is contained in the kernel of f and f factors via the projection map $p\colon M \to M/S$, say $f = f' \circ p$. By assumption, $p = p' \circ g_i$ for some i, and therefore $f = f' \circ p = f' \circ p' \circ g_i$. On the other hand, if f is a monomorphism, then there is some finite length submodule N' of N such that the map $f\colon M \to N'$ does not split, since f is not a pure monomorphism. By assumption, we know that this $f\colon M \to N'$ factors via some g_i.)

The Ziegler spectrum. Denote by $\mathcal{Z}(R)$ the set of isomorphism classes of R-modules which are indecomposable and algebraically compact[19]. As Ziegler [Z]

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[18] This means that μ is a pure monomorphism, $\overline{M}$ is algebraically compact, and μ is left minimal (any endomorphism ϕ of $\overline{M}$ with $\phi\mu = \mu$ is an automorphism).

[19] The number of isomorphism classes of indecomposable algebraically compact R-module is bounded by 2^λ, where $\lambda = \max(|R|, \aleph_0)$, see [JL],7.57.

has pointed out, the set $\mathcal{Z}(R)$ carries a natural topology which is useful for many considerations. There are several ways to define the Ziegler topology. Here we use the following approach: Given a class $\mathcal{X}$ of maps $f\colon X \to X'$ between finitely presented modules X, X', let $\mathcal{A}(\mathcal{X})$ be the indecomposable algebraically compact modules M such that $\operatorname{Hom}(f, M)$ is surjective for all $f \in \mathcal{X}$. As closed sets in $\mathcal{Z}(R)$, one takes the subsets of the form $\mathcal{A}(\mathcal{X})$. As one can show the closed sets are just the sets of isomorphism classes in $\mathcal{Z}(R)$ which belong to some definable subcategory; here, a subcategory $\mathcal{U}$ of $\operatorname{Mod} R$ is said to be *definable* provided it is closed under direct limits, products and pure submodules, or equivalently, if $\mathcal{U}$ is defined by the vanishing of a collection of functors $F\colon \operatorname{Mod} R \to \operatorname{Mod} \mathbb{Z}$ which commute with direct limits and products [CB4]. Given an indecomposable algebraically compact module M, we denote by $\operatorname{cl}(M) = \operatorname{cl}(\{M\})$ the Ziegler closure of the one-element set $\{M\}$. In general there do exist non-isomorphic indecomposable algebraically compact modules M, M' such that $\operatorname{cl}(M)$ and $\operatorname{cl}(M')$ coincide (thus the Ziegler spectrum is not necessarily a T_0-space). For example, if R is a simple von Neumann regular ring, then the only Ziegler closed sets are the empty set and $\mathcal{Z}(R)$ itself. We say that the modules M, M' are topologically equivalent provided $\operatorname{cl}(M) = \operatorname{cl}(M')$.

Elementary duality. The category $\operatorname{Mod} R$ of all R-modules where R is a ring is a Grothendieck category, the dual of this category is not. The dual of a full subcategory $\mathcal{U}$ of a module category is equivalent to a full subcategory of some other module category only in case there are severe restrictions on the size of the modules in $\mathcal{U}$. If we consider, as we do, full subcategories of module categories, the existence of a contravariant equivalence is very rare. The so called elementary duality which we are going to discuss does not concern maps, but only collections of objects in the Ziegler spectrum. It has been observed by Herzog [He] that there is a bijection D between the collection of closed sets of $\mathcal{Z}(R)$ and the collection of closed sets of $\mathcal{Z}(R^{\text{op}})$ which respects finite unions and arbitrary intersections, in particular, it preserves and reflects inclusions, see also [P2,P6]. This duality is based on the duality between the categories of finitely presented functors on $\operatorname{mod} R$ and on $\operatorname{mod}(R^{\text{op}})$ and it can be interpreted as well in terms of the so called positive primitive formulae in model theory. As Krause [K6] has pointed out, if R is a k-algebra, the elementary dual $D\mathcal{A}$ of any closed set $\mathcal{A}$ in $\mathcal{Z}(R)$ is obtained as the set of indecomposable direct summands of modules which belong to the definable subcategory generated by the modules $A^* = \operatorname{Hom}(A, k)$, with $A \in \mathcal{A}$.

Clearly, D induces a bijection between the equivalence classes with respect to topological equivalence. Unfortunately, even for a finite-dimensional k-algebra, where k is some field, a pointwise description of the elementary duality does not seem to be available: given an indecomposable algebraically compact R-module M, one may expect that all the modules N in $\mathcal{Z}(R^{\text{op}})$ such that $D\operatorname{cl}(M) = \operatorname{cl}(N)$ are direct summands of the k-dual M^* of M, and then and one would like to have an effective procedure for obtaining such a direct summand. Of course, if M is finite-dimensional and indecomposable, then M^* has the same dimension, is indecomposable and is just the required module N. But if M is infinite-dimensional and indecomposable, then M^* may be decomposable. As Krause has shown [K3, K6], there is a class $\mathcal{R}(R)$ of indecomposable algebraically compact R-modules M, the so-called simple reflexive ones, such that one can define a duality $D\colon \mathcal{R}(R) \to \mathcal{R}(R^{\mathrm{op}})$ with the following properties: for any $M \in \mathcal{R}(R)$, the R^{op}-module DM is a direct summand of M^* and $D\,\mathrm{cl}(M) = \mathrm{cl}(DM)$. Of course, $DDM \simeq M$.

The denomination "elementary duality" refers to the fact that it is based on the elementary language of R-modules. It has to be stressed that the elementary duality does not preserve Σ-algebraic compactness. We will discuss below the Ziegler spectrum of a finite-dimensional algebra which is tame and hereditary; for these algebras, the Prüfer modules are Σ-algebraically compact whereas the adic modules are not, and the elementary dual of a Prüfer module is just an adic module.

Generic modules. Detailed examples of indecomposable algebraically compact modules will be presented in the next section. Here we want to mention only the most prominent class, namely the generic modules: these are those indecomposable modules which are of finite length when considered as modules over their endomorphism ring[20]. Of importance is the following observation of Krause [K4]: *The closure of any tube contains at least one generic module of infinite length.* One obtains such a generic module as follows [R9]: Take a ray $M_1 \to M_2 \to \cdots$ and its direct limit $P = \varinjlim M_i$ and form a countable product of copies of P; this will be a direct sum of copies of P and of copies of finitely many generic modules of infinite length. One conjectures that the closure of a tube contains precisely one generic module of infinite length[21]. Also, as Crawley-Boevey has shown, if R is a tame k-algebra, where k is an algebraically closed field, then *any generic module of infinite length is obtained in this way.* Namely, according to [CB1], such a module is of the form $M \otimes_{k[T]} k(T)$, where M is an R-$k[T]$-bimodule which is

[20] Some authors require in addition that the module itself is not of finite length. This may be a reasonable convention in case one deals with a tame algebra, but it is odd in general: The word "generic" refers to the fact that such a module G serves to parameterize an algebraic family of indecomposable modules, just in the same way, as in classical geometry generic points were used. To exclude the possibility for G to be of finite length would correspond to the requirement that only irreducible varieties of dimension at least 1 should have a generic point.

[21] In the general setting as considered in [R9], one deals with a module P with a locally nilpotent endomorphism ϕ such that ϕ has finite-dimensional kernel. Of course, the direct limit module for a ray in a tube has these properties, but there are other examples. A string algebra with a contracting $\mathbb{Z}$-word z provides the example of such a module $P = M(z)$ such that the infinite direct products of copies of P have two non-isomorphic generic modules as direct summands.

finitely generated and free as a $k[T]$-module and such that for almost all elements $\lambda \in k$, the modules $M \otimes_{k[T]} k[R]/(T-\lambda)^n$ form a tube.

If R is a finite-dimensional k-algebra which is connected, hereditary and tame, there is precisely one generic module of infinite length; details will be given below in the case where k is an algebraically closed field. There are other classes of algebras where all the generic modules are known, let us mention at least the tubular algebras, see for example [Le], and also the string algebras. For the relevance of maps between generic modules we refer to Bautista [Bt]. It has been shown in [R11] that for many string algebras one can find sequences $G_1, G_2, \dots$ of generic modules such that for every i there do exist both proper monomorphisms $G_i \to G_{i+1}$ and proper epimorphisms $G_{i+1} \to G_i$.

4. Finite-dimensional algebras

The impetus for this collection of surveys came from a maybe surprising, but apparent need of using infinite-dimensional modules in order to understand the behaviour of finite-dimensional modules over a finite-dimensional algebra R. Much effort has been spent in order to define the representation type of such an algebra: this concerns the category of finite-dimensional R-modules, but the usual approaches involve infinite-dimensional R-modules, thus infinite length modules. Let us recall some of these developments.

Products of finite-dimensional modules. Of interest is Couchot's characterization [Ct] of the algebraically compact modules: *For a finite-dimensional k-algebra R, an R-module M is algebraically compact if and only if it is a direct summand of a product of finite length modules.* (Proof: On the one hand, the class of algebraically compact modules is closed under products and direct summands, for any ring R, and it includes, for any k-algebra R, all the finite-dimensional R-modules. Conversely, assume that R is a finite-dimensional k-algebra and take any R-module M. Consider the R^{op}-module $M^* = \mathrm{Hom}(M,k)$. It is well-known that there exists a pure exact sequence $0 \to N' \to N \to M^* \to 0$ such that N is a direct sum of finite-dimensional R^{op}-modules, and that the dual of a pure exact sequence is split exact; see for example [Fc2] 1.23 and 1.28. Thus M^{**} is isomorphic to a direct summand of N^*. But the canonical inclusion $M \to M^{**}$ is a pure monomorphism. If M is algebraically compact, then M is a direct summand of M^{**} and thus also of N^*. But since N is a direct sum of finite-dimensional R^{op}-modules, N^* is the direct product of these modules.) Thus, in order to deal with all possible algebraically compact modules, no fancy constructions are needed: it is sufficient to form products and to take direct summands.

But we may interprete this result also differently: after all, as we will see, there do exist quite complicated algebraically compact modules. Thus we see that the process of forming products[22] of modules is not at all easy to control, see

[22] A general discussion of products in Grothendieck categories should be very worthwhile. The products we have considered here are always cartesian products,

also [HO]. Forming products of finite-dimensional modules is a very effective way in order to obtain new types of modules. In particular, we note the following well-known result: *Let* $(M_i)_i$ *be a set of finite-dimensional modules and assume that any indecomposable module is a direct summand of at most finitely many* M_i. *Then the module* $\prod M_i / \bigoplus M_i$ *has no finite-dimensional indecomposable direct summand.*

Representation types. We assume that R is a finite-dimensional algebra (or, more generally, an artin algebra). In case R is representation-finite, the structure of all the R-modules is known: **Theorem:** *If* R *is a representation-finite algebra, and* M *is any* R*-module, then* M *is a direct sum of indecomposable* R*-modules of finite length* [T,RT,A2]. Of course, such a direct sum is essentially unique, according to the Azumaya Theorem. On the other hand, *if* R *is not representation-finite, then there are always indecomposable* R*-modules of infinite length* [A3].

In [A1], Auslander has introduced the notion of a representation equivalence. A *representation equivalence* (or an *epivalence* [GR]) is a functor which is full, dense and reflects isomorphisms, and two categories $\mathcal{A}$ and $\mathcal{A}'$ were said to be *representation equivalent* (or to have equivalent representations) provided there is a sequence of representation equivalences $\mathcal{A} = \mathcal{A}_0 \to \mathcal{A}_1 \leftarrow \mathcal{A}_2 \to \cdots \mathcal{A}_n = \mathcal{A}'$. Actually, this equivalence relation is not very useful, for the following reason: If k is an algebraically closed field, then all the representation-infinite k-algebras R are representation-equivalent: the category $\operatorname{mod} R/\operatorname{rad}(\operatorname{mod} R)$ is the direct sum of copies of $\operatorname{mod} k$, the number of copies is just the cardinality of k (note that the number of isomorphism classes of indecomposables is equal to the cardinality of the field k).

The concept of a representation embedding as introduced by Crawley-Boevey in [CB2] is more appropriate and avoids such difficulties: a k-linear functor $F\colon \operatorname{Mod} S \to \operatorname{Mod} R$ is said to be a *representation embedding* provided it is exact, preserves direct sums and products, preserves indecomposability and non-isomorphy. Prest shows in [P4] that a representation embedding from $\operatorname{Mod} S$ to $\operatorname{Mod} R$ induces a homeomorphic embedding of $\mathcal{Z}(S)$ into $\mathcal{Z}(R)$.

On the basis of examples considered by Corner, by Brenner and Butler and others, there had been a vague feeling concerning a possible distinction between tame and wild algebras; a challenging conjecture was formulated by Donovan and Freislich at the Bonn workshop 1973 and proved by Drozd [Dd]. Let k be an

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but one should be aware that a full subcategory $\mathcal{C}'$ of a Grothendieck category $\mathcal{C}$ which is closed under kernels, cokernels and direct limits (and thus a Grothendieck category on its own) does not have to be closed under products, a typical example is the subcategory $\mathcal{C}'$ of all abelian p-groups in the category of all abelian groups. In such a situation, the products in $\mathcal{C}'$ are subobjects of the products formed in $\mathcal{C}$ (in our example, the product in $\mathcal{C}'$ is the torsion subgroup of the cartesian product; note that for non-bounded p-groups the cartesian product is no longer a torsion group).

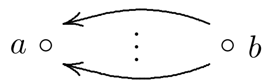
algebraically closed field. A k-algebra R is said to be *tame* provided for every dimension d, there are finitely many R-$k[T]$-bimodules M_i which are finitely generated free as $k[T]$-modules, such that almost all indecomposable R-modules of dimension d are isomorphic to modules of the form $M_i \otimes_{k[T]} N$, where N is an indecomposable finite length $k[T]$-module. Let us call R to be t*-domestic* provided t bimodules $M_1, \dots, M_t$, but not less, are needed altogether (for all d), and *non-domestic,* in case infinitely many bimodules M_i are needed. If we fix such a bimodule M_i and consider the set of all the R-modules $M_i \otimes_{k[T]} N$, where N runs through the indecomposable $k[T]$-modules say of dimension n, we obtain what may be called a (rational) one-parameter family of R-modules. Thus, an algebra is tame provided for every dimension d, almost all the indecomposable R-modules belong to a finite number of one-parameter families. In this way, one can reformulate the notion of tameness in terms of algebraic geometry. Since these bimodules M_i are free as $k[T]$-modules (and non-zero), they are infinite-dimensional over k, thus, as R-modules they also have infinite length. Only recently, Krause [K6,K7] gave the first characterization of tameness which only relies on the category of R-modules of finite length, without reference to an infinite length R-module, or the (external) algebraic geometrical structure.

It seems that an algebra R is tame (or representation-finite) if and only if any non-zero algebraically compact module has an indecomposable direct summand. If this is true, this would provide a very convenient and easy way for defining tameness, using only the notions of indecomposability and of algebraic compactness. Actually, taking into account Couchot's characterization, we may even avoid the notion of algebraically compactness, thus we arrive at the following reformulation: An artin algebra R should be tame if and only if any product of finite length modules is a discrete module.

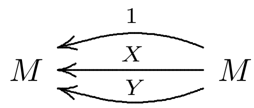
The representation-finite algebras are quite well-understood (see [GR]): We recall that a representation-finite algebra with a faithful indecomposable representation is standard, so that one may use covering theory in order to recover all the indecomposables from a suitable representation-directed algebra, and there are very effective algorithms known in order to deal with the indecomposable representations of a representation-directed algebra. For algebras which are not representation-finite, no general theory is available at present: there does not yet exist a structure theory even for the 1-domestic algebras, the algebras nearest to the representation-finite ones.

Drozd's definition of wildness involves, as that of tameness, infinite length R-modules; here one uses an R-$k\langle X, Y\rangle$-bimodule M which is finitely generated free as $k\langle X, Y\rangle$-module.

Strictly wild and controlled wild algebras. Let us start with a strictly wild algebra R, here one requires that for any k-algebra S, there is a full and exact embedding $\operatorname{Mod} S \to \operatorname{Mod} R$ which sends finitely presented S-modules to finitely presented R-modules. Of course, as soon one knows one strictly wild algebra S_0, it is sufficient to find an embedding $\operatorname{Mod} S_0 \to \operatorname{Mod} R$ as required in order to know that also R is strictly wild. A typical example of a strictly wild algebra is the free k-algebra $k\langle X, Y\rangle$ in two generators X, Y. Also all the generalized Kronecker algebras $kK(n)$ with $n \geq 3$ are strictly wild; by definition, $kK(n)$ is the path algebra of the quiver $K(n)$ with two vertices, say a, b and n arrows $b \to a$



For example, for $n = 3$ one obtains an embedding as required $\operatorname{Mod} k\langle X, Y\rangle \to \operatorname{Mod} kK(3)$ by sending the $\langle X, Y\rangle$-module (M, X, Y) to the following representation of $K(3)$



(here, M is a vector space, $X\colon M \to M$ denotes the multiplication by X, and similarly $Y\colon M \to M$ that by Y).

Why is it of interest to know that a k-algebra R is strictly wild? Of course, this implies that every k-algebra S occurs as the endomorphism ring of a suitable R-module, thus all module theoretical phenomena which can be read off from the endomorphism rings of a module occur for R-modules; in particular, this applies to all kinds of possible direct decompositions, since the direct decompositions of a module are encoded into the set of idempotents of its endomorphism ring. Strict wildness does not concern only individual modules, but also sets or even classes of R-modules. For example, for certain considerations it is good to have large sets of pairwise orthogonal modules at hand, a set $(M_i)_{i\in I}$ being called *pairwise orthogonal* provided $\operatorname{Hom}(M_i, M_j) = 0$ for all pairs of indices $i \neq j$, and this is the case for any strictly wild algebra.

It is well-known that there do exist k-algebras R which are not strictly wild, but which have the weaker property that any k-algebra S can be realized as a (nice) factor algebra of the endomorphism ring of an R-module. For example, consider the polynomial ring $k[X, Y, Z]$ in three variables and its factor algebra $R = k[X, Y, Z]/(X, Y, Z)^2$ modulo the square of the ideal generated by X, Y, Z. This is a local algebra, thus the only division ring which can be realized as an endomorphism ring of an R-module is k itself (the only module M with endomorphism ring being a division ring is the simple module k). Also, if $(M_i)_{i\in I}$ is a set of pairwise orthogonal R-modules, then the index set consists of at most one element! On the other hand, given any k-algebra S, there does exist an R-module M such that $S = \operatorname{End}(M)/J$, where J is an ideal of $\operatorname{End}(M)$, it is even a nicely defined ideal, namely the set of all endomorphisms of M with semisimple image. Such "wild" algebras were studied by Corner, Brenner and others. When Drozd formulated and proved his celebrated tame-and-wild theorem, he introduced a definition of wildness which deviated from the older intuitive notion: an algebra R is *wild* in the sense of Drozd provided there exists an R-$k\langle X, Y\rangle$-bimodule W which is finitely generated and projective as a $k\langle X, Y\rangle$-module such that the functor $F(-) = \big({}_RW \otimes_{k\langle X,Y\rangle} -\big)$ preserves indecomposability and non-isomorphy. Of course, in this definition, we may replace the (infinite-dimensional) algebra $k\langle X,Y\rangle$ by the (five dimensional) algebra $kK(3)$, considering a bimodule ${}_RW_{kK(3)}$ which is finitely generated and projective as a $kK(3)$-module and the functor

$$F(-) = \big({}_RW \otimes_{k\langle X,Y\rangle} -\big)\colon \operatorname{Mod} kK(3) \to \operatorname{Mod} R,$$

or at least its restriction to $\operatorname{mod} kK(3)$. Now, given a k-algebra S, there is a $kK(3)$-module N with $\operatorname{End}(N) = S$ and we may consider now the R-module $F(N) = W \otimes_{kK(3)} N$. Its endomorphism ring $\operatorname{End}(F(N))$ has S as a subring, but there is no reason that in this way S can be realized as the factor ring of an endomorphism ring. The problem we are dealing with is the following: given a R-$kK(3)$-bimodule W which is finitely generated and free as a $kK(3)$-module, such that the functor $F(-) = \big(W \otimes_{kK(3)} -\big)$ is faithful, then $F(-)$ is not necessarily full. In which way is it possible to control the subspaces

$$F(\operatorname{Hom}_{kK(3)}(N_1,N_2)) \subseteq \operatorname{Hom}_R(F(N_1),F(N_2))$$

We say that R is *controlled wild* provided the subspace $F(\operatorname{Hom}_{kK(3)}(N_1,N_2))$ of $\operatorname{Hom}_R(F(N_1),F(N_2))$ is complemented by the set $\operatorname{Hom}_R(F(N_1),F(N_2))_{\mathcal U}$ of homomorphisms $F(N_1) \to F(N_2)$ which factor through a prescribed additive subcategory $\mathcal U$ of R-modules (and $\mathcal U$ may be called the corresponding *control class*):

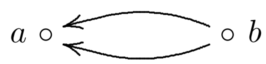
$$(*) \quad \operatorname{Hom}_R(F(N_1),F(N_2)) = F(\operatorname{Hom}_{kK(3)}(N_1,N_2)) \oplus \operatorname{Hom}_R(F(N_1),F(N_2))_{\mathcal U}.$$

For example, for the local algebra $R = k[X,Y,Z]/(X,Y,Z)^2$ the class $\mathcal U$ of all semisimple modules serves as such a control class. Recent investigations by Rosenthal and Han [Ha] support the conjecture that all finite-dimensional wild k-algebras (k is algebraically closed) are controlled wild. We can reformulate this concept[23] as follows: Let $\mathcal V$ be the class of R-modules which are images under F and consider the ideal $\langle\mathcal U\rangle \cap \mathcal V$ of $\mathcal V$. According to $(*)$, we see that the factor category $\mathcal V/\big(\langle\mathcal U\rangle \cap \mathcal V\big)$ is equivalent to the category $\operatorname{mod} k\langle X,Y\rangle$.

Tame is Wild. If we consider for a finite-dimensional k-algebra all modules and not only those of finite length, the difference between "tame" and "wild" vanishes. This is usually formulated as follows: tame algebras are Wild [R3,R12], here the small "t" in *tame* refers to tameness with respect to modules of finite length, the capital "W" in *Wild* refers to wildness with respect to arbitrary modules which

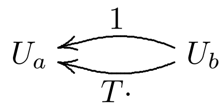
[23] This seems to be an appropriate setting for discussing the wildness of many categories; for example, we may take as $\mathcal V$ the category of all abelian p-groups and as $\mathcal U$ the subcategory of all bounded ones. Or, let $\mathcal V$ be the category of all abelian groups which are slender, and $\mathcal U$ the subcategory of all free abelian groups of finite rank.

are not necessarily of finite length. To be more precise, the quoted papers show that the Kronecker quiver $K(2)$



is strictly Wild: they provide an explicit full exact embedding of the category of representations of $K(3)$ into the category of representations of $K(2)$, and, of course, $K(3)$ is a typical strictly wild quiver. Similar to the notion of a strictly wild algebra, one may call a k-algebra R *strictly tame* provided there is a full exact embedding of the category $\operatorname{Mod} kK(2)$ of all $kK(2)$-modules into the category $\operatorname{Mod} R$ of all R-modules. The precise formulation is: *All strictly tame algebras are Wild.* There is a strong belief that for all tame finite-dimensional k-algebras, where k is an algebraically closed field, the one-parameter families of indecomposables can be obtained using functors $\operatorname{Mod} kK(2) \to \operatorname{Mod} R$ which are quite well-behaved also with respect to infinite-dimensional modules. If this turns out to be true, then one will see that really all tame algebras are Wild. In this way, for the global behaviour of finite-dimensional algebras, the only relevant distinction seems to be that between finite representation type and infinite representation type.

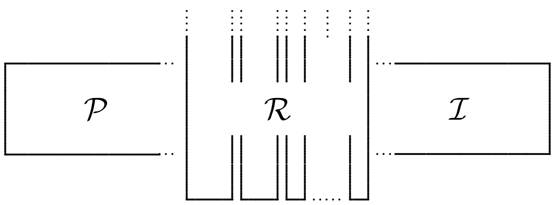
The main observation behind the tame-is-Wild theorem concerns the existence of (many) infinite-dimensional $kK(2)$-modules with endomorphism ring k. Here is a recipe in case k is infinite: let $k(T)$ be the field of rational functions in one variable, let I be an infinite subset of k. Let U_b be the subspace of $k(T)$ generated by the elements $\frac{1}{T-\lambda}$ with $\lambda \in I$ and $U_a = U_b + k1$. The representation $U = U(I)$ we are interested in is



This is a subrepresentation of the generic one $(k(T), k(T); 1, T\cdot)$, and it is not difficult to check that its endomorphism ring is k. Moreover, if I, J are disjoint infinite sets then $\operatorname{Hom}(U(I), U(J)) = 0$, but $\operatorname{Ext}^1(U(I), U(J)) \neq 0$. Using such representations $U(I)$, it is easy to construct many different full exact embeddings of strictly wild categories into $\operatorname{Mod} kK(2)$.

We have mentioned above that for any dualizing k-variety, the category $\operatorname{mod} \mathcal{C}$ has Auslander-Reiten sequences. As a consequence, one can consider the corresponding Auslander-Reiten quiver. It incorporates the basic concepts developed by Auslander and Reiten, in particular the notion of an irreducible map and that of the Auslander-Reiten translate τ ("dual of the transpose"), in order to provide a first overview over the category $\operatorname{mod} \mathcal{C}$. Here we will use this theory in the classical case where R is a finite-dimensional algebra over some field (or, more generally, an artin algebra) and denote by Γ_R its *Auslander-Reiten quiver.* This is a quiver with vertex set the isomorphism classes $[X]$ of the indecomposable R-modules of finite length, and with an arrow $[X] \to [Y]$, where X, Y are indecomposable, provided there exists an irreducible map $X \to Y$. In addition, this quiver is endowed with the partial translation τ, it has the following property: $\tau[Z]$ is defined if and only if Z is not projective, and then there is an arrow $[Y] \to [Z]$ if and only if there is an arrow $\tau[X] \to [Y]$. If R is connected and representation-finite, then Γ_R is connected. It has been conjectured that also the converse is true (this is known to be true in case k is algebraically closed). A component of Γ_R which does not contain any projective or injective module is said to be a *stable* component.

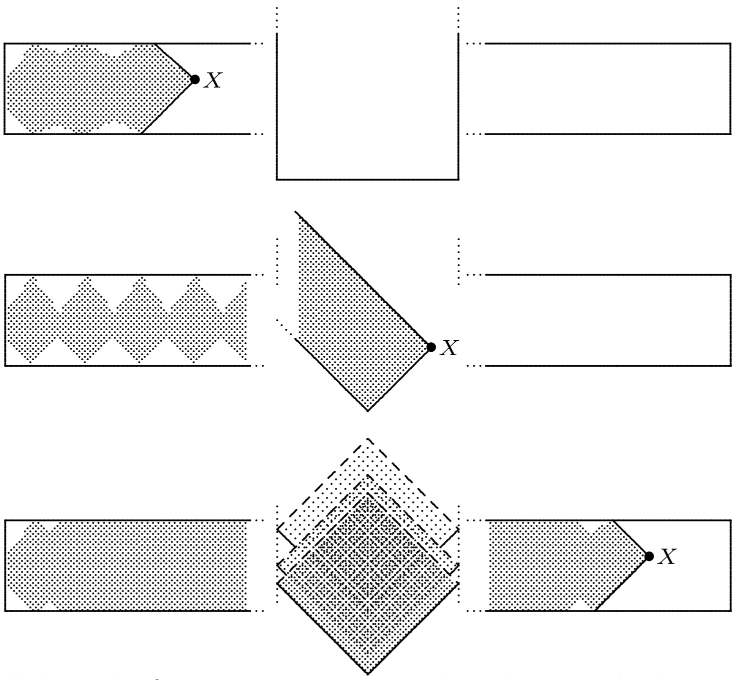
The tame hereditary algebras. One class of domestic algebras is very well-known, the tame hereditary k-algebras, where k is an algebraically closed field: these are just the path algebras of quivers of type $\widetilde{A}_n, \widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$. Let us recall the structure of the module category of such an algebra: There are two components of the Auslander-Reiten quiver which are not stable, one consists of the indecomposable projective modules and their τ^{-1}-translates (it is called the *preprojective* component $\mathcal{P}$), the other contains the injective modules and their τ-translates (the *preinjective* component $\mathcal{I}$). The remaining components are stable, all these components are "tubes", and the set of these components is indexed in a natural way by the projective line $\mathbb{P}_1 k$. The modules belonging to these stable components and their direct sums are said to be *regular.* The regular modules form an abelian subcategory $\mathcal{R}$, thus one may speak of simple regular representations (these are the regular representations which are simple objects when considered as objects in $\mathcal{R}$). Any regular module M is τ-periodic (this means that $\tau^p M \simeq M$ for some $p \geq 1$), the indecomposable modules in all but at most three of the tubes are *homogeneous* (this means that $\tau M \simeq M$); the remaining tubes are said to be *exceptional* and the minimal period for all the modules in a tube is said to be its *rank.* The global structure of the category $\operatorname{mod} R$ is as follows:



Globally, all the maps go from left to right: there are no maps from $\mathcal{R}$ to $\mathcal{P}$ and no maps from $\mathcal{I}$ to $\mathcal{P}$ or $\mathcal{R}$. (Actually, the subcategories $\mathcal{P}$ and $\mathcal{I}$ are also directed: the indecomposable modules in these subcategories can be arranged in such a way that there are no maps from right to left; but inside $\mathcal{R}$, there do exist cycles of maps.)

For a tame hereditary algebra R, the structure of the category $\operatorname{mod}\operatorname{mod} R$ is well-known, see [Gg]. Let us recall the shape of the indecomposable projective objects in $\operatorname{mod}\operatorname{mod} R$, these are the functors $\operatorname{Hom}(-,X)$ with X an indecomposable R-module of finite length. The top of $\operatorname{Hom}(-,X)$ is the simple functor $S_X = \operatorname{Hom}(-,X)/\operatorname{rad}(-,X)$, its socle is of the form $\bigoplus_i S_{P_i}$, where $\bigoplus_i P_i$ is a projective cover of $\operatorname{soc} X$. Of course, $\operatorname{Hom}(-,X)$ is of finite length if and only if X

is preprojective. If X is not preprojective, then the socle sequence $\mathrm{soc}_i \mathrm{Hom}(-,X)$ and the radical sequence $\mathrm{rad}_i \mathrm{Hom}(-,X)$ are controlled by the Auslander-Reiten structure of the category $\mathrm{mod}\, R$. It is instructive to specify suitable subobjects of $\mathrm{Hom}(-,X)$ in $\mathrm{Mod}\,\mathrm{mod}\, R$, namely the restriction to $\mathcal{P}$ and to $\mathcal{P}\vee\mathcal{R}$. The restriction $\mathrm{Hom}(-,X)|\mathcal{P}$ is just the union $\bigcup_i \mathrm{soc}_i \mathrm{Hom}(-,X)$, in particular this is an artinian functor. In case X is regular, $\mathrm{Hom}(-,X)/(\mathrm{Hom}(-,X)|\mathcal{P})$ is noetherian. Of special interest is the case where $X = E$ is simple regular. In this case the shape of $\mathrm{Hom}(-,E)$ (or the corresponding dual configuration) has been studied quite carefully in [R5] as the patterns which are relevant for tubular one-point extensions. If X is preinjective and $F = \mathrm{Hom}(-,X)$, let F'' be the restriction of F to $\mathcal{P}\vee\mathcal{R}$ and F'' the restriction to $\mathcal{P}$. Then $0 \subset F'' \subset F' \subset F$, the functor F'' is artinian, F/F' is noetherian, and F'/F'' is the direct sum of infinitely many functors (the summands correspond to the stable components, thus to the elements of $\mathbb{P}_1 k$), and none of these summands is artinian or noetherian.

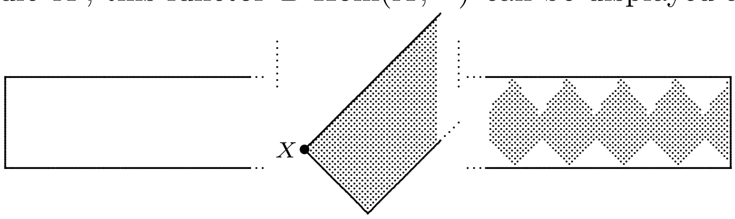


X preprojective

X regular

X preinjective

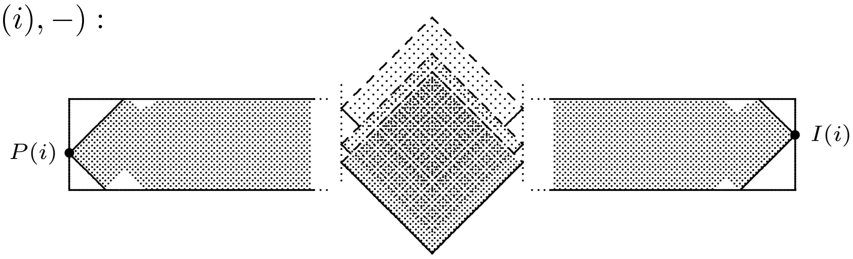
Recall that the functor category $\mathrm{mod}\,\mathrm{mod}\, R$ is a dualizing k-variety. In particular, this means that given any R-module X of finite length, the functor $D\,\mathrm{Hom}(X,-)$ is finitely presented. For example, starting with an indecomposable regular R-module X, this functor $D\,\mathrm{Hom}(X,-)$ can be displayed as follows:



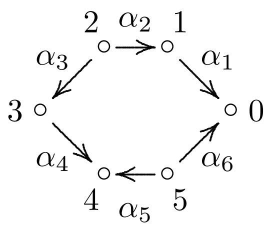
X regular

How do we obtain a finite presentation of $D\,\mathrm{Hom}(X,-)$? If $X = P(i)$ is an indecomposable projective R-module, then $D\,\mathrm{Hom}(P(i),-) \simeq \mathrm{Hom}(-,I(i))$, thus

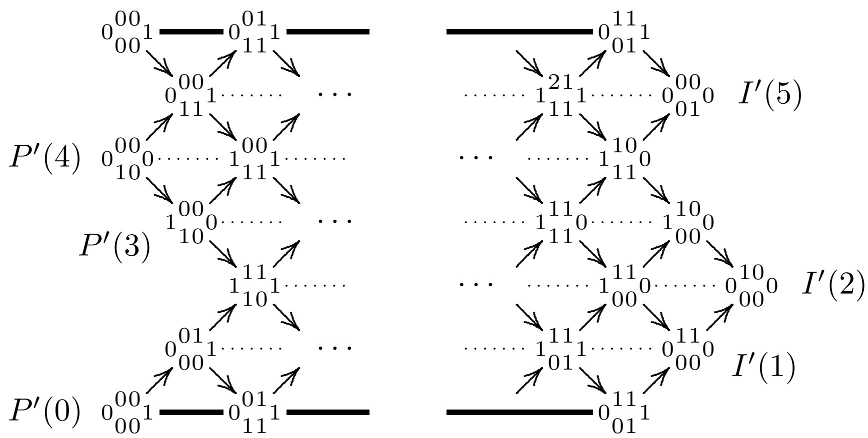
$D\,\mathrm{Hom}(P(i),-)$ is an indecomposable projective functor. Otherwise we can use the Auslander-Reiten formula $D\,\mathrm{Hom}(-,X) \simeq \mathrm{Ext}^1(-,\tau X)$; we embed τX into an injective module I, the exact sequence $0 \to \tau X \to I \to I/\tau X \to 0$ yields the exact sequence $\mathrm{Hom}(-,I) \to \mathrm{Hom}(-,I/\tau X) \to \mathrm{Ext}^1(-,\tau X) \to 0$, thus a finite presentation of $\mathrm{Ext}^1(-,\tau X)$. Of course, since our ring R is hereditary, the module $I/\tau X$ is injective again. — Finally, let us single out also the shape of the projective-injective objects: recall that these are just the functors $\mathrm{Hom}(-,I(i)) \simeq D\,\mathrm{Hom}(P(i),-)$:



Let us deal with a specific example of a tame hereditary algebra. We consider the following quiver Q' of type $\widetilde{A}_5$

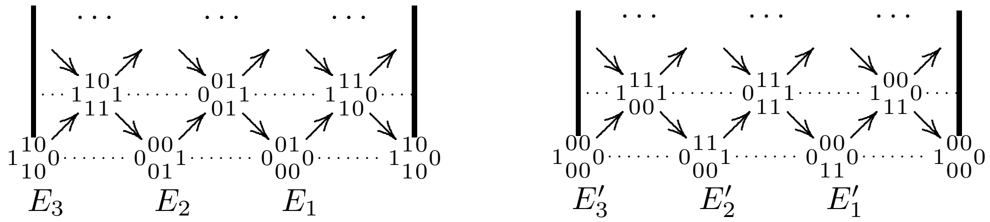


First of all, let us indicate, on the left, part of the preprojective component, and, on the right, part of the preinjective component. Here, the vertices are replaced by the corresponding dimension vectors[24], some of the modules are also labeled: Given a vertex i of Q', the corresponding indecomposable projective kQ'-module will be denoted by $P'(i)$; the corresponding indecomposable injective kQ'-module by $I'(i)$. Note that in both pictures, the solid horizontal lines have to be identified.



[24] The dimension vector of a quiver representation M exhibits the dimensions of the different vector spaces involved. For a directed quiver, these dimensions are just the Jordan-Hölder multiplicities $[M : S(i)]$ of the corresponding simple representations $S(i)$.

For this algebra, there are two exceptional tubes, both of rank 3. We are going to present also part of these tubes, again using dimension vectors and some labels.



It is easy to list the indecomposable algebraically compact modules: Of course, all the indecomposable modules of finite length have to be mentioned. In addition, every simple regular module E gives rise to two indecomposable modules which are algebraically compact: the *Prüfer module* $E[\infty]$ and the *adic module* $\widehat{E}$. They are obtained as follows: Consider all the finite-dimensional indecomposable regular R-modules M with $\mathrm{Hom}(E, M) \neq 0$. These modules can be labeled in the form $M = E[s]$ and arranged as a so called *ray*:

$$E = E[1] \subset E[2] \subset \cdots \subset E[s] \subset \cdots$$

where all the inclusion maps $E[s] \subset E[s+1]$ are irreducible maps, and

$$E[\infty] = \bigcup E[s].$$

Similarly, consider all the finite-dimensional indecomposable regular R-modules M with $\mathrm{Hom}(M, E) \neq 0$. These modules can be labeled in the form $M = [s]E$ and arranged as a *coray*:

$$\cdots \to [s]E \to \cdots \to [2]E \to [1]E = E$$

where all the maps $E[s{+}1] \to E[s]$ are irreducible epimorphisms. The adic module $\widehat{E}$ is defined as the inverse limit

$$\widehat{E} = \lim_{\leftarrow} \, [s]E$$

and we denote by $\pi_s \colon \widehat{E} \to [s]E$ the canonical projection (the names Prüfer module and adic module are parallel to the use of the corresponding names for abelian groups, where one speaks of a Prüfer group and the p-adic integers; the p in "p-adic" specifies the simple top $\mathbb{Z}/\mathbb{Z}p$, in a similar way, we may call $\widehat{E}$ the E-adic module).

Let us indicate why *both modules* $E[\infty]$ *and* $\widehat{E}$ *are indecomposable.* It is well-known that for any $s \geq 1$, any non-zero map $E \to E[s]$ (and therefore also $E \to E[\infty]$) is the composition of an automorphism of E and the inclusion map. Thus, given a direct decomposition $E[\infty] = U \oplus U'$, we may assume that $E \subseteq U$

and $\operatorname{Hom}(E,U')=0$. Inductively, one sees that $E[s]\subseteq U$ for all s, thus $U'=0$. Similarly, we want to see that any non-zero map $\widehat{E}\to E$ is the composition of π_1 and an automorphism of E. Note that $\operatorname{Ext}^1(\tau^{-1}E,\widehat{E})\simeq \varprojlim \operatorname{Ext}^1(\tau^{-1}E,[s]E)$, and the maps $[s+1]E\to[s]E$ induce isomorphisms $\operatorname{Ext}^1(\tau^{-1}E,[s+1]E)\to \operatorname{Ext}^1(\tau^{-1}E,[s]E)\to \operatorname{Ext}^1(\tau^{-1}E,E)$. Now, we use the Auslander-Reiten formula which yields $\operatorname{Hom}(\widehat{E},E)\simeq D\operatorname{Ext}^1(\tau^{-1}E,\widehat{E})\simeq D\operatorname{Ext}^1(\tau^{-1}E,E)\simeq \operatorname{Hom}(E,E)$. This shows that any map $\widehat{E}\to E$ vanishes on the kernel of π_1. Using induction, it follows that any map $\widehat{E}\to[s]E$ vanishes on the kernel of π_s, and this implies that for any decomposition $\widehat{E}=U\oplus U'$, one of the summands has to be contained in the intersection of all these kernels, thus has to be zero.

Also, *both modules $E[\infty]$ and $\widehat{E}$ are algebraically compact.* This is trivial for $\widehat{E}$, since it is the inverse limit of finite-dimensional modules, and such an inverse limit is always algebraically compact. But it is also clear for the Prüfer module $E[\infty]$, since it is artinian when considered as a module over its endomorphism ring.

There is just one additional R-module which is indecomposable and algebraically compact, the *generic module* G of infinite length, see [R4]. There are several ways to construct G. Starting with a Prüfer module $E[\infty]$, note that there is an epimorphism $(\tau E)[\infty]\to E[\infty]$ whose kernel is simple regular. We obtain a sequence of maps

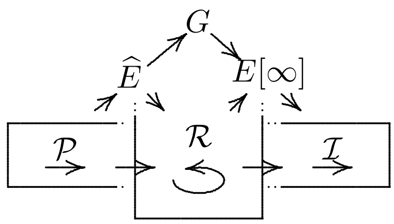
$$\cdots\to(\tau^nE)[\infty]\to\cdots\to(\tau E)[\infty]\to E[\infty]$$

and we may form the inverse limit $\varprojlim\,(\tau^nE)[\infty]$. This inverse limit is a direct sum of copies of G. Dually, starting with an adic module $\widehat{E}$, note that there is an embedding $\widehat{E}\subset(\tau^{-1}E)^\frown$ with simple regular cokernel. We obtain a sequence of inclusion maps

$$\widehat{E}\subset(\tau^{-1}E)^\frown\subset\cdots\subset(\tau^{-n}E)^\frown\subset\cdots$$

This time, we have to form the direct limit and obtain a module which again is the direct sum of copies of G.

We have constructed the infinite-dimensional indecomposable algebraically compact modules using limits and colimits (and direct summands) of regular modules. It is also possible to construct the adic modules as well as the generic module of infinite length as direct limits of preprojective modules, and the Prüfer modules as well as this generic module as direct summands of inverse limits of preinjective modules. We are going to insert the additional algebraically compact modules into our global picture of $\operatorname{mod}R$, the arrows indicate the directions of all the possible maps.



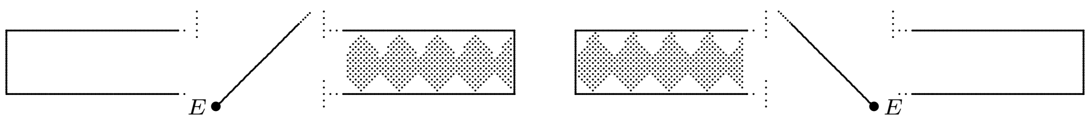
We recall that any algebraically compact R-module occurs as a direct summand of a product $X = \prod_{i\in I} X_i$ of finite-dimensional modules X_i. Since we only are interested in algebraically compact modules which are indecomposable, we may assume that we start with a collection of modules X_i where all these modules are preprojective, or regular, or preinjective. First, consider the case where all X_i are preinjective. Then X is a divisible module[25], thus it is a direct sum of indecomposable divisible modules. An indecomposable divisible module is either finite-dimensional and then preinjective, or else a Prüfer module or the generic module G of infinite length. Next, consider the case where all the modules X_i are regular. Then X will contain a submodule which is a direct sum of indecomposable modules which are finite-dimensional (and regular), or adic modules, and such that X/X' is a direct sum of copies of G. Usually X' will be a proper submodule of X, thus it will not be possible to write X as a direct sum of indecomposable modules. Finally, in case all the modules X_i are preprojective, then the only indecomposable summands of X are the finite-dimensional ones. Let us assume that all the modules X_i are indecomposable, let $P(1), P(2), \dots$ be a complete list of all the indecomposable preprojective modules, one from each isomorphism class, and denote by $I(s)$ the set of indices i with X_i isomorphic to $P(s)$. Note that the product $\prod_{i\in I(s)} X_i$ can be written as a direct sum of copies of $P(s)$. The submodule $X' = \bigoplus_s \prod_{i\in I(s)} X_i$ of X is a direct sum of finite-dimensional indecomposable modules, and it is maximal with this property. Of course, in case $I(s)$ is non-empty for infinitely many s, then X' is a proper submodule of X. — It is of interest to compare the two extreme cases when dealing with preinjective or preprojective modules X_i. In the preinjective case, several new types of indecomposable direct summands occur, but X can be written as the direct sum of indecomposable modules. On the other hand, in the preprojective case, no new isomorphism classes of indecomposable direct summands do occur, but usually X cannot be written as a direct sum of indecomposable modules. — The case where all X_i are regular is intermediate and is similar to the well-known situation of analyzing reduced algebraically compact abelian groups [Fu1].

Having determined all the indecomposable algebraically compact modules, let us describe the Ziegler topology (see [Gr], [P5] and [R10]). A subset $\mathcal{X}$ of $\mathcal{Z}(R)$ is closed if and only if the following conditions are satisfied: First, if E is a simple regular R-module and if there are infinitely many finite length modules $X \in \mathcal{X}$ with $\mathrm{Hom}(E, X) \neq 0$, then $E[\infty]$ belongs to $\mathcal{X}$. Second, the dual condition, if E is a simple regular R-module and if there are infinitely many finite length modules $X \in \mathcal{X}$ with $\mathrm{Hom}(X, E) \neq 0$, then $\widehat{E}$ belongs to $\mathcal{X}$. And third, if there are infinitely many finite length modules in $\mathcal{X}$ or if there exists at least one module in $\mathcal{X}$ which is not of finite length, then the generic module of infinite length belongs to $\mathcal{X}$.

We see that the Ziegler closed subsets of $\mathcal{Z}(R)$ are related to the support

[25] For the notion of divisibility as well as the structure of products of preprojective modules, we refer to [R3], sections 5 und 2, respectively.

of the functors $\operatorname{Hom}(E,-)$ and $\operatorname{Hom}(-,E)$, where E is simple regular. These functors play an important role for many questions in the representation theory of tame hereditary algebras, for example for one-point extensions and coextensions.



String algebras. We are going to describe the structure of some additional indecomposable algebraically compact modules. They are similar to the Prüfer modules and the adic modules, or to pairs of such modules. The algebras we will consider are string algebras, this is a class of tame algebras whose indecomposable modules can be constructed with bare hands. We are going to outline how to determine all the indecomposable algebraically compact modules for a string algebra R; in case R is domestic, we will give a complete description of these modules.

We recall that for a connected tame hereditary algebra[26] the indecomposable algebraically compact modules which are not of finite length are the Prüfer modules, the adic modules and one endofinite module of infinite length. These kinds of modules do also exist for any string algebra, more precisely: for any primitive cyclic word (or better: its equivalence class) there are Prüfer modules, adic modules and one endofinite module of infinite length, but there may be additional ones which are built up using Prüfer modules, adic modules and finite dimensional ones. For the almost periodic $\mathbb{N}$-words, one Prüfer module or one adic module is used, for the biperiodic $\mathbb{Z}$-words two such modules are used, and all three possible combinations occur. Of special interest seems to be the mixed case which involves at the same time a Prüfer module and an adic module.

Let us start with the case of R being domestic, even 1-domestic. First, let us mention an easy way for constructing some 1-domestic string algebras (following [R7]), this should help to illustrate some of the phenomena occurring for representation-infinite algebras. We start with a quiver Q' of type $\widetilde{A}_{n-1}$, thus there are n vertices, say labeled by the integers modulo n, and n arrows α_i, with $0 \le i < n$, such that $\{s(\alpha_i), t(\alpha_i)\} = \{i-1, i\}$. We assume that n is even and we assume that there is given also a fix point free involution $\omega\colon i \mapsto i'$ on the set Q'_0 of vertices of Q'. We form the algebra $R(Q',\omega) = kQ\langle\rho_0,\dots,\rho\rangle$, where the quiver Q is obtained from Q' by adding vertices a_ι for any ω-orbit ι of Q'_0 and arrows β_i with $\{s(\beta_i), t(\beta_i)\} = \{i, a_\iota\}$, for any $0 \le i < n$ with $i \in \iota$. The orientation of β_i is chosen so that the arrows α_i and β_i can be composed in order to form a path labeled ρ_i; thus, if $s(\alpha_i) = i$, let $t(\beta_i) = i$, and let $\rho_i = \alpha_i\beta_i$; similarly, if $t(\alpha_i) = i$, let $s(\beta_i) = i$, and let $\rho_i = \beta_i\alpha_i$. As an example, consider the quiver Q' of type $\widetilde{A}_5$ exhibited above and the map $\omega\colon Q'_0 \to Q'_0$ which exchanges 1 and 2; and 3 and 5; and 4 and 0. We obtain the following quiver Q, the dotted lines

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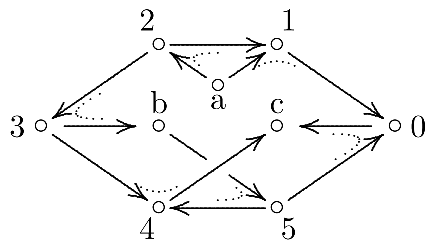
[26] and similarly for a Dedekind ring.

indicate the relations:



here, $a = a_{\{1,2\}}, b = a_{\{3,5\}}$ and $c = a_{\{0,4\}}$, and the six new arrows are those labeled β_i. This algebra $R(Q', \omega)$ is a string algebra [BuR], thus it is easy to list all the indecomposable modules of finite length. Since there is (up to equivalence) only one primitive cyclic word, namely $w = \alpha_1\alpha_2\alpha_3^{-1}\alpha_4^{-1}\alpha_5\alpha_6^{-1}$, we see that $R(Q', \omega)$ is 1-domestic. As an intuitive description of the new algebra $R(Q', \omega)$ we may say that we have added some "bridges" to the quiver Q' (connecting a vertex i with its image ωi), these bridges correspond to the orbits of ω.

Of course, the full subcategory of all kQ'-modules in mod $R(Q', \omega)$ is known, and it consists of all representations M of Q for which all the vector spaces M_{a_ι} are zero.

Let us describe some interesting $R(Q', \omega)$-modules. Recall the following: A string algebra has two kinds of finite-dimensional indecomposable modules, the string modules and the band modules. In our case, all the band modules are actually kQ'-modules, thus let us concentrate on string modules. They are obtained by choosing a (finite) walk w in the quiver Q, this is a word $w = l_1l_2 \dots l_n$, where we use as letters l_i the arrows and their formal inverses, subject to the requirement that never an arrow and its inverse are neighbors, and that consecutive arrows are composable. The kQ-module $M(w)$ is given by a k-space (here of dimension $n+1$, since w is supposed to have length n), and the word w describes the operation of the arrows of Q on this space. In order to obtain an $R(Q', \omega)$-module, we have to require in addition that w avoids the relations ρ_i; this means that these relations do not occur as a subword or as the inverse of a subword of w. We also may work with words using as letters the vertices of Q, say replacing the sequence $l_1l_2 \dots l_n$ by the sequence $t(l_1)t(l_2)\dots t(l_n)s(l_n)$, provided this does not lead to confusion. This has the advantage that now the letters of the word w correspond bijectively to basis elements of $M(w)$.

In order to construct infinite-dimensional indecomposable modules, we will use $\mathbb{N}$-words $w = l_0l_2 \dots l_n \dots$ or $\mathbb{Z}$-words $\dots l_{-1}l_0l_1l_2 \dots$ (here $\mathbb{N}$ and $\mathbb{Z}$ refer to the sets of indices used). For example, for the algebra R exhibited above, there are precisely six $\mathbb{Z}$-words, namely the words

$$z(a) = {}^{\infty}(105432)\, a\, (123450)^{\infty},$$
$$z(b) = {}^{\infty}(210543)\, b\, (501234)^{\infty},$$
$$z(c) = {}^{\infty}(321054)\, c\, (012345)^{\infty},$$

and their inverses[27]. All the possible $\mathbb{N}$-words occur as subwords of these $\mathbb{Z}$-words. As for finite words, we can attach to every $\mathbb{Z}$-word or $\mathbb{N}$-word w a corresponding string module $M(w)$. In addition, we also may consider suitable completions of $M(w)$, see [R8], in order to obtain algebraically compact modules.

To be more precise, consider any $\mathbb{N}$-word $x = l_1 l_2 \dots$, where the letters l_i are arrows or inverses of arrows. The string module $M(x)$ is constructed as follows: take a (countably dimensional) vector space with basis $e_0, e_1, \dots$ and let R act according to the word x: in case the letter l_i is an arrow, say $l_1 = \alpha$, then define $\alpha(e_i) = e_{i-1}$, otherwise l_i is the inverse of an arrow, say $l_i = \alpha^{-1}$, then let $\alpha(e_{i-1}) = e_i$. In this way, we have defined the action of some arrows on some basis elements, the action of arrows on all other basis elements should be zero. Besides $M(x) = \bigoplus ke_i$, we also consider the product $\overline{M}(x) = \prod ke_i$ with a similarly defined action. It is well-known that the string modules $M(x)$ are indecomposable, and it is obvious that the modules $\overline{M}(x)$ are algebraically compact. Observe that at most one of the two modules can be both indecomposable and algebraically compact: The embedding $\iota\colon M(x) \to \overline{M}(x)$ is a pure embedding, thus, if $M(x)$ is algebraically compact, then ι splits and $\overline{M}(x)$ cannot be indecomposable.

Let us assume that x is almost periodic (this means that $x = x'w^\infty$, where x' and w are finite words). We claim that in this case, one of the two modules is both indecomposable and algebraically compact, and thus this is the module $C(x)$ we are interested in. An almost periodic $\mathbb{N}$-word x is either contracting or expanding, using the terminology of [R8]. There, it has been shown that for x contracting, the string module $M(x)$ is algebraically compact. But it is not difficult to see that for x expanding, the module $\overline{M}(x)$ is indecomposable[28].

Let us now consider $\mathbb{Z}$-words. In our example, the module $C(z(c))$ is constructed as follows: Again, we start with the string module $M(z(c)) = \bigoplus_{i\in\mathbb{Z}} ke_i$, where e_i, $i \in \mathbb{Z}$ is the defining basis following the letters of the $\mathbb{Z}$-word $z(c)$. Now, take the corresponding product module $C(z(c)) = \prod_{i\in\mathbb{Z}} ke_i$. This module $C(z(c))$ still has the simple module $S(c)$ as a submodule, and $C(z(c))/S(c)$ is the direct sum of two adic kQ'-modules, namely those corresponding to the simple regular modules E_2 and E_1. Second, consider the module $C(z(b))$, it is obtained from $M(z(b))$ by a partial completion ("on the left"), so that $C(z(b))$ has submodules $N' \subset N$, where N is the Prüfer kQ'-module with regular socle E'_2,

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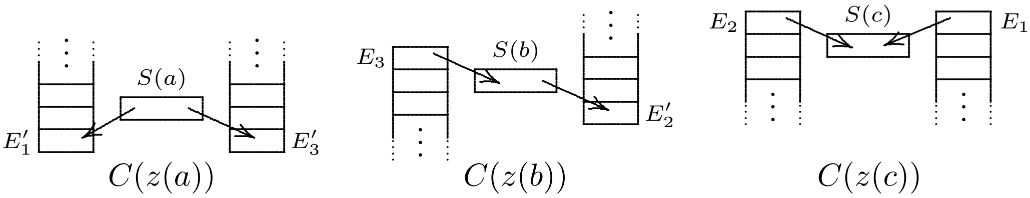
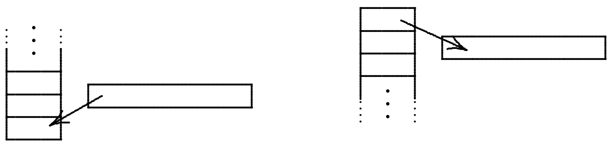
[27] If u, v, w are finite words, we write vw^∞ instead of $vww\cdots$ and ${}^\infty uv$ instead of $\cdots uuv$. For quite obvious reasons we say that vw^∞ as well as the inverse of ${}^\infty uv$ are "almost periodic $\mathbb{N}$-words".

[28] For any j, we consider the subspace $C_j = \prod_{i\neq j} ke_i$ of $\overline{M}(x)$. Note that the intersection $\bigcap_j C_j$ is zero. We use the functorial filtration of the forgetful functor F as in [R1]. The word x defines a sequence of intervals $G_i \subset F_i$ of subfunctors of F. Now suppose there is given a direct decomposition $\overline{M}(x) = N \oplus N'$. Clearly, $(F_0/G_0)(\overline{M}(x))$ is one-dimensional, thus we may assume that $(F_0/G_0)(N') = 0$, and therefore $G_0(N') \subseteq C_0$. Inductively one shows that $G_i(N') \subseteq C_i$, but this implies that $N' = 0$, thus $\overline{M}(x)$ is indecomposable.

$N/N' = S(b)$ and $C(z(b))/N$ is an adic module with regular top E_3. And finally, consider $C(z(a)) = M(z(a))$. In this case $S(a)$ occurs as a factor module, the corresponding maximal submodule is the direct sum of two Prüfer modules with regular socles E'_1, E'_3. Here are some schematic pictures:



Similar pictures can be drawn in the case of an almost periodic $\mathbb{N}$-word x; to the left, we exhibit the case where x is contracting (there is a submodule which is a Prüfer module such that the factor module is indecomposable and of finite length), to the right, x is expanding (there is an indecomposable submodule of finite length such that the factor module is an adic module):



If R is a string algebra, then all the indecomposable algebraically compact modules can be determined and one can show that the only algebraically compact module which is superdecomposable is the zero module [R14]. For the proof, one determines R-modules M such that the functor $(- \otimes M)$ is $\mathbb{N}$-uniform; this implies that the pure injective envelope $\overline{M(z)}$ of $M(z)$ is indecomposable. *If z is a $\mathbb{Z}$-word which has no expanding end, then the functor $(- \otimes M(z))$ is $\mathbb{N}$-uniform.* For the proof, write $z = xy$, and further $x = \cdots x_2x_1x_0$, where the x_i are finite words with last letter being an arrow, and $y = y_0y_1y_2\cdots$, where the first letter of y_i is the inverse of an arrow. Observe that for all i, the two modules $N'_i = M(xy_0\cdots y_i)$ and $N''_i = M(x_i\cdots x_0y)$ are factor modules of $M = M(z)$. Let $g_i\colon M \to N_i = N'_i \oplus N''_i$ be given by the canonical maps. One has to check that for any simple submodule S of M, the canonical projection $f\colon M \to M/S$ can be factored through one of the maps g_i, and also that any inclusion map $f\colon M \to N$ whose cokernel is finite-dimensional and indecomposable can be factored through some g_i. Similarly, one shows that *if y is a non-expanding $\mathbb{N}$-word, then the functor $(- \otimes M(y))$ is $\mathbb{N}$-uniform.*

5. Hammocks and quilts.

Hammocks. As mentioned above, the hammocks have been introduced in the realm of representation-finite algebras R by S. Brenner in order to obtain a combinatorial characterization of the translation quivers which occur as Auslander-Reiten quivers. The word "hammock" describes in a very intuitive way the shape of the representable functors $\mathrm{Hom}(-, I(S)) \simeq D\,\mathrm{Hom}(P(S), -)$, where S is a simple R-module, $I(S)$ its injective envelope, $P(S)$ its projective cover. One can attach to such a functor a translation quiver $\Gamma(S)$ whose vertices are equivalence classes of non-zero maps from $P(S)$ to indecomposable R-modules M, thus equivalence classes of composition factors of indecomposable R-modules which are isomorphic to S. These translation quivers $\Gamma(S)$ have a unique source (corresponding to the top of $P(S)$) and a unique sink (corresponding to the socle of $I(S)$); in addition, there is a function $\Gamma(S)_0 \to \mathbb{N}_1$, the hammock function, which plays a role: it counts the number of composition factors of the form S in suitable layers of the module in question; this function is additive on meshes. An axiomatic treatment of such translation quivers with hammock functions has been given in [RV], and there it has been shown that one obtains in this way precisely the Auslander-Reiten quivers of the categories of Ω-spaces[29], where Ω is a finite poset. Altogether the hammock philosophy uses three different approaches: a functorial one, dealing with functors which are both projective and injective, a combinatorial one, dealing with translation quivers and additive functions, and a linear one, dealing with subspace configurations in vector spaces.

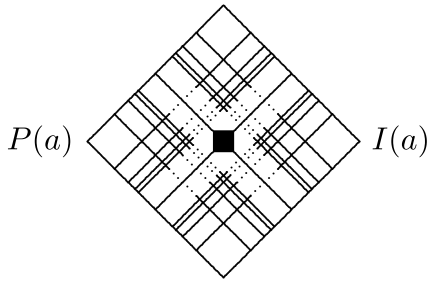
An extension of these considerations to finite-dimensional algebras which are representation-infinite is needed. Of course, there is a strong interest to be able to extend all three approaches, but it seems clear that at present a combinatorial procedure can serve only as an auxiliary device: in dealing with representation-infinite algebras one cannot avoid to take the infinite radical rad^{ω} into account and too little is known about the possibilities to handle it combinatorially. The main problem presently concerns the question under what conditions hammock functors $\mathrm{Hom}(-, I(S)) \simeq D\,\mathrm{Hom}(P(S), -)$ can be desribed in terms of S-space categories or related categories.

Note that these hammock functors are objects in a diamond category. To deal with objects in a diamond category has to be rated as a strong finiteness condition: such objects may be arbitrarily large, but the local structure of their subobject lattices is coined by their finite subfactors.

Tiles. Let us return to the algebras $R = R(Q', \omega)$ obtained from a quiver of type $\widetilde{A}_{n-1}$ by adding bridges. We consider in more detail whole parts of the category of all R-modules. Let a be a label of such a bridge, let $S(a)$ be the corresponding simple module. Let us consider all the words (finite words, as well as $\mathbb{N}$-words and $\mathbb{Z}$-words) w which contain the fixed letter a. For every such word,

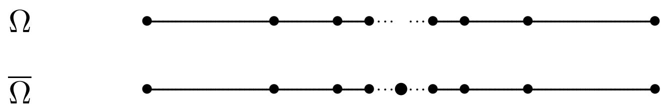
[29] Given a poset Ω, an Ω-space $(V; V_s)_s$ is given by a k-space V and subspaces V_s of V indexed by the elements $s \in \Omega$, such that for $s \le s'$ one has $V_s \subseteq V_{s'}$. By definition, the dimension of $(V; V_s)_s$ is that of V. Given two Ω-spaces $(V; V_s)_s$ and $(W; W_s)_s$, a map $f : (V; V_s)_s \to (W; W_s)_s$ is given by a k-linear map $f \colon V \to W$ such that $f(V_s) \subseteq W_s$ for all $s \in \Omega$. The category of all Ω-spaces is an exact category. In case Ω is finite, the category of finite-dimensional Ω-spaces has Auslander-Reiten sequences.

there is given an indecomposable algebraically compact module $C(w)$ (for finite words, $C(w) = M(w)$, otherwise $C(w)$ may be a proper completion of $M(w)$). Given two such words w, w', write $w \le w'$ provided there exists a homomorphism $f\colon C(w) \to C(w')$ with $f_a \neq 0$. We obtain a rectangle (a *tile*) of the following form (here, small elements with respect to the ordering are on the left, large ones on the right), and we denote it by $\mathcal{T}(a)$:



There is a smallest element, namely the word w with $C(w) = P(a)$, the projective cover of $S(a)$, and a largest element, the word w with $C(w) = I(a)$, the injective envelope of $S(a)$. The remaining two corners of the rectangle are given by suitable serial modules. The simple module $S(a)$ occurs in one of the four corners: if a is a sink, then $S(a) = P(a)$; if it is a source, then $S(a) = I(a)$; otherwise $S(a)$ is one of the other two corners. The center of this rectangle is the unique $\mathbb{Z}$-word which contains the letter a, on the two diagonals through the center, we have in addition just all the corresponding $\mathbb{N}$-words.

Given two posets Ω', Ω'', write $\Omega' \sqcup \Omega''$ for the disjoint union of these posets and $\Omega' \times \Omega''$ for the product. Instead of $\Omega' \sqcup \Omega'$ we also write $2\Omega'$. In addition, we also need the ordered sum $\Omega' \triangleleft \Omega''$, it is obtained from the disjoint union of Ω' and Ω'' by adding the relations $s' < s''$ for all $s' \in \Omega', s'' \in \Omega''$. When we consider subsets of $\mathbb{Z}$ as posets, then we use the natural ordering. Let $\Omega = \mathbb{N} \triangleleft (-\mathbb{N})$ and consider also its completion $\overline{\Omega} = \mathbb{N} \triangleleft \{*\} \triangleleft (-\mathbb{N})$:

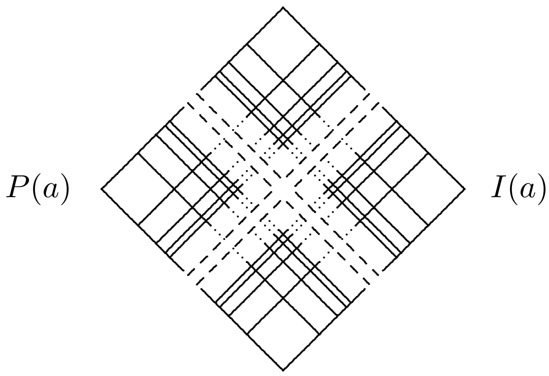


We may describe $\mathcal{T}(a)$ as the product $\overline{\Omega} \times \overline{\Omega}$ and, as Dräxler has pointed out, this product is the just Auslander-Reiten quiver $\Gamma_{2\Omega}$ of the category of all (2Ω)-spaces, thus

$$\mathcal{T}(a) = \overline{\Omega} \times \overline{\Omega} = \Gamma_{2\Omega}.$$

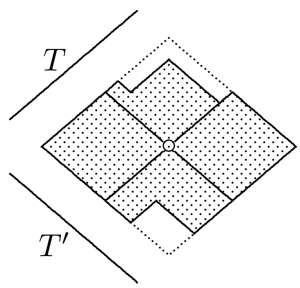
Indeed, the category which we consider when dealing with $\mathcal{T}(a)$ can be identified with the category of all (2Ω)-spaces: Let $I(a)$ be the ideal of $\operatorname{Mod} R$ of all maps f with $f_a = 0$. Then $(\operatorname{Mod} R)/I(a)$ is the category in question and it is equivalent to the category of (2Ω)-spaces. On the other hand, all the indecomposable (2Ω)-spaces are one-dimensional. The following fact should be stressed: Whereas some of the vertices of $\mathcal{T}(a)$ are infinite words, thus correspond to infinite-dimensional R-modules, the (2Ω)-spaces which are the vertices of $\Gamma_{2\Omega}$ are one-dimensional.

Hammock functors. Let us remove from $\mathcal{T}(a)$ for a while the modules of infinite length. What we obtain in this way is:



and this is just the product $\Omega \times \Omega$. On the other hand, this clearly describes the hammock functor $\operatorname{Hom}(-, I(S))$ which corresponds to the simple module $S = S(a)$.

In general, Schröer [Sc1] has described the structure of such hammock functors for all simple modules S over string algebras. The description is rather easy if one restricts the attention just to string modules. This part $\mathcal{H}(S)_0$ of the hammock functor can be visualized by a subset of the product $T \times T'$ of two totally ordered sets T, T' which are similar, but usually more complicated than $\mathbb{N} \triangleleft (-\mathbb{N})$.

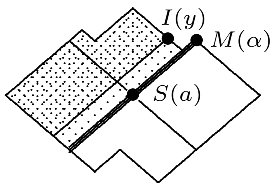


The chains T, T' are related to those introduced by Gelfand and Ponomarev ([GP], see also [R1]) in order to identify indecomposable modules by using functorial filtrations, but here we take into account only string modules. As before, the ordering $\leq$ goes from left to right; the unique minimal element is the projective cover of S, the unique maximal element the injective envelope of S. Note that the center is given by the module S itself. The position of S is important for describing the subset of $\big(T \times T'\big)$ which corresponds to $\mathcal{H}(S)_0$, this subset is determined by the (finite!) set of words w such that $M(w)$ is serial, and is contained in the two quarters which contain elements which are incomparable with S. Let us stress that in general the structure of the chains T, T' is quite complicated, whereas it is easy to locate $\mathcal{H}(S)_0$ inside $T \times T'$.

How can we recover T and T' from the hammock $\mathcal{H}(S)$? Let us determine a subfactor as follows: Let $S = S(a)$, and suppose there are given arrows $x \xrightarrow{\alpha} a \xrightarrow{\alpha'} y$ such that $\alpha'\alpha = 0$. These arrows yield maps

$$I(x) \xleftarrow{I(\alpha)} I(a) \xleftarrow{I(\alpha')} I(y)$$

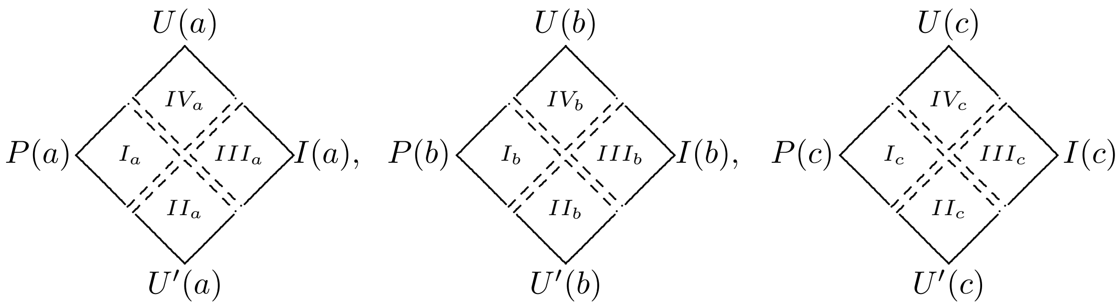
with zero composition. Let us denote by $M(\alpha)$ the kernel of $I(\alpha)$. The zero composition gives a map $I(y) \to M(\alpha)$, and we are interested in the cokernel $\operatorname{Hom}(-, M(\alpha))/\operatorname{Hom}(-, I(y))$. This is the functor which produces just one of the crossing lines through $S(a)$. Here is its support:



The support of $\operatorname{Hom}(-, M(\alpha))$ are the shaded parts, that of $\operatorname{Hom}(-, I(y))$ is shaded more heavily. As factor $F = \operatorname{Hom}(-, M(\alpha))/\operatorname{Hom}(-, I(y))$, there remains the bold line. This functor F is a serial functor. For a general discussion of serial subfactors of representable functors in the case of a string algebra we refer to [Sc2] and [PSc].

Quarters and Auslander-Reiten components. We return to the 1-domestic algebras $R = R(Q', \omega)$. As we have removed the modules of infinite length from a tile $\mathcal{T}(a)$, four connected parts (*quarters*) remain and these are actually parts of the usual Auslander-Reiten quiver Γ_R (since all the small rectangles occurring in $\mathcal{T}(S)$ are just usual meshes in Γ_R).

Let us consider the special case of the $\widetilde{A}_5$-quiver and the choice of ω as discussed above in greater detail. There are three tiles $\mathcal{T}(a), \mathcal{T}(b), \mathcal{T}(c)$ and they give rise to altogether 12 quarters. We label them as follows and indicate always the corresponding corner modules:

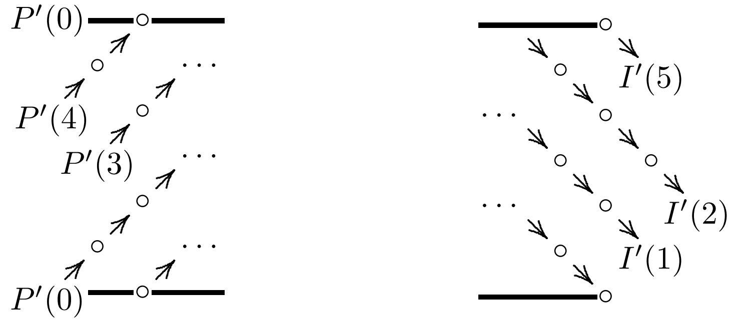


The modules labeled $U(-), U'(-)$ all are serial, they are determined by their composition factors: Here is the list of the factors:

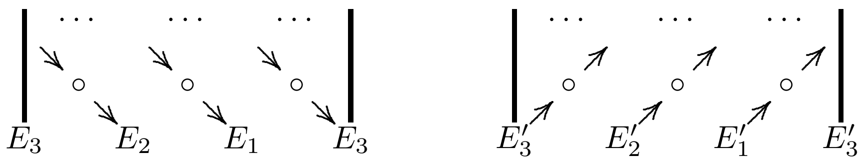
$U(a)$	$U'(a)$	$U(b)$	$U'(b)$	$U(c)$	$U'(c)$
$a, 2, 3, 4$	$a, 1$	b	$3, b, 5, 0$	$5, 4, c$	$2, 1, 0, c$

In order to describe Γ_R, we have to see how these quarters as well as the kQ'-modules which are string modules have to be fitted together. The four components of $\Gamma_{kQ'}$ which contain string modules have to be cut into rays or corays and then we have to rearrange these pieces. Let us consider in more detail our special example.

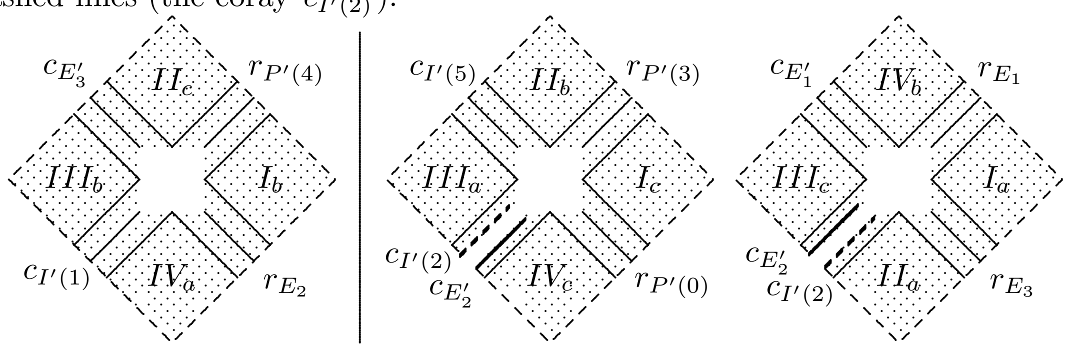
We cut the preprojective component in rays starting at the kQ'-projective modules $P'(0), P'(3), P'(4)$, such that the corresponding cokernels are of the form E_1, E_2, E_3. Similarly, we cut the preinjective component in corays ending at $I'(1)$, $I'(2), I'(5)$ such that the corresponding kernels are of the form E_1', E_2', E_3'. Here are these rays and corays:



The two exceptional tubes also have to be cut; the one containing E_1, E_2, E_3 into corays, the other one into rays:

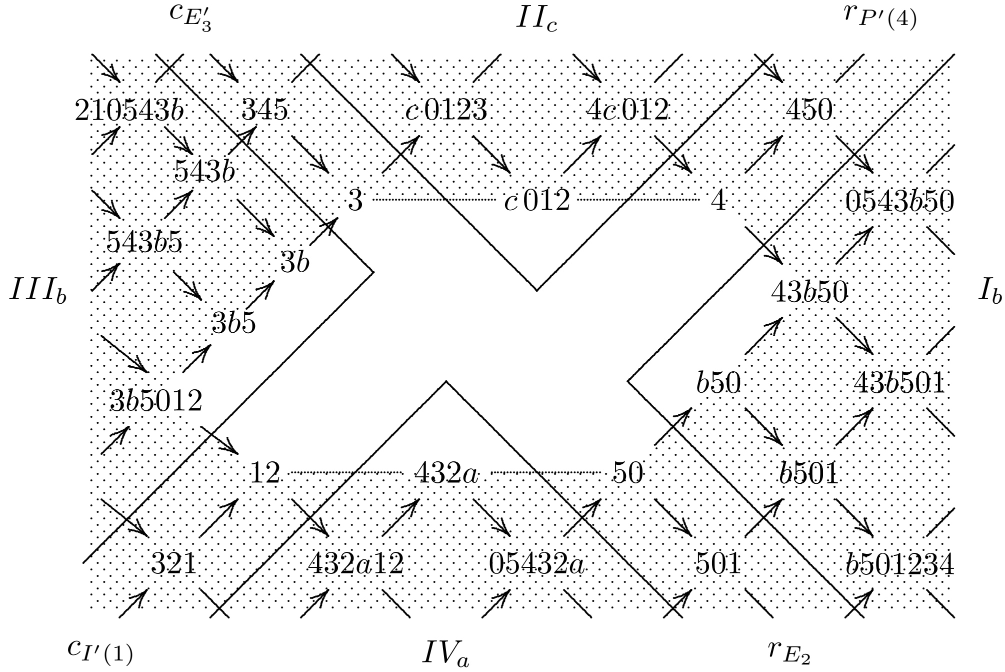


Altogether, we deal with the 12 quarters obtained from the three tiles $\mathcal{T}(a)$, $\mathcal{T}(b)$ and $\mathcal{T}(c)$, and in addition with 6 rays and 6 corays; if necessary, a fixed ray starting at a module M will be denoted by r_M, a coray ending in M by c_M. It is not difficult to see how these pieces fit together and that they yield two components $\mathcal{C}_1, \mathcal{C}_2$ of the Auslander-Reiten quiver Γ_R; the first of the following three pictures shows the structure of one of these components, the remaining two pictures still have to be put together in order to obtain the second component, this has to be done in the same way as one constructs the Riemann surface of the square root: one has to identify the bold solid lines (this is the coray $c_{E_2'}$) as well as the bold dashed lines (the coray $c_{I'(2)}$).



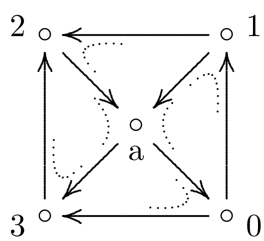
Topologically, both components are surfaces with a boundary; the left one is a punched plane, the right one is a twofold covering of a punched plane.

Note that the construction of non-stable components of a string algebra is easy, since we know all the modules lying on the boundary of these components [BuR]. Here is the boundary region of the first of the two components (the punched plane) exhibited above, it shows in which way rays, corays and quarters may be connected:



It seems to be of interest to compare this procedure of joining rays, corays and quarters with the well-understood process of ray insertions and coray insertions in tubes. For example, to deal with a ray insertion means to cut between two rays to insert there a new ray. Here we also cut between rays and we cut between corays, but we insert not rays or corays but quarters, for example the quarter I_b in between the two rays $r_{P'(4)}$ and r_{E_2} (but these are rays of quite different nature: the second is a ray coming from a tube, whereas the first one comes from a preprojective component), the quarter IV_a in between the ray r_{E_2} and the coray $c_{I'(1)}$ and so on. A clear recipe for this procedure of quarter insertion is not yet known.

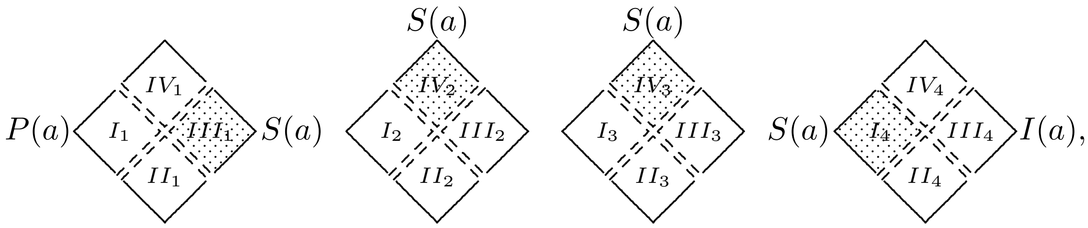
In the example, the only stable components of Γ_R were tubes. A minor modification allows to produce also stable components of the form $\mathbb{Z}A_\infty^\infty$. Consider the following algebra



There are (up to inversion) four $\mathbb{Z}$-words:

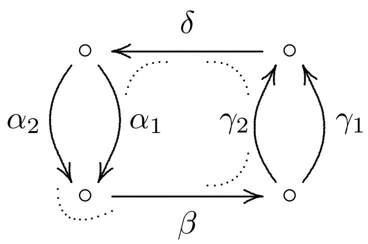
$$z_1 = {}^{\infty}(2103)\, a\, (0123)^{\infty}, \quad z_2 = {}^{\infty}(1032)\, a\, (0123)^{\infty},$$
$$z_3 = {}^{\infty}(2103)\, a\, (1230)^{\infty}, \quad z_4 = {}^{\infty}(1032)\, a\, (1230)^{\infty}.$$

For any of these $\mathbb{Z}$-words z_i, we may consider all the (finite or infinite) subwords which contain the letter a, what we obtain in this way are again tiles. And, deleting the infinite-dimensional modules from the tiles, every tile is cut into four quarters:



We see that the four quarters containing the module $S(a)$, namely III_1, IV_2, IV_3 and I_4, fit together and form an Auslander-Reiten component of the form $\mathbb{Z}A_\infty^\infty$. In general [BuR], stable components of a string algebra are either tubes or of the form $\mathbb{Z}A_\infty^\infty$, and Geiß [G] has shown that a component of the form $\mathbb{Z}A_\infty^\infty$ always contains a unique module of smallest length[30]. Of course, in our example $S(a)$ is such a Geiß module.

Note that a *1-domestic string algebra has only finitely many $\mathbb{Z}$-words,* thus only finitely many tiles and only finitely many components of the form $\mathbb{Z}A_\infty^\infty$. (Proof: write such a word in the form $(w^\infty)^{-1}uw^\infty$ with u minimal, where w is a fixed primitive cyclic word. Then there are only finitely many possibilities for u.) On the other hand, it is easy to construct examples of 2-domestic string algebras with infinitely many $\mathbb{Z}$-words, for example:



[30] A finite word w will be called a Geiß word provided there do exist $\mathbb{N}$-words x_1, x_2 which start with a direct letter and $\mathbb{N}$-words y_1, y_2 which start with an inverse letter such that $y_1^{-1}wx_1$ and $x_2^{-1}wy_2$ are $\mathbb{Z}$-words. If w is a Geiß word, then the string module $M(w)$ is contained in a component of the form $\mathbb{Z}A_\infty^\infty$ and all the other modules in this component have larger length. Also the converse is true: if $M(w)$ is a module of smallest length in a component of the form $\mathbb{Z}A_\infty^\infty$, then w is a Geiß word. Note that in case w contains both direct and inverse letters, then it is sufficient to require the existence of $\mathbb{N}$-words of the form $wx_1, w^{-1}y_1, w^{-1}x_2$ and wy_2, such that x_1, x_2 start with a direct letter and y_1, y_2 with an inverse letter.

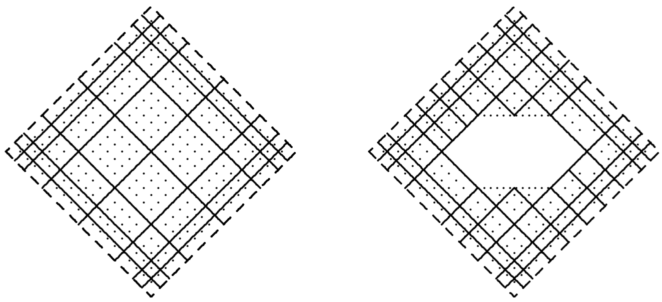
Up to inversion, the $\mathbb{Z}$-words are of the form

$$z_n = {}^{\infty}\big(\alpha_1^{-1}\alpha_2\big)\,\delta\,\gamma_1\big(\gamma_2^{-1}\gamma_1\big)^n\beta\,\big(\alpha_1\alpha_2^{-1}\big)^{\infty}$$

for $n \geq 0$. Note that by adding bridges to the cycle $\gamma_2^{-1}\gamma_1$ we will obtain 2-domestic algebras which have even more $\mathbb{Z}$-words.

Quilts. Recall that we have removed the infinite-dimensional modules from the tiles $\mathcal{T}(a)$, $\mathcal{T}(b)$ and $\mathcal{T}(c)$ in order to obtain pieces of Γ_R. Of course, we may reinsert them. What we obtain in this way is a topological space which is connected and compact: a very nice compactification of the union $\mathcal{C}_1 \sqcup \mathcal{C}_2$.

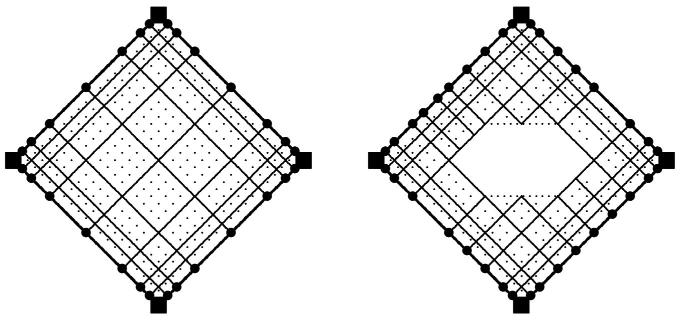
Let us analyze this compactification process further: We start with a planar component of a string algebra, say either with a component of the form $\mathbb{Z}A_\infty^\infty$ or with a punched plane (similar considerations apply in the case of a finite covering of a punched plane). The usual rule for drawing components is that all meshes should have equal size (in the case of components with holes a slight squeezing may be necessary). However, the focus of such a visualization is the central part of the component and our aim is to understand also the relationship to other components. Thus, the first step is a very innocent one: to change the metric in such a way that the component fits into a finite region. We may think of a planar component as a square or a lozenge shape, standing on one of its corners:



To the left, you see the case of a $\mathbb{Z}A_\infty^\infty$ component (the center is its Geiß module, the uniquely determined module of smallest length), to the right, the punched plane component considered before. This process of **shrinking** has mainly a psychological meaning[31], but it draws our attention to the endpoints of the rays and corays and to the four corners.

[31] A mathematical interpretation of such a metric could be as follows: fix a real number λ with $0 < \lambda < 1$ and define the length of an arrow $\alpha\colon [X] \to [Y]$ by $\sigma(\alpha) = \lambda^{|X|+|Y|}$. Then the length of a ray $[X_0] \to [X_1] \to [X_2] \to \cdots$ (and similarly that of a coray) will be bounded. When applied to a tube, such a metric realizes it as a punched disk, and it can be used for the following consideration: Call a cyclic walk *inessential* provided it is homotopic to walks of arbitrarily small length (with respect to this metric). Given a tame hereditary algebra R, all cyclic walks in the Auslander-Reiten quiver Γ_R are inessential.

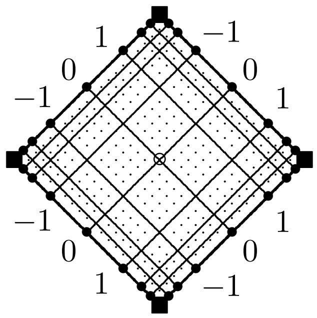
Now comes the second step of **compactifying** these components by adding the boundary. We can formulate this purely combinatorially as follows: recall that the components of Γ_R we deal with have as vertices string modules $M(w)$, or, equivalently, just the strings w, and the arrows correspond to the operations on words of adding hooks or deleting cohooks (see [BuR]). Now, we also take into account $\mathbb{N}$-words (the small bullets on the four sides) and $\mathbb{Z}$-words (the black squares at the corners).



Recall that a countable sequence of arrows $w_0 \to w_1 \to w_2 \to \cdots$ is called a *ray*, provided $\tau w_{i+1} \neq w_{i-1}$, for all $i \geq 1$. Two such sequences are called *equivalent* provided they only differ by finitely many arrows. We may assume that we work with words w_i such that w_{i+1} is obtained from w_i by a change on the right hand side, for all i, and then x is constructed as follows: for every natural number n, almost all the words w_i will be of the form $w_i = x_n w_{in}$, where x_n is a fixed word of length n, and then the first n letters of x will just be x_n. In case no vertex w_i is injective, we may form the sequence $w'_1 \to w'_2 \to w'_3 \to \cdots$ where $w'_i = \tau^{-1} w_{i-1}$. This is again a ray, let x' be the corresponding $\mathbb{N}$-word. Note that there are arrows $w_i \to w'_i$ and these arrows assert that w'_i is obtained from w_i by a change on the left (addition of a hook or deletion of a cohook). For all indices i, the same change on the left occurs, and also x' is obtained from x by this change on the left, thus we will draw an arrow $x \to x'$. For every component, we obtain in this way two sequences of consecutive $\mathbb{N}$-words x_i (where $i \in \mathbb{Z}$) and arrows $x_i \to x_{i+1}$. There is the dual procedure for obtaining an $\mathbb{N}$-word for any equivalence class of corays (the dual of a ray), which again is almost periodic, and also arrows between these $\mathbb{N}$-words. Again, we obtain two sequences of consecutive $\mathbb{N}$-words x_i (where $i \in \mathbb{Z}$) and arrows $x_i \to x_{i+1}$; altogether we obtain in this way the four boundary lines of our lozenge. Finally, note that these procedures combine and yield four biperiodic $\mathbb{Z}$-words, corresponding to the corners of the lozenge.

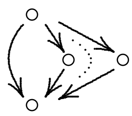
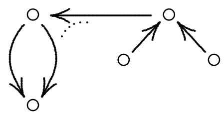
Here is the module theoretical interpretation: The vertices we have added are almost periodic $\mathbb{N}$-words x and biperiodic $\mathbb{Z}$-words z. In [R8], we have constructed corresponding indecomposable algebraically compact modules $C(x)$ and $C(z)$. Also, the new arrows $x_i \to x_{i+1}$ between $\mathbb{N}$-words indicate that x_{i+1} is obtained from x_i by adding a hook or deleting a cohook, and this corresponds to an irreducible map $C(x_i) \to C(x_{i+1})$.

In the case of a $\mathbb{Z}A_\infty^\infty$ component, the Geiß module may be considered as the origin, in case we need to index the modules. Note that any boundary line contains precisely one word which starts with a Geiß word, and it will be convenient to use the index 0 for this $\mathbb{N}$-word:

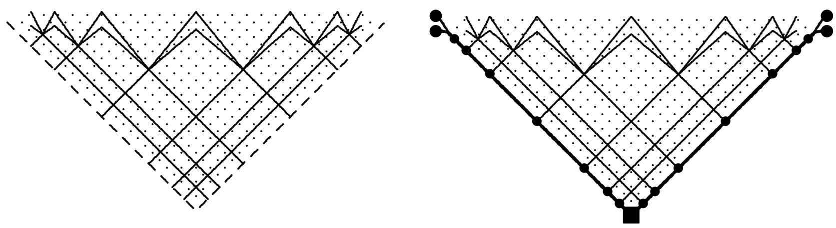


A similar process of shrinking and compactifying can be achieved in the case of components of the form $\mathbb{Z}D_\infty$. Such components do not occur for string algebras but they do for a related class of algebras, the so called clan algebras[32].

[32] Here are two typical such algebras:

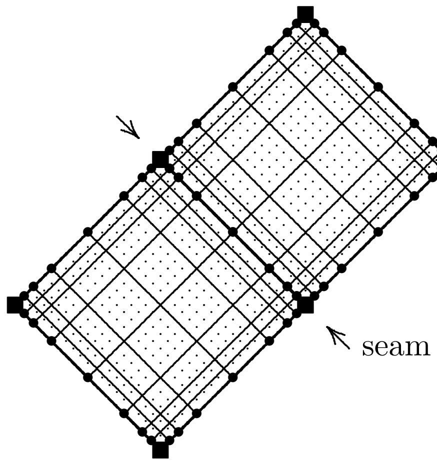
 and 

(the relation on the left is a commutativity relation). Let us exhibit a $\mathbb{Z}D_\infty$ component and its compactification: As for the $\mathbb{Z}A_\infty^\infty$ components, the boundary lines will be given by $\mathbb{N}$-words, the corners by $\mathbb{Z}$-words. However, at least two of these $\mathbb{Z}$-words yield decomposable modules, namely modules which are the direct sums of two indecomposable modules, thus we have inserted pairs $\substack{\bullet \\ \bullet}$ of bullets:



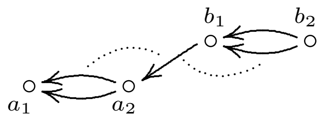
Many components of this form arise for the algebra above depicted left; for example the component containing the two simple modules which are neither projective nor injective is of this kind. In general, the third corner of the compactification (indicated by the black square ■) may also correspond to a decomposable module; in this case, it is more appropriate to replace also ■ by a pair of bullets $\substack{\bullet \\ \bullet}$. This happens for the algebra depicted right, namely for the component which contains the two simple injective modules; of course, this is a non-regular component, it is similar to a $\mathbb{Z}D_\infty$ component with a boundary part being missing — in the same way as our punched plane was similar to a $\mathbb{Z}A_\infty^\infty$ component.

Sewing. It may happen that the same boundary line (the same sequence $(x_i)_i$ of $\mathbb{N}$-words) is obtained twice: once by dealing with rays, the second time by dealing with corays. Then this boundary line can be considered as a **sewing** together of the two components (or of two different sides of one component):

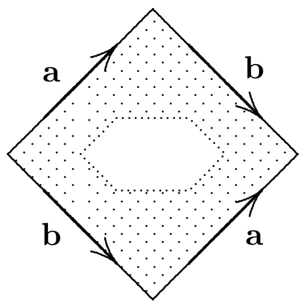


What we obtain in this way, by sewing together various components of a string algebra R may be called the *Auslander-Reiten quilt* of R. Let us stress that the sewing together is achieved by the infinite length modules which have been added.

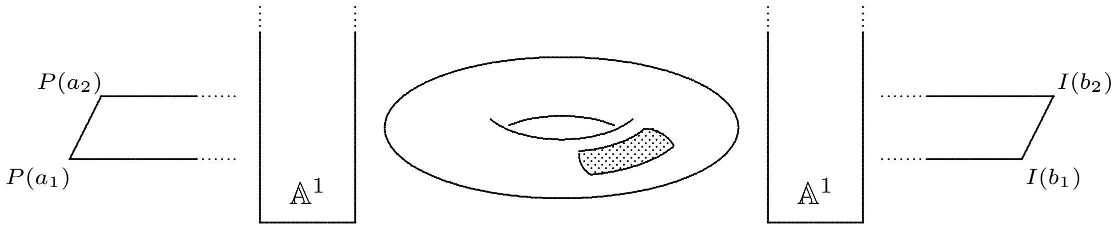
Let us consider the following 2-domestic algebra:



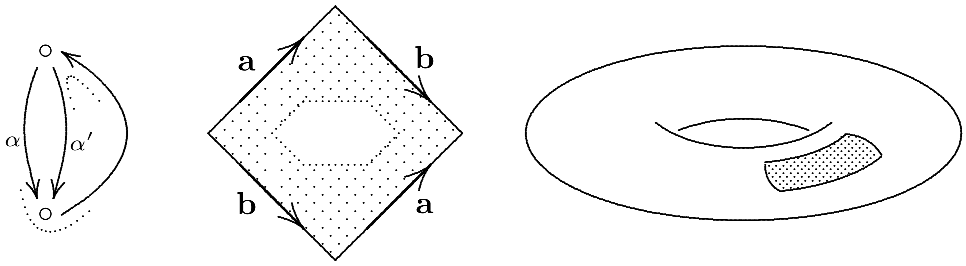
This algebra R has a an interesting Auslander-Reiten component, that containing the projective modules $P(b_1), P(b_2)$ and the injective modules $I(a_1), I(a_2)$. This component is a $\mathbb{Z}A_\infty^\infty$-component with a hole:



Using the sewing process, pairs of the sides of the square have to be ientified and we have indicated in which way: the two sides labeled **a** have to be identified, similarly, the two sides labeled **b** have to be identified, always in the direction of the arrows (but note that these arrows indicate the direction of the irreducible maps, thus no other identification would be possible). Clearly, in this way we obtain a torus with a hole. — We denote by K_a the Kronecker quiver with vertices a_1, a_2, by K_b that with vertices b_1, b_2. The preprojective component of K_a as well as a family of tubes of K_a-modules indexed by the affine line $\mathbb{A}^1$ remain components of Γ_R. Similarly, the preinjective component of K_b as well as a family of tubes of K_b-modules again indexed by the affine line $\mathbb{A}^1$ remain components of Γ_R. Let us exhibit the Auslander-Reiten quiver Γ_R. To the left, there are those components of the Kronecker quiver K_a with which remain components in Γ_R, to the right, those of K_b. In the middle, you see the torus[33]:



It is easy to see that such a torus and a similar Kleinian bottle occurs already for 1-domestic algebras: Let us present two typical examples, both being obtained from an $\widetilde{A}$-algebra by adding one bridge. Here is the first example: the algebra R', the punched plane with the labels $\mathbf{a}, \mathbf{b}$ for identification and the torus:

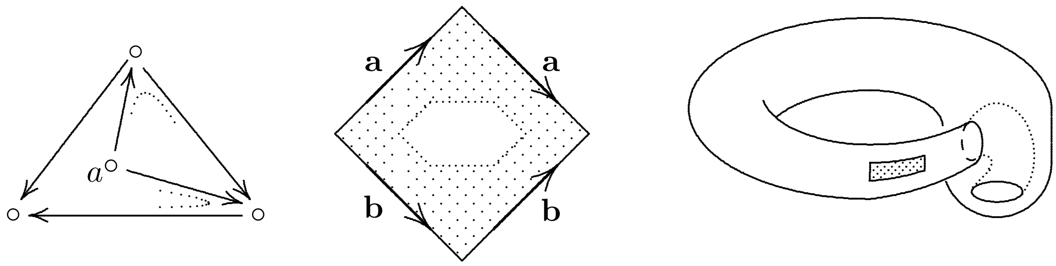


In addition to the torus, there is a one-parameter family of homogeneous tubes, indexed by $\mathbb{P}_1 k \setminus \{0, \infty\}$ (these are those components of the Kronecker quiver given by the arrows α, α' which remain components of $\Gamma_{R'}$). Actually, using covering

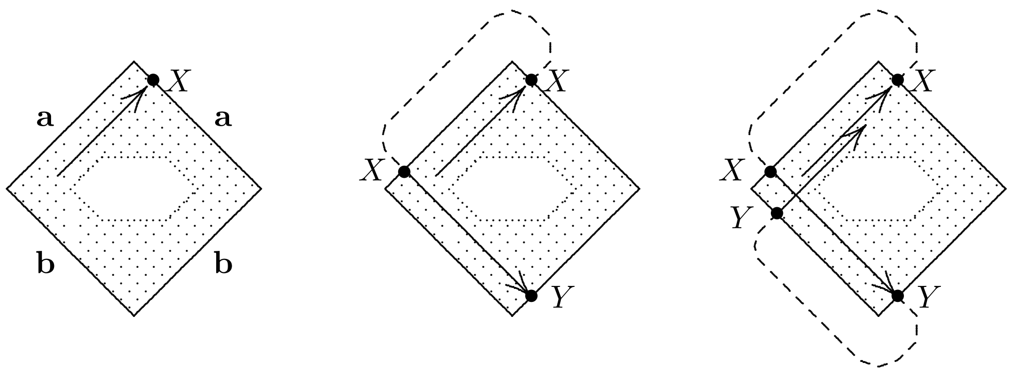
[33] The reader, or better here: the viewer, should be aware that this torus (as well as similar pictures which follow) is misleading in at least one respect: the torus concerns just a part of the given module category, here just one Auslander-Reiten component, but does not take into account other parts of the category. Indeed, the component $\mathcal{C}$ presented here contains the injective R-modules $I(a_1)$ and $I(a_2)$. Thus, for any non-zero R-module M whose support is not contained in K_b, there are non-zero homomorphisms f from M to modules in $\mathcal{C}$. In the same way, given a module M whose support is not contained in K_a, there are non-zero homomorphisms g from modules in $\mathcal{C}$ to M. But the picture exhibited here does not care about the nature of these maps, it does not even give a hint where such a map f arrives in the component or where a g will leave the component.

theory, one can reduce the study of the module category $\mathrm{mod}\, A'$ to the previous example $\mathrm{mod}\, A$.

Next, consider the second example, in this case the sewed component is a Kleinian bottle with a hole. As additional components, there is a preprojective component as well as a one-parameter family of homogeneous tubes, but here indexed by the affine line $\mathbb{A}^1 = \mathbb{P}_1 k \setminus \{\infty\}$. Again, to the left, we present the quiver with relations, in the middle the punched plane with the labels $\mathbf{a}, \mathbf{b}$ for the sewing process, to the right the Kleinian bottle which ones obtains in this way (see [HC]):



These quilts are convenient tools for the visualization of hammocks and for distinguishing the different behaviour of representable functors. The difference between the torus and the Kleinian bottle lies in the fact that the torus is orientable whereas the Kleinian bottle is not. To consider rays and corays on the torus amounts to consider foliations. What happens on the Kleinian bottle? Let us start with a ray, say in northeastern direction. We travel along the ray until we leave the component; thus we pass through an infinite dimensional module, say X, and return to the component, via first a coray then a ray, but now in southeastern direction. Again, we leave the component via an infinite-dimensional module, say Y, and return to a coray and a ray, in northeastern direction, parallel to the ray we have started with:



We hope that the sewing procedure of Auslander-Reiten components will help to get a better understanding of some hammocks, thus of some typical objects in diamond categories.

Let us reformulate the process of sewing together components and the way infinite length modules are used. There is given a ray of irreducible maps $X_0 \to$

$X_1 \to X_2 \to \cdots$, the direct limit $\varinjlim X_i$ being a string module for an $\mathbb{N}$-word, say the $\mathbb{N}$-word x. Second, there is given a coray of irreducible maps $\cdots \to X_2' \to X_1' \to X_0'$, and its inverse limit $\varprojlim X_i'$ is the product module corresponding to the same $\mathbb{N}$-word x; in particular, there is a canonical embedding $\iota\colon \varinjlim X_i \to \varprojlim X_i'$. Altogether, we deal with the following configuration of modules:

$$X_0 \to X_1 \to X_2 \to \cdots \quad \to \varinjlim X_i \xrightarrow{\iota} \varprojlim X_i' \to \quad \cdots \to X_2' \to X_1' \to X_0'.$$

Now, the direct limit module $\varinjlim X_i$ is always indecomposable, whereas the inverse limit module $\varprojlim X_i'$ always is algebraically compact, and as we have mentioned, one of the two modules (and only one) will be both indecomposable and algebraically compact. This is the module to be selected.

As the Auslander-Reiten quiver itself, also the Auslander-Reiten quilt is a purely combinatorial object which yields generators and relations for describing an additive category.

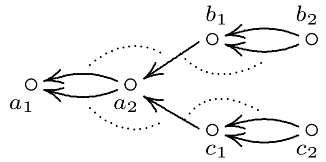
Unfortunately, the sewing procedure using rays and corays of irreducible maps can be applied only in very special cases, since usually there are not enough such rays and corays available. This is already the case for most of the tame hereditary algebras: the rays in the tubes produce all the Prüfer modules, their corays give rise to the adic modules; but for the adic modules, we also would need rays in the preprojective component, and for the Prüfer modules, we would need corays in the preinjective component.

Note that the Kronecker quiver $K(2)$ has enough rays and corays of irreducible maps, but the sewing of the preprojective component via the adic modules with the tubes, and dually the sewing of the tubes via the Prüfer modules with the preinjective component gives a rather involved and unintelligible picture.

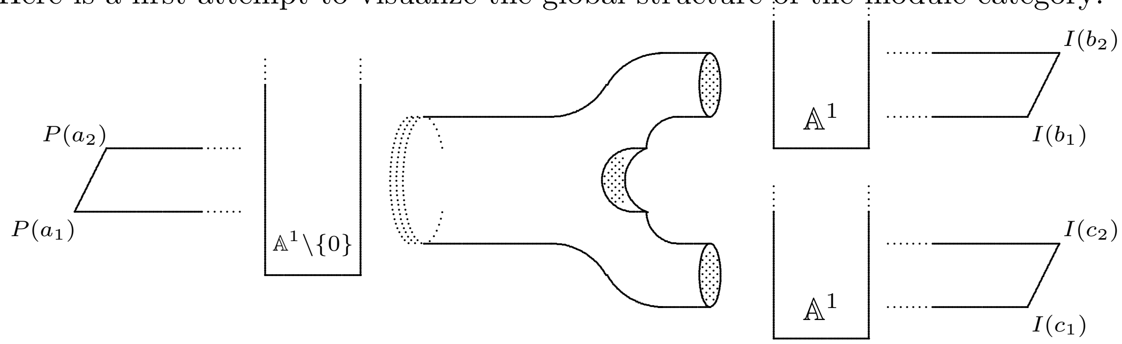
The algebras R of type $\widetilde{D}_n, \widetilde{E}_6, \widetilde{E}_7, \widetilde{E}_8$ have no rays in the preprojective component, and no corays in the preinjective component. Of course, in all these cases, the category $\operatorname{mod} R$ contains as a full subcategory the category $\operatorname{mod} K(2)$, thus we could use the preprojective rays and the preinjective corays in this subcategory, but this adds new difficulties. On the other hand, we should stress that the dissertation of Geigle [Gg] has to be named as first investigation dealing with the process of sewing of components, and he discusses precisely this complicated case: the tame hereditary algebras.

In the case of a tame hereditary algebra, the missing rays (and similarly, the missing corays) concern indecomposable modules which belong to one component. In general, we will have to deal with sequences $X_0 \to X_1 \to \cdots$ or $\cdots \to X_1 \to X_0$ where all the modules X_i may belong to pairwise different components. Usually, it should be hopeless to keep track of all sequences needed. But there are special cases of algebras where one obtains a quite satisfactory description of the module category.

As our last example, let us consider the 3-domestic algebra R with the following quiver and relations:

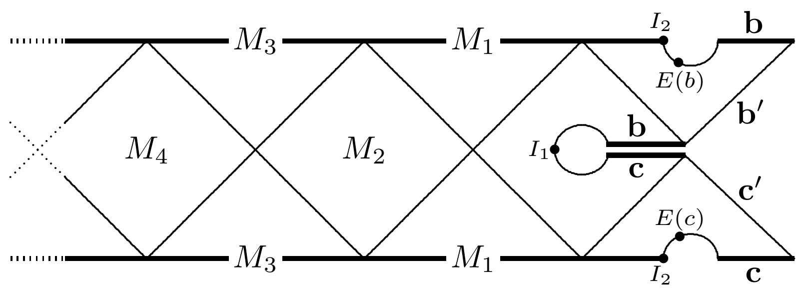


Here is a first attempt to visualize the global structure of the module category:



This picture exhibits all the Auslander-Reiten components, but taking into account the sewing procedure for $\mathbb{Z}A_\infty^\infty$ components as outlined above. To the left, there are those components of the Kronecker quiver K_a with vertices a_1, a_2 which remain components in Γ_R: there is the preprojective component of K_a as well as a family of tubes indexed by the punched affine line $\mathbb{A}^1 \setminus \{0\}$. Let K_b, K_c be the Kronecker quivers with vertices b_1, b_2 or c_1, c_2, respectively. To the right of the picture, there are those components of K_b and K_c, which remain components in Γ_R: the preinjective components as well as tubular families indexed by the affine line $\mathbb{A}^1$.

Of interest are the remaining components and there are countably many. There is one additional non-regular component $\mathcal{C}$, it contains the projective modules $P(b_1), P(b_2), P(c_1), P(c_2)$, and the injective modules $I_1 = I(a_1), I_2 = I(a_2)$. Its shape is obtained from the Riemann surface of the square root by cutting a central hole (in the same way as one of the components of the 1-domestic algebras discussed above). For later use, decompose $I_2/\operatorname{soc} I_2 = E(b) \oplus E(c)$, such that the support of $E(b)$ is K_b, that of $E(c)$ is K_c. For $i \geq 1$, we denote by M_i the preinjective K_a-module of dimension $2i-1$. It is easy to see that this is a Geiß module, thus we obtain countably many $\mathbb{Z}A_\infty^\infty$ components $\mathcal{C}_i$. The components $\mathcal{C}$ and $\mathcal{C}_i$ are all the remaining ones, they can be sewed together to a big connected quilt. What one obtains in this way are the “Chinese baby trousers” seen in the middle. Here is another view of the trousers, this time using three cuts:



one cut yields the solid line through the modules M_i with odd index i, the other two cuts are labelled $\mathbf{b}$ and $\mathbf{c}$. The position of the Geiß modules indicate the $\mathbb{Z}A_\infty^\infty$ components $\mathcal{C}_i$. On the right we see the non-regular component $\mathcal{C}$, there are three pieces which are sewed along two seams; we also provide the position of the injective modules I_1, I_2 and of their factor modules $E(b), E(c)$.

It remains to consider the boundary lines labelled $\mathbf{b}'$ and $\mathbf{c}'$ (the openings for the legs in the trousers). There are indeed infinite-dimensional indecomposable algebraically compact modules which live on these lines and as usually they are approached by rays of irreducible maps. The modules for the seam $\mathbf{c}'$ are given by $\mathbb{N}$-words and the unique biperiodic $\mathbb{Z}$-word ${}^\infty(a_1a_2)(c_1c_2)^\infty$ with support on K_a and K_c; those for the seam $\mathbf{b}'$ are similarly given by $\mathbb{N}$-words and the unique biperiodic $\mathbb{Z}$-word ${}^\infty(a_1a_2)(b_1b_2)^\infty$ with support on K_a and K_b. Since the support of the boundary line $\mathbf{b}'$ is related to K_b, we are tended to direct the corresponding leg of the trousers towards the K_b-components, and similarly, we have directed the leg $\mathbf{c}'$ towards the K_c-components, but as we will see this is questionable.

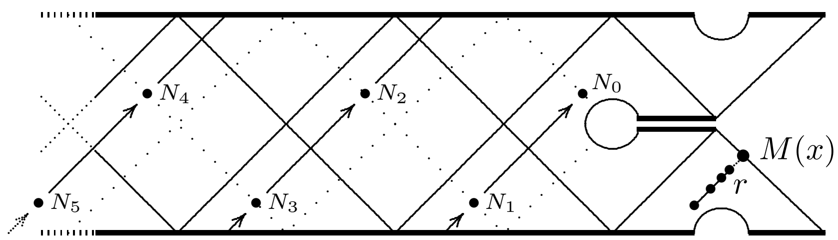
The modules occurring on the boundary lines $\mathbf{b}'$ and $\mathbf{c}'$ are approached by rays of irreducucible maps, but, in contrast to previous situations, not by corays of irreducible maps. For example, consider $x = {}^\infty(a_1a_2)(c_1c_2)^n$, this is (the inverse of) an $\mathbb{N}$-word lying on the boundary line $\mathbf{c}'$. As we have mentioned, the R-module $C(x) = M(x)$ is approached by a ray, namely it is the union of the sequence

$$M((c_1c_2)^n) \subset M(a_1a_2(c_1c_2)^n) \subset M((a_1a_2)^2(c_1c_2)^n) \subset \cdots$$

of inclusions and these inclusions are irreducible maps. On the other hand, we may consider the modules $N_i = M((a_2a_1)^i a_2(c_1c_2)^n)$ and the canonical surjective maps

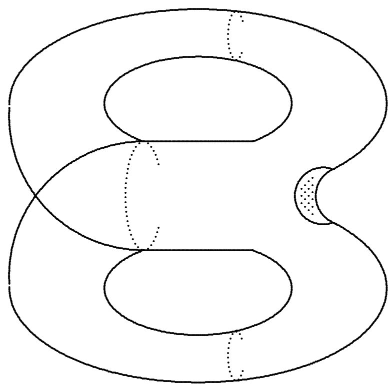
$$\cdots \to N_2 \to N_1 \to N_0.$$

However, these modules belong to pairwise different Auslander-Reiten components: the module N_0 belongs to $\mathcal{C}$, the module N_i to $\mathcal{C}_i$. But note that the maps can be controlled quite well: they are jumps from one component to a neighboring one and are compositions of a ray and a coray of irreducible maps; the corresponding arrows all point in northeastern direction. The following picture indicates a ray r approaching the module $M(x)$ as well as the modules N_i and the maps $N_{i+1} \to N_i$.



In the same way, we may approach any module on the boundary line $\mathbf{b}'$ by a sequence of modules in the various components $\mathcal{C}_i$, this time using sequences of arrows which point in southeastern direction.

The sequences of modules needed approach the waist of the trousers, thus we sew a second time, now joining the legs of the trousers with the waist:



Since we join both legs to the waist, the waist becomes a branching locus. We stress that also the second sewing procedure is achieved by infinite length modules which we have added.

Conclusion. When dealing with finite length modules, there are two essentially different ways in which infinite length modules or similar objects in related abelian categories play a decisive role: on the one hand, as objects which can be used in order to describe infinite families of R-modules (two typical ways: a Prüfer group incorporates all indecomposable p-groups as all its proper non-zero subgroups; Crawley-Boevey introduced the generic modules for tame algebras in order to parameterize the one-parameter families), on the other hand in order to describe in a module-theoretical language the behaviour of functors on $\operatorname{mod} R$. But these two ways turn out to be just two sides of one and the same coin, it is a challenging demand to descibe the correspondences.

Epilogue. A clear misunderstanding.

There are by now several books available which deal with questions in the representation theory of finite dimensional algebras, starting with Curtis and Reiner: *Representation Theory of Finite Groups and Associative Algebras (1962)* and its successor *Methods of Representation Theory I* (1981) and *II* (1987); then the volume 73 of the Encyclopaedia of Mathematical Sciences by Gabriel and Roiter, with the title *Representations of Finite-Dimensional Algebras* (1992) and finally the book *Representation Theory of Artin Algebras* (1995) by Auslander, Reiten and Smalø. All these treatises restrict their attention to **finite-dimensional** representations and take it for granted that a title which does not mention this specification will not be considered as misleading. We do not object that the books are confined to a specific class of representations with nice properties, namely the finite-dimensional ones, but we wonder why the authors do not see the necessity to mention this in the title. Actually, the orientation on these titles had the effect that some mathematicians now seem to distinguish between the *representation theory* and the *module theory* of finite-dimensional algebras, meaning that the first one concerns finite-dimensional representations, in contrast to the second. But, of course, this does not correspond to the usual use of the word representation theory, say when dealing with representations of Lie algebras or algebraic groups, where the study of infinite-dimensional representations is considered as an essential part of the theory. There seems to be the attitude that dealing with finite-dimensional algebras (or also with finite groups), the most natural representations are the finite-dimensional ones, but there should be real doubts! The belief that the natural representations of a finite-dimensional algebra are just the finite-dimensional representations has to be rated as very naive. In the same vein, the natural setting for considering finite Galois groups should be the realm of finite fields only.

The examples discussed up to now were motivated from within representation theory. Here, let us draw the attention to an application, the possible use of Kronecker modules in order to deal with operators on a vector space. The Kronecker modules are the representations of the Kronecker quiver $K(2)$, the corresponding path algebra $kK(2)$ is four dimensional, thus really a very small algebra. Looking at the algebra, one may be inclined to consider its indecomposable projective representations (their dimension is 1 and 3) or its indecomposable injective representations (again dimension 1 and 3) as the most natural ones. But is this justified? The Kronecker algebra $kK(2)$ is one of the tame hereditary algebras, thus the structure of the category $\operatorname{mod} kK(2)$ is as displayed above: besides the preprojective and the preinjective modules, there are the indecomposable regular modules, they belong to tubes, and these tubes are indexed by the projective line $\mathbb{P}_1(k)$. If we delete one of these tubes, the remaining regular representations form a category which is equivalent to the category $\operatorname{fin} k[T]$ of all finite-dimensional $k[T]$-modules, where $k[T]$ is the polynomial ring in one variable. As usual, the additional tube is indexed by the symbol ∞. Of course, $k[T]$-modules are nothing else than pairs (V,β), where V is a vector space over k and $\beta\colon V \to V$ a linear transformation (where one writes $T \cdot v$ instead of $\beta(v)$, for $v \in V$), or, as it is also called, a linear operator on V. The embedding

$$\iota\colon\ \operatorname{Mod} k[T] \longrightarrow \operatorname{Mod} kK(2)$$

is achieved by sending the pair (V,β) to the Kronecker module

$$V \underset{\beta}{\overset{1}{\leftleftarrows}} V$$

The image under this functor ι are all the representations of $K(2)$ for which the upper map α is an identity map; up to a categorical equivalence, these are all the representations of $K(2)$ for which α is an isomorphism. Of course, for this subcategory one tends to rate the module $\iota(k[T])$ as being a very natural representation. The indecomposable finite-dimensional $k[T]$-modules are (in case k is algebraically closed) of the form $k[T]/(T-\lambda)^n$, and we may assume that $\iota(k[T]/(T-\lambda)^n)$ belongs to the tube with index λ, thus the labeling of the tubes reflects the eigenvalue behaviour of the operator. The label ∞ refers to situations where the map α is not invertible. Of particular interest is the module $G = \iota(k(T))$, the only infinite-dimensional generic representation of $K(2)$. We may consider the one-dimensional projective representation $P(2)$ of $K(2)$ as a submodule of G (up to automorphisms of G there is just one such embedding), and $G/P(2)$ is just the direct sum of all possible Prüfer modules, each one occurring with multiplicity one.

What are typical operators? The vector spaces dealt with in functional analysis are usually function spaces, the operators are differential or integral operators. Let us look at a very elementary example: consider a space F of functions closed under differentiation, and let $\beta = \frac{d}{dT}$. If we take for F the space of all polynomial functions, the module we obtain is a very interesting one, it is the Prüfer module corresponding to the eigenvalue 0, provided we work with a field k of characteristic zero. For an example concerning the use of infinite-dimensional Kronecker modules in studying perturbations of differential operators we refer to [AB].

A more detailed look at the development of the representation theory of finite-dimensional algebras reveals an interest in arbitrary representations from its beginning, at least in some papers. For example, when dealing with serial finite-dimensional algebras, Nakayama and later again Eisenbud and Griffith were very proud to be able to decompose all the modules, not just the finite-dimensional ones, as a direct sum of indecomposables. There has been a parallel development in the representation theory of finite groups: until recently, the main emphasis was lying on finite-dimensional representations. New approaches which are based on the use of infinite-dimensional representations are outlined in the paper by Benson [Be].

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